Journal of Computational Mathematics
Vol.38, No.6, 2020, 879–902.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1906-m2018-0131

TWO-VARIABLE JACOBI POLYNOMIALS FOR SOLVING SOME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

Jafar Biazar[1)] and Khadijeh Sadri
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 41335-1914, Guilan, Rasht, Iran
E-mail: biazar@guilan.ac.ir, jafar.biazar@gmail.com, kh.sadri@uma.ac.ir

Abstract

Two-variable Jacobi polynomials, as a two-dimensional basis, are applied to solve a class of temporal fractional partial differential equations. The fractional derivative operators are in the Caputo sense. The operational matrices of the integration of integer and fractional orders are presented. Using these matrices together with the Tau Jacobi method converts the main problem into the corresponding system of algebraic equations. An error bound is obtained in a two-dimensional Jacobi-weighted Sobolev space. Finally, the efficiency of the proposed method is demonstrated by implementing the algorithm to several illustrative examples. Results will be compared with those obtained from some existing methods.

Mathematics subject classification: 35R11, 65M15, 65M70.
Key words: Fractional partial differential equation, Two-variable Jacobi polynomials, Caputo derivative, Error bound.

1. Introduction

Fractional partial differential equations (FPDEs) are used as modeling tools of various phenomena in different branches of science. For example, diffusive processes associated with subdiffusion (fractional in time), super-diffusion (fractional in space), or both, advection-diffusion, and convection-diffusion processes can be modeled by FPDEs [1–5]. The advantage of these equations in compared to integer-order partial differential equations is the ability of natural simulation of physical processes and dynamical systems more accurately [6]. For instance, some phenomena in fluid and continuum mechanics [7], viscoplastic and viscoelastic flows [8], biology, and acoustics [9], describing chemical and pollute transport in heterogeneous aquifers [10–12], pricing mechanisms and heavy stochastic processes in finance [13], and describing convection process of liquid in medium [14]. Therefore, it helps mathematicians and engineers in the better understanding of the nature and behavior of physical phenomena. For this reason, FPDEs are increasingly studied, but their analytic solving is difficult. Hence, mathematicians have been attracted to solve this class of equations numerically. For example, in [14], the normalized and rational Bernstein polynomials are applied to solve a kind of time-space fractional diffusive equation. The finite difference method is used to solve the fractional reaction-subdiffusion equation in [15]. Authors in [16] propose a wavelet method to solve a class of fractional convection-diffusion equation with variable coefficients. Chen and et al. use generalized fractional-order Legendre functions to obtain numerical solutions of FPDEs with variable coefficients [17]. Ding

* Received June 25, 2018 / Revised version received February 21, 2019 / Accepted June 20, 2019 / Published online July 29, 2019 /
1) Corresponding author

introduces a general Pade approximation method for time-space fractional diffusive equations in [18]. Also, Heydari and et al. apply the Legendre wavelet method for solving the time fractional diffusion-wave equation [19]. In [20], a two-dimensional wavelets collocation method uses to solve electromagnetic waves in dielectric media.

In this paper, an operational Tau method, based on two-variable Jacobi polynomials (TVJP-s), is proposed to deal with a class of FPDEs which involves equations such as diffusion and advection-diffusion equations. The derivative operators appeared in these equations are in the Caputo sense. First, the TVJPs, on the domain $\Omega = [0,1] \times [0,1]$, are obtained as a generalization of the classic one-variable Jacobi polynomials (OVJPs) on the interval $\Omega_0 = [0,1]$. A given continuous function $u(x,t)$, defined on Ω, can be approximated in terms of the two-variable presented basis. In order to approximate the terms including the derivative operators in the equation under study, the operational matrices of the integration of fractional and integer orders are derived for the one-variable Jacobi basis, then the resultant matrices are applied to construct the two-dimensional integral operational matrices for both two independent variables x and t. Applying these matrices together with the Tau method leads to reduce the given equation to the corresponding system of the algebraic equations which is a Sylvester equation. Solving the resulting system leads to determine the vector of unknown coefficients, therefore, an approximate solution is obtained. Also, the convergence of the proposed approach is investigated in a two-dimensional Jacobi-weighted Sobolev space and an error bound is computed for an approximate solution. Finally, the suggested algorithm is implemented to several illustrative examples.

The outline of the paper is as follows: Section 2 gives some elementary definitions and concepts of the fractional calculus. In Section 3, the TVJPs are constructed with help of the OVJPs. The integral operational matrices of fractional and integer orders are derived in Section 4, which are used to construct the operational matrices corresponding to the fractional partial derivative operators. In Section 5, an error bound is given in a two-dimensional Sobolev space. The applicability and efficiency of the proposed approach are demonstrated by implementing the method on several illustrative examples in Section 6. Finally, a conclusion is presented in Section 7.

2. Elementary Definitions of Fractional Calculus

The two most used fractional operators are the Caputo derivative and the Riemann-Liouville integral operators.

Definition 2.1. *If $\gamma \in \mathbb{R}$ and $n = \lceil \gamma \rceil$, the Caputo derivative operator is defined as,*

$$\begin{aligned} D^{\gamma} u(t) &= \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1}\ u(s)\ ds, \qquad t \in \Omega_0, \\ D^0 u(t) &= u(t). \end{aligned} \tag{2.1}$$

Definition 2.2. *If $\nu \in \mathbb{R}$, the Riemann-Liouville integral operator is defined as,*

$$\begin{aligned} J^{\nu} u(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\ ds, \qquad t \in \Omega_0, \\ J^0 u(t) &= u(t). \end{aligned} \tag{2.2}$$