Journal of Computational Mathematics
Vol.38, No.6, 2020, 827–838.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1905-m2018-0256

AN EXTENDED BLOCK RESTRICTED ISOMETRY PROPERTY FOR SPARSE RECOVERY WITH NON-GAUSSIAN NOISE*

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Abstract

We study the recovery conditions of weighted mixed ℓ_2/ℓ_p minimization for block sparse signal reconstruction from compressed measurements when partial block support information is available. We show theoretically that the extended block restricted isometry property can ensure robust recovery when the data fidelity constraint is expressed in terms of an ℓ_q norm of the residual error, thus establishing a setting wherein we are not restricted to Gaussian measurement noise. We illustrate the results with a series of numerical experiments.

Mathematics subject classification: 94A12, 94A20.
Key words: Compressed sensing, Block sparsity, Partial support information, Signal reconstruction, Convex optimization.

1. Introduction

Recovering an unknown signal from significantly fewer measurements is a fundamental aspect in computational sciences today. The key ingredient here is the sparsity of the unknown signal – a realisation that has led to the theory of compressed sensing (CS) [1–3] through which successful recovery of high dimensional (approximately) sparse signals is now possible at a rate significantly lower than the Nyquist sampling rate. This allows an unknown signal $x \in \mathbb{R}^N$ to be successfully recovered via $y = Ax + e \in \mathbb{R}^m, m \ll N$, if x is (approximately) sparse in some transform domain, and the noise e satisfies $\|e\|_2 \leq \epsilon$, for $\epsilon > 0$. In short, recovery is possible if the measurement matrix $A \in \mathbb{R}^{m\times N}$ satisfies the restricted isometry property (RIP): $(1-\delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_k)\|x\|_2^2$ for any k-sparse x and some $\delta_k \in [0,1]$ [1]. Under such conditions, stable and robust recovery is guaranteed via the ℓ_1 minimization

$$\min_z \|z\|_1 \quad \text{s.t.} \quad \|y - Az\|_2 \leq \epsilon. \tag{1.1}$$

The question of how few measurements one might use was answered when it was shown that Gaussian random matrices satisfy the RIP with high probability, provided that $m \geq Ck\log(eN/$

* Received November 19, 2018 / Revised version received March 3, 2019 / Accepted May 13, 2019 / Published online September 17, 2019 /
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k), for some $C > 0$ [4]. Today, an interesting challenge lies in customizing the recovery process to take into account prior knowledge about e.g. signal structure and properties of noise present. However, so far no unified framework has been proposed for this purpose - something we aim to do in this paper.

In addition to pure sparsity, nonzero signal components may appear in clustered regions, either naturally or as a result of some sparsifying transformation. These 'blocks' occur in many real worlds scenarios such as genetics and image processing [5–7]. Incorporating the block structure into a CS recovery algorithm provides some immediate benefits in terms of reduction of the number of required measurements for stable recovery, and a more robust recovery via better differentiation of recovery artifacts [8].

Let $x[i]$ define the ith block of a vector $x \in \mathbb{R}^N$ over the block index set $\mathcal{I} = \{d_1, \ldots, d_n\}$ such that $N = \sum_{i=1}^n d_i$, and let the blocks be formed sequentially with length d_i of block i

$$x = (\underbrace{x_1 \cdots x_{d_1}}_{x^T[1]} \underbrace{x_{d_1+1} \cdots x_{d_1+d_2}}_{x^T[2]} \cdots \underbrace{x_{N-d_n+1} \cdots x_N}_{x^T[n]})^T. \tag{1.2}$$

We define a signal $x \in \mathbb{R}^N$ as block k-sparse over $\mathcal{I}$ if $x[i]$ is nonzero for at most k indices i, i.e., if $\|x\|_{0,\mathcal{I}} \le k$, where $\|x\|_{0,\mathcal{I}} = \sum_{i=1}^n \mathbb{I}(\|x[i]\|_2 > 0)$. The block structure of the unknown signal can be incorporated into the recovery process via a mixed minimization scheme using e.g. the ℓ_2/ℓ_1 norm [9], or its nonconvex generalization, the ℓ_2/ℓ_p norm $(0 < p \le 1)$ [5, 6], where the mixed ℓ_2/ℓ_p norm is defined as $\|x\|_{2,p} = (\sum_{i=1}^n \|x[i]\|_2^p)^{1/p}$. The sufficient condition for the existence of an exact solution to (1.1) provided by the RIP has been generalized into the block sparse setting, thus guaranteeing exact and robust recovery of block sparse signals via both mixed ℓ_2/ℓ_1 and ℓ_2/ℓ_p minimization [5, 9].

It may furthermore be possible to draw an estimate of the support of the largest block components of a signal, e.g., when working with recursive reconstruction of time sequences of sparse spatial signals where support estimates of previous instances can be used to estimate the present ones [10–12]. Given a support estimate $\tilde{T} \subset \{1, \ldots, N\}$, one can incorporate the prior support information via a weighted minimization approach with weights $\omega_i = \omega \in [0, 1]$ when $i \in \tilde{T}$ and $\omega_i = 1$ otherwise [10].

From a Bayesian point of view, the ℓ_2 fidelity constraint in (1.1) corresponds to a conditional loglikelihood associated with Gaussian white noise. The measurement noise might however not be Gaussian. This motivates an extension of the existing CS theory to one with a data fidelity constraint expressed in the ℓ_q norm of the residual error. The case with k-sparse signals has been studied in [4, 13, 14] for $q \ge 2$ and in [15] for $0 \le q < 2$. A sufficient condition for sparse recovery from noisy measurements with non-Gaussian noise is given by an extension of the RIP [13], wherein the measurement matrix $A \in \mathbb{R}^{m \times N}$ is said to satisfy the extended restricted isometry property ($\text{RIP}_{q,2}$) of order k if there exists a $\delta_k \in (0, 1)$ such that $\mu_{q,2}^2(1 - \delta_k)\|x\|_2^2 \le \|Ax\|_q^2 \le \mu_{q,2}^2(1 + \delta_k)\|x\|_2^2$, for any k-sparse vector x and some $\mu_{q,2} > 0$. A natural question is whether one can find an optimal ℓ_q constraint for specific noise types. We expand the existing results to investigate possible q-optimality for block-sparse signals with partially known block support.

Consider an arbitrary signal $x \in \mathbb{R}^N$, defined as (1.2), with x^k as its best block k-sparse approximation. Let T_0 be the block support of x^k, where $T_0 \subset \{1, \ldots, n\}$ and $|T_0| \le k$. Let $\tilde{T} \subset \{1, \ldots, n\}$ be the block support estimate, where $|\tilde{T}| = \rho k$ and $0 \le \rho \le a$ for some $a > 1$ and $|\tilde{T} \cap T_0| = \alpha\rho k$ (for interpretation of ρ and α see [10]). We define the weighted mixed ℓ_2/ℓ_p