

On Function Spaces with Mixed Norms — a Survey

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Abstract. The targets of this article are threefold. The first one is to give a survey on the recent developments of function spaces with mixed norms, including mixed Lebesgue spaces, iterated weak Lebesgue spaces, weak mixed-norm Lebesgue spaces and mixed Morrey spaces as well as anisotropic mixed-norm Hardy spaces. The second one is to provide a detailed proof for a useful inequality about mixed Lebesgue norms and the Hardy–Littlewood maximal operator and also to improve some known results on the maximal function characterizations of anisotropic mixed-norm Hardy spaces and the boundedness of Calderón–Zygmund operators from these anisotropic mixed-norm Hardy spaces to themselves or to mixed Lebesgue spaces. The last one is to correct some errors and seal some gaps existing in the known articles.

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1 Introduction

In 1961, the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$, with $\vec{p} \in (0, \infty]^n$, as a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$ via replacing the constant exponent p by an exponent vector \vec{p} , was investigated by Benedek and Panzone [9]. Indeed, the origin of these mixed Lebesgue spaces can be traced back to the interesting article of Hörmander [44] on the estimates for translation invariant operators, in 1960. Later on, in 1965, Galmarino and Panzone [35] extended the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ to the mixed Lebesgue space $L^P(\mathbb{R}^n)$ with the exponent P being a sequence, namely, an ∞ -tuple. Since the early 1960s, lots of nice work have been done in the study of the boundedness of operators on mixed norm spaces; see, for instance, Benedek *et al.* [8], Lizorkin [64], Adams and Bagby [1], Schmeisser [74], Rubio de Francia *et al.* [73] and Fernandez [33] as well as Stefanov and

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Torres [76]. Recently, the Plancherel–Polya inequality on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and the wavelet characterization of $L^{\vec{p}}(\mathbb{R}^n)$ were studied by Torres and Ward [84]; the smoothing properties of bilinear operators and Leibniz-type rules in mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ were considered by Hart *et al.* [41]; the boundedness of the multilinear strong maximal operator from the product of mixed Lebesgue spaces to mixed Lebesgue spaces was obtained by Liu *et al.* [61]. In addition, more recently, Córdoba and Latorre Crespo in [26] revisited some classical conjectures in harmonic analysis in the setting of mixed norm spaces. To be exact, they established the sharp boundedness for the restriction of the Fourier transform to compact hypersurfaces of revolution and studied an extension of the disc multiplier in the mixed norm setting. For more progresses about the mixed Lebesgue space, we refer the reader to [3, 4, 19, 20, 27, 43, 48, 49, 65, 75].

On another hand, motivated by the aforementioned work of Benedek and Panzone [9] on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, numerous other function spaces with mixed norms were introduced and studied. For instance, Besov spaces, Sobolev spaces and Bessel potential spaces with mixed norms were investigated by Besov *et al.* [11, 12] in the 1970s; inhomogeneous Triebel–Lizorkin spaces with mixed norms were also studied by Besov *et al.* in [13]; parabolic function spaces with mixed norms were considered by Gopala Rao [39]. Particularly, in 1977, Fernandez [32] first introduced the Lorentz spaces with mixed norms. Later, an interpolation result on these Lorentz spaces with mixed norms was obtained by Milman [69]. Moreover, Lorentz–Marcinkiewicz spaces with mixed norms and Orlicz spaces with mixed norms were considered by Milman in [67] and [68], respectively; Banach function spaces with mixed norms were studied by Blozinski [14]; anisotropic mixed-norm Hardy spaces were introduced by Clenathous *et al.* [23]; mixed-norm α -modulation spaces were researched by Cleanthous and Georgiadis [21]; mixed Lebesgue spaces with variable exponents were considered by Ho [43]; Morrey spaces with mixed norms were investigated by Nogayama [71, 72]; mixed martingale Hardy spaces were studied by Szarvas and Weisz [81]. Indeed, the function spaces with mixed norms have attracted considerable attention and have rapidly been developed. For more progresses about various function spaces with mixed norms and their applications in the boundedness of different operators, we refer the reader to [19, 22, 24, 25, 36, 37, 41, 42, 51–54].

In the last two decades, due to the wider usefulness of function spaces with mixed norms within the context of partial differential equations, there has been a renewed interest in the study of them. More precisely, since the function spaces with mixed norms have finer structures than the corresponding classical function spaces, they naturally arise in the studies on the solutions of partial differential equations used to model physical processes involving in both space and time variables, such as the heat or the wave equations (particularly, the very useful Strichartz estimates); see, for instance, [3, 56–58, 83]. This is also based on the fact that, while treating some linear or nonlinear equations, functions with different orders of integrability in different variables give more precise information on the parameters involved in the estimates and further induce a better regularity (of traces) of solutions; see, for instance, [28, 38, 85]. Another recent interest in developing the theory of function spaces with mixed norms comes from bilinear estimates and their

vector-valued extensions which have proved useful in partial differential equations involving functions in n dimension space variable x and one dimension time variable t ; see, for instance, [6,7,34,41,84]. In particular, in order to obtain the smoothing properties of bilinear operators and Leibniz-type rules in mixed Lebesgue spaces, Hart *et al.* [41] introduced the mixed-norm Hardy space $H^{p,q}(\mathbb{R}^{n+1})$ with $p, q \in (0, \infty)$ via the Littlewood–Paley g -function. As was mentioned in their article [41, p. 8586], the space $H^{p,q}(\mathbb{R}^{n+1})$ plays an important role in overcoming the difficulty caused by full derivatives both in the space variable x and the time variable t in the mixed Lebesgue spaces. For more progresses about the applications of function spaces with mixed norms in partial differential equations, we refer the reader to [17, 58, 59, 66].

The purposes of this article are threefold. The first one is to give a survey on the recent developments of function spaces with mixed norms, including mixed Lebesgue spaces, iterated weak Lebesgue spaces, weak mixed-norm Lebesgue spaces and mixed Morrey spaces as well as anisotropic mixed-norm Hardy spaces. To be precise, the main results that we review include: the interpolation theorems, the dual inequality of Stein type, the Fefferman–Stein vector-valued inequality as well as the boundedness of fractional integral operators and geometric inequalities on these three kinds of (weak) Lebesgue spaces with mixed norms, the boundedness of maximal operators, Calderón–Zygmund operators and fractional integral operators on mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$, a necessary and sufficient condition for the boundedness of the commutators of fractional integral operators on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$, various real-variable characterizations of the anisotropic mixed-norm Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, with $\vec{a} \in [1, \infty)^n$ and $\vec{p} \in (0, \infty)^n$, and their dual spaces as well as applications to the boundedness of Calderón–Zygmund operators. The second purpose is to provide a detailed proof for an extended inequality on the central Hardy–Littlewood maximal operator on mixed Lebesgue norms stated by Bagby in [5] but without giving a proof and also improve some known results on the maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ given in [23, Theorem 3.1] and the boundedness of Calderón–Zygmund operators in [45, Theorems 6.8 and 6.9]. The last purpose is to correct some errors and seal some gaps existing in the proof of the Lusin area function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, namely, the proof of the sufficiency of [45, Theorem 4.1].

The organization of this survey is as follows.

In Section 2, we first recall the notions of the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in (0, \infty]^n$, the iterated weak Lebesgue space

$$L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m) \quad \text{with } q_1, q_2 \in (0, \infty),$$

the weak mixed-norm Lebesgue space $L^{\vec{q}, \infty}(\mathbb{R}^{2n})$ with $\vec{q} \in (0, \infty]^2$, and their basic properties which include the completeness, the corresponding Hölder inequalities and interpolation theorems. Then we present an extended very useful inequality about mixed Lebesgue norms and the central Hardy–Littlewood maximal operator, which was stated by Bagby in [5] but without giving a proof. We provide a detailed proof of this important inequality in Subsection 2.2 below. Finally, we give a survey on applications of these

Lebesgue spaces with mixed norms, which include the dual inequality of Stein type and Fefferman–Stein vector-valued inequality on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ proved by Nogayama in [71] as well as the boundedness of fractional integral operators and geometric inequalities on iterated weak Lebesgue spaces $L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})$ and weak mixed-norm Lebesgue spaces $L^{\vec{q}, \infty}(\mathbb{R}^{2n})$ obtained by Chen and Sun in [19].

The aim of Section 3 is the summarization of mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$, with $\vec{q} \in (0, \infty)^n$ and $p \in (0, \infty]$, and their applications to the boundedness of operators. To this end, we first recall the notion and some examples of mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ from Nogayama [71] and then we further show some basic properties about these spaces, including the completeness as well as the embedding theorem, obtained in the same aforementioned article. Moreover, the behaviors of the Hardy–Littlewood maximal operator, the iterated maximal operator, Calderón–Zygmund operators and fractional integral operators on mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ are discussed. At the end of Section 3, we review a necessary and sufficient condition for the boundedness of commutators of fractional integral operators I_α with $\alpha \in (0, n)$ on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ established by Nogayama in [72].

In Section 4, via recalling the notions of anisotropic quasi-homogeneous norms in [10, 29] (see also [79]) and anisotropic non-tangential grand maximal functions in [23], we first give the definition of anisotropic mixed-norm Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ appearing in [23] (see also [45]), where $\vec{a} \in [1, \infty)^n$ and $\vec{p} \in (0, \infty)^n$, and present some basic facts about them. Then various real-variable characterizations of the spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, respectively, in terms of various maximal functions, atoms, finite atoms and the Lusin area function as well as the Littlewood–Paley g -function or the Littlewood–Paley g_λ^* -function, established in [23, 45], are displayed. Furthermore, as the applications of these various real-variable characterizations, the dual spaces of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, a criterion on the boundedness of sublinear operators from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ into a quasi-Banach space and the boundedness of anisotropic convolutional δ -type and non-convolutional β -order Calderón–Zygmund operators from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ to itself [or to $L^{\vec{p}}(\mathbb{R}^n)$], obtained in [45], are presented. Some errors and gaps existing in the proof of the sufficiency of [45, Theorem 4.1] are also corrected and sealed in Subsection 4.2.1. In addition, by providing a new proof, we improve the maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ given in [23, Theorem 3.1]. The revised versions of [45, Theorems 6.8 and 6.9] on the boundedness of anisotropic β -order Calderón–Zygmund operators from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ to itself [or to $L^{\vec{p}}(\mathbb{R}^n)$] are also obtained.

Finally, we make some conventions on notation. We always let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\vec{0}_n$ be the origin of \mathbb{R}^n . For any multi-index $\alpha := (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}$. We denote by C a positive constant which is independent of the main parameters, but may vary from line to line. The notation $f \lesssim g$ means $f \leq Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. We also use the following convention: If $f \leq Cg$ and $g = h$ or $g \leq h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. For any $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. Moreover, if

$\vec{p} := (p_1, \dots, p_n) \in [1, \infty]^n$, we denote by $\vec{p}' := (p'_1, \dots, p'_n)$ its *conjugate index*, namely, for any $i \in \{1, \dots, n\}$, $\frac{1}{p_i} + \frac{1}{p'_i} = 1$. In addition, for any set $E \subset \mathbb{R}^n$, we denote by E^c the set $\mathbb{R}^n \setminus E$, by $\mathbf{1}_E$ its *characteristic function* and $|E|$ its *n-dimensional Lebesgue measure*. For any $\ell \in \mathbb{R}$, we denote by $\lfloor \ell \rfloor$ (resp., $\lceil \ell \rceil$) the *largest* (resp., *least*) *integer not greater* (resp., *less*) *than* ℓ . In what follows, the symbol $C^\infty(\mathbb{R}^n)$ denotes the set of all *infinitely differentiable functions* on \mathbb{R}^n .

2 (Weak) Lebesgue spaces with mixed norms

The aims of this section are twofold. The first one is devoted to summarizing three kinds of (weak) Lebesgue spaces with mixed norms, which include the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in (0, \infty]^n$, the iterated weak Lebesgue space $L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$ with $q_1, q_2 \in (0, \infty)$, and the weak mixed-norm Lebesgue space $L^{\vec{q}, \infty}(\mathbb{R}^{2n})$ with $\vec{q} \in (0, \infty]^2$ (see Subsection 2.1 below), then we further review some properties of these spaces as well as their applications to the dual inequality of Stein type, the Fefferman–Stein vector-valued inequality, the boundedness of fractional integrals and geometric inequalities (see Subsection 2.3 below). The second one is to recall an extended inequality about mixed Lebesgue norms and the central Hardy–Littlewood maximal operator, and further to provide a detailed proof for it (see Subsection 2.2 below).

2.1 (Weak) Lebesgue spaces with mixed norms

In this subsection, we first recall the definitions of three kinds of (weak) Lebesgue spaces with mixed norms, and then display some basic properties of them, including the completeness, the corresponding Hölder inequalities (see Subsection 2.1.1 below) as well as interpolation theorems (see Subsection 2.1.2 below). To this end, we first present the notion of mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in (0, \infty]^n$, which was originally introduced in [9].

Definition 2.1. Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$. The *mixed Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that their quasi-norms

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \cdots dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \dots, n\}$.

Remark 2.1. Obviously, when $\vec{p} := (\overbrace{p, \dots, p}^{n \text{ times}})$ with some $p \in (0, \infty]$, the space $L^{\vec{p}}(\mathbb{R}^n)$ coincides with the classical Lebesgue space $L^p(\mathbb{R}^n)$ and, in this case, they have the same quasi-norms.

If $\vec{p} \in [1, \infty]^n$, Benedek and Panzone proved that the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is complete in [9, p. 304, Theorem 1.b)], which is stated as follows.

Theorem 2.1. *Let $\vec{p} \in [1, \infty]^n$, then $(L^{\vec{p}}(\mathbb{R}^n), \|\cdot\|_{L^{\vec{p}}(\mathbb{R}^n)})$ becomes a Banach space.*

Moreover, it was also shown by Benedek and Panzone in [9, p. 304, Theorem 1.a)] that the dual space of $L^{\vec{p}}(\mathbb{R}^n)$ with any given $\vec{p} \in [1, \infty]$ is $L^{\vec{p}'}(\mathbb{R}^n)$ as follows.

Theorem 2.2. *Let $\vec{p} \in [1, \infty]^n$ and \vec{p}' denote the conjugate exponent of \vec{p} , namely, for any $i \in \{1, \dots, n\}$, $1/p_i + 1/p'_i = 1$. Then the dual space of $L^{\vec{p}}(\mathbb{R}^n)$, denoted by $(L^{\vec{p}}(\mathbb{R}^n))^*$, is $L^{\vec{p}'}(\mathbb{R}^n)$ in the following sense: J is a continuous linear functional on $L^{\vec{p}}(\mathbb{R}^n)$ if and only if there exists a uniquely $h \in L^{\vec{p}'}(\mathbb{R}^n)$ such that, for any $f \in L^{\vec{p}}(\mathbb{R}^n)$,*

$$J(f) = \int_{\mathbb{R}^n} f(x)h(x)dx$$

and $\|J\|_{(L^{\vec{p}}(\mathbb{R}^n))^*} = \|h\|_{L^{\vec{p}'}(\mathbb{R}^n)}$.

Now, we recall the notions of iterated weak Lebesgue spaces and weak mixed-norm Lebesgue spaces given in [19, Definition 1.1].

Definition 2.2. Let $\vec{p} := (p_1, p_2) \in (0, \infty]^2$. The iterated weak Lebesgue space $L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$ with $n, m \in \mathbb{N}$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)} := \sup_{\beta \in (0, \infty)} \beta \left| \left\{ y \in \mathbb{R}^m : \sup_{\alpha \in (0, \infty)} \alpha |\{x \in \mathbb{R}^n : |f(x, y)| > \alpha\}|^{1/p_1} > \beta \right\} \right|^{1/p_2} < \infty$$

and the weak mixed-norm Lebesgue space $L^{\vec{p}, \infty}(\mathbb{R}^n)$ is defined to be the set of all measurable functions g such that

$$\|g\|_{L^{\vec{p}, \infty}(\mathbb{R}^n)} := \sup_{\alpha \in (0, \infty)} \alpha \left\| \mathbf{1}_{\{x \in \mathbb{R}^n : |g(x)| > \alpha\}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty.$$

The completeness of iterated weak Lebesgue spaces and weak mixed-norm Lebesgue spaces was proved in [19, Theorem 2.10] as follows.

Theorem 2.3. *Let $\vec{p} := (p_1, p_2) \in (0, \infty]^2$. Then the weak spaces $L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$ and $L^{\vec{p}, \infty}(\mathbb{R}^n)$ are both complete.*

The iterated weak Lebesgue space and the weak mixed-norm Lebesgue space are two different spaces and one of them can not cover each other, which was obtained in [19, Theorem 2.2].

Theorem 2.4. *Let $\vec{p} := (p_1, p_2) \in (0, \infty)^2$. Then, for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, one has*

- (i) $L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)$.
- (ii) $L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$.
- (iii) $L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{p_1, \infty}(L^{p_2, \infty})(\mathbb{R}^m \times \mathbb{R}^n)$.
- (iv) $L^{p_1, \infty}(L^{p_2, \infty})(\mathbb{R}^m \times \mathbb{R}^n) \not\subseteq L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$.
- (v) $L^{\vec{p}}(\mathbb{R}^n \times \mathbb{R}^m) \subsetneq L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m) \cap L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$.

For any given $\vec{p} := (p_1, p_2) \in (0, \infty)^2$, the space $L^{p_2, \infty}(L^{p_1})(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{p_2, \infty}(L^{p_1})(\mathbb{R}^n \times \mathbb{R}^m)} := \sup_{\alpha \in (0, \infty)} \alpha \left| \left\{ y \in \mathbb{R}^m : \|f(\cdot, y)\|_{L^{p_1}(\mathbb{R}^n)} > \alpha \right\} \right|^{1/p_2} < \infty$$

and the space $L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the set of all measurable functions g such that

$$\|g\|_{L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)} := \left\| \sup_{\alpha \in (0, \infty)} \alpha \left| \left\{ x \in \mathbb{R}^n : |g(x, \cdot)| > \alpha \right\} \right|^{1/p_1} \right\|_{L^{p_2}(\mathbb{R}^m)} < \infty.$$

The relations among these three mixed norms of $L^{p_2, \infty}(L^{p_1})(\mathbb{R}^n \times \mathbb{R}^m)$, $L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)$ and $L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)$ are as follows, which is just [19, Theorem 2.4].

Theorem 2.5. Let $\vec{p} := (p_1, p_2) \in (0, \infty)^2$. Then

- (i) $L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m) \subset L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)$ and, for any measurable function f defined on $\mathbb{R}^n \times \mathbb{R}^m$,

$$\|f\|_{L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}.$$

- (ii) $L^{p_2, \infty}(L^{p_1})(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)$ and $L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{p_2, \infty}(L^{p_1})(\mathbb{R}^n \times \mathbb{R}^m)$.

2.1.1 Hölder inequalities

It is known that the Hölder inequality holds true for classical Lebesgue spaces and weak Lebesgue spaces. We now discuss the Hölder inequality on the above three kinds of (weak) Lebesgue spaces. First, for mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, we have the following conclusion, which was obtained by Benedek and Panzone in [9, (1)].

Theorem 2.6. Let $\vec{p} \in [1, \infty]^n$. Then, for any measurable function f and g , one has

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^{\vec{p}}(\mathbb{R}^n)} \|g\|_{L^{\vec{p}' }(\mathbb{R}^n)},$$

where \vec{p}' denotes the conjugate vector of \vec{p} , namely, for any $i \in \{1, \dots, n\}$, $1/p_i + 1/p'_i = 1$.

Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$, $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$ and $\vec{r} := (r_1, \dots, r_n) \in (0, \infty]^n$ satisfy that, for any $i \in \{1, \dots, n\}$,

$$\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}.$$

Then, via the inequality in Theorem 2.6, we find that, for any measurable functions f and g ,

$$\|fg\|_{L^{\vec{r}}(\mathbb{R}^n)} \leq \|f\|_{L^{\vec{p}}(\mathbb{R}^n)} \|g\|_{L^{\vec{q}}(\mathbb{R}^n)}. \quad (2.1)$$

Recall that, applying the Hölder inequality for classical weak Lebesgue spaces, the corresponding extended inequality (2.1) for iterated weak Lebesgue spaces was shown by Chen and Sun in [19, Theorem 2.16].

Theorem 2.7. Let $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2)$ and $\vec{r} := (r_1, r_2)$ satisfy that, for any $i \in \{1, 2\}$, $p_i, q_i, r_i \in (0, \infty]$ and

$$\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}.$$

Then there exists a positive constant C , depending on \vec{p} , \vec{q} and \vec{r} , such that, for any measurable functions f and g defined on $\mathbb{R}^n \times \mathbb{R}^m$,

$$\|fg\|_{L^{r_2, \infty}(L^{r_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)} \|g\|_{L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)},$$

where $C := \prod_{i=1}^2 (p_i/r_i)^{1/p_i} (q_i/r_i)^{1/q_i}$ if $p_i, q_i \neq \infty$ for any $i \in \{1, 2\}$ and $C := 1$ if $p_i = \infty$ or $q_i = \infty$ for some $i \in \{1, 2\}$.

However, for weak mixed-norm Lebesgue spaces, the corresponding Hölder inequality usually does not hold true. Particularly, we state a result in [19, Theorem 2.17] as follows.

Theorem 2.8. Let $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2)$ and $\vec{r} := (r_1, r_2)$ satisfy that, for any $i \in \{1, 2\}$, $p_i, q_i, r_i \in (0, \infty]$ and

$$\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}.$$

Then there exists a positive constant C , depending on \vec{p} , \vec{q} and \vec{r} , such that, for any measurable functions f and g defined on \mathbb{R}^n ,

$$\|fg\|_{L^{\vec{r}, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L^{\vec{p}, \infty}(\mathbb{R}^n)} \|g\|_{L^{\vec{q}, \infty}(\mathbb{R}^n)}$$

holds true if and only if $p_1 q_2 = p_2 q_1$. Moreover, when this condition holds true, then

$$C := \begin{cases} \max \left\{ 1, 2^{1/r_1 - 1/r_2} \right\} \frac{p_2^{1/p_2} q_2^{1/q_2}}{r_2^{1/r_2}} & \text{when } p_1, p_2, q_1, q_2 \in (0, \infty), \\ \max \left\{ 1, 2^{1/r_1 - 1} \right\} \frac{p_1^{r_1/p_1} q_1^{r_1/q_1}}{r_1} & \text{when } p_2 = q_2 = \infty, p_1, q_1 \in (0, \infty), \\ \frac{p_2^{1/p_2} q_2^{1/q_2}}{r_2^{1/r_2}} & \text{when } p_1 = q_1 = \infty, p_2, q_2 \in (0, \infty), \\ 1 & \text{when } \vec{p} = (\infty, \infty) \text{ or } \vec{q} = (\infty, \infty). \end{cases}$$

2.1.2 Interpolations

Interpolation theorem has proved an important and useful tool in many applications due to the fact that it is applicable to allow one to pass from hypotheses for certain exponents p (for instance $p = 2$ and $p = \infty$) to conclusions involving a range of p [for instance $p \in (2, \infty)$]. In this subsection, we mainly review some interpolation results about the above (weak) Lebesgue spaces with mixed norms.

Let the symbol $V(\mathbb{R}^n)$ denote the set of all simple functions on \mathbb{R}^n . Note that $V(\mathbb{R}^n) \subset L^{\vec{p}}(\mathbb{R}^n)$ and $V(\mathbb{R}^n)$ is dense in $L^{\vec{p}}(\mathbb{R}^n)$ for any given $\vec{p} \in [1, \infty)^n$ (see [9, p. 313]). We first display the Riesz–Thorin interpolation theorem on mixed Lebesgue spaces, which was established by Benedek and Panzone in [9, p. 316, Theorem 2].

Theorem 2.9. *Let $\theta \in (0, 1)$, $\vec{p} := (p_1, \dots, p_n)$, $\vec{q} := (q_1, \dots, q_n)$, $\vec{p}^{(j)} := (p_1^{(j)}, \dots, p_n^{(j)})$ and $\vec{q}^{(j)} := (q_1^{(j)}, \dots, q_n^{(j)})$ for any $j \in \{1, 2\}$ satisfy that, for any $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$, $p_i, q_i, p_i^{(j)}, q_i^{(j)} \in [1, \infty]$,*

$$\frac{1}{p_i} = \frac{\theta}{p_i^{(1)}} + \frac{1-\theta}{p_i^{(2)}} \quad \text{and} \quad \frac{1}{q_i} = \frac{\theta}{q_i^{(1)}} + \frac{1-\theta}{q_i^{(2)}}.$$

Let T be a linear operator, defined on $V(\mathbb{R}^n)$, satisfying that there exist two positive constants M_1 and M_2 such that, for any $j \in \{1, 2\}$ and $f \in V(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}^{(j)}}(\mathbb{R}^n)} \leq M_j \|f\|_{L^{\vec{q}^{(j)}}(\mathbb{R}^n)}.$$

Then, for any $f \in V(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq M_1^{1-\theta} M_2^\theta \|f\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

Furthermore, if $\vec{q} \in [1, \infty)^n$, the linear operator T can be uniquely extended to the space $L^{\vec{q}}(\mathbb{R}^n)$.

Remark 2.2. We point out that Theorem 2.9 plays a curial role in the proof of Theorem 2.12 below.

For weak mixed-norm Lebesgue spaces, Chen and Sun obtained the following interpolation theorem in [19, Theorem 2.21].

Theorem 2.10. *Let $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2)$ and $\vec{r} := (r_1, r_2)$ satisfy that, for any $i \in \{1, 2\}$, $p_i, q_i, r_i \in (0, \infty)$ and*

$$\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$$

with constant $\theta \in (0, 1)$. Then, for any measurable function f defined on $\mathbb{R}^n \times \mathbb{R}^m$, one has

$$\|f\|_{L^{\vec{r}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^{\vec{p}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)}^\theta \|f\|_{L^{\vec{q}, \infty}(\mathbb{R}^n \times \mathbb{R}^m)}^{1-\theta}$$

and

$$\|f\|_{L^{\vec{r}}(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left(\frac{r_1}{r_1 - p_1} + \frac{r_1}{q_1 - r_1} \right)^{1/r_1} \|f\|_{L^{p_2}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^\theta \|f\|_{L^{q_2}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^{1-\theta}.$$

However, if $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, then, for any multiple index $\vec{r} \in (0, \infty)^2$, $L^{\vec{r}, \infty}(\mathbb{R}^n \times \mathbb{R}^m) \cap L^{\vec{q}, \infty}(\mathbb{R}^n \times \mathbb{R}^m) \not\subseteq L^{\vec{r}}(\mathbb{R}^n \times \mathbb{R}^m)$.

Moreover, in [19, Theorem 2.22], Chen and Sun provided the following interpolation theorem for iterated weak Lebesgue spaces.

Theorem 2.11. Let $\vec{r} := (r_1, r_2) \in (0, \infty)^2$ and $p_1, q_1, p_{2,1}, p_{2,2}, q_{2,1}, q_{2,2} \in (0, \infty)$ satisfy that

$$\frac{1}{r_1} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}$$

and

$$\frac{1}{r_2} = \frac{\theta \xi}{p_{2,1}} + \frac{(1-\theta)\xi}{p_{2,2}} + \frac{\theta(1-\xi)}{q_{2,1}} + \frac{(1-\theta)(1-\xi)}{q_{2,2}},$$

where the constants $\theta, \xi \in (0, 1)$. Then there exists a positive constant C such that, for any measurable function f defined on $\mathbb{R}^n \times \mathbb{R}^m$,

$$\begin{aligned} \|f\|_{L^{\vec{r}}(\mathbb{R}^n \times \mathbb{R}^m)} &\leq C \|f\|_{L^{p_{2,1}, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^{\xi \theta} \|f\|_{L^{p_{2,2}, \infty}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^{\xi(1-\theta)} \\ &\quad \times \|f\|_{L^{q_{2,1}, \infty}(L^{p_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^{(1-\xi)\theta} \|f\|_{L^{q_{2,2}, \infty}(L^{q_1, \infty})(\mathbb{R}^n \times \mathbb{R}^m)}^{(1-\xi)(1-\theta)}. \end{aligned}$$

2.2 An extended inequality on the central Hardy–Littlewood maximal operator

In this subsection, we first recall an extended inequality about mixed Lebesgue norms and the central Hardy–Littlewood maximal operator, which was stated by Bagby [5] without giving a proof. For its importance and also the convenience of the reader, we provide a detailed proof for it in this subsection.

To begin with, we state this inequality as follows. In what follows, the symbol $L^1_{\text{loc}}(E)$ denotes the collection of all locally integrable functions on set E .

Theorem 2.12. Let $m \in \mathbb{Z}_+$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^m)$. For any $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^m$, define

$$f^*(s, t) := \sup_{r \in (0, \infty)} \frac{1}{|B(s, r)|} \int_{B(s, r)} |f(y, t)| dy,$$

here and thereafter, for any $s \in \mathbb{R}^n$ and $r \in (0, \infty)$, $B(s, r) := \{z \in \mathbb{R}^n : |z - s| < r\}$. If $m \in \mathbb{N}$, then, for any given $\vec{p}_m := (p_1, \dots, p_m) \in (1, \infty)^m$ and any $s \in \mathbb{R}^n$, define

$$T_{L^{\vec{p}_m}(\mathbb{R}^m)}(f)(s) := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(s, t_1, \dots, t_m)|^{p_1} dt_1 \right]^{\frac{p_2}{p_1}} \cdots dt_m \right\}^{\frac{1}{p_m}},$$

where, if $m \equiv 0$, then let $\mathbb{R}^0 := \emptyset$ and $T_{L^{\vec{p}_m}(\mathbb{R}^m)}(f)(s) := |f(s)|$ for any $s \in \mathbb{R}^n$. Thus, for any given $q \in (1, \infty)$, there exists a positive constant C , depending on q , but being independent of f , such that

$$\int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_m}(\mathbb{R}^m)}(f^*)(s) \right]^q ds \leq C \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_m}(\mathbb{R}^m)}(f)(s) \right]^q ds. \tag{2.2}$$

Remark 2.3. We should point out that Theorem 2.12 plays a key role in the proofs of both the boundedness of the Hardy–Littlewood maximal operator on mixed Lebesgue spaces (see [45, Lemma 3.5] or [71, Theorem 1.2]) and the Fefferman–Stein vector-valued inequality on mixed Lebesgue spaces (see [55, p. 679] or [71, Theorem 1.7]), which are known to be fundamental tools in developing a real-variable theory of related function spaces.

Recall that the *centered Hardy–Littlewood maximal function* f^* of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^*(x) := \sup_{r \in (0, \infty)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy. \tag{2.3}$$

The succeeding two conclusions are just, respectively, [9, p. 304, Theorem 2] and [30, Lemma 1], which are used later to show Theorem 2.12.

Theorem 2.13. Let $\vec{p} \in [1, \infty)^n$ and f be a measurable function on \mathbb{R}^n . Then

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} = \sup_{g \in U_{\vec{p}'}} \left| \int_{\mathbb{R}^n} f(x)g(x) dx \right|,$$

where \vec{p}' denotes the conjugate vector of \vec{p} , namely, for any $i \in \{1, \dots, n\}$, $1/p_i + 1/p'_i = 1$ and $U_{\vec{p}'}$ the unit sphere of $L^{\vec{p}'}(\mathbb{R}^n)$.

Theorem 2.14. For any $r \in (1, \infty)$, there exists a positive constant C , depending only on r , such that, for any positive real-valued functions f and ϕ on \mathbb{R}^n ,

$$\int_{\mathbb{R}^n} [f^*(x)]^r \phi(x) dx \leq C \int_{\mathbb{R}^n} [f(x)]^r \phi^*(x) dx,$$

where f^* and ϕ^* respectively denote the centered Hardy–Littlewood maximal functions of f and ϕ as in (2.3).

Via Theorems 2.13 and 2.14, we now show Theorem 2.12.

Proof of Theorem 2.12. We prove this theorem by induction. Let $m \in \mathbb{Z}_+$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^m)$. We perform induction on m . If $m := 0$, then $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the desired inequality (2.2) becomes, for any given $q \in (1, \infty)$,

$$\int_{\mathbb{R}^n} |f^*(s)|^q ds \lesssim \int_{\mathbb{R}^n} |f(s)|^q ds.$$

This is just the well-known boundedness of the centered Hardy–Littlewood maximal operator on $L^q(\mathbb{R}^n)$ with any given $q \in (1, \infty)$ (see, for instance, [77, p. 13, Theorem 1.(c)]).

Now assume (2.2) holds true for $m := k$ with some fixed $k \in \mathbb{N}$, namely, for any given $\vec{p}_k := (p_1, \dots, p_k) \in (1, \infty)^k$ and $q \in (1, \infty)$,

$$\int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(f^*)(s) \right]^q ds \lesssim \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(f)(s) \right]^q ds. \quad (2.4)$$

To complete the proof of Theorem 2.12, it suffices to show that, for $m := k+1$, (2.2) also holds true. To this end, we only need to prove, for any given $\vec{p}_k \in (1, \infty)^k$, $r \in (1, \infty]$ and $q \in (1, \infty)$,

$$\int_{\mathbb{R}^n} \left\{ T_{L^{\vec{p}_k}(\mathbb{R}^k)} \left(\left[\int_{\mathbb{R}} |f^*(s, y)|^r dy \right]^{1/r} \right) \right\}^q ds \lesssim \int_{\mathbb{R}^n} \left\{ T_{L^{\vec{p}_k}(\mathbb{R}^k)} \left(\left[\int_{\mathbb{R}} |f(s, y)|^r dy \right]^{1/r} \right) \right\}^q ds \quad (2.5)$$

with the usual modifications made when $r = \infty$. Indeed, for any given $r \in (1, \infty)$, any $s \in \mathbb{R}^n$ and $\tilde{t} \in \mathbb{R}^k$, let

$$J_r(f)(s, \tilde{t}) := \left[\int_{\mathbb{R}} |f(s, y, \tilde{t})|^r dy \right]^{1/r}$$

and $J_\infty(f)(s, \tilde{t}) := \text{esssup}_{y \in \mathbb{R}} |f(s, y, \tilde{t})|$. Then we can rewrite (2.5) simply as

$$\int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f^*))(s) \right]^q ds \lesssim \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f))(s) \right]^q ds. \quad (2.6)$$

We now prove (2.6) by three steps.

Step 1. In this step, we show that (2.6) holds true for $r = \infty$. To do this, first notice that, for any $s \in \mathbb{R}^n$, $\tilde{t} \in \mathbb{R}^k$ and almost every $y \in \mathbb{R}$,

$$|f(s, y, \tilde{t})| \leq J_\infty(f)(s, \tilde{t}).$$

Thus, for any $s \in \mathbb{R}^n$, $\tilde{t} \in \mathbb{R}^k$ and almost every $y \in \mathbb{R}$, we have

$$f^*(s, y, \tilde{t}) \leq [J_\infty(f)]^*(s, \tilde{t})$$

and hence

$$J_\infty(f^*)(s, \tilde{t}) \leq [J_\infty(f)]^*(s, \tilde{t}).$$

From this and (2.4), it follows that, for any given $\vec{p}_k \in (1, \infty)^k$ and $q \in (1, \infty)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_\infty(f^*))(s) \right]^q ds &\leq \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}([J_\infty(f)]^*)(s) \right]^q ds \\ &\lesssim \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_\infty(f))(s) \right]^q ds, \end{aligned}$$

which implies that (2.6) holds true for $r = \infty$.

Step 2. In this step, we prove that (2.6) holds true for any given $r \in (1, \min\{q, p_1, \dots, p_k\})$. Indeed, note that, for any $s \in \mathbb{R}^n$,

$$T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f^*))(s) = T_{L^{\vec{p}_k}(\mathbb{R}^k)}\left(\left\{\int_{\mathbb{R}} [f^*(s, y)]^r dy\right\}^{1/r}\right) = \left[T_{L^{\vec{p}_k/r}(\mathbb{R}^k)}\left(\int_{\mathbb{R}} [f^*(s, y)]^r dy\right)\right]^{1/r},$$

where $\vec{p}_k/r := (p_1/r, \dots, p_k/r) \in (1, \infty)^k$. Therefore, for any given $q \in (1, \infty)$, by the fact that $r \in (1, \min\{q, p_1, \dots, p_k\})$ and Theorem 2.13, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f^*))(s)\right]^q ds \\ &= \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k/r}(\mathbb{R}^k)}\left(\int_{\mathbb{R}} [f^*(s, y)]^r dy\right)\right]^{q/r} ds \\ &= \left\| \left\| \int_{\mathbb{R}} [f^*(\cdot, y, \cdot)]^r dy \right\|_{L^{\vec{p}_k/r}(\mathbb{R}^k)} \right\|_{L^{q/r}(\mathbb{R}^n)}^{q/r} \\ &= \sup_{\phi} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left\{ \int_{\mathbb{R}} [f^*(s, y, \tilde{t})]^r dy \right\} \phi(s, \tilde{t}) d\tilde{t} ds \right|^{q/r}, \end{aligned} \tag{2.7}$$

where the supremum is taken over all ϕ belonging to

$$\left\{ \zeta \in L^{(q/r)'}(L^{(\vec{p}_k/r)'}) (\mathbb{R}^{n+k}) : \int_{\mathbb{R}^n} \left[T_{L^{(\vec{p}_k/r)' }(\mathbb{R}^k)}(\zeta)(s)\right]^{\frac{q}{q-r}} ds = 1 \right\}.$$

This, together with the Tonelli theorem and Theorem 2.14, implies that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left\{ \int_{\mathbb{R}} [f^*(s, y, \tilde{t})]^r dy \right\} \phi(s, \tilde{t}) d\tilde{t} ds \right| \\ & \leq \int_{\mathbb{R}^k} \int_{\mathbb{R}} \int_{\mathbb{R}^n} [f^*(s, y, \tilde{t})]^r |\phi(s, \tilde{t})| ds dy d\tilde{t} \lesssim \int_{\mathbb{R}^k} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |f(s, y, \tilde{t})|^r |\phi^*(s, \tilde{t})| ds dy d\tilde{t}. \end{aligned}$$

From this, the Tonelli theorem again, the fact that $r \in (1, \min\{q, p_1, \dots, p_k\})$, Theorem 2.6 and the Hölder inequality, it follows that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left\{ \int_{\mathbb{R}} [f^*(s, y, \tilde{t})]^r dy \right\} \phi(s, \tilde{t}) d\tilde{t} ds \right| \\ & \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}} |f(s, y, \tilde{t})|^r dy \right] \phi^*(s, \tilde{t}) d\tilde{t} ds \\ & \lesssim \int_{\mathbb{R}^n} T_{L^{\vec{p}_k/r}(\mathbb{R}^k)}\left(\int_{\mathbb{R}} |f(s, y)|^r dy\right) T_{L^{(\vec{p}_k/r)' }(\mathbb{R}^k)}(\phi^*)(s) ds \\ & \lesssim \left\{ \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k/r}(\mathbb{R}^k)}\left(\int_{\mathbb{R}} |f(s, y)|^r dy\right)\right]^{q/r} ds \right\}^{r/q} \\ & \quad \times \left\{ \int_{\mathbb{R}^n} \left[T_{L^{(\vec{p}_k/r)' }(\mathbb{R}^k)}(\phi^*)(s)\right]^{q/(q-r)} ds \right\}^{1-r/q}, \end{aligned}$$

which, combined with (2.4) and the fact that $\int_{\mathbb{R}^n} \left[T_{L^{(\tilde{p}_k/r)'(\mathbb{R}^k)}}(\phi)(s) \right]^{\frac{q}{q-r}} ds = 1$, further implies that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left\{ \int_{\mathbb{R}} [f^*(s,y,\tilde{t})]^r dy \right\} \phi(s,\tilde{t}) d\tilde{t} ds \right| \\ & \lesssim \left\{ \int_{\mathbb{R}^n} \left[T_{L^{\tilde{p}_k/r}(\mathbb{R}^k)} \left(\int_{\mathbb{R}} |f(s,y)|^r dy \right) \right]^{q/r} ds \right\}^{r/q} \\ & \quad \times \left\{ \int_{\mathbb{R}^n} \left[T_{L^{(\tilde{p}_k/r)'(\mathbb{R}^k)}}(\phi)(s) \right]^{q/(q-r)} ds \right\}^{1-r/q} \\ & \lesssim \left\{ \int_{\mathbb{R}^n} \left[T_{L^{\tilde{p}_k/r}(\mathbb{R}^k)} \left(\int_{\mathbb{R}} |f(s,y)|^r dy \right) \right]^{q/r} ds \right\}^{r/q} \\ & \sim \left\{ \int_{\mathbb{R}^n} \left[T_{L^{\tilde{p}_k}(\mathbb{R}^k)}(J_r(f))(s) \right]^q ds \right\}^{r/q}. \end{aligned}$$

By this and (2.7), we conclude that (2.6) holds true for any given $r \in (1, \min\{q, p_1, \dots, p_k\})$.

Step 3. In this step, based on the obtained results in Steps 1 and 2 above, we complete the proof of (2.6) via an interpolation procedure. To this end, recall that Theorem 2.9, the Riesz–Thorin interpolation theorem on mixed Lebesgue spaces, has been established by Benedek and Panzone in [9, p. 316, Theorem 2]. However, this interpolation theorem is only applicable to linear operators and, obviously, the Hardy–Littlewood maximal operator as in (2.3) is only sublinear. Thus, for any $i \in \mathbb{Z}$, we define a linear operator Γ by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$, $s \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\tilde{t} \in \mathbb{R}^k$,

$$(\Gamma(f))_i(s,y,\tilde{t}) := \frac{1}{|B(s,2^i)|} \int_{B(s,2^i)} f(z,y,\tilde{t}) dz,$$

where, for any $s \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $B(s,2^i) := \{z \in \mathbb{R}^n : |z-s| < 2^i\}$, and let

$$I_\infty[\{\Gamma(f)\}](s,y,\tilde{t}) := \sup_{i \in \mathbb{Z}} |(\Gamma f)_i(s,y,\tilde{t})|.$$

Then it is easy to see that, for any $s \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\tilde{t} \in \mathbb{R}^k$,

$$I_\infty[\{\Gamma(|f|)\}](s,y,\tilde{t}) \leq f^*(s,y,\tilde{t}). \tag{2.8}$$

In addition, notice that, for any $r \in (0, \infty)$, there exists some $i_r \in \mathbb{Z}$ such that $r \in [2^{i_r-1}, 2^{i_r})$. Therefore, for any $r \in (0, \infty)$, $s \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\tilde{t} \in \mathbb{R}^k$,

$$\begin{aligned} \frac{1}{|B(s,r)|} \int_{B(s,r)} |f(z,y,\tilde{t})| dz & \leq \frac{1}{r^n v_n} \int_{B(s,2^{i_r})} |f(z,y,\tilde{t})| dz \leq \frac{1}{2^{(i_r-1)n} v_n} \int_{B(s,2^{i_r})} |f(z,y,\tilde{t})| dz \\ & = 2^n I_\infty[\{\Gamma(|f|)\}](s,y,\tilde{t}), \end{aligned}$$

here and thereafter, v_n denotes the Lebesgue measure of the unit ball on \mathbb{R}^n , namely, $v_n := |B(\vec{0}_n, 1)|$ with $B(\vec{0}_n, 1) := \{y \in \mathbb{R}^n : |y| < 1\}$. Thus, for any $s \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $\tilde{t} \in \mathbb{R}^k$,

$$f^*(s, y, \tilde{t}) \leq 2^n I_\infty[\{\Gamma(|f|)\}](s, y, \tilde{t}),$$

which, together with (2.8), further implies that

$$I_\infty[\{\Gamma(|f|)\}](s, y, \tilde{t}) \sim f^*(s, y, \tilde{t}).$$

Thus, (2.6) is equivalent to, for any given $\vec{p}_k \in (1, \infty)^k$, $r \in (1, \infty]$ and $q \in (1, \infty)$,

$$\int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(I_\infty[\{\Gamma(|f|)\}]))(s) \right]^q ds \lesssim \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f))(s) \right]^q ds. \tag{2.9}$$

For any given $\vec{p}_k \in (1, \infty)^k$, $r \in (1, \infty]$ and $q \in (1, \infty)$, let

$$\|\Gamma(|f|)\|_{\vec{Q}_r(\vec{p}_k, q)} := \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(I_\infty[\{\Gamma(|f|)\}]))(s) \right]^q ds$$

and

$$\|f\|_{\vec{V}_r(\vec{p}_k, q)} := \int_{\mathbb{R}^n} \left[T_{L^{\vec{p}_k}(\mathbb{R}^k)}(J_r(f))(s) \right]^q ds.$$

Then (2.9) becomes, for any given $\vec{p}_k \in (1, \infty)^k$, $r \in (1, \infty]$ and $q \in (1, \infty)$,

$$\|\Gamma(|f|)\|_{\vec{Q}_r(\vec{p}_k, q)} \lesssim \|f\|_{\vec{V}_r(\vec{p}_k, q)}.$$

On another hand, by the conclusions of Steps 1 and 2, we know that, for any given $\vec{p}_k \in (1, \infty)^k$ and $q \in (1, \infty)$,

$$\|\Gamma(|f|)\|_{\vec{Q}_\infty(\vec{p}_k, q)} \lesssim \|f\|_{\vec{V}_\infty(\vec{p}_k, q)}$$

and, for any given $r \in (1, \min\{q, p_1, \dots, p_k\})$,

$$\|\Gamma(|f|)\|_{\vec{Q}_r(\vec{p}_k, q)} \lesssim \|f\|_{\vec{V}_r(\vec{p}_k, q)},$$

which, combined with Theorem 2.9, further implies that, for any given $r \in (1, \infty]$,

$$\|\Gamma(|f|)\|_{\vec{Q}_r(\vec{p}_k, q)} \lesssim \|f\|_{\vec{V}_r(\vec{p}_k, q)}.$$

This implies that (2.9) holds true and hence finishes the proof of Theorem 2.12. □

2.3 Applications

This subsection is devoted to a survey of some applications which include the dual inequality of Stein type, the Fefferman–Stein vector-valued inequality on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ proved by Nogayama in [71] as well as the boundedness of fractional integrals and geometric inequalities on iterated weak Lebesgue spaces $L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})$ and

weak mixed-norm Lebesgue spaces $L^{\vec{q},\infty}(\mathbb{R}^{2n})$ obtained by Chen and Sun in [19]. For this purpose, we first present the notion of iterated maximal operators.

For any $k \in \{1, \dots, n\}$, the maximal function $M_k(f)$ of any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ for the k th variable is defined by setting, for any $x := (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$M_k(f)(x) := \sup_{I \in \mathbb{I}_{x_k}} \frac{1}{|I|} \int_I |f(x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n)| dy_k, \tag{2.10}$$

where, for any $k \in \{1, \dots, n\}$, \mathbb{I}_{x_k} denotes the set of all intervals in \mathbb{R}_{x_k} containing x_k . Moreover, for any given $t \in (0, \infty)$, the iterated maximal function $\mathfrak{M}_t(f)$ of any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathfrak{M}_t(f)(x) := [M_n \cdots (M_1(|f|^t))(x)]^{1/t}. \tag{2.11}$$

Now we state a result given in [71, Theorem 1.6] about the dual inequality of Stein type on mixed Lebesgue spaces, which extends the corresponding result of Fefferman and Stein [30, Lemma 1]. To this end, we first recall the following notion of Muckenhoupt weights.

Definition 2.3. Let $p \in (1, \infty)$. The weight class $\mathbf{A}_p(\mathbb{R}^n)$ is defined to be the set of all non-negative locally integrable functions ω on \mathbb{R}^n such that

$$[\omega]_{\mathbf{A}_p(\mathbb{R}^n)} := \sup_{Q \in \mathbb{R}^n} \frac{1}{|Q|^p} \int_Q \omega(x) dx \left(\int_Q [\omega(y)]^{-p'/p} dy \right)^{p/p'} < \infty,$$

where the supremum is taken over all closed cubes $Q \subset \mathbb{R}^n$ and $1/p + 1/p' = 1$.

When $p = 1$, the weight class $\mathbf{A}_1(\mathbb{R}^n)$ is defined to be the set of all non-negative locally integrable functions ω on \mathbb{R}^n such that

$$[\omega]_{\mathbf{A}_1(\mathbb{R}^n)} := \sup_{Q \in \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(x) dx \left(\text{esssup}_{y \in Q} [\omega(y)]^{-1} \right) < \infty,$$

where the supremum is taken over all closed cubes $Q \subset \mathbb{R}^n$. Moreover, for any $E \subset \mathbb{R}^n$, let $\omega(E) := \int_E \omega(x) dx$.

Theorem 2.15. Let $\vec{p} := (p_1, \dots, p_n) \in [1, \infty)^n$, $t \in (0, \min\{p_1, \dots, p_n\})$ and, for any $j \in \{1, \dots, n\}$, $(\omega_j)^t \in \mathbf{A}_{p_j}(\mathbb{R})$. Then there exists a positive constant C such that, for any measurable function f ,

$$\left\| \mathfrak{M}_t(f) \cdot \bigotimes_{j=1}^n (\omega_j)^{\frac{1}{p_j}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \left\| f \cdot \bigotimes_{j=1}^n [M_j(\omega_j)]^{\frac{1}{p_j}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $(\bigotimes_{j=1}^n \omega_j)(x) := \prod_{j=1}^n \omega_j(x_j)$ and M_j as well as \mathfrak{M}_t are, respectively, as in (2.10) and (2.11).

Moreover, the following Fefferman–Stein vector-valued inequality of iterated maximal operators \mathfrak{M}_t on mixed Lebesgue spaces was also shown by Nogayama in [71, Theorem 1.7] (see also [55, p. 679]).

Theorem 2.16. *Let $\vec{p} \in (0, \infty)^n$, $u \in (0, \infty]$ and $t \in (0, \min\{p_1, \dots, p_n, u\})$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of measurable functions,*

$$\left\| \left(\sum_{j \in \mathbb{N}} [\mathfrak{M}_t(f_j)]^u \right)^{\frac{1}{u}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where \mathfrak{M}_t is as in (2.11).

To present some applications of iterated weak Lebesgue spaces and weak mixed-norm Lebesgue spaces, we recall the following conclusions given in [19, Theorem 3.1], which imply some boundedness of non-negative measurable functions from $L^\infty(\mathbb{R}^{2n})$ to $L^{\vec{q}, \infty}(\mathbb{R}^{2n})$ or to $L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})$.

Theorem 2.17. *Let $\vec{q} := (q_1, q_2) \in (0, \infty)^2$. Then there exists a positive constant C , depending only on \vec{q} and n , such that, for any non-negative measurable function F on \mathbb{R}^{2n} ,*

$$\|F\|_{L^{\vec{q}, \infty}(\mathbb{R}^{2n})} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ F(x, y) (|x+y| + |x-y|)^{n/q_1 + n/q_2} \right\}$$

and

$$\|F\|_{L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ F(x, y) (|x+y| + |x-y|)^{n/q_1 + n/q_2} \right\}.$$

Next we state the following conclusions of [19, Theorem 3.2], which further induce two geometric inequalities as Theorem 2.19 below.

Theorem 2.18. *Let $\vec{q} := (q_1, q_2) \in (0, \infty)^2$. Then there exists a constant C , depending only on \vec{q} and n , such that, for any non-negative measurable function F on \mathbb{R}^{2n} ,*

$$\|F\|_{L^{q_2, \infty}(L^{q_1})(\mathbb{R}^{2n})} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ F(x, y) (|x+y| + |x-y|)^{n/q_1 + n/q_2} \right\}.$$

Moreover, on the endpoint cases, for any given $q_1 \in (0, \infty]$, there exists a positive constant C , depending only on q_1 and n , such that, for any non-negative measurable function F on \mathbb{R}^{2n} ,

$$\|F\|_{L^{q_1, \infty}(L^\infty)(\mathbb{R}^{2n})} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ F(x, y) (|x+y| + |x-y|)^{n/q_1} \right\}$$

and

$$\|F\|_{L^\infty(L^{q_1, \infty})(\mathbb{R}^{2n})} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ F(x, y) (|x+y| + |x-y|)^{n/q_1} \right\}.$$

As a consequence of Theorem 2.18, the following geometric inequalities were obtained by Chen and Sun in [19, Corollary 3.3].

Theorem 2.19. Let $\vec{p} := (p_1, p_2) \in (0, \infty)^2$. Then there exists a positive constant C , depending only on \vec{p} and n , such that, for any non-negative function $f \in L^{p_1, \infty}(\mathbb{R}^n)$ and non-negative function $g \in L^{p_2, \infty}(\mathbb{R}^n)$,

$$\|f\|_{L^{p_1, \infty}(\mathbb{R}^n)} \|g\|_{L^{p_2, \infty}(\mathbb{R}^n)} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ f(x)g(y) |x+y|^{n/p_1+n/p_2} \right\}.$$

Moreover, there exists a positive constant C such that, for any non-negative function $f \in L^{p_1}(\mathbb{R}^n)$ and non-negative function $g \in L^{p_2}(\mathbb{R}^n)$,

$$\|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \leq C \sup_{x, y \in \mathbb{R}^n} \left\{ f(x)g(y) |x-y|^{n/p_1+n/p_2} \right\}. \tag{2.12}$$

Remark 2.4. Notice that, if $f \equiv g \equiv \mathbf{1}_E$ with some measurable set $E \in \mathbb{R}^n$, then (2.12) becomes

$$|E| \leq C \sup_{x, y \in E} |x-y|^n,$$

which means the “volume” of the set E , namely, $|E|$, can be controlled by its “diameter” $\sup_{x, y \in E} |x-y|$.

Let $\gamma \in (0, \infty)$ and f be a measurable function defined on \mathbb{R}^{2n} . The linear operators L_γ and T_γ are defined, respectively, by setting, for any $x, y \in \mathbb{R}^n$,

$$L_\gamma f(x, y) := \frac{f(x, y)}{|x-y|^\gamma}$$

and

$$T_\gamma f(x, y) := \frac{f(x, y)}{(|x+y| + |x-y|)^\gamma}.$$

Recall that, in [19, Theorem 3.7], the boundedness of T_γ and T_γ^{-1} on iterated weak Lebesgue spaces and weak mixed-norm Lebesgue spaces was shown as follows.

Theorem 2.20. Let $\gamma \in (0, \infty)$, $\vec{p} := (p_1, p_2)$ and $\vec{q} := (q_1, q_2)$.

(i) If $0 < q_1 \leq p_1 \leq \infty$ and $0 < q_2 \leq p_2 \leq \infty$ satisfy the homogeneity condition

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{\gamma}{n}, \tag{2.13}$$

then there exists a positive constant C , depending only on \vec{p} , \vec{q} and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\|T_\gamma f\|_{L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})} \leq C \|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^{2n})}.$$

(ii) If $0 < p_1 \leq q_1 \leq \infty$ and $0 < p_2 \leq q_2 \leq \infty$ satisfy the homogeneity condition

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{\gamma}{n}, \quad (2.14)$$

then there exists a positive constant C , depending only on \vec{p} , \vec{q} and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\left\| T_\gamma^{-1} f \right\|_{L^{q_2, \infty}(L^{q_1, \infty})(\mathbb{R}^{2n})} \leq C \|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^{2n})}.$$

(iii) If $0 < q_1 \leq p_1 \leq \infty$ and $0 < q_2 \leq p_2 \leq \infty$ satisfy $p_1 q_2 = p_2 q_1$ and the homogeneity condition (2.13), then there exists a positive constant C , depending only on \vec{p} , \vec{q} and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\|T_\gamma f\|_{L^{\vec{q}, \infty}(\mathbb{R}^{2n})} \leq C \|f\|_{L^{\vec{p}, \infty}(\mathbb{R}^{2n})}.$$

(iv) If $0 < p_1 \leq q_1 \leq \infty$ and $0 < p_2 \leq q_2 \leq \infty$ satisfy $p_1 q_2 = p_2 q_1$ and the homogeneity condition (2.14), then there exists a positive constant C , depending only on \vec{p} , \vec{q} and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\left\| T_\gamma^{-1} f \right\|_{L^{\vec{q}, \infty}(\mathbb{R}^{2n})} \leq C \|f\|_{L^{\vec{p}, \infty}(\mathbb{R}^{2n})}.$$

Using Theorems 2.5 and 2.20, Chen and Sun [19, Theorem 3.12] also obtained the following inequalities.

Theorem 2.21. Let $\gamma \in (0, \infty)$. If $0 < r < p_1 \leq \infty$ and $p_2 \in (0, \infty]$ satisfy the homogeneity condition $\frac{1}{r} = \frac{1}{p_1} + \frac{\gamma}{n}$, then there exists a positive constant C , depending only on p_1 , p_2 , r and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\|L_\gamma f\|_{L^{p_2}(L^{r, \infty})(\mathbb{R}^{2n})} \leq C \|f\|_{L^{p_2}(L^{p_1, \infty})(\mathbb{R}^{2n})}$$

and

$$\|L_\gamma f\|_{L^{p_2, \infty}(L^{r, \infty})(\mathbb{R}^{2n})} \leq C \|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^{2n})}.$$

If $0 < p_1 < r \leq \infty$ and $p_2 \in (0, \infty]$ satisfy the homogeneity condition $\frac{1}{p_1} = \frac{1}{r} + \frac{\gamma}{n}$, then there exists a positive constant C , depending only on p_1 , p_2 , r and n , such that, for any non-negative measurable function f on \mathbb{R}^{2n} ,

$$\left\| L_\gamma^{-1} f \right\|_{L^{p_2}(L^{r, \infty})(\mathbb{R}^{2n})} \geq C \|f\|_{L^{p_2}(L^{p_1, \infty})(\mathbb{R}^{2n})}$$

and

$$\left\| L_\gamma^{-1} f \right\|_{L^{p_2, \infty}(L^{r, \infty})(\mathbb{R}^{2n})} \geq C \|f\|_{L^{p_2, \infty}(L^{p_1, \infty})(\mathbb{R}^{2n})}.$$

In addition, it was shown, via Theorem 2.20, by Chen and Sun in [19, Corollary 3.8] that the Hardy–Littlewood–Sobolev inequality and its reverse version hold true as follows.

Theorem 2.22. *Let $\vec{p} := (p_1, p_2) \in (1, \infty)^2$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$. Then there exists a positive constant C , depending only on \vec{p} and n , such that, for any non-negative functions $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x)|x-y|^{-n(2-1/p_1-1/p_2)} dx dy \leq C \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

If $\vec{p} \in (0, 1)^2$, then there exists a positive constant C , depending only on \vec{p} and n , such that, for any non-negative functions $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x)|x-y|^{n(1/p_1+1/p_2-2)} dx dy \geq C \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

3 Mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$

In this section, we first recall the notion of mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$, with $\vec{q} \in (0, \infty]^n$ and $p \in (0, \infty]$, and then discuss some basic properties of them (see Subsection 3.1 below) as well as the boundedness of maximal operators on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ (see Subsection 3.2 below). Finally, we review the boundedness of Calderón–Zygmund operators and fractional integral operators on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ as well as a necessary and sufficient condition for the boundedness of the commutators of fractional integral operators on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ (see Subsection 3.3 below).

3.1 Definition and some basic properties

In this subsection, we present the definition and some examples of mixed Morrey spaces and then recall some basic properties about these spaces. To this end, we begin with recalling the notion of mixed Morrey spaces given in [71, Definition 1.3].

Definition 3.1. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$ and $p \in (0, \infty]$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}.$$

The *mixed Morrey space* $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that their quasi-norms

$$\|f\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)} := \sup \left\{ |Q|^{\frac{1}{p} - \frac{1}{n}(\sum_{j=1}^n \frac{1}{q_j})} \|f \mathbf{1}_Q\|_{L^{\vec{q}}(\mathbb{R}^n)} : Q \text{ is a cube in } \mathbb{R}^n \right\} < \infty.$$

Remark 3.1. Obviously, when $\vec{q} := (\overbrace{q, \dots, q}^{n \text{ times}})$ with some $q \in (0, \infty]$, then the mixed Morrey space $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ coincides with the classical Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ of [70] and, in this case, they have the same quasi-norms. In addition, when $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$ and $p \in (0, \infty]$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} = \frac{n}{p},$$

then $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n)$ with equivalent quasi-norms (see [71, Section 3]).

We review the following completeness of mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$, which was proved by Nogayama [71, Remark 3.1].

Theorem 3.1. Let $\vec{q} \in [1, \infty]^n$ and $p \in [1, \infty]$. Then the mixed Morrey space $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ becomes a Banach space.

Next we borrow two examples given in [71, Examples 3.3 and 3.5] as follows to show some simple elements in $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$.

Example 3.1. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$, $p \in (0, \infty]$ and $q_+ := \max\{q_1, \dots, q_n\}$ satisfy $q_+ \in (0, p)$. Then, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, let $f(x) := |x|^{-\frac{n}{p}}$; then

$$f \in \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n).$$

Example 3.2. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$, $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$ and $p \in (0, \infty]$ satisfy that, for any $j \in \{1, \dots, n\}$, $q_j \in (0, p_j]$ if $p_j \in (0, \infty)$, and $q_j \in (0, \infty]$ if $p_j = \infty$ as well as

$$\sum_{j=1}^n \frac{1}{p_j} = \frac{n}{p}.$$

Then, for any $x := (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, let $f(x) := \prod_{j=1}^n |x_j|^{-\frac{1}{p_j}}$; then

$$f \in \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n).$$

We now state the embedding properties of mixed Morrey spaces given in [71, Proposition 3.2] as follows.

Theorem 3.2. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$, $\vec{r} := (r_1, \dots, r_n) \in (0, \infty]^n$ and $p \in (0, \infty)$ satisfy that, for any $j \in \{1, \dots, n\}$, $q_j \in (0, r_j]$ and

$$\sum_{j=1}^n \frac{1}{r_j} \geq \frac{n}{p}.$$

Then

$$\mathcal{M}_{\vec{r}}^p(\mathbb{R}^n) \subset \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$$

and the embedding is continuous.

Remark 3.2. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty)^n$ and $p \in [\max\{q_1, \dots, q_n\}, \infty)$. Then, from Theorem 3.2, it follows that

$$\mathcal{M}_{\max\{q_1, \dots, q_n\}}^p(\mathbb{R}^n) \subset \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n) \subset \mathcal{M}_{\min\{q_1, \dots, q_n\}}^p(\mathbb{R}^n).$$

3.2 Some maximal inequalities on $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$

The aim of this subsection is the summarization of conclusions on the boundedness of several maximal operators on mixed Morrey spaces. We first present the following notion of the uncentered Hardy–Littlewood maximal operator.

The *uncentered Hardy–Littlewood maximal function* $M_{\text{HL}}(f)$ of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{\text{HL}}(f)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{3.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

Then we display the boundedness of the iterated maximal operator \mathfrak{M}_t on mixed Morrey spaces $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ established in [71, Theorem 1.4] as follows.

Theorem 3.3. *Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty)^n$ and $p \in (0, \infty)$ satisfy*

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p} \quad \text{and} \quad \frac{n-1}{n} p < \max\{q_1, \dots, q_n\}.$$

If $t \in (0, \min\{q_1, \dots, q_n, p\})$, then there exists a positive constant C such that, for any $f \in \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$,

$$\|\mathfrak{M}_t(f)\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)},$$

where \mathfrak{M}_t is as in (2.11).

Remark 3.3. Observe that, in [71, Theorem 1.4], $\vec{q} := (q_1, \dots, q_n) \in (0, \infty]^n$. However, it was pointed out to us by Professor Ferenc Weisz that, when some $q_{i_1} \in (0, \infty)$ and $q_{i_2} := \infty$ with $i_1, i_2 \in \{1, \dots, n\}$ and $i_1 < i_2$, [71, Theorem 1.4] is not correct. Indeed, Professor Ferenc Weisz showed that [71, Lemma 4.8], which was used in the proof of [71, Theorem 1.4], does not hold true if $n := 2$, $p_1 \in (1, \infty)$ and $p_2 := \infty$. Thus, in Theorem 3.3, the range of \vec{q} should be $(0, \infty)^n$.

As a consequence of Theorem 3.3, Nogayama [71, Corollary 1.5] also obtained the following boundedness of the iterated maximal operator \mathfrak{M}_t on classical Morrey spaces.

Corollary 3.1. Let

$$0 < \frac{n-1}{n} p < q \leq p < \infty.$$

If $t \in (0, q)$, then there exists a positive constant C such that, for any $f \in \mathcal{M}_q^p(\mathbb{R}^n)$,

$$\|\mathfrak{M}_t(f)\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)},$$

where \mathfrak{M}_t is as in (2.11).

Moreover, Nogayama in [71, Theorems 1.8 and 1.9] established the following two succeeding Fefferman–Stein vector-valued inequalities of the uncentered Hardy–Littlewood maximal operator M_{HL} and the iterated maximal operator \mathfrak{M}_t on mixed Morrey spaces.

Theorem 3.4. Let $\vec{q} := (q_1, \dots, q_n) \in (1, \infty)^n$, $u \in (1, \infty]$ and $p \in (1, \infty]$ satisfy

$$\frac{n}{p} \leq \sum_{j=1}^n \frac{1}{q_j}.$$

Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j \in \mathbb{N}}$ of measurable functions,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [M_{\text{HL}}(f_j)]^u \right\}^{\frac{1}{u}} \right\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)},$$

where M_{HL} is as in (3.1).

Theorem 3.5. Let $\vec{q} := (q_1, \dots, q_n) \in (0, \infty)^n$, $u \in (0, \infty]$ and $p \in (0, \infty)$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p} \quad \text{and} \quad \frac{n-1}{n} p < \max\{q_1, \dots, q_n\}.$$

If $t \in (0, \min\{q_1, \dots, q_n, u\})$, then there exists a positive constant C such that, for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} [\mathfrak{M}_t(f_j)]^u \right\}^{\frac{1}{u}} \right\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)},$$

where \mathfrak{M}_t is as in (2.11).

Remark 3.4. Observe that, in [71, Theorem 1.9], $\vec{q} := (q_1, \dots, q_n) \in (0, \infty)^n$. However, similarly to Remark 3.3, when some $q_{i_1} \in (0, \infty)$ and $q_{i_2} := \infty$ with $i_1, i_2 \in \{1, \dots, n\}$ and $i_1 < i_2$, [71, Theorem 1.9] is not correct, which was also pointed out to us by Professor Ferenc Weisz. Therefore, in Theorem 3.5, the range of \vec{q} should also be $(0, \infty)^n$.

As a corollary of Theorem 3.5, the following Fefferman–Stein vector-valued inequality of the iterated maximal operator \mathfrak{M}_t on classical Morrey spaces was obtained by Nogayama in [71, Theorem 1.10].

Corollary 3.2. Let $u \in (0, \infty]$ and

$$0 < \frac{n-1}{n}p < q \leq p < \infty.$$

If $t \in (0, \min\{q, u\})$, then there exists a positive constant C such that, for any $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}_q^p(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j \in \mathbb{N}} [\mathfrak{M}_t(f_j)]^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \leq C \left\| \left(\sum_{j \in \mathbb{N}} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^p(\mathbb{R}^n)}.$$

3.3 Boundedness of operators on $\mathcal{M}_q^p(\mathbb{R}^n)$

In this subsection, we discuss the boundedness of Calderón–Zygmund operators T and fractional integral operators I_α on mixed Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$. Then we review a necessary and sufficient condition for the boundedness of commutators of fractional integral operators I_α on $\mathcal{M}_q^p(\mathbb{R}^n)$. To this end, we first recall the following notion of Calderón–Zygmund operators (see, for instance, [71]).

Definition 3.2. A linear operator T is called a *Calderón–Zygmund operator* if its kernel

$$k: \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\} \rightarrow \mathbb{C}$$

satisfies that

- (i) there exists a positive constant C such that, for any $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$|k(x, y)| \leq \frac{C}{|x - y|^n};$$

- (ii) there exist positive constants C and ε such that, for any $x, y \in \mathbb{R}^n$ with $|x - y| \geq 2|x - z| \neq 0$,

$$|k(x, y) - k(z, y)| + |k(y, x) - k(y, z)| \leq C \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}};$$

- (iii) if $f \in L^\infty(\mathbb{R}^n)$ with compact support, then, for any $x \notin \text{supp}(f)$,

$$Tf(x) := \int_{\mathbb{R}^n} k(x, y)f(y)dy.$$

It was shown in [71, Theorem 1.12] that the Calderón–Zygmund operator is bounded on mixed Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$ as follows.

Theorem 3.6. Let $\vec{q} := (q_1, \dots, q_n) \in (1, \infty)^n$, $p \in (1, \infty)$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}$$

and T be a Calderón–Zygmund operator defined on $\mathcal{M}_{\min\{q_1, \dots, q_n\}}^p(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any $f \in \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$,

$$\|Tf\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)}.$$

Remark 3.5. We should point out that the statement of Theorem 3.6 contains the following fact that

$$\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n) \subset \mathcal{M}_{\min\{q_1, \dots, q_n\}}^p(\mathbb{R}^n).$$

Indeed, this embedding follows from Remark 3.2.

Next we present the notion of fractional integral operators in [71] as follows.

Definition 3.3. Let $\alpha \in (0, n)$. The fractional integral operator I_α of order α is defined by setting, for any $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Now we state the following result about the boundedness of fractional integral operators on mixed Morrey spaces, which was shown by Nogayama in [71, Theorem 1.11].

Theorem 3.7. Let $\alpha \in (0, n)$, $\vec{q} := (q_1, \dots, q_n) \in (1, \infty)^n$, $\vec{s} := (s_1, \dots, s_n) \in (1, \infty)^n$ and $p, r \in (1, \infty)$. Assume that

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}, \quad \sum_{j=1}^n \frac{1}{s_j} \geq \frac{n}{r}, \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$$

and, for any $j \in \{1, \dots, n\}$, $\frac{q_j}{p} = \frac{s_j}{r}$. Then there exists a positive constant C such that, for any $f \in \mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$,

$$\|I_\alpha(f)\|_{\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)}.$$

We next introduce the notion of the commutators of fractional integral operators in [72] as follows.

Definition 3.4. Let $\alpha \in (0, n)$, $b \in L_{\text{loc}}^1(\mathbb{R}^n)$ and I_α be a fractional integral operator of order α . The commutator $[b, I_\alpha]$ of I_α is defined by setting, for any $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$[b, I_\alpha](f)(x) := b(x)I_\alpha(f)(x) - I_\alpha(bf)(x).$$

Via Theorem 3.7 and a sharp maximal inequality on mixed Morrey spaces, Nogayama [72, Theorem 1.2] gave a necessary and sufficient condition for the boundedness of the commutators of fractional integral operators on mixed Morrey spaces. To present this result, we first recall the notion of $\text{BMO}(\mathbb{R}^n)$.

Definition 3.5. The space $\text{BMO}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that their quasi-norms

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes Q in \mathbb{R}^n and

$$f_Q := \frac{1}{|Q|} \int_Q f(y) dy.$$

Theorem 3.8. Let $\alpha \in (0, n)$, $\vec{q} := (q_1, \dots, q_n) \in (1, \infty)^n$, $\vec{s} := (s_1, \dots, s_n) \in (1, \infty)^n$, $p \in (1, n/\alpha)$ and $r \in (1, \infty)$. Assume that

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}, \quad \sum_{j=1}^n \frac{1}{s_j} \geq \frac{n}{r}, \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$$

and, for any $j \in \{1, \dots, n\}$, $\frac{q_j}{p} = \frac{s_j}{r}$. Then the following statements are mutually equivalent:

- (i) $b \in \text{BMO}(\mathbb{R}^n)$;
- (ii) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ to $\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)$;
- (iii) $[b, I_\alpha]$ is bounded from $\overline{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)}$ to $\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)$;
- (iv) $[b, I_\alpha]$ is bounded from $\overline{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)}$ to $\mathcal{M}_1^r(\mathbb{R}^n)$.

Here, $\overline{\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)}$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ in the norm of $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$.

4 Anisotropic mixed-norm Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$

In this section, we first present the definition of anisotropic mixed-norm Hardy spaces and some basic facts of them (see Subsection 4.1 below). Then, various real-variable characterizations of these Hardy spaces, respectively, in terms of the maximal functions, atoms, finite atoms and Lusin area functions as well as Littlewood–Paley g -functions or g_λ^* -functions, are displayed (see Subsection 4.2.1 below). As the applications of these various real-variable characterizations, the dual spaces of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ (see Subsection 4.3 below), and the boundedness of anisotropic Calderón–Zygmund operators (see Subsection 4.4 below) are presented. Some errors and gaps existing in the proof of [45, Theorem 4.1] are also corrected and sealed (see Subsection 4.2.1 below). In addition, by providing a

new proof, we improve the maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ given in [23, Theorem 3.1] (see Subsection 4.2.2 below). The revised versions of the boundedness of anisotropic Calderón–Zygmund operators are obtained (see Subsection 4.4 below).

4.1 Definitions and basic properties

This subsection is devoted to recalling the notion of anisotropic quasi-homogeneous norms and anisotropic mixed-norm Hardy spaces as well as some basic properties of them. We begin with stating the notion of anisotropic quasi-homogeneous norms given in [10, 29] (see also [79]) as follows. For any $b := (b_1, \dots, b_n) \in \mathbb{R}^n$, $x := (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t \in [0, \infty)$, let $t^b x := (t^{b_1} x_1, \dots, t^{b_n} x_n)$.

Definition 4.1. Let $\vec{a} := (a_1, \dots, a_n) \in [1, \infty)^n$. The *anisotropic quasi-homogeneous norm* $|\cdot|_{\vec{a}}$, associated with \vec{a} , is a non-negative measurable function on \mathbb{R}^n defined by setting $|\vec{0}_n|_{\vec{a}} := 0$ and, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, $|x|_{\vec{a}} := t_0$, where t_0 is the unique positive number such that $|t_0^{-\vec{a}} x| = 1$, namely,

$$\frac{x_1^2}{t_0^{2a_1}} + \dots + \frac{x_n^2}{t_0^{2a_n}} = 1.$$

We also present the following notions of the anisotropic bracket and the homogeneous dimension (see, for instance, [79]).

Definition 4.2. Let $\vec{a} := (a_1, \dots, a_n) \in [1, \infty)^n$. The *anisotropic bracket*, associated with \vec{a} , is defined by setting, for any $x \in \mathbb{R}^n$,

$$\langle x \rangle_{\vec{a}} := |(1, x)|_{(1, \vec{a})}.$$

Furthermore, the *homogeneous dimension* v is defined by setting

$$v := |\vec{a}| := a_1 + \dots + a_n.$$

For any $\vec{a} \in [1, \infty)^n$, $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, we define the *anisotropic ball* $B_{\vec{a}}(x, r)$, with center x and radius r , by setting

$$B_{\vec{a}}(x, r) := \{y \in \mathbb{R}^n : |y - x|_{\vec{a}} < r\}.$$

Then $B_{\vec{a}}(x, r) = x + r^{\vec{a}} B_{\vec{a}}(\vec{0}_n, 1)$ and $|B_{\vec{a}}(x, r)| = v_n r^v$, where $v_n := |B(\vec{0}_n, 1)|$. In what follows, we always let $B_0 := \{y \in \mathbb{R}^n : |y| < 1\} = B_{\vec{a}}(\vec{0}_n, 1)$ (see [45, Lemma 2.4 (ii)]) and \mathfrak{B} be the set of all anisotropic balls, namely,

$$\mathfrak{B} := \{B_{\vec{a}}(x, r) : x \in \mathbb{R}^n, r \in (0, \infty)\}. \tag{4.1}$$

For any $B \in \mathfrak{B}$ centered at $x \in \mathbb{R}^n$ with radius $r \in (0, \infty)$, and $\delta \in (0, \infty)$, let

$$B^{(\delta)} := B_{\vec{a}}^{(\delta)}(x, r) := B_{\vec{a}}(x, \delta r). \tag{4.2}$$

In addition, for any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, the *anisotropic cube* $Q_{\vec{a}}(x, r)$ is defined by setting $Q_{\vec{a}}(x, r) := x + r^{\vec{a}}(-1, 1)^n$, whose Lebesgue measure $|Q_{\vec{a}}(x, r)|$ equals $2^n r^v$. Denote by \mathfrak{Q} the set of all anisotropic cubes, namely,

$$\mathfrak{Q} := \{Q_{\vec{a}}(x, r) : x \in \mathbb{R}^n, r \in (0, \infty)\}. \tag{4.3}$$

On another hand, recall that a *Schwartz function* is a $C^\infty(\mathbb{R}^n)$ function φ satisfying, for any $N \in \mathbb{Z}_+$ and multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|\varphi\|_{N, \alpha} := \sup_{x \in \mathbb{R}^n} \left\{ (1 + |x|)^N |\partial^\alpha \varphi(x)| \right\} < \infty.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the set of all Schwartz functions, equipped with the topology determined by $\{\|\cdot\|_{N, \alpha}\}_{N \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^n}$, and $\mathcal{S}'(\mathbb{R}^n)$ the *dual space* of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-* topology. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \left[\langle x \rangle_{\vec{a}}^N \sup_{|\alpha| \leq N} |\partial^\alpha \varphi(x)| \right] \leq 1 \right\}.$$

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $t \in (0, \infty)$, let $\varphi_t(\cdot) := t^{-v} \varphi(t^{-\vec{a}} \cdot)$.

To introduce the mixed-norm Hardy spaces, we first recall the following notions of radial maximal functions, non-tangential maximal functions and non-tangential grand maximal functions (see, for instance, [23]).

Definition 4.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *radial maximal function* $M_\varphi^0(f)$ of f associated to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi^0(f)(x) := \sup_{t \in (0, \infty)} |\varphi_t * f(x)|,$$

and the *non-tangential maximal function* $M_\varphi(f)$ of f associated to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{y \in B_{\vec{a}}(x, t), t \in (0, \infty)} |\varphi_t * f(y)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of f associated to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x).$$

In what follows, for any $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$, we always let

$$p_- := \min\{p_1, \dots, p_n\}, \quad p_+ := \max\{p_1, \dots, p_n\} \quad \text{and} \quad \underline{p} \in (0, \min\{1, p_-\}). \tag{4.4}$$

Similarly, for any $\vec{a} := (a_1, \dots, a_n) \in [1, \infty)^n$, let

$$a_- := \min\{a_1, \dots, a_n\} \quad \text{and} \quad a_+ := \max\{a_1, \dots, a_n\}. \tag{4.5}$$

We now present the notion of anisotropic mixed-norm Hardy spaces as follows, which was first introduced by Cleanthous *et al.* [23, Definition 3.3].

Definition 4.4. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $N_{\vec{p}} := \lfloor \nu \frac{a_+}{a_-} (\frac{1}{\min\{1, p_-\}} + 1) + \nu + 2a_+ \rfloor + 1$ and

$$N \in \mathbb{N} \cap [N_{\vec{p}}, \infty), \tag{4.6}$$

where a_-, a_+ are as in (4.5) and p_- as in (4.4). The *anisotropic mixed-norm Hardy space* $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is defined by setting

$$H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : M_N(f) \in L^{\vec{p}}(\mathbb{R}^n) \right\}$$

and, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, let $\|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} := \|M_N(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}$.

Remark 4.1. (i) Observe that the quasi-norm of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ in Definition 4.4 depends on N . However, by Theorem 4.9 below, we know that the space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is independent of the choice of N as long as N same as in Theorem 4.9.

(ii) Recall that Ho [43] introduced the mixed Lebesgue spaces with variable exponents. Then, based on those spaces, the corresponding variable mixed-norm Hardy spaces may also be worth studying.

The following completeness of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is a consequence of Proposition 4.2 and [47, Proposition 3.7] with A therein being as

$$A := \begin{pmatrix} 2^{a_1} & 0 & \dots & 0 \\ 0 & 2^{a_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 2^{a_n} \end{pmatrix}. \tag{4.7}$$

Theorem 4.1. Let \vec{p} and N be as in Definition 4.4. Then $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is complete.

The following proposition established in [23, Theorem 6.1] shows the relation between the mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and anisotropic mixed-norm Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$.

Proposition 4.1. Let $\vec{p} \in (1, \infty)^n$. Then $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n) = L^{\vec{p}}(\mathbb{R}^n)$ with equivalent norms.

4.2 Real-variable characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$

The goals of this subsection are twofold. The first one is in Subsection 4.2.1 below to display various real-variable characterizations of anisotropic mixed-norm Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, respectively, in terms of the maximal functions, atoms, finite atoms, Lusin area functions as well as Littlewood–Paley g -functions or g_{λ}^* -functions, and also correct some errors or seal some gaps existing in the proof of [45, Theorem 4.1]. The second one is in Subsection 4.2.2 below to provide a new proof of the maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, which allows the exponent N to have a weaker restriction than [23, Theorem 3.4] which is re-stated in Theorem 4.2 below.

4.2.1 Various real-variable characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$

To begin with, we first recall the following maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ established by Cleanthous *et al.* [23, Theorem 3.4].

Theorem 4.2. *Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$ and N be as in (4.6). Then, for any given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x) dx \neq 0$, the following statements are mutually equivalent:*

- (i) $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$;
- (ii) $f \in \mathcal{S}'(\mathbb{R}^n)$ and $M_{\varphi}(f) \in L^{\vec{p}}(\mathbb{R}^n)$;
- (iii) $f \in \mathcal{S}'(\mathbb{R}^n)$ and $M_{\varphi}^0(f) \in L^{\vec{p}}(\mathbb{R}^n)$.

Moreover, there exist two positive constants C_1 and C_2 , independent of f , such that

$$\|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C_1 \left\| M_{\varphi}^0(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C_1 \|M_{\varphi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C_2 \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

To complete the real-variable theory of the Hardy spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, Huang *et al.* [45, Theorem 3.16] established the atomic characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$. To state this atomic characterizations, we first introduce the notions of anisotropic mixed-norm (\vec{p}, r, s) -atoms and anisotropic mixed-norm atomic Hardy spaces as follows, which are, respectively, [45, Definitions 3.1 and 3.2]. In what follows, for any $q \in (0, \infty]$, denote by $L^q(\mathbb{R}^n)$ the space of all measurable functions f such that

$$\|f\|_{L^q(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^q dx \right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$.

Definition 4.5. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$, $r \in (1, \infty]$ and

$$s \in \left[\left\lfloor \frac{\nu}{a_-} \left(\frac{1}{p_-} - 1 \right) \right\rfloor, \infty \right) \cap \mathbb{Z}_+, \tag{4.8}$$

where a_- is as in (4.5) and p_- as in (4.4). An *anisotropic mixed-norm (\vec{p}, r, s) -atom* a is a measurable function on \mathbb{R}^n satisfying

- (i) $\text{supp } a := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset B$, where $B \in \mathfrak{B}$ with \mathfrak{B} as in (4.1);
- (ii) $\|a\|_{L^r(\mathbb{R}^n)} \leq \frac{|B|^{1/r}}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}}$;
- (iii) $\int_{\mathbb{R}^n} a(x) x^{\alpha} dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

In what follows, we always call an anisotropic mixed-norm (\vec{p}, r, s) -atom simply by a (\vec{p}, r, s) -atom.

Definition 4.6. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $r \in (1, \infty]$ and s be as in (4.8). The *anisotropic mixed-norm atomic Hardy space* $H_{\vec{a}}^{\vec{p}, r, s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_i\}_{i \in \mathbb{N}}$ of (\vec{p}, r, s) -atoms supported, respectively, in $\{B_i\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H_{\vec{a}}^{\vec{p}, r, s}(\mathbb{R}^n)$, let

$$\|f\|_{H_{\vec{a}}^{\vec{p}, r, s}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where \underline{p} is as in (4.4) and the infimum is taken over all decompositions of f as above

Remark 4.2. Recall that, in [45, Definition 3.2], $\underline{p} := \min\{1, p_-\}$ with p_- as in (4.4). However, in Definition 4.6 above, we correct the range of \underline{p} to be $(0, \min\{1, p_-\})$ due to the fact that Lemma 4.3 below holds true only for $\vec{p} \in (1, \infty)^n$.

It is well known that the Calderón–Zygmund decomposition is a key tool in the real-variable theory of function spaces. The idea behind this decomposition is that it is often useful to split a function or distribution into its “good” and “bad” part, and then use different techniques to analysis each part. Recall that Huang *et al.* [45, Lemma 3.12] obtained the following adapted Calderón–Zygmund decomposition, which plays a curial role in the proof of atomic characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$. Indeed, as has been demonstrated in the proof of the atomic decomposition for classical Hardy spaces, we need to use this lemma to break down functions or distributions into atoms.

Let Φ be some fixed $C^\infty(\mathbb{R}^n)$ function satisfying $\text{supp } \Phi \subset B(\vec{0}_n, 1)$ and $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we always let

$$M_0(f)(x) := M_\Phi^0(f)(x), \tag{4.9}$$

where $M_\Phi^0(f)$ is as in Definition 4.3 with ϕ replaced by Φ . In what follows, for any given $s \in \mathbb{Z}_+$, the symbol $\mathbb{P}_s(\mathbb{R}^n)$ denotes the linear space of all polynomials on \mathbb{R}^n with degree not greater than s .

Lemma 4.1. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $s \in \mathbb{Z}_+$ and N be as in (4.6). For any $\sigma \in (0, \infty)$ and $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, let

$$\mathcal{O} := \{x \in \mathbb{R}^n : M_N(f)(x) > \sigma\},$$

where M_N is as in Definition 4.3. Then the following statements hold true:

- (i) There exists a sequence $\{B_k^*\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ with \mathfrak{B} as in (4.1), which has finite intersection property, such that

$$\mathcal{O} = \bigcup_{k \in \mathbb{N}} B_k^*.$$

- (ii) There exist two distributions g and b such that $f = g + b$ in $\mathcal{S}'(\mathbb{R}^n)$.
- (iii) For the distribution g as in (ii) and any $x \in \mathbb{R}^n$,

$$M_0(g)(x) \lesssim M_N(f)(x) \mathbf{1}_{\mathcal{O}^c}(x) + \sum_{k \in \mathbb{N}} \frac{\sigma r_k^{\nu+(s+1)a_-}}{(r_k + |x - x_k|_{\bar{a}})^{\nu+(s+1)a_-}},$$

where a_- and M_0 are as in (4.5), respectively, (4.9), and the implicit positive constant is independent of f and g . Moreover, for any $k \in \mathbb{N}$, x_k denotes the center of B_k^* and there exists a constant $A^* \in (1, \infty)$, independent of k , such that $A^* - 1$ is small enough and $A^* r_k$ equals the radius of B_k^* .

- (iv) If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then the distribution g as in (ii) belongs to $L^\infty(\mathbb{R}^n)$ and $\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim \sigma$ with the implicit positive constant independent of f and g .
- (v) If s is as in (4.8) and b as in (ii), then $b = \sum_{k \in \mathbb{N}} b_k$ in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $k \in \mathbb{N}$, $b_k := (f - c_k)\eta_k$, $\{\eta_k\}_{k \in \mathbb{N}}$ is a partition of unity with respect to $\{B_k^*\}_{k \in \mathbb{N}}$, namely, for any $k \in \mathbb{N}$, $\eta_k \in C_c^\infty(\mathbb{R}^n)$, $\text{supp } \eta_k \subset B_k^*$, $0 \leq \eta_k \leq 1$ and

$$\mathbf{1}_{\mathcal{O}} = \sum_{k \in \mathbb{N}} \eta_k,$$

and $c_k \in \mathbb{P}_s(\mathbb{R}^n)$ is a polynomial such that, for any $q \in \mathbb{P}_s(\mathbb{R}^n)$,

$$\langle f - c_k, q \eta_k \rangle = 0.$$

Moreover, for any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$M_0(b_k)(x) \lesssim M_N(f)(x) \mathbf{1}_{B_k^*}(x) + \frac{\sigma r_k^{\nu+(s+1)a_-}}{|x - x_k|_{\bar{a}}^{\nu+(s+1)a_-}} \mathbf{1}_{(B_k^*)^c}(x),$$

where a_- and M_0 are as in (4.5), respectively, (4.9), and the implicit positive constant is independent of f and k .

Now we state the atomic characterizations of $H_{\bar{a}}^{\vec{p}}(\mathbb{R}^n)$ as follows, which was established by Huang *et al.* in [45, Theorem 3.16].

Theorem 4.3. Let $\bar{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $r \in (\max\{p_+, 1\}, \infty]$ with p_+ as in (4.4), N be as in (4.6) and s as in (4.8). Then

$$H_{\bar{a}}^{\vec{p}}(\mathbb{R}^n) = H_{\bar{a}}^{\vec{p}, r, s}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Combining Proposition 4.1 and Theorem 4.3, the following result was obtained in [45, Corollary 3.18].

Corollary 4.1. Let \vec{a} and s be as in Theorem 4.3, $\vec{p} \in (1, \infty)^n$ and $r \in (p_+, \infty]$ with p_+ as in (4.4). Then

$$L^{\vec{p}}(\mathbb{R}^n) = H_{\vec{a}}^{\vec{p}, r, s}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Moreover, the finite atomic characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ was also shown by Huang *et al.* in [45, Theorem 5.9] as follows. To state this theorem, we first recall the following anisotropic mixed-norm finite atomic Hardy space given in [45, Definition 5.1].

Definition 4.7. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $r \in (1, \infty]$ and s be as in (4.8). The *anisotropic mixed-norm finite atomic Hardy space* $H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist an $I \in \mathbb{N}$, a sequence $\{\lambda_i\}_{i \in [1, I] \cap \mathbb{N}} \subset \mathbb{C}$ and a finite sequence $\{a_i\}_{i \in [1, I] \cap \mathbb{N}}$ of (\vec{p}, r, s) -atoms supported, respectively, in $\{B_i\}_{i \in [1, I] \cap \mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i=1}^I \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)$, let

$$\|f\|_{H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i=1}^I \left[\frac{|\lambda_i| \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\frac{p}{p_-}} \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where \underline{p} is as in (4.4) and the infimum is taken over all decompositions of f as above.

Remark 4.3. Similarly to Remark 4.2, in Definition 4.7 above, we correct the range of \underline{p} to be $(0, \min\{1, p_-\})$ due to the fact that Lemma 4.3 below holds true only for $\vec{p} \in (1, \infty)^n$.

Theorem 4.4. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$ and s be as in (4.8).

- (i) If $r \in (\max\{p_+, 1\}, \infty)$ with p_+ as in (4.4), then $\|\cdot\|_{H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)$;
- (ii) $\|\cdot\|_{H_{\vec{a}, \text{fin}}^{\vec{p}, \infty, s}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{\vec{a}, \text{fin}}^{\vec{p}, \infty, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, here and thereafter, $C(\mathbb{R}^n)$ denotes the set of all continuous functions on \mathbb{R}^n .

To establish the Littlewood–Paley function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$. Huang *et al.* [45, Lemma 4.13] first established the anisotropic Calderón reproducing formula as

follows. Indeed, it is known that the Calderón reproducing formulae are bridges to connect the theory of function spaces and the boundedness of operators. In what follows, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\phi}$ denotes its Fourier transform, namely, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x) e^{-2\pi i x \cdot \xi} dx,$$

where $i := \sqrt{-1}$.

Lemma 4.2. *Let $\vec{a} \in [1, \infty)^n$ and $s \in \mathbb{Z}_+$. For any $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\text{supp } \phi \subset B_0$,*

$$\int_{\mathbb{R}^n} x^\gamma \phi(x) dx = 0 \text{ for any } \gamma \in \mathbb{Z}_+^n \text{ with } |\gamma| \leq s,$$

$|\widehat{\phi}(\xi)| \geq C$ for any $\xi \in \{x \in \mathbb{R}^n : 2^{-(1+a_+)} \leq |x| \leq 1\}$, where $C \in (0, \infty)$ is a constant, there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

- (i) *$\text{supp } \widehat{\psi}$ is compact and away from the origin;*
- (ii) *for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, $\sum_{k \in \mathbb{Z}} \widehat{\psi}(2^{k\vec{a}} \xi) \widehat{\phi}(2^{k\vec{a}} \xi) = 1$.*

*Moreover, for any $f \in L^2(\mathbb{R}^n)$, $f = \sum_{k \in \mathbb{Z}} f * \psi_k * \phi_k$ in $L^2(\mathbb{R}^n)$. The same holds true in $\mathcal{S}'(\mathbb{R}^n)$ for any $f \in \mathcal{S}'_0(\mathbb{R}^n)$.*

Let $\vec{a} \in [1, \infty)^n$. Assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the same assumptions as ϕ in Lemma 4.2 with s as in (4.8). Then, for any $\lambda \in (0, \infty)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the anisotropic Lusin area function $S(f)$, the anisotropic Littlewood–Paley g -function $g(f)$ and the anisotropic Littlewood–Paley g_λ^* -function $g_\lambda^*(f)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$S(f)(x) := \left[\sum_{k \in \mathbb{Z}} 2^{-k\nu} \int_{B_{\vec{a}}(x, 2^k)} |f * \phi_k(y)|^2 dy \right]^{1/2},$$

$$g(f)(x) := \left[\sum_{k \in \mathbb{Z}} |f * \phi_k(x)|^2 \right]^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left\{ \sum_{k \in \mathbb{Z}} 2^{-k\nu} \int_{\mathbb{R}^n} \left[\frac{2^k}{2^k + |x - y|_{\vec{a}}} \right]^{\lambda\nu} |f * \phi_k(y)|^2 dy \right\}^{1/2},$$

where, for any $k \in \mathbb{Z}$, $\phi_k(\cdot) := 2^{-k\nu} \phi(2^{-k\vec{a}} \cdot)$.

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_k \rightarrow 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \rightarrow \infty$. In what follows, we always let $\mathcal{S}'_0(\mathbb{R}^n)$ be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity.

As an application of the atomic characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, Huang *et al.* in [45, Theorems 4.1 through 4.3] obtained the following Littlewood–Paley function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ with the help of Lemma 4.2.

Theorem 4.5. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$ and N be as in (4.6). Then $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{\vec{p}}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$C^{-1} \|S(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|S(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

Theorem 4.6. Let \vec{a} , \vec{p} and N be as in Theorem 4.5. Then $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g(f) \in L^{\vec{p}}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$C^{-1} \|g(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|g(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

Theorem 4.7. Let \vec{a} , \vec{p} and N be as in Theorem 4.5 and $\lambda \in (1 + \frac{2}{\min\{p_-, 2\}}, \infty)$, where p_- is as in (4.4). Then $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g_{\lambda}^*(f) \in L^{\vec{p}}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$C^{-1} \|g_{\lambda}^*(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|g_{\lambda}^*(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

We point out that, in the proof of Theorem 4.5, namely, in the proof of [45, Theorem 4.1], we first need to show that the $L^{\vec{p}}(\mathbb{R}^n)$ quasi-norms of the anisotropic Lusin area function $S(f)$ are independent of the choices of φ and ψ as in Lemma 4.2. For this purpose, we denote by $S_{\varphi}(f)$ and $S_{\psi}(f)$ the anisotropic Lusin area functions defined, respectively, by φ and ψ .

Theorem 4.8. Let $\vec{p} \in (0, \infty)^n$, φ and ψ be as in Lemma 4.2 with s as in (4.8). Then there exists a positive constant C such that, for any $f \in \mathcal{S}'_0(\mathbb{R}^n)$,

$$C^{-1} \|S_{\varphi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq \|S_{\psi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|S_{\varphi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

To prove Theorem 4.8, we first recall the notion of the anisotropic Hardy–Littlewood maximal operator as follows.

Definition 4.8. The anisotropic Hardy–Littlewood maximal function $M_{\text{HL}}^{\vec{a}}(f)$ of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{\text{HL}}^{\vec{a}}(f)(x) := \sup_{x \in Q \in \Omega} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{4.10}$$

where Ω is as in (4.3).

The following boundedness of the anisotropic Hardy–Littlewood maximal operator $M_{\text{HL}}^{\vec{a}}$ on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ is just [45, Lemma 3.5].

Lemma 4.3. *Let $\vec{p} \in (1, \infty)^n$. Then there exists a positive constant C , depending on \vec{p} , such that, for any $f \in L^{\vec{p}}(\mathbb{R}^n)$,*

$$\left\| M_{\text{HL}}^{\vec{a}}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{L^{\vec{p}}(\mathbb{R}^n)}, \tag{4.11}$$

where $M_{\text{HL}}^{\vec{a}}$ is as in (4.10).

Remark 4.4. Observe that, in [45, Lemma 3.5], $\vec{p} \in (1, \infty]^n$. However, Professor Ferenc Weisz pointed out to us that, when $n := 2$ and $\vec{p} := (p_1, \infty)$ with $p_1 \in (1, \infty)$, [45, Lemma 3.5] is not correct. To show this, we find the following counterexample.

Let $\vec{a} := (1, 1)$, $I := \{(x_1, x_2) \in (0, \infty)^2 : x_1 \geq x_2\}$, $\delta := 1 - \frac{1}{p_1}$ and, for any $(x_1, x_2) \in I$, $f(x_1, x_2) := \frac{x_2^\delta}{x_1}$; otherwise, $f \equiv 0$. Then, for any $x_2 \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}} |f(x_1, x_2)|^{p_1} dx_1 = x_2^{\delta p_1} \int_{x_2}^{\infty} \frac{dx_1}{x_1^{p_1}} = \frac{1}{p_1 - 1}.$$

Thus,

$$\|f\|_{L^{(p_1, \infty)}(\mathbb{R}^2)} = \left(\frac{1}{p_1 - 1} \right)^{1/p_1}. \tag{4.12}$$

On another hand, for any $(x_1, x_2) \in I$, let $Q := [x_1, 2x_1] \times [0, x_1]$. Obviously, $(x_1, x_2) \in Q \subset I$. Therefore, for any $(x_1, x_2) \in I$,

$$M_{\text{HL}}^{\vec{a}}(f)(x_1, x_2) \geq \frac{1}{|Q|} \int_Q |f(t_1, t_2)| dt_1 dt_2 = \frac{1}{x_1^2} \int_0^{x_1} t_2^\delta dt_2 \int_{x_1}^{2x_1} \frac{1}{t_1} dt_1 \sim x_1^{\delta - 1},$$

which further implies that

$$\int_{\mathbb{R}} \left| M_{\text{HL}}^{\vec{a}}(f)(x_1, x_2) \right|^{p_1} dx_1 \gtrsim \int_{x_2}^{\infty} x_1^{(\delta - 1)p_1} dx_1 \sim \int_{x_2}^{\infty} x_1^{-1} dx_1 \sim \infty.$$

From this and (4.12), it follows that, in this case, (4.11) does not hold true.

Thus, in Lemma 4.3, we restrict the range of \vec{p} to be $(1, \infty)^n$. Moreover, due to the fact that Lemma 4.3 does not hold true for some $p_{i_1} \in (1, \infty)$ and $p_{i_2} = \infty$ with $i_1 < i_2$, which was used in the proof of [45, Lemma 3.15], the range of exponent \vec{p} in [45], namely, \underline{p} in this survey, should be corrected to be $(0, \min\{1, p_-\})$. In addition, we point out that, if $n := 2$ and $\vec{p} := (\infty, p_2)$ with $p_2 \in (1, \infty)$, then (4.11) still holds true; its proof is similar to the proof of [45, Lemma 3.5].

To show Theorem 4.8, the following three succeeding technical lemmas are also necessary. Lemmas 4.4 and 4.5 are just, respectively, [45, Lemma 3.7] and a consequence of [16, Lemma 5.4] with A therein as in (4.7).

Lemma 4.4. Let $\vec{p} \in (1, \infty)^n$ and $u \in (1, \infty]$. Then there exists a positive constant C such that, for any sequence $\{f_k\}_{k \in \mathbb{N}} \subset L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{k \in \mathbb{N}} [M_{\text{HL}}^{\vec{a}}(f_k)]^u \right\}^{1/u} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \left\| \left(\sum_{k \in \mathbb{N}} |f_k|^u \right)^{1/u} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}$$

with the usual modification made when $u = \infty$, where $M_{\text{HL}}^{\vec{a}}$ is as in (4.10).

Lemma 4.5. Let a_- and s be, respectively, as in (4.5) and (4.8) and $\varphi, \phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, $\int_{\mathbb{R}^n} \varphi(x) x^\alpha dx = 0$ and $\int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0$. Then there exists a positive constant C such that, for any $k, \ell \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|\varphi_k * \phi_\ell(x)| \leq C 2^{-(s+1)|k-\ell|a_-} \frac{2^{(k \vee \ell)(s+1)va_-}}{[2^{(k \vee \ell)} + |x|_{\vec{a}}]^{(s+1)va_- + v}},$$

here and thereafter, for any $k, \ell \in \mathbb{Z}$, $k \vee \ell := \max\{k, \ell\}$.

The following lemma plays a key role in the proof of Theorem 4.8.

Lemma 4.6. Let s be as in (4.8) and $r \in (\frac{v}{v+(s+1)a_-}, 1]$. Then there exists a positive constant C such that, for any $k, \ell \in \mathbb{Z}$, $\{c_{Q_{\vec{a}}}\}_{Q_{\vec{a}} \in \mathfrak{Q}} \subset [0, \infty)$ with \mathfrak{Q} as in (4.3), and $x \in \mathbb{R}^n$,

$$\begin{aligned} & \sum_{l(Q_{\vec{a}}) = 2^{k-1}} |Q_{\vec{a}}| \frac{2^{(k \vee \ell)(s+1)a_-}}{[2^{(k \vee \ell)} + |x - z_{Q_{\vec{a}}}|_{\vec{a}}]^{(s+1)a_- + v}} c_{Q_{\vec{a}}} \\ & \leq C 2^{-[k - (k \vee \ell)](1/r - 1)v} \left\{ M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}}) = 2^{k-1}} [c_{Q_{\vec{a}}}]^r \mathbf{1}_{Q_{\vec{a}}} \right) (x) \right\}^{1/r}, \end{aligned}$$

where $l(Q_{\vec{a}})$ denotes the side-length of $Q_{\vec{a}}$ and $z_{Q_{\vec{a}}} \in Q_{\vec{a}}$.

Proof. We show this lemma by the following two cases.

Case I) $\ell \geq k$. In this case, applying the well-known inequality that, for any $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and $\theta \in [0, 1]$,

$$\left(\sum_{i \in \mathbb{N}} |\lambda_i| \right)^\theta \leq \sum_{i \in \mathbb{N}} |\lambda_i|^\theta, \tag{4.13}$$

and the Tonelli theorem, we conclude that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
 & \left\{ \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{\ell(s+1)a_-}}{[2^\ell + |x - z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_- + v}} c_{Q_{\bar{a}}} \right\}^r \\
 & \leq \sum_{l(Q_{\bar{a}})=2^{k-1}} \int_{Q_{\bar{a}}} |Q_{\bar{a}}|^{r-1} \left[\frac{1}{2^\ell + |x - z_{Q_{\bar{a}}}|_{\bar{a}}} \right]^{vr} \left[\frac{2^\ell}{2^\ell + |x - z_{Q_{\bar{a}}}|_{\bar{a}}} \right]^{(s+1)ra_-} (c_{Q_{\bar{a}}})^r dy \\
 & \lesssim \int_{\mathbb{R}^n} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^\ell + |x - y|_{\bar{a}}} \right]^{vr} \left[\frac{2^\ell}{2^\ell + |x - y|_{\bar{a}}} \right]^{(s+1)ra_-} \\
 & \quad \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy \\
 & \sim I_1 + I_2,
 \end{aligned} \tag{4.14}$$

where, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
 I_1 := & \int_{|x-y|_{\bar{a}} \leq 2^\ell} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^\ell + |x-y|_{\bar{a}}} \right]^{vr} \left[\frac{2^\ell}{2^\ell + |x-y|_{\bar{a}}} \right]^{(s+1)ra_-} \\
 & \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy
 \end{aligned}$$

and

$$\begin{aligned}
 I_2 := & \sum_{j \in \mathbb{N}} \int_{2^{j+\ell-1} < |x-y|_{\bar{a}} \leq 2^{j+\ell}} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^\ell + |x-y|_{\bar{a}}} \right]^{vr} \left[\frac{2^\ell}{2^\ell + |x-y|_{\bar{a}}} \right]^{(s+1)ra_-} \\
 & \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy.
 \end{aligned}$$

For I_1 , we have

$$\begin{aligned}
 I_1 & \leq \int_{|x-y|_{\bar{a}} \leq 2^\ell} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left(\frac{1}{2^\ell} \right)^{vr} \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy \\
 & = \frac{1}{|B_{\bar{a}}(x, 2^\ell)|} \int_{|x-y|_{\bar{a}} \leq 2^\ell} \left[\frac{|B_{\bar{a}}(x, 2^\ell)|}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy \\
 & \leq 2^{(\ell-k)(1-r)v} M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}] \right) (x).
 \end{aligned} \tag{4.15}$$

For I_2 , from the fact that $2^{j+\ell-1} < |x-y|_{\bar{a}} \leq 2^{j+\ell}$, it follows that, for any $x \in \mathbb{R}^n$,

$$I_2 \lesssim \sum_{j \in \mathbb{N}} \int_{2^{j+\ell-1} < |x-y|_{\bar{a}} \leq 2^{j+\ell}} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left(\frac{1}{2^{j+\ell}} \right)^{vr} \left(\frac{1}{2^j} \right)^{(s+1)ra_-} \\ \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy.$$

Thus,

$$I_2 \lesssim \sum_{j \in \mathbb{N}} 2^{-j(s+1)ra_-} \frac{1}{|B_{\bar{a}}(x, 2^{j+\ell})|} \int_{2^{j+\ell-1} < |x-y|_{\bar{a}} \leq 2^{j+\ell}} \left[\frac{|B_{\bar{a}}(x, 2^{j+\ell})|}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \\ \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy \\ \lesssim \sum_{j \in \mathbb{N}} 2^{-j[(s+1)ra_- - (1-r)v]} 2^{(\ell-k)(1-r)v} M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} [c_{Q_{\bar{a}}}]^r \mathbf{1}_{Q_{\bar{a}}} \right) (x),$$

which, together with (4.14), (4.15) and the fact that $r > \frac{v}{v+(s+1)a_-}$, further implies that

$$\left\{ \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{\ell(s+1)a_-}}{[2^\ell + |x-z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_-+v}} c_{Q_{\bar{a}}} \right\}^r \\ \lesssim 2^{(\ell-k)(1-r)v} M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} [c_{Q_{\bar{a}}}]^r \mathbf{1}_{Q_{\bar{a}}} \right) (x). \tag{4.16}$$

Case II) $\ell < k$. In this case, by (4.13) and the Tonelli theorem again, we conclude that, for any $x \in \mathbb{R}^n$,

$$\left\{ \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{k(s+1)a_-}}{[2^k + |x-z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_-+v}} c_{Q_{\bar{a}}} \right\}^r \\ \lesssim \sum_{l(Q_{\bar{a}})=2^{k-1}} \int_{Q_{\bar{a}}} |Q_{\bar{a}}|^{r-1} \left[\frac{1}{2^k + |x-z_{Q_{\bar{a}}}|_{\bar{a}}} \right]^{vr} \left[\frac{2^k}{2^k + |x-z_{Q_{\bar{a}}}|_{\bar{a}}} \right]^{(s+1)ra_-} (c_{Q_{\bar{a}}})^r dy \\ \lesssim \int_{\mathbb{R}^n} \left[\frac{1}{|B_{\bar{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^k + |x-y|_{\bar{a}}} \right]^{vr} \left[\frac{2^k}{2^k + |x-y|_{\bar{a}}} \right]^{(s+1)ra_-} \\ \times \sum_{l(Q_{\bar{a}})=2^{k-1}} [(c_{Q_{\bar{a}}})^r \mathbf{1}_{Q_{\bar{a}}}(y)] dy \\ \sim I_3 + I_4, \tag{4.17}$$

where, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} I_3 := & \int_{|x-y|_{\vec{a}} \leq 2^k} \left[\frac{1}{|B_{\vec{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^k + |x-y|_{\vec{a}}} \right]^{vr} \left[\frac{2^k}{2^k + |x-y|_{\vec{a}}} \right]^{(s+1)ra_-} \\ & \times \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy \end{aligned}$$

and

$$\begin{aligned} I_4 := & \sum_{j \in \mathbb{N}} \int_{2^{k+j-1} < |x-y|_{\vec{a}} \leq 2^{k+j}} \left[\frac{1}{|B_{\vec{a}}(x, 2^k)|} \right]^{1-r} \left[\frac{1}{2^k + |x-y|_{\vec{a}}} \right]^{vr} \left[\frac{2^k}{2^k + |x-y|_{\vec{a}}} \right]^{(s+1)ra_-} \\ & \times \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy. \end{aligned}$$

For I_3 , we have

$$\begin{aligned} I_3 & \lesssim \int_{|x-y|_{\vec{a}} \leq 2^k} \left[\frac{1}{|B_{\vec{a}}(x, 2^k)|} \right]^{1-r} \left(\frac{1}{2^k} \right)^{vr} \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy \\ & \sim \frac{1}{|B_{\vec{a}}(x, 2^k)|} \int_{|x-y|_{\vec{a}} \leq 2^k} \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy \\ & \lesssim M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}] \right) (x). \end{aligned} \quad (4.18)$$

For I_4 , from the facts that $2^{j+k-1} < |x-y|_{\vec{a}} \leq 2^{j+k}$ and $r > \frac{\nu}{\nu + (s+1)a_-}$, it follows that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} I_4 & \lesssim \sum_{j \in \mathbb{N}} \int_{2^{j+k-1} < |x-y|_{\vec{a}} \leq 2^{j+k}} \left[\frac{1}{|B_{\vec{a}}(x, 2^k)|} \right]^{1-r} \left(\frac{1}{2^{j+k}} \right)^{vr} \left(\frac{1}{2^j} \right)^{(s+1)ra_-} \\ & \quad \times \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy \\ & \sim \sum_{j \in \mathbb{N}} 2^{-j(s+1)ra_-} \frac{1}{|B_{\vec{a}}(x, 2^{j+k})|} \int_{2^{j+k-1} < |x-y|_{\vec{a}} \leq 2^{j+k}} \left[\frac{|B_{\vec{a}}(x, 2^{j+k})|}{|B_{\vec{a}}(x, 2^k)|} \right]^{1-r} \\ & \quad \times \sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}(y)] dy \\ & \lesssim \sum_{j \in \mathbb{N}} 2^{-j[(s+1)ra_- - (1-r)\nu]} M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}] \right) (x) \\ & \lesssim M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}}) = 2^{k-1}} [(c_{Q_{\vec{a}}})^r \mathbf{1}_{Q_{\vec{a}}}] \right) (x). \end{aligned} \quad (4.19)$$

From (4.17)–(4.19), we deduce that, for any $x \in \mathbb{R}^n$,

$$\left\{ \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{k(s+1)a_-}}{[2^k + |x - z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_- + \nu}} c_{Q_{\bar{a}}} \right\}^r \lesssim M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} [c_{Q_{\bar{a}}}]^r \mathbf{1}_{Q_{\bar{a}}} \right) (x).$$

Combining this and (4.16), we further conclude that, for any given $r \in (\frac{\nu}{\nu + (s+1)a_-}, 1]$, any $k, \ell \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} & \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{(k\vee\ell)(s+1)a_-}}{[2^{(k\vee\ell)} + |x - z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_- + \nu}} c_{Q_{\bar{a}}} \\ & \lesssim 2^{-[k - (k\vee\ell)](1/r - 1)\nu} \left\{ M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} [c_{Q_{\bar{a}}}]^r \mathbf{1}_{Q_{\bar{a}}} \right) (x) \right\}^{1/r}. \end{aligned}$$

This finishes the proof of Lemma 4.6. □

Now we prove Theorem 4.8.

Proof of Theorem 4.8. By symmetry, to finish the proof of this theorem, we only need to show that, for any $f \in S'_0(\mathbb{R}^n)$,

$$\|S_\varphi(f)\|_{L^{\bar{p}}(\mathbb{R}^n)} \lesssim \|S_\psi(f)\|_{L^{\bar{p}}(\mathbb{R}^n)}. \tag{4.20}$$

To this end, for any $k \in \mathbb{Z}$ and $z \in \mathbb{R}^n$, let

$$m_{\psi_k}(f)(z) := \left[2^{-k\nu} \int_{B_{\bar{a}}(z, 2^k)} |f * \psi_k(y)|^2 dy \right]^{1/2}$$

and, for any $\ell \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $y \in B_{\bar{a}}(x, 2^\ell)$,

$$E_{\varphi_\ell}(f)(y) := f * \varphi_\ell(y).$$

Then, from Lemma 4.2 and the Lebesgue dominated convergence theorem, it follows that, for any $\ell \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $y \in B_{\bar{a}}(x, 2^\ell)$,

$$\begin{aligned} E_{\varphi_\ell}(f)(y) &= \sum_{k \in \mathbb{Z}} f * \psi_k * \varphi_k * \varphi_\ell(y) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} f * \psi_k(z) \varphi_k * \varphi_\ell(y - z) dz \\ &= \sum_{k \in \mathbb{Z}} \sum_{l(Q_{\bar{a}})=2^{k-1}} \int_{Q_{\bar{a}}} f * \psi_k(z) \varphi_k * \varphi_\ell(y - z) dz \end{aligned} \tag{4.21}$$

in $S'(\mathbb{R}^n)$, where, for any $Q_{\bar{a}} \in \Omega$ with Ω as in (4.3), $l(Q_{\bar{a}})$ denotes the side-length of $Q_{\bar{a}}$.

On another hand, by Lemma 4.5, we know that, for any $k, \ell \in \mathbb{Z}$, $z \in Q_{\bar{a}}$, $x \in \mathbb{R}^n$ and $y \in B_{\bar{a}}(x, 2^\ell)$,

$$|\varphi_k * \varphi_\ell(y-z)| \lesssim 2^{-(s+1)|k-\ell|a_-} \frac{2^{(k\vee\ell)(s+1)a_-}}{[2^{(k\vee\ell)} + |y-z|_{\bar{a}}]^{(s+1)a_- + \nu}},$$

where s is as in (4.8). From this and the fact that $y \in B_{\bar{a}}(x, 2^\ell)$, we further deduce that there exists a $z_{Q_{\bar{a}}} \in Q_{\bar{a}}$ such that, for any $k, \ell \in \mathbb{Z}$ and $z \in Q_{\bar{a}}$,

$$|\varphi_k * \varphi_\ell(y-z)| \lesssim 2^{-(s+1)|k-\ell|a_-} \frac{2^{(k\vee\ell)(s+1)a_-}}{[2^{(k\vee\ell)} + |x-z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_- + \nu}}. \quad (4.22)$$

In addition, observe that, for any $z \in Q_{\bar{a}}$, $Q_{\bar{a}} \subset B_{\bar{a}}(z, 2^k)$ and $|Q_{\bar{a}}| \sim |B_{\bar{a}}(z, 2^k)|$ and, therefore, via the Hölder inequality, we have

$$\left| \frac{1}{|Q_{\bar{a}}|} \int_{Q_{\bar{a}}} f * \psi_k(y) dy \right| \lesssim \left[\frac{1}{|B_{\bar{a}}(z, 2^k)|} \int_{B_{\bar{a}}(z, 2^k)} |f * \psi_k(y)|^2 dy \right]^{1/2} \sim m_{\psi_k}(f)(z),$$

which further implies that

$$\left| \frac{1}{|Q_{\bar{a}}|} \int_{Q_{\bar{a}}} f * \psi_k(y) dy \right| \lesssim \inf_{z \in Q_{\bar{a}}} m_{\psi_k}(f)(z).$$

By this, (4.21), (4.22) and Lemma 4.6, we find that, for any given $r \in (\frac{\nu}{\nu+(s+1)a_-}, 1]$, any $\ell \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $y \in B_{\bar{a}}(x, 2^\ell)$,

$$\begin{aligned} |E_{\varphi_\ell}(f)(y)| &\lesssim \sum_{k \in \mathbb{Z}} 2^{-(s+1)|k-\ell|a_-} \sum_{l(Q_{\bar{a}})=2^{k-1}} |Q_{\bar{a}}| \frac{2^{(k\vee\ell)(s+1)a_-}}{[2^{(k\vee\ell)} + |x-z_{Q_{\bar{a}}}|_{\bar{a}}]^{(s+1)a_- + \nu}} \inf_{z \in Q_{\bar{a}}} m_{\psi_k}(f)(z) \\ &\lesssim \sum_{k \in \mathbb{Z}} 2^{-(s+1)|k-\ell|a_-} 2^{-[k-(k\vee\ell)](1/r-1)\nu} \\ &\quad \times \left\{ M_{\text{HL}}^{\bar{a}} \left(\sum_{l(Q_{\bar{a}})=2^{k-1}} \inf_{z \in Q_{\bar{a}}} [m_{\psi_k}(f)(z)]^r \mathbf{1}_{Q_{\bar{a}}} \right) (x) \right\}^{1/r}. \end{aligned} \quad (4.23)$$

Due to the fact that s is as in (4.8), we can choose r such that

$$r \in \left(\frac{\nu}{\nu+(s+1)a_-}, \min\{1, p_-\} \right)$$

with p_- as in (4.4). Then, from (4.23), we deduce that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
 [S_\varphi(f)(x)]^2 &= \sum_{\ell \in \mathbb{Z}} 2^{-\ell v} \int_{B_{\vec{a}}(x, 2^\ell)} |E_{\varphi_\ell}(f)(y)|^2 dy \\
 &\lesssim \sum_{\ell \in \mathbb{Z}} \left[\sum_{k \in \mathbb{Z}} 2^{-(s+1)|k-\ell|a_-} 2^{-[k-(k \vee \ell)](1/r-1)v} \right. \\
 &\quad \left. \times \left\{ M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}})=2^{k-1}} \inf_{z \in Q_{\vec{a}}} [m_{\psi_k}(f)(z)]^r \mathbf{1}_{Q_{\vec{a}}} \right) (x) \right\}^{1/r} \right]^2,
 \end{aligned}$$

which, together with the Hölder inequality and the fact that $r > \frac{v}{v+(s+1)a_-}$, implies that

$$\begin{aligned}
 [S_\varphi(f)(x)]^2 &\lesssim \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-(s+1)|k-\ell|a_-} 2^{-[k-(k \vee \ell)](1/r-1)v} \\
 &\quad \times \left\{ M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}})=2^{k-1}} \inf_{z \in Q_{\vec{a}}} [m_{\psi_k}(f)(z)]^r \mathbf{1}_{Q_{\vec{a}}} \right) (x) \right\}^{2/r} \\
 &\lesssim \sum_{k \in \mathbb{Z}} \left\{ M_{\text{HL}}^{\vec{a}} \left(\sum_{l(Q_{\vec{a}})=2^{k-1}} \inf_{z \in Q_{\vec{a}}} [m_{\psi_k}(f)(z)]^r \mathbf{1}_{Q_{\vec{a}}} \right) (x) \right\}^{2/r} \\
 &\lesssim \sum_{k \in \mathbb{Z}} \left\{ M_{\text{HL}}^{\vec{a}} \left([m_{\psi_k}(f)]^r \right) (x) \right\}^{2/r}.
 \end{aligned}$$

Therefore, by the fact that $r < p_-$ and Lemma 4.4, we find that

$$\begin{aligned}
 \|S_\varphi(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} \left\{ M_{\text{HL}}^{\vec{a}} \left([m_{\psi_k}(f)]^r \right) \right\}^{2/r} \right)^{r/2} \right\|_{L^{\vec{p}/r}(\mathbb{R}^n)}^{1/r} \\
 &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} [m_{\psi_k}(f)]^2 \right)^{1/2} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \|S_\psi(f)\|_{L^{\vec{p}}(\mathbb{R}^n)},
 \end{aligned}$$

which implies (4.20) holds true and hence completes the proof of Theorem 4.8. □

We point out that, in the proof of the sufficiency of [45, Theorem 4.1], there exist some errors and gaps. Therefore we give a revised proof as follows.

To show the sufficiency of Theorem 4.5, the following lemma is necessary.

Lemma 4.7. *Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $\beta \in (0, \infty)$, $\varepsilon \in (0, \infty)$, $\kappa \in (0, \infty)$ and $r \in [1, \infty] \cap (p_+, \infty]$ with p_+ as in (4.4). Assume that $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, $\{B_i\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ and $\{m_i^{\varepsilon, \kappa}\}_{i \in \mathbb{N}} \subset L^r(\mathbb{R}^n)$ satisfy*

that, for any $\varepsilon \in (0, \infty)$, $\kappa \in (0, \infty)$ and $i \in \mathbb{N}$, $\text{supp } m_i^{\varepsilon, \kappa} \subset B_i^{(\beta)}$ with $B_i^{(\beta)}$ as in (4.2),

$$\|m_i^{\varepsilon, \kappa}\|_{L^r(\mathbb{R}^n)} \leq \frac{|B_i|^{1/r}}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \quad (4.24)$$

and

$$\left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} < \infty.$$

Then, for any $\kappa \in (0, \infty)$,

$$\left\| \liminf_{\varepsilon \rightarrow 0^+} \left[\sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}|^p \right]^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}, \quad (4.25)$$

where \underline{p} is as in (4.4) and C is a positive constant independent of $\{\lambda_i\}_{i \in \mathbb{N}}$, $\{B_i\}_{i \in \mathbb{N}}$, $\{m_i^{\varepsilon, \kappa}\}_{i \in \mathbb{N}}$, ε and κ .

Proof. By Theorem 2.13, we find that there exists some $g \in L^{(\bar{p}/\underline{p})'}(\mathbb{R}^n)$ with $\|g\|_{L^{(\bar{p}/\underline{p})'}(\mathbb{R}^n)} = 1$ such that, for any $\kappa \in (0, \infty)$,

$$\begin{aligned} & \left\| \liminf_{\varepsilon \rightarrow 0^+} \left[\sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}|^p \right]^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \\ &= \left\| \liminf_{\varepsilon \rightarrow 0^+} \sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}|^p \right\|_{L^{\bar{p}/\underline{p}}(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \liminf_{\varepsilon \rightarrow 0^+} \sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}(x)|^p |g(x)| dx. \end{aligned}$$

From this, the Fatou lemma, the Tonelli theorem, the Hölder inequality and (4.24), we deduce that, for any $\kappa \in (0, \infty)$ and $r \in [1, \infty] \cap (p_+, \infty]$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \liminf_{\varepsilon \rightarrow 0^+} \sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}(x)|^p |g(x)| dx \\ & \leq \liminf_{\varepsilon \rightarrow 0^+} \sum_{i \in \mathbb{N}} |\lambda_i|^p \|m_i^{\varepsilon, \kappa}\|_{L^r(\mathbb{R}^n)}^p \|g\|_{L^{(r/p)'}(B_i^{(\beta)})} \leq \sum_{i \in \mathbb{N}} \frac{|\lambda_i|^p |B_i|^{p/r}}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}^p} \|g\|_{L^{(r/p)'}(B_i^{(\beta)})} \\ & \lesssim \sum_{i \in \mathbb{N}} \frac{|\lambda_i|^p |B_i^{(\beta)}|}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}^p} \inf_{z \in B_i^{(\beta)}} \left[M_{\text{HL}}^{\vec{a}} \left(|g|^{(r/p)'} \right) (z) \right]^{1/(r/p)'} \\ & \lesssim \int_{\mathbb{R}^n} \sum_{i \in \mathbb{N}} \frac{|\lambda_i|^p \mathbf{1}_{B_i^{(\beta)}}(x)}{\|\mathbf{1}_{B_i}\|_{L^{\bar{p}}(\mathbb{R}^n)}^p} \left[M_{\text{HL}}^{\vec{a}} \left(|g|^{(r/p)'} \right) (x) \right]^{1/(r/p)'} dx, \end{aligned}$$

which, together with Theorem 2.6, [45, Remark 3.8], Lemma 4.3 and the fact that $r \in (p_+, \infty]$, further implies that

$$\begin{aligned} & \int_{\mathbb{R}^n} \liminf_{\varepsilon \rightarrow 0^+} \sum_{i \in \mathbb{N}} |\lambda_i m_i^{\varepsilon, \kappa}(x)|^p |g(x)| dx \\ & \lesssim \left\| \sum_{i \in \mathbb{N}} \frac{|\lambda_i|^p \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\vec{p}}(\mathbb{R}^n)}^p} \right\|_{L^{\vec{p}/p}(\mathbb{R}^n)} \left\| \left[M_{\text{HL}}^{\vec{a}}(|g|^{(r/p)'}) \right]^{1/(r/p)'} \right\|_{L^{(\vec{p}/p)'(\mathbb{R}^n)}} \\ & \lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B_i}}{\|\mathbf{1}_{B_i}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \|g\|_{L^{(\vec{p}/p)'(\mathbb{R}^n)}}. \end{aligned} \tag{4.26}$$

By this and the fact that $\|g\|_{L^{(\vec{p}/p)'(\mathbb{R}^n)}} = 1$, we know that (4.25) holds true. This finishes the proof of Lemma 4.7. \square

Remark 4.5. Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$. Notice that, if $\underline{p} := \min\{1, p_-\}$ with p_- as in (4.4), then there may exist some p_{i_0} such that $p_{i_0}/\underline{p} = 1$ and hence $(p_{i_0}/\underline{p})' = \infty$. By this and the fact that Lemma 4.3 holds true only for $\vec{p} \in (1, \infty)^n$, we know that (4.26) does not hold true in this case. Thus, we need to restrict the range of \underline{p} to be $(0, \min\{1, p_-\})$.

To prove the sufficiency of Theorem 4.5, we also need the following lemma, which is just [45, Lemma 4.9].

Lemma 4.8. Let $\vec{a} \in [1, \infty)^n$. Then there exists a set

$$\mathcal{Q} := \left\{ Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in E_k \right\}$$

of open subsets, where E_k is some index set, such that

- (i) for any $k \in \mathbb{Z}$, $|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k| = 0$ and, when $\alpha \neq \beta$, $Q_\alpha^k \cap Q_\beta^k = \emptyset$;
- (ii) for any α, β, k, ℓ with $\ell \geq k$, either $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ or $Q_\alpha^\ell \subset Q_\beta^k$;
- (iii) for any (ℓ, β) and $k < \ell$, there exists a unique α such that $Q_\beta^\ell \subset Q_\alpha^k$;
- (iv) there exist some $w \in \mathbb{Z} \setminus \mathbb{Z}_+$ and $u \in \mathbb{N}$ such that, for any Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in E_k$, there exists an $x_{Q_\alpha^k} \in Q_\alpha^k$ such that, for any $x \in Q_\alpha^k$,

$$x_{Q_\alpha^k} + 2^{(wk-u)\vec{a}} B_0 \subset Q_\alpha^k \subset x + 2^{(wk+u)\vec{a}} B_0,$$

where B_0 denotes the unit ball of \mathbb{R}^n .

In what follows, we call $\mathcal{Q} := \{Q_\alpha^k\}_{k \in \mathbb{Z}, \alpha \in E_k}$ from Lemma 4.8 *dyadic cubes* and k the *level*, denoted by $\ell(Q_\alpha^k)$, of the dyadic cube Q_α^k for any $k \in \mathbb{Z}$ and $\alpha \in E_k$.

We now prove the sufficiency of Theorem 4.5.

Proof of the sufficiency of Theorem 4.5. Let ψ and φ be as in Lemma 4.2, $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{\vec{p}}(\mathbb{R}^n)$ with $\vec{p} \in (0, \infty)^n$. Then, from Theorem 4.8, it follows that $S_\psi(f) \in L^{\vec{p}}(\mathbb{R}^n)$.

Thus, we only need to prove that $f \in H^{\vec{p}}_a(\mathbb{R}^n)$ and

$$\|f\|_{H^{\vec{p}}_a(\mathbb{R}^n)} \lesssim \|S_\psi(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}. \tag{4.27}$$

To this end, for any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : S_\psi(f)(x) > 2^k\}$ and

$$\mathcal{Q}_k := \left\{ Q \in \mathcal{Q} : |Q \cap \Omega_k| > \frac{|Q|}{2} \text{ and } |Q \cap \Omega_{k+1}| \leq \frac{|Q|}{2} \right\}.$$

It is easy to see that, for any $Q \in \mathcal{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{Q}_k$. For any given $k \in \mathbb{Z}$, denote by $\{Q_i^k\}_i$ the collection of all *maximal dyadic cubes* in \mathcal{Q}_k , namely, there exists no $Q \in \mathcal{Q}_k$ such that $Q_i^k \subsetneq Q$ for any i .

For any $Q \in \mathcal{Q}$, let

$$\widehat{Q} := \left\{ (y, t) \in \mathbb{R}_+^{n+1} : y \in Q \text{ and } t \sim 2^{w\ell(Q)+u} \right\}, \tag{4.28}$$

here and thereafter, $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ and $t \sim 2^{w\ell(Q)+u}$ always means

$$2^{w\ell(Q)+u+1} \leq t < 2^{w[\ell(Q)-1]+u+1}, \tag{4.29}$$

where w and u are as in Lemma 4.8 (iv) and $\ell(Q)$ denotes the level of Q . Clearly, $\{\widehat{Q}\}_{Q \in \mathcal{Q}}$ are mutually disjoint and

$$\mathbb{R}_+^{n+1} = \bigcup_{k \in \mathbb{Z}} \bigcup_i B_{k,i} \tag{4.30}$$

where, for any $k \in \mathbb{Z}$ and i , $B_{k,i} := \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \widehat{Q}$. Then, by Lemma 4.8(ii), we easily know that $\{B_{k,i}\}_{k \in \mathbb{Z}, i}$ are mutually disjoint.

In addition, by Lemma 4.2, the properties of tempered distributions (see [40, Theorem 2.3.20] or [80, Theorem 3.13]), we find that, for any $f \in \mathcal{S}'_0(\mathbb{R}^n)$ with $S_\psi(f) \in L^{\vec{p}}(\mathbb{R}^n)$, and for any $x \in \mathbb{R}^n$,

$$f(x) = \sum_{k \in \mathbb{Z}} f * \psi_k * \varphi_k(x) = \int_{\mathbb{R}_+^{n+1}} f * \psi_t(y) \varphi_t(x-y) dy dm(t) \tag{4.31}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where m denotes the *integer counting measure* on \mathbb{R} , namely, for any set $E \subset \mathbb{R}$, $m(E)$ is the number of integers contained in E . For any $k \in \mathbb{Z}$, i and $x \in \mathbb{R}^n$, let

$$h_i^k(x) := \int_{B_{k,i}} f * \psi_t(y) \varphi_t(x-y) dy dm(t).$$

Next we show that

$$\sum_{k \in \mathbb{Z}} \sum_i h_i^k \text{ converges in } \mathcal{S}'(\mathbb{R}^n). \tag{4.32}$$

To this end, we first claim that

(i) for any given $r \in (1, \infty)$, any $k \in \mathbb{Z}$, i and $x \in \mathbb{R}^n$,

$$h_i^k(x) = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\hat{Q}} f * \psi_t(y) \varphi_t(x-y) dy dm(t) =: \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q(x)$$

converges in $L^r(\mathbb{R}^n)$ and hence in $S'(\mathbb{R}^n)$;

(ii) for any $k \in \mathbb{Z}$ and i , h_i^k is a multiple of a (\vec{p}, r, s) -atom.

Indeed, using [63, (3.23)] with the dilation A as in (4.7), we conclude that, for any $x \in \mathbb{R}^n$,

$$\left[S_\varphi \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q \right) (x) \right]^2 \lesssim \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left[M_{\text{HL}}^{\vec{a}}(c_Q \mathbf{1}_Q)(x) \right]^2, \tag{4.33}$$

where, for any $Q \subset Q_i^k$ and $Q \in \mathcal{Q}_k$,

$$c_Q := \left[\int_{\hat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{t^v} \right]^{1/2}.$$

We now show the above two assertions and we first show assertion (i). To this end, for any $k \in \mathbb{Z}$, $Q \in \mathcal{Q}_k$ and $x \in Q$, by Lemma 4.8 (iv), we find that

$$M_{\text{HL}}^{\vec{a}}(\mathbf{1}_{\Omega_k})(x) \gtrsim \frac{1}{2^{[w\ell(Q)+u]v}} \int_{x_Q+2^{[w\ell(Q)+u]\vec{a}}B_0} \mathbf{1}_{\Omega_k}(z) dz \gtrsim 2^{-2uv} \frac{|\Omega_k \cap Q|}{|Q|} \gtrsim 2^{-2uv-1},$$

which implies that

$$\bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} Q \subset \hat{\Omega}_k := \left\{ x \in \mathbb{R}^n : M_{\text{HL}}^{\vec{a}}(\mathbf{1}_{\Omega_k})(x) \gtrsim 2^{-2uv-1} \right\}.$$

In addition, for any $k \in \mathbb{Z}$, $Q \in \mathcal{Q}_k$ and $x \in Q$, by $Q \subset \hat{\Omega}_k$, we know that

$$M_{\text{HL}}^{\vec{a}}(\mathbf{1}_{Q \cap (\hat{\Omega}_k \setminus \Omega_{k+1})})(x) \geq \frac{1}{|Q|} \int_Q \mathbf{1}_{Q \cap (\hat{\Omega}_k \setminus \Omega_{k+1})}(z) dz \geq \frac{|Q| - |\Omega_{k+1}|}{|Q|} \geq \frac{1}{2}.$$

From this, [16, Theorem 3.2] with the dilation A as in (4.7), (4.33), [45, Lemma 3.7] and an argument similar to that used in the proof of [63, (3.26)], it follows that, for any given $r \in (1, \infty)$, any $k \in \mathbb{Z}$ and i ,

$$\left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q \right\|_{L^r(\mathbb{R}^n)} \lesssim \left\| \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \mathbf{1}_{Q \cap (\hat{\Omega}_k \setminus \Omega_{k+1})} \right]^{1/2} \right\|_{L^r(\mathbb{R}^n)}. \tag{4.34}$$

On another hand, for any $k \in \mathbb{Z}$, $Q \in \mathcal{Q}_k$, $x \in Q$ and $(y, t) \in \widehat{Q}$, by [45, Lemmas 4.9 (iv) and 2.5 (ii)] and (4.29), we easily know that

$$x - y \in 2^{[w\ell(Q)+u]\bar{a}}B_0 + 2^{[w\ell(Q)+u]\bar{a}}B_0 \subset 2^{[w\ell(Q)+u+1]\bar{a}}B_0 \subset t^{\bar{a}}B_0,$$

here and thereafter, $B_0 := B_{\bar{a}}(\vec{0}_n, 1)$. From this and the disjointness of $\{\widehat{Q}\}_{Q \subset Q_i^k}$, we deduce that, for any $k \in \mathbb{Z}$, i and $x \in \mathbb{R}^n$,

$$\begin{aligned} & \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \mathbf{1}_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \\ &= \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{t^v} \mathbf{1}_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \\ &\lesssim [S_\psi(f)(x)]^2 \mathbf{1}_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x). \end{aligned} \tag{4.35}$$

Combining this and Lemma 4.8 (iv), we further conclude that, for any given $r \in (1, \infty)$, any $k \in \mathbb{Z}$ and i ,

$$\begin{aligned} & \left\| \left\{ \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \mathbf{1}_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right\}^{1/2} \right\|_{L^r(\mathbb{R}^n)}^r \\ &\leq \int_{\mathbb{R}^n} [S_\psi(f)(x)]^r \mathbf{1}_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) dx \\ &\lesssim 2^{kr} |Q_i^k| \lesssim 2^{kr} |2^{(wk+u)\bar{a}}B_0| \lesssim 2^{kr} 2^{(wk+u)v} < \infty. \end{aligned} \tag{4.36}$$

For any $N \in \mathbb{N}$, let $\mathcal{Q}_{k,N} := \{Q \in \mathcal{Q}_k : |\ell(Q)| > N\}$. Then, replacing $\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$ by $\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q$ in (4.34), we obtain, for any $N \in \mathbb{N}$, $k \in \mathbb{Z}$ and i ,

$$\left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q \right\|_{L^r(\mathbb{R}^n)}^r \lesssim \left\| \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} (c_Q)^2 \mathbf{1}_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right]^{1/2} \right\|_{L^r(\mathbb{R}^n)}^r.$$

From this, (4.36) and the Lebesgue dominated convergence theorem, we deduce that, for any given $r \in (1, \infty)$, any $k \in \mathbb{Z}$ and i ,

$$\left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q \right\|_{L^r(\mathbb{R}^n)} \rightarrow 0$$

as $N \rightarrow \infty$, and hence

$$\left\| \int_{\cup_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} \widehat{Q}} f * \psi_t(y) \varphi_t(x-y) dy dm(t) \right\|_{L^r(\mathbb{R}^n)} \rightarrow 0$$

as $N \rightarrow \infty$. Thus, $h_i^k = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$ in $L^r(\mathbb{R}^n)$. This finishes the proof of assertion (i) above. By this, (4.34), the estimation of (4.36) and Lemma 4.8 (iv), we know that

$$\begin{aligned} \|h_i^k\|_{L^r(\mathbb{R}^n)} &\lesssim \left\{ \int_{\mathbb{R}^n} [S_\psi(f)(x)]^r \mathbf{1}_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) dx \right\}^{1/r} \\ &\lesssim 2^k |Q_i^k|^{1/r} \leq \widetilde{C}_1 2^k |B_i^k|^{1/r}, \end{aligned} \tag{4.37}$$

where \widetilde{C}_1 is a positive constant independent of f, k and i and, for any $k \in \mathbb{Z}$ and i ,

$$B_i^k := x_{Q_i^k} + 2^{(w[\ell(Q_i^k)-1]+u+3)\bar{a}} B_0.$$

We now show assertion (ii). To this end, observe that, for any $x \in \text{supp } h_i^k$ with $k \in \mathbb{Z}$, $h_i^k(x) \neq 0$ implies that there exists a $Q \subset Q_i^k$ and $Q \in \mathcal{Q}_k$ such that $e_Q(x) \neq 0$. Then there exists a $(y, t) \in \widehat{Q}$ such that $t^{-\bar{a}}(x-y) \in B_0$. By this, Lemma 4.8 (iv), (4.29) and [45, Lemma 2.5 (ii)], we have

$$x \in y + t^{\bar{a}} B_0 \subset x_Q + 2^{(w[\ell(Q)+u]\bar{a})} B_0 + 2^{(w[\ell(Q)-1]+u+1)\bar{a}} B_0 \subset x_Q + 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0.$$

Thus,

$$\text{supp } e_Q \subset x_Q + 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0.$$

From this, the fact that $h_i^k = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$, (ii) and (iv) of Lemma 4.8 and [45, Lemma 2.5 (ii)], we further deduce that, for any $k \in \mathbb{Z}$ and i ,

$$\begin{aligned} \text{supp } h_i^k &\subset \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left(x_Q + 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0 \right) \\ &\subset x_{Q_i^k} + 2^{w[\ell(Q_i^k)+u]\bar{a}} B_0 + 2^{(w[\ell(Q_i^k)-1]+u+2)\bar{a}} B_0 \subset B_i^k. \end{aligned} \tag{4.38}$$

Recall that φ has the vanishing moments up to order $s \geq \lfloor \frac{\nu}{a_-} (\frac{1}{p_-} - 1) \rfloor$ and so does e_Q . For any $k \in \mathbb{Z}$, i , $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$, and $x \in \mathbb{R}^n$, let $g(x) := x^\gamma \mathbf{1}_{B_i^k}(x)$. Clearly, $g \in L^{r'}(\mathbb{R}^n)$ with $r \in (1, \infty)$ and $1/r + 1/r' = 1$. Thus, by (4.38) and the facts that $(L^{r'}(\mathbb{R}^n))^* = L^r(\mathbb{R}^n)$ and

$$\text{supp } e_Q \subset x_Q + 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0,$$

we conclude that, for any k, i and γ as above,

$$\int_{\mathbb{R}^n} h_i^k(x) x^\gamma dx = \langle h_i^k, g \rangle = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \langle e_Q, g \rangle = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\mathbb{R}^n} e_Q(x) x^\gamma dx = 0,$$

namely, h_i^k has the vanishing moments up to order s , which, combined with (4.37) and (4.38), implies that h_i^k is a multiple of a (\bar{p}, r, s) -atom supported in B_i^k . This finishes the proof of assertion (ii) above.

Now we prove (4.32). For any $k \in \mathbb{Z}$ and i , let

$$\lambda_i^k := \widetilde{C}_1 2^k \left\| \mathbf{1}_{B_i^k} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \quad \text{and} \quad a_i^k := (\lambda_i^k)^{-1} h_i^k, \tag{4.39}$$

where \widetilde{C}_1 is as in (4.37). Then it is easy to see that, for any $k \in \mathbb{Z}$ and i , a_i^k is a (\vec{p}, r, s) -atom. Notice that $i \in \mathbb{N}$ or there exists a $D \in \mathbb{N}$ such that $i \in \{1, \dots, D\}$. When $i \in \mathbb{N}$, by (4.39), to prove (4.32), it suffices to show that

$$\lim_{l \rightarrow \infty} \left\| \sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right\|_{H_a^{\vec{p}}(\mathbb{R}^n)} = 0. \tag{4.40}$$

Assuming that (4.40) holds true for the moment, then, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, let $\widetilde{\phi}(\cdot) := \phi(-\cdot)$. Obviously, for any $l, m \in \mathbb{N}$,

$$\left| \left\langle \sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k, \phi \right\rangle \right| = \left| \left(\sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right) * \widetilde{\phi}(\vec{0}_n) \right|.$$

Combining this, the proof of [45, Lemma 4.8] with f, ϕ and k therein replaced, respectively, by $\sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k, \widetilde{\phi}$ and 0, and (4.40), we further conclude that

$$\begin{aligned} \lim_{l \rightarrow \infty} \left| \left\langle \sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k, \phi \right\rangle \right| &\lesssim \lim_{l \rightarrow \infty} \left\| M_N \left(\sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ &\sim \lim_{l \rightarrow \infty} \left\| \sum_{l \leq |k| \leq ml} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right\|_{H_a^{\vec{p}}(\mathbb{R}^n)} \rightarrow 0, \end{aligned}$$

where N is as in (4.6). From this and the completeness of $\mathcal{S}'(\mathbb{R}^n)$, we deduce that (4.32) holds true. Therefore, to show (4.32), it remains to prove (4.40). To do this, for any $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, by the fact that $|Q_i^k \cap \Omega_k| \geq \frac{|Q_i^k|}{2}$, we find that, for any $x \in \mathbb{R}^n$,

$$M_{\text{HL}}^{\vec{a}} \left(\mathbf{1}_{Q_i^k \cap \Omega_k} \right) (x) \gtrsim \frac{1}{|Q_i^k|} \int_{Q_i^k} \mathbf{1}_{Q_i^k \cap \Omega_k}(y) dy \sim \frac{|Q_i^k \cap \Omega_k|}{|Q_i^k|} \gtrsim \frac{1}{2},$$

which, together with [45, (2.5) and Lemma 3.5], implies that

$$\begin{aligned} \left\| \mathbf{1}_{Q_i^k} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} &= \left\| \mathbf{1}_{Q_i^k} \right\|_{L^{2\vec{p}/p_-}(\mathbb{R}^n)}^{2/p_-} \lesssim \left\| M_{\text{HL}}^{\vec{a}} \left(\mathbf{1}_{Q_i^k \cap \Omega_k} \right) \right\|_{L^{2\vec{p}/p_-}(\mathbb{R}^n)}^{2/p_-} \\ &\lesssim \left\| \mathbf{1}_{Q_i^k \cap \Omega_k} \right\|_{L^{2\vec{p}/p_-}(\mathbb{R}^n)}^{2/p_-} \sim \left\| \mathbf{1}_{Q_i^k \cap \Omega_k} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}. \end{aligned} \tag{4.41}$$

From the fact that, for any $l, m \in \mathbb{N}$, $\sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \in H_a^{\vec{p}, r, s}(\mathbb{R}^n)$, Theorem 4.3, the mutual disjointness of $\{Q_i^k\}_{k \in \mathbb{Z}, i \in \mathbb{N}}$ and [45, Lemma 5.9 (iv)], it follows that

$$\begin{aligned} & \left\| \sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right\|_{H_a^{\vec{p}}(\mathbb{R}^n)} \\ & \lesssim \left\| \left\{ \sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} \left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \left[\sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} (2^k \mathbf{1}_{B_i^k})^p \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ & \sim \left\| \left[\sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} (2^k \mathbf{1}_{Q_i^k})^p \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} (2^k \mathbf{1}_{Q_i^k})^p \right\|_{L^{\vec{p}/p}(\mathbb{R}^n)}^{1/p}, \end{aligned}$$

which, combined with (4.41), implies that

$$\begin{aligned} & \left\| \sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} \lambda_i^k a_i^k \right\|_{H_a^{\vec{p}}(\mathbb{R}^n)} \\ & \sim \left\| \sum_{l \leq |k| \leq m} \sum_{l \leq i \leq m} (2^k \mathbf{1}_{Q_i^k \cap \Omega_k})^p \right\|_{L^{\vec{p}/p}(\mathbb{R}^n)}^{1/p} \lesssim \left\| \left[\sum_{l \leq |k| \leq m} (2^k \mathbf{1}_{\Omega_k})^p \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ & \sim \left\| \left[\sum_{l \leq |k| \leq m} (2^k \mathbf{1}_{\Omega_k \setminus \Omega_{k+1}})^p \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| S_\psi(f) \left(\sum_{l \leq |k| \leq m} \mathbf{1}_{\Omega_k \setminus \Omega_{k+1}} \right) \right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{1/p} \rightarrow 0 \end{aligned}$$

as $l \rightarrow \infty$. This further implies that (4.40) holds true and so does (4.32) in the case $i \in \mathbb{N}$. When $i \in \{1, \dots, D\}$, to prove (4.32), it suffices to show that

$$\lim_{l \rightarrow \infty} \left\| \sum_{l \leq |k| \leq m} \sum_{i=1}^D \lambda_i^k a_i^k \right\|_{H_a^{\vec{p}}(\mathbb{R}^n)} = 0. \tag{4.42}$$

Applying a similar argument to that used in the proof of (4.40) above, we know that (4.42) also holds true. Thus,

$$\sum_{k \in \mathbb{Z}} \sum_i h_i^k = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k \text{ converges in } \mathcal{S}'(\mathbb{R}^n).$$

Now, for any $x \in \mathbb{R}^n$, let

$$\eta(x) := \sum_{k \in \mathbb{Z}} \sum_i h_i^k(x) = \sum_{k \in \mathbb{Z}} \sum_i \int_{B_{k,i}} f * \psi_t(y) \varphi_t(x-y) dy dm(t) \text{ in } \mathcal{S}'(\mathbb{R}^n),$$

here and thereafter, for any $k \in \mathbb{Z}$ and i , $B_{k,i}$ is as in (4.30). To finish the proof of the sufficiency of Theorem 4.5, we next show that

$$f = \eta \quad \text{in } \mathcal{S}'(\mathbb{R}^n). \quad (4.43)$$

Indeed, by assertion (i) above, (4.28) and (4.29), we know that, for any given $r \in (1, \infty)$, any $k \in \mathbb{Z}$, i and $x \in \mathbb{R}^n$,

$$\begin{aligned} h_i^k(x) &= \lim_{N \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{\cup_{\substack{Q \subset Q_k^i, Q \in \mathcal{Q}_k \\ |\ell(Q)| \leq N}} \hat{Q}}(y,t) dy dm(t) \\ &= \lim_{N \rightarrow \infty} \int_{\alpha(N)}^{\beta(N)} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{B_{k,i}}(y,t) dy dm(t) \end{aligned} \quad (4.44)$$

converges in $L^r(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $N \in \mathbb{N}$, $\alpha(N) := 2^{wN+u+1}$ and $\beta(N) := 2^{-w(N+1)+u+1}$ with w and u as in Lemma 4.8 (iv). For the convenience of the notation, we rewrite η as, for any $x \in \mathbb{R}^n$,

$$\eta(x) = \sum_{j \in \mathbb{N}} \int_{R^{(j)}} f * \psi_t(y) \varphi_t(x-y) dy dm(t),$$

where $\{R^{(j)}\}_{j \in \mathbb{N}}$ is any rearrangement of $\{B_{k,i}\}_{k \in \mathbb{Z}, i}$. For any $M \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$\eta_M(x) := f(x) - \sum_{j=1}^M \int_{R^{(j)}} f * \psi_t(y) \varphi_t(x-y) dy dm(t).$$

Then, using (4.30), (4.31) and (4.44), we have, for any $M \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \eta_M(x) &= \lim_{N \rightarrow \infty} \int_{\alpha(N)}^{\beta(N)} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{\cup_{j=1}^\infty R^{(j)}}(y,t) dy dm(t) \\ &\quad - \lim_{N \rightarrow \infty} \int_{\alpha(N)}^{\beta(N)} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{\cup_{j=1}^M R^{(j)}}(y,t) dy dm(t) \\ &= \lim_{N \rightarrow \infty} \int_{\alpha(N)}^{\beta(N)} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{\cup_{j=M+1}^\infty R^{(j)}}(y,t) dy dm(t) \end{aligned} \quad (4.45)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

Note that $H_a^{\vec{p}}(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$ (see [47, Lemma 3.6]). Thus, to prove (4.43), it suffices to show that $\|\eta_M\|_{H_a^{\vec{p}}(\mathbb{R}^n)} \rightarrow 0$ as $M \rightarrow \infty$. To do this, we borrow some ideas from the proof of the atomic characterizations of $H_a^{\vec{p}}(\mathbb{R}^n)$ (see [45, Theorem 3.16]). Indeed, for any $\varepsilon \in (0, \infty)$, $M \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$\eta_M^{\varepsilon, \kappa}(x) := \int_\varepsilon^{\kappa/\varepsilon} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{\cup_{j=M+1}^\infty R^{(j)}}(y,t) dy dm(t),$$

where κ is any given positive constant. Therefore, by the Lebesgue dominated convergence theorem, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$, $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we have

$$\eta_M^{\varepsilon, \kappa}(x) = \sum_{j=M+1}^{\infty} \int_{\varepsilon}^{\kappa/\varepsilon} \int_{\mathbb{R}^n} f * \psi_t(y) \varphi_t(x-y) \mathbf{1}_{R^{(j)}}(y, t) dy dm(t) =: \sum_{j=M+1}^{\infty} h_j^{\varepsilon, \kappa}(x)$$

in $\mathcal{S}'(\mathbb{R}^n)$.

In addition, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $Q \in \mathcal{Q}$, let

$$\widehat{Q}_{\varepsilon, \kappa} := \left\{ (y, t) \in \mathbb{R}^n \times (\varepsilon, \kappa/\varepsilon) : y \in Q \text{ and } t \sim 2^{w\ell(Q)+u} \right\},$$

where w and u are as in Lemma 4.8 (iv) and $\ell(Q)$ denotes the level of Q . Obviously, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$, $\{\widehat{Q}_{\varepsilon, \kappa}\}_{Q \in \mathcal{Q}}$ are mutually disjoint and

$$\mathbb{R}^n \times (\varepsilon, \kappa/\varepsilon) = \bigcup_{k \in \mathbb{Z}} \bigcup_i B_{k,i}^{\varepsilon, \kappa},$$

where, for any $k \in \mathbb{Z}$ and i , $B_{k,i}^{\varepsilon, \kappa} := \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \widehat{Q}_{\varepsilon, \kappa}$. Then, by Lemma 4.8 (ii), we easily know that, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$, $\{B_{k,i}^{\varepsilon, \kappa}\}_{k \in \mathbb{Z}, i}$ are mutually disjoint.

Now we claim that, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\left[S_{\varphi} \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q^{\varepsilon, \kappa} \right) (x) \right]^2 \lesssim \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left[M_{\text{HL}}^{\vec{a}}(c_Q^{\varepsilon, \kappa} \mathbf{1}_Q)(x) \right]^2, \tag{4.46}$$

where, for any $Q \subset Q_i^k$, $Q \in \mathcal{Q}_k$ and $x \in \mathbb{R}^n$,

$$e_Q^{\varepsilon, \kappa}(x) := \int_{\widehat{Q}_{\varepsilon, \kappa}} f * \psi_t(y) \varphi_t(x-y) dy dm(t)$$

and

$$c_Q^{\varepsilon, \kappa} := \left[\int_{\widehat{Q}_{\varepsilon, \kappa}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{t^v} \right]^{1/2}.$$

Assuming this inequality holds true for the moment, then, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$, by some arguments similar to these used in the proofs of the above assertions (i) and (ii) with \widehat{Q} and (4.33) therein replaced, respectively, by $\widehat{Q}_{\varepsilon, \kappa}$ and (4.46), we conclude that, for any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$h_j^{\varepsilon, \kappa}(x) = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q^{\varepsilon, \kappa}(x)$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and, for any given $r \in (1, \infty)$, $h_j^{\varepsilon, \kappa}$ is a multiple of a (\vec{p}, r, s) -atom, namely, there exist a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of (\vec{p}, r, s) -atoms, $\{a_j^{\varepsilon, \kappa}\}_{j \in \mathbb{N}}$,

supported, respectively, in $\{B_j\}_{j \in \mathbb{N}} \subset \mathfrak{B}$ such that, for any $j \in \mathbb{N}$, $h_j^{\varepsilon, \kappa} = \lambda_j a_j^{\varepsilon, \kappa}$, where, for any $j \in \mathbb{N}$, λ_j and B_j are independent of ε and κ . Thus, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$, $M \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\eta_M^{\varepsilon, \kappa}(x) = \sum_{j=M+1}^{\infty} \lambda_j a_j^{\varepsilon, \kappa}(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{4.47}$$

$$\left\| \left\{ \sum_{j=M+1}^{\infty} \left[\frac{|\lambda_j| \mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty. \tag{4.48}$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let $M_0(f)$ be as in (4.9). Then, by the fact that, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$, $\{a_j^{\varepsilon, \kappa}\}_{j \in \mathbb{N}}$ is a sequence of (\vec{p}, r, s) -atoms and [45, (3.41)], we know that, for any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$M_0(a_j^{\varepsilon, \kappa})(x) \lesssim M_{\text{HL}}^{\vec{a}}(a_j^{\varepsilon, \kappa})(x) \mathbf{1}_{B_j^{(2)}}(x) + \frac{1}{\|\mathbf{1}_{B_j}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \left[M_{\text{HL}}^{\vec{a}}(\mathbf{1}_{B_j})(x) \right]^{\frac{v+(s+1)a_-}{v}}, \tag{4.49}$$

where, for any $j \in \mathbb{N}$, $B_j^{(2)}$ is as in (4.2) with $\delta := 2$. Let $r \in (\max\{p_+, 1\}, \infty)$. Then, by [45, (3.38)] and Lemma 4.3, we find that, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$,

$$\left\| M_0(a_j^{\varepsilon, \kappa}) \mathbf{1}_{B_j^{(2)}} \right\|_{L^r(\mathbb{R}^n)} \lesssim \left\| M_{\text{HL}}^{\vec{a}}(a_j^{\varepsilon, \kappa}) \mathbf{1}_{B_j^{(2)}} \right\|_{L^r(\mathbb{R}^n)} \lesssim \frac{|B_j|^{1/r}}{\|\mathbf{1}_{B_j}\|_{L^{\vec{p}}(\mathbb{R}^n)}},$$

which, combined with Lemma 4.7, further implies that

$$\begin{aligned} & \left\| \liminf_{\varepsilon \rightarrow 0^+} \left\{ \sum_{j=M+1}^{\infty} \left[|\lambda_j| M_0(a_j^{\varepsilon, \kappa}) \mathbf{1}_{B_j^{(2)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ & \lesssim \left\| \left\{ \sum_{j=M+1}^{\infty} \left[\frac{|\lambda_j| \mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}. \end{aligned} \tag{4.50}$$

Let $\varepsilon := \alpha(N)$ and

$$\kappa := 2^{-w+2(u+1)}$$

with $N \in \mathbb{N}$, w and u as in Lemma 4.8 (iv). Then, by (4.45), we know that

$$\begin{aligned} M_0(\eta_M) &= M_0 \left(\lim_{N \rightarrow \infty} \eta_M^{\alpha(N), \kappa} \right) = \sup_{t \in (0, \infty)} \left| \lim_{N \rightarrow \infty} \Phi_t * \eta_M^{\alpha(N), \kappa} \right| \\ &\leq \liminf_{N \rightarrow \infty} \sup_{t \in (0, \infty)} \left| \Phi_t * \eta_M^{\alpha(N), \kappa} \right| = \liminf_{N \rightarrow \infty} M_0(\eta_M^{\alpha(N), \kappa}), \end{aligned}$$

where Φ is as in (4.9). From this, Theorem 4.2, (4.47) and (4.49), we deduce that

$$\begin{aligned} \|\eta_M\|_{H_a^{\bar{p}}(\mathbb{R}^n)} &\leq \left\| \liminf_{N \rightarrow \infty} M_0 \left(\eta_M^{\alpha(N), \kappa} \right) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \leq \left\| \liminf_{N \rightarrow \infty} \sum_{j=M+1}^{\infty} |\lambda_j| M_0(a_j^{\alpha(N), \kappa}) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \\ &\lesssim \left\| \liminf_{N \rightarrow \infty} \sum_{j=M+1}^{\infty} |\lambda_j| M_0(a_j^{\alpha(N), \kappa}) \mathbf{1}_{B_j^{(2)}} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \\ &\quad + \left\| \sum_{j=M+1}^{\infty} \frac{|\lambda_j|}{\|\mathbf{1}_{B_j^{(2)}}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \left[M_{\text{HL}}^{\bar{a}}(\mathbf{1}_{B_j}) \right]^{\frac{v+(s+1)a_-}{v}} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}, \end{aligned}$$

which, together with (4.50), Lemma 4.4 and (4.48), further implies that

$$\begin{aligned} \|\eta_M\|_{H_a^{\bar{p}}(\mathbb{R}^n)} &\lesssim \left\| \liminf_{N \rightarrow \infty} \left\{ \sum_{j=M+1}^{\infty} \left[|\lambda_j| M_{\text{HL}}^{\bar{a}}(a_j^{\alpha(N), \kappa}) \mathbf{1}_{B_j^{(2)}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \\ &\quad + \left\| \sum_{j=M+1}^{\infty} \left\{ \frac{|\lambda_j|}{\|\mathbf{1}_{B_j}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \left[M_{\text{HL}}(\mathbf{1}_{B_j}) \right]^{\frac{v+(s+1)a_-}{v}} \right\}^{\frac{v}{v+(s+1)a_-}} \right\|_{L^{\frac{v+(s+1)a_-}{v} \bar{p}}(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{j=M+1}^{\infty} \left[\frac{|\lambda_j| \mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \rightarrow 0 \end{aligned}$$

as $M \rightarrow \infty$. This implies that (4.43) holds true. Therefore,

$$f = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

By this, Theorem 4.3, the mutual disjointness of $\{Q_i^k\}_{k \in \mathbb{Z}, i}$ and [45, Lemma 5.9 (iv)], we conclude that

$$\begin{aligned} \|f\|_{H_a^{\bar{p}}(\mathbb{R}^n)} &\lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_i \left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_{L^{\bar{p}}(\mathbb{R}^n)}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \sim \left\| \left[\sum_{k \in \mathbb{Z}} \sum_i \left(2^k \mathbf{1}_{B_i^k} \right)^p \right]^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \\ &\sim \left\| \left[\sum_{k \in \mathbb{Z}} \sum_i \left(2^k \mathbf{1}_{Q_i^k} \right)^p \right]^{1/p} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}, \end{aligned}$$

which, combined with (4.41), further implies that

$$\begin{aligned} \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} &\sim \left\| \sum_{k \in \mathbb{Z}} \sum_i \left(2^k \mathbf{1}_{Q_i^k} \right)^{\frac{p}{p}} \right\|_{L^{\vec{p}/p}(\mathbb{R}^n)}^{1/p} \sim \left\| \sum_{k \in \mathbb{Z}} \sum_i \left(2^k \mathbf{1}_{Q_i^k \cap \Omega_k} \right)^{\frac{p}{p}} \right\|_{L^{\vec{p}/p}(\mathbb{R}^n)}^{1/p} \\ &\lesssim \left\| \left[\sum_{k \in \mathbb{Z}} \left(2^k \mathbf{1}_{\Omega_k} \right)^{\frac{p}{p}} \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \left[\sum_{k \in \mathbb{Z}} \left(2^k \mathbf{1}_{\Omega_k \setminus \Omega_{k+1}} \right)^{\frac{p}{p}} \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ &\sim \left\| S_{\psi}(f) \left[\sum_{k \in \mathbb{Z}} \mathbf{1}_{\Omega_k \setminus \Omega_{k+1}} \right]^{1/p} \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \|S_{\psi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}. \end{aligned}$$

Thus, $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ and hence (4.27) holds true, which then completes the proof of the sufficiency of Theorem 4.5.

To finish the whole proof, it remains to prove (4.46). Indeed, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \left[S_{\varphi} \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q^{\varepsilon, \kappa} \right) (x) \right]^2 &= \int_{\Gamma(x)} \left| \varphi_t * \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q^{\varepsilon, \kappa} \right) (y) \right|^2 \frac{dy dm(t)}{t^{\nu}} \\ &\leq \sum_{P \in \mathcal{Q}, \widehat{P} \cap \Gamma(x) \neq \emptyset} \int_{\widehat{P}} \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left| \varphi_t * e_Q^{\varepsilon, \kappa}(y) \right| \right]^2 \frac{dy dm(t)}{t^{\nu}}, \end{aligned}$$

where, for any $x \in \mathbb{R}^n$,

$$\Gamma(x) := \left\{ (y, t) \in \mathbb{R}_+^{n+1} : |y - x|_{\vec{a}} < t \right\}$$

and, for any $P \in \mathcal{Q}$, \widehat{P} is as in (4.28).

Let $x \in \text{supp } e_Q^{\varepsilon, \kappa}$. Note that $\text{supp } \varphi \subset B_0$. Then there exists a $(y, t) \in \widehat{Q}_{\varepsilon, \kappa}$ such that $t^{-\vec{a}}(x - y) \in B_0$. From this, Lemma 4.8 (iv), (4.29) and [45, Lemma 2.5 (ii)], it follows that

$$x \in y + t^{\vec{a}} B_0 \subset x_Q + 2^{(w\ell(Q)+u)\vec{a}} B_0 + 2^{(w[\ell(Q)-1]+u+1)\vec{a}} B_0 \subset x_Q + 2^{(w[\ell(Q)-1]+u+2)\vec{a}} B_0.$$

Thus,

$$\text{supp } e_Q^{\varepsilon, \kappa} \subset x_Q + 2^{(w[\ell(Q)-1]+u+2)\vec{a}} B_0 =: R_Q.$$

Moreover, by the fact that $\text{supp } \varphi \subset B_0$, Lemma 4.8 (iv), (4.29) and [45, Lemma 2.5 (ii)], we know that, for any $(y, t) \in \widehat{P}$,

$$\begin{aligned} \text{supp } \varphi_t(y - \cdot) &\subset y + t^{\vec{a}} B_0 \subset x_P + 2^{(w\ell(P)+u)\vec{a}} B_0 + 2^{(w[\ell(P)-1]+u+1)\vec{a}} B_0 \\ &\subset x_P + 2^{(w[\ell(P)-1]+u+2)\vec{a}} B_0 =: R_P. \end{aligned}$$

For any $x \in \mathbb{R}^n$ and $(y, t) \in \widehat{P}$ with $\widehat{P} \cap \Gamma(x) \neq \emptyset$, we will show below that, for any given $\kappa \in (0, \infty)$ and any $\varepsilon \in (0, \infty)$, when $R_Q \cap R_P = \emptyset$, $\varphi_t * e_Q^{\varepsilon, \kappa}(y) \equiv 0$, otherwise,

$$\left| \varphi_t * e_Q^{\varepsilon, \kappa}(y) \right| \lesssim c_Q^{\varepsilon, \kappa} M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x) 2^{(s+1)w|\ell(P)-\ell(Q)|a_-}. \tag{4.51}$$

Assuming that this holds true for the moment, then, via the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left| \varphi_t * e_Q^{\varepsilon, \kappa}(y) \right| \right]^2 \\ & \lesssim \left[\sum_{\substack{Q \subset Q_i^k, Q \in \mathcal{Q}_k \\ R_Q \cap R_P \neq \emptyset}} c_Q^{\varepsilon, \kappa} M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x) 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \right]^2 \\ & \lesssim \sum_{\substack{Q \subset Q_i^k, Q \in \mathcal{Q}_k \\ R_Q \cap R_P \neq \emptyset}} (c_Q^{\varepsilon, \kappa})^2 \left[M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x) \right]^2 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \sum_{\substack{Q \subset Q_i^k, Q \in \mathcal{Q}_k \\ R_Q \cap R_P \neq \emptyset}} 2^{(s+1)w|\ell(P)-\ell(Q)|a_-}. \end{aligned}$$

In addition, from [16, (4.18)] with A as in (4.7), we deduce that, for any $P \in \mathcal{Q}$,

$$\sum_{\substack{Q \subset Q_i^k, Q \in \mathcal{Q}_k \\ R_Q \cap R_P \neq \emptyset}} 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \lesssim 1$$

and, for any $Q \subset Q_i^k$ and $Q \in \mathcal{Q}_k$,

$$\sum_{P \in \mathcal{Q}, R_Q \cap R_P \neq \emptyset} 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \lesssim 1.$$

Therefore, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} & \left[S_\varphi \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q^{\varepsilon, \kappa} \right) (x) \right]^2 \\ & \lesssim \sum_{\substack{P \in \mathcal{Q} \\ \widehat{P} \cap \Gamma(x) \neq \emptyset}} \int_{\widehat{P}} \sum_{\substack{Q \subset Q_i^k, Q \in \mathcal{Q}_k \\ R_Q \cap R_P \neq \emptyset}} (c_Q^{\varepsilon, \kappa})^2 \left[M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x) \right]^2 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \frac{dy dm(t)}{t^v} \\ & \lesssim \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q^{\varepsilon, \kappa})^2 \left[M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x) \right]^2 \left[\sum_{P \in \mathcal{Q}, R_Q \cap R_P \neq \emptyset} 2^{(s+1)w|\ell(P)-\ell(Q)|a_-} \right] \\ & \lesssim \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left[M_{\text{HL}}^{\vec{a}}(c_Q^{\varepsilon, \kappa} \mathbf{1}_Q)(x) \right]^2, \end{aligned}$$

which implies (4.46).

To finish the whole proof, we still need to show (4.51). To this end, observe that, when $R_Q \cap R_P = \emptyset$, from Lemma 4.8 (iv), it follows that, for any $(y, t) \in \widehat{P}$,

$$y \in P \subset x_P + 2^{[w\ell(P)+u]\bar{a}} B_0 \subset R_P,$$

which implies that, for any $z \in R_Q$,

$$|y - z|_{\bar{a}} \geq 2^{w[\ell(P)-1]+u+2} > t.$$

Thus, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $(y, t) \in \widehat{P}$,

$$\varphi_t * e_Q^{\varepsilon, \kappa}(y) = \int_{R_Q} \varphi_t(y - z) e_Q^{\varepsilon, \kappa}(z) dz = 0.$$

Now we deal with the non-trivial case $R_Q \cap R_P \neq \emptyset$ by considering the following two subcases.

Case I $\ell(P) \leq \ell(Q)$. In this case, for any $(y, t) \in \widehat{P}$ and $z \in R_Q$, we have

$$z' := t^{-\bar{a}} z \in t^{-\bar{a}} x_Q + t^{-\bar{a}} 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0 \subset t^{-\bar{a}} x_Q + 2^{(w[\ell(Q)-\ell(P)-1]+1)\bar{a}} B_0 =: R_{QP}.$$

By this, Lemma 4.15 and the facts that $-w > 0$ and $w[\ell(Q) - \ell(P)] \leq 0$, we find that, for any $z' \in R_{QP}$,

$$\begin{aligned} |z' - t^{-\bar{a}} x_Q| &= \left| 2^{(-w+1)\bar{a}} \left[2^{(w-1)\bar{a}} (z' - t^{-\bar{a}} x_Q) \right] \right| \\ &\leq 2^{(-w+1)a_+} \left| 2^{(w-1)\bar{a}} (z' - t^{-\bar{a}} x_Q) \right| \lesssim 2^{[\ell(Q) - \ell(P)]w a_-}. \end{aligned} \tag{4.52}$$

On another hand, from the Hölder inequality, it follows that, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\left| e_Q^{\varepsilon, \kappa}(x) \right|^2 \leq (c_Q^{\varepsilon, \kappa})^2 \int_{\widehat{Q}_{\varepsilon, \kappa}} |\varphi_\tau(x - y)|^2 \tau^\nu dy dm(\tau) \lesssim (c_Q^{\varepsilon, \kappa})^2 \sum_{\tau \sim 2^{w\ell(Q)+u}} |Q| \tau^{-\nu} \lesssim (c_Q^{\varepsilon, \kappa})^2,$$

which, together with the vanishing moments of $e_Q^{\varepsilon, \kappa}$ and (4.52), further implies that, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $(y, t) \in \widehat{P}$,

$$\begin{aligned} \left| e_Q^{\varepsilon, \kappa} * \varphi_t(y) \right| &= \left| \int_{\mathbb{R}^n} e_Q^{\varepsilon, \kappa}(z) \varphi_t(y - z) dz \right| = \left| \int_{\mathbb{R}^n} e_Q^{\varepsilon, \kappa}(t^{\bar{a}} z) \varphi(t^{-\bar{a}} y - z) dz \right| \\ &= \left| \int_{R_{QP}} e_Q^{\varepsilon, \kappa}(t^{\bar{a}} z) \left[\varphi(t^{-\bar{a}} y - z) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \varphi(t^{-\bar{a}} y - t^{-\bar{a}} x_Q)}{\alpha!} (t^{-\bar{a}} x_Q - z)^\alpha \right] dz \right| \\ &\lesssim \int_{R_{QP}} \left| e_Q^{\varepsilon, \kappa}(t^{\bar{a}} z) \right| \left| t^{-\bar{a}} x_Q - z \right|^{s+1} dz \lesssim c_Q^{\varepsilon, \kappa} 2^{[\ell(Q) - \ell(P)](s+1)w a_-} 2^{[\ell(Q) - \ell(P)]w \nu}. \end{aligned} \tag{4.53}$$

Note that $\ell(P) \leq \ell(Q)$ and $R_Q \cap R_P \neq \emptyset$. Thus,

$$R_Q := x_Q + 2^{(w[\ell(Q)-1]+u+2)\bar{a}} B_0 \subset x_P + 2^{(w[\ell(P)-1]+u+4)\bar{a}} B_0 =: R'_P. \tag{4.54}$$

Moreover, for any $x \in \mathbb{R}^n$ such that $\widehat{P} \cap \Gamma(x) \neq \emptyset$, we know that there exists a $(y_0, t_0) \in \widehat{P} \cap \Gamma(x)$ such that $|x - y_0|_{\bar{a}} < t_0 < 2^{w[\ell(P)-1]+u+1}$; thus, from Lemma 4.8 (iv) and [45, Lemma 2.5 (ii)], we deduce that

$$\begin{aligned} x &\in y_0 + 2^{(w[\ell(P)-1]+u+1)\bar{a}} B_0 \subset x_P + 2^{[w\ell(P)+u]\bar{a}} B_0 + 2^{(w[\ell(P)-1]+u+1)\bar{a}} B_0 \\ &\subset x_P + 2^{(w[\ell(P)-1]+u+2)\bar{a}} B_0 = R_P \subset R'_P. \end{aligned}$$

By this, (4.54) and [16, Lemma 2.7], we conclude that, for any $x \in \mathbb{R}^n$ such that $\widehat{P} \cap \Gamma(x) \neq \emptyset$,

$$2^{[\ell(Q)-\ell(P)]wv} \sim \frac{|R_Q|}{|R'_P|} \sim \frac{|R_Q \cap R'_P|}{|R'_P|} \lesssim M_{\text{HL}}^{\bar{a}}(\mathbf{1}_{R_Q})(x) \lesssim M_{\text{HL}}^{\bar{a}}(\mathbf{1}_Q)(x),$$

which, combined with (4.53), implies (4.51) holds true in this case.

Case II) $\ell(P) > \ell(Q)$. In this case, for any $(y, t) \in \widehat{P}$ and $z \in R_P$, it is easy to see that

$$z' := 2^{[-w\ell(Q)-u]\bar{a}} z \in 2^{[-w\ell(Q)-u]\bar{a}} x_P + 2^{(w[\ell(P)-\ell(Q)-1]+2)\bar{a}} B_0 =: R_{PQ}.$$

From this, Lemma 4.15 and the facts that $-w > 0$ and $w[\ell(P) - \ell(Q)] < 0$, it follows that, for any $z' \in R_{PQ}$,

$$\begin{aligned} \left| z' - 2^{[-w\ell(Q)-u]\bar{a}} x_P \right| &= \left| 2^{(-w+2)\bar{a}} \left[2^{(w-2)\bar{a}} \left(z' - 2^{[-w\ell(Q)-u]\bar{a}} x_P \right) \right] \right| \\ &\leq 2^{(-w+2)a_+} \left| 2^{(w-2)\bar{a}} \left(z' - 2^{[-w\ell(Q)-u]\bar{a}} x_P \right) \right| \lesssim 2^{[\ell(P)-\ell(Q)]wa_-}. \end{aligned} \tag{4.55}$$

Let $\widetilde{e}_Q^{\varepsilon, \kappa}(\cdot) := e_Q^{\varepsilon, \kappa}(2^{[w\ell(Q)+u]\bar{a}} \cdot)$. For any $\alpha \in \mathbb{Z}_+^n$, using the Hölder inequality, we find that, for any given $\kappa \in (0, \infty)$, any $\varepsilon \in (0, \infty)$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} \left| \partial^\alpha \widetilde{e}_Q^{\varepsilon, \kappa}(z) \right|^2 &= \left| \int_{\widehat{Q}_{\varepsilon, \kappa}} f * \psi_\tau(y) \partial^\alpha \varphi_\tau(2^{[w\ell(Q)+u]\bar{a}} z - y) dy dm(\tau) \right|^2 \\ &\leq (c_Q^{\varepsilon, \kappa})^2 \int_{\widehat{Q}_{\varepsilon, \kappa}} \left| \partial^\alpha \varphi_\tau(2^{[w\ell(Q)+u]\bar{a}} z - y) \right|^2 \tau^v dy dm(\tau) \\ &\lesssim (c_Q^{\varepsilon, \kappa})^2 \sum_{\tau \sim 2^{w\ell(Q)+u}} |\tau| \tau^{-v} \lesssim (c_Q^{\varepsilon, \kappa})^2. \end{aligned}$$

By this, the vanishing moments of φ and (4.55), we know that, for any given $\kappa \in (0, \infty)$,

any $\varepsilon \in (0, \infty)$ and $(y, t) \in \widehat{P}$,

$$\begin{aligned}
 \left| e_Q^{\varepsilon, \kappa} * \varphi_t(y) \right| &= \left| 2^{[w\ell(Q)+u]v} \int_{\mathbb{R}^n} e_Q^{\varepsilon, \kappa}(2^{[w\ell(Q)+u]\vec{a}}z) \varphi_t(y - 2^{[w\ell(Q)+u]\vec{a}}z) dz \right| \\
 &= \left| 2^{[w\ell(Q)+u]v} \int_{\mathbb{R}^n} \widetilde{e}_Q^{\varepsilon, \kappa}(z) \varphi_t(y - 2^{[w\ell(Q)+u]\vec{a}}z) dz \right| \\
 &= \left| 2^{[w\ell(Q)+u]v} \int_{R_{PQ}} \left[\widetilde{e}_Q^{\varepsilon, \kappa}(z) - \sum_{|\alpha| \leq s} \frac{\partial^\alpha \widetilde{e}_Q^{\varepsilon, \kappa}(2^{[-w\ell(Q)-u]\vec{a}}x_P)}{\alpha!} \left(z - 2^{[-w\ell(Q)-u]\vec{a}}x_P \right)^\alpha \right] \right. \\
 &\quad \left. \times \varphi_t(y - 2^{[w\ell(Q)+u]\vec{a}}z) dz \right| \\
 &= 2^{[w\ell(Q)+u]v} c_Q^{\varepsilon, \kappa} \int_{R_{PQ}} \left| z - 2^{[-w\ell(Q)-u]\vec{a}}x_P \right|^{s+1} \left| \varphi_t(y - 2^{[w\ell(Q)+u]\vec{a}}z) \right| dz \\
 &\lesssim c_Q^{\varepsilon, \kappa} 2^{[\ell(P)-\ell(Q)](s+1)w\vec{a}_-}. \tag{4.56}
 \end{aligned}$$

Note that $\ell(P) > \ell(Q)$ and $R_Q \cap R_P \neq \emptyset$. Thus,

$$R_P := x_P + 2^{(w[\ell(P)-1]+u+2)\vec{a}}B_0 \subset x_Q + 2^{(w[\ell(Q)-1]+u+4)\vec{a}}B_0 =: R'_Q. \tag{4.57}$$

Moreover, for any $x \in \mathbb{R}^n$ such that $\widehat{P} \cap \Gamma(x) \neq \emptyset$, we know that there exists a $(y_1, t_1) \in \widehat{P} \cap \Gamma(x)$ such that $|x - y_1|_{\vec{a}} < t_1 < 2^{w[\ell(P)-1]+u+1}$. Thus, by Lemma 4.8 (iv) and [45, Lemma 2.5 (ii)], we find that, for any $x \in \mathbb{R}^n$ such that $\widehat{P} \cap \Gamma(x) \neq \emptyset$,

$$\begin{aligned}
 x &\in y_1 + 2^{(w[\ell(P)-1]+u+1)\vec{a}}B_0 \subset x_P + 2^{[w\ell(P)+u]\vec{a}}B_0 + 2^{(w[\ell(P)-1]+u+1)\vec{a}}B_0 \\
 &\subset x_P + 2^{(w[\ell(P)-1]+u+2)\vec{a}}B_0 = R_P.
 \end{aligned}$$

By this, (4.57) and [16, Lemma 2.7], we conclude that, for any $x \in \mathbb{R}^n$ such that $\widehat{P} \cap \Gamma(x) \neq \emptyset$,

$$1 \sim \frac{|R_P|}{|R_P|} \sim \frac{|R'_Q \cap R_P|}{|R_P|} \lesssim M_{\text{HL}}^{\vec{a}}(\mathbf{1}_{R'_Q})(x) \lesssim M_{\text{HL}}^{\vec{a}}(\mathbf{1}_Q)(x),$$

which, together with (4.56), implies (4.51) also holds true in this case. This finishes the proof of (4.46) and hence of the sufficiency of Theorem 4.5. \square

Using Theorem 4.6, in [45, Proposition 4.5], Huang *et al.* further established the relation between $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ and $H_A^p(\mathbb{R}^n)$ as follows, where $H_A^p(\mathbb{R}^n)$ denotes the anisotropic Hardy space introduced by Bownik in [15, p. 17, Definition 3.11].

Proposition 4.2. Let $\vec{a} := (a_1, \dots, a_n) \in [1, \infty)^n$, $\vec{p} := (\overbrace{p, \dots, p}^{n \text{ times}})$ with $p \in (0, \infty)$, and A be as in (4.7). Then $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ and the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

4.2.2 A new proof for maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$

This subsection is devoted to providing a new proof of maximal function characterizations of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, which improves Theorem 4.2. We first state the main results of this subsection as follows.

Theorem 4.9. *Let $\vec{p} \in (0, \infty)^n$, $N \in \mathbb{N} \cap [\lfloor \frac{1}{p_-} \rfloor + 2\nu + 3, \infty)$ and φ be as in Theorem 4.2. Then the following statements are mutually equivalent:*

- (i) $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$;
- (ii) $f \in \mathcal{S}'(\mathbb{R}^n)$ and $M_{\varphi}(f) \in L^{\vec{p}}(\mathbb{R}^n)$;
- (iii) $f \in \mathcal{S}'(\mathbb{R}^n)$ and $M_{\varphi}^0(f) \in L^{\vec{p}}(\mathbb{R}^n)$.

Moreover, there exist two positive constants C_3 and C_4 , independent of f , such that

$$\|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C_3 \|M_{\varphi}^0(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C_3 \|M_{\varphi}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C_4 \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Remark 4.6. Note that, in Theorem 4.2, the exponent N belongs to

$$\mathbb{N} \cap \left[\left\lfloor \nu \frac{a_+}{a_-} \left(\frac{1}{\min\{1, p_-\}} + 1 \right) + \nu + 2a_+ \right\rfloor + 1, \infty \right),$$

which is a proper subset of

$$\mathbb{N} \cap \left[\left\lfloor \frac{1}{p_-} \right\rfloor + 2\nu + 3, \infty \right).$$

In this sense, Theorem 4.9 improves Theorem 4.2.

To show Theorem 4.9, we need several technical lemmas. We begin with introducing some notation as follows. In what follows, for any $x \in \mathbb{R}^n$, let

$$\rho_{\vec{a}}(x) := \sum_{j \in \mathbb{Z}} 2^{\nu j} \mathbf{1}_{2^{(j+1)\vec{a}} B_0 \setminus 2^{j\vec{a}} B_0}(x) \quad \text{when } x \neq \vec{0}_n, \quad \text{or else } \rho_{\vec{a}}(\vec{0}_n) := 0.$$

We first recall the following notions of some maximal functions, which are used later to prove Theorem 4.9.

Definition 4.9. Let $K \in \mathbb{Z}$, $L \in [0, \infty)$ and $N \in \mathbb{N}$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the maximal functions $M_{\varphi}^{0(K,L)}(f)$, $M_{\varphi}^{(K,L)}(f)$ and $T_{\varphi}^{N(K,L)}(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$M_{\varphi}^{0(K,L)}(f)(x) := \sup_{k \in \mathbb{Z}, k \leq K} |(f * \varphi_k)(x)| \left[\max \left\{ 1, \rho_{\vec{a}} \left(2^{-\vec{a}K} x \right) \right\} \right]^{-L} \left[1 + 2^{-\nu(k+K)} \right]^{-L},$$

$$M_{\varphi}^{(K,L)}(f)(x) := \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in B_{\bar{a}}(x, 2^k)} |(f * \varphi_k)(y)| \left[\max \left\{ 1, \rho_{\bar{a}} \left(2^{-\bar{a}k} y \right) \right\} \right]^{-L} \left[1 + 2^{-\nu(k+K)} \right]^{-L}$$

and

$$T_{\varphi}^{N(K,L)}(f)(x) := \sup_{k \in \mathbb{Z}, k \leq K} \sup_{y \in \mathbb{R}^n} \frac{|(f * \varphi_k)(y)|}{\left[\max \left\{ 1, \rho_{\bar{a}} \left(2^{-\bar{a}k} (x-y) \right) \right\} \right]^N} \frac{\left[1 + 2^{-\nu(k+K)} \right]^{-L}}{\left[\max \left\{ 1, \rho_{\bar{a}} \left(2^{-\bar{a}k} y \right) \right\} \right]^L},$$

here and thereafter, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\varphi_k(\cdot) := 2^{-k\nu} \varphi(2^{-k\bar{a}} \cdot)$. Moreover, the maximal functions $M_N^{0(K,L)}(f)$ and $M_N^{(K,L)}(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} M_N^{0(K,L)}(f)(x) &:= \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} M_{\varphi}^{0(K,L)}(f)(x), \\ M_N^{(K,L)}(f)(x) &:= \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} M_{\varphi}^{(K,L)}(f)(x). \end{aligned}$$

The following Lemmas 4.9 and 4.10 are just [2, Lemma 2.3] with A as in (4.7), and [45, Remark 2.8 (iii)], respectively.

Lemma 4.9. *There exists a positive constant C such that, for any $K \in \mathbb{Z}$, $L \in [0, \infty)$, $\lambda \in (0, \infty)$, $N \in \mathbb{N} \cap [\frac{1}{\lambda}, \infty)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\left[T_{\varphi}^{N(K,L)}(f)(x) \right]^{\lambda} \leq C M_{\text{HL}}^{\bar{a}} \left(\left[M_{\varphi}^{(K,L)}(f) \right]^{\lambda} \right) (x),$$

where $T_{\varphi}^{N(K,L)}$ and $M_{\varphi}^{(K,L)}$ are as in Definition 4.1 and $M_{\text{HL}}^{\bar{a}}$ denotes the anisotropic Hardy–Littlewood maximal operator as in (4.10).

Lemma 4.10. *Let $\vec{p} \in [1, \infty)^n$. Then, for any $r \in (0, \infty)$ and $f \in L^{\vec{p}}(\mathbb{R}^n)$,*

$$\| |f|^r \|_{L^{\vec{p}}(\mathbb{R}^n)} = \| f \|_{L^{r\vec{p}}(\mathbb{R}^n)}^r.$$

In addition, for any $\mu \in \mathbb{C}$, $\theta \in [0, \min\{1, p_{-}\}]$ with p_{-} as in (4.4), and $f, g \in L^{\vec{p}}(\mathbb{R}^n)$, $\| \mu f \|_{L^{\vec{p}}(\mathbb{R}^n)} = |\mu| \| f \|_{L^{\vec{p}}(\mathbb{R}^n)}$ and

$$\| f + g \|_{L^{\vec{p}}(\mathbb{R}^n)}^{\theta} \leq \| f \|_{L^{\vec{p}}(\mathbb{R}^n)}^{\theta} + \| g \|_{L^{\vec{p}}(\mathbb{R}^n)}^{\theta}.$$

Applying the monotone convergence theorem (see [78, p. 62, Corollary 1.9]) and [9, p. 304, Theorem 2], we have the following monotone convergence property of $L^{\vec{p}}(\mathbb{R}^n)$ and we omit the details.

Lemma 4.11. *Let $\vec{p} \in [1, \infty)^n$ and $\{g_i\}_{i \in \mathbb{N}} \subset L^{\vec{p}}(\mathbb{R}^n)$ be any sequence of non-negative functions satisfying that g_i , as $i \rightarrow \infty$, increases pointwisely almost everywhere to some $g \in L^{\vec{p}}(\mathbb{R}^n)$. Then*

$$\| g - g_i \|_{L^{\vec{p}}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

By Lemmas 4.3, 4.9 and 4.10, we easily obtain the following conclusion; the details are omitted.

Lemma 4.12. *Let $\vec{p} \in (0, \infty)^n$. Then there exists a positive constant C such that, for any $K \in \mathbb{Z}$, $L \in [0, \infty)$, $N \in \mathbb{N} \cap (\frac{1}{p_-}, \infty)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\| T_\varphi^{N(K,L)}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where $T_\varphi^{N(K,L)}$ and $M_\varphi^{(K,L)}$ are as in Definition 4.1.

To prove Theorem 4.9, we also need the following three technical lemmas, which are just [15, p. 45, Lemma 7.5, p. 46, Lemma 7.6 and p. 11, Lemma 3.2] with A as in (4.7), respectively.

Lemma 4.13. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x) dx \neq 0$. Then, for any given $N \in \mathbb{N}$ and $L \in [0, \infty)$, there exist an $I = N + 2(\nu + 1) + \nu L$ and a positive constant C , depending on N and L , such that, for any $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M_I^{0(K,L)}(f)(x) \leq C T_\varphi^{N(K,L)}(f)(x),$$

where $M_I^{0(K,L)}$ and $T_\varphi^{N(K,L)}$ are as in Definition 4.1.

Lemma 4.14. *Let φ be as in Lemma 4.13. Then, for any given $\lambda \in (0, \infty)$ and $K \in \mathbb{Z}_+$, there exist an $L \in (0, \infty)$ and a positive constant C , depending on K and λ , such that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M_\varphi^{(K,L)}(f)(x) \leq C [\max\{1, \rho_{\vec{a}}(x)\}]^{-\lambda}, \tag{4.58}$$

where $M_\varphi^{(K,L)}$ is as in Definition 4.1.

Lemma 4.15. *There exists a positive constant C such that, for any $x \in \mathbb{R}^n$,*

$$C^{-1} [\rho_{\vec{a}}(x)]^{a-/\nu} \leq |x| \leq C [\rho_{\vec{a}}(x)]^{a+/\nu} \quad \text{when } \rho_{\vec{a}}(x) \in [1, \infty),$$

and

$$C^{-1} [\rho_{\vec{a}}(x)]^{a+/\nu} \leq |x| \leq C [\rho_{\vec{a}}(x)]^{a-/\nu} \quad \text{when } \rho_{\vec{a}}(x) \in [0, 1).$$

We now prove Theorem 4.9.

Proof of Theorem 4.9. Clearly, (i) implies (ii) and (ii) implies (iii). Therefore, to prove this theorem, it suffices to show that (ii) implies (i) and that (iii) implies (ii).

We first prove that (ii) implies (i). To this end, let $\vec{p} \in (0, \infty)^n$, $I \in \mathbb{N} \cap [\lfloor \frac{1}{p_-} \rfloor + 2\nu + 3, \infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$, φ be as in Lemma 4.13 and $M_\varphi(f) \in L^{\vec{p}}(\mathbb{R}^n)$. We now show that $M_I(f) \in$

$L^{\bar{p}}(\mathbb{R}^n)$. Indeed, from Lemma 4.13 with $N := \frac{1}{p_-} + 1$ and $L := 0$, it follows that, for any given $I \in \mathbb{N} \cap [\lfloor \frac{1}{p_-} \rfloor + 2\nu + 3, \infty)$ and any $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_I^{0(K,0)}(f)(x) \lesssim T_\varphi^{N(K,0)}(f)(x).$$

This, combined with Lemma 4.12, implies that, for any $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\left\| M_I^{0(K,0)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \lesssim \left\| M_\varphi^{(K,0)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)}. \quad (4.59)$$

Letting $K \rightarrow \infty$ in (4.59) and using Lemma 4.11, we easily know that

$$\left\| M_I^0(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \lesssim \left\| M_\varphi(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)}.$$

By this and [15, p. 17, Proposition 3.10] with A as in (4.7), we further conclude that, if (ii) holds true, then (i) also holds true.

Now we show that (iii) implies (ii). To this end, let $M_\varphi^0(f) \in L^{\bar{p}}(\mathbb{R}^n)$. By Lemma 4.14 with $\lambda \in (a_+ / p_-, \infty)$ and $K \in \mathbb{Z}_+$, we know that there exists some $L \in (0, \infty)$ such that (4.58) holds true. Therefore, for any $K \in \mathbb{Z}_+$, $M_\varphi^{(K,L)}(f) \in L^{\bar{p}}(\mathbb{R}^n)$. Indeed, when $\lambda \in (a_+ / p_-, \infty)$, from Lemma 4.10 with $\theta := \underline{p}$, and Lemma 4.15, we deduce that

$$\begin{aligned} \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} &\leq \left\| M_\varphi^{(K,L)}(f) \mathbf{1}_{2^{\bar{a}}B_0} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} + \sum_{k \in \mathbb{N}} \left\| M_\varphi^{(K,L)}(f) \mathbf{1}_{2^{(k+1)\bar{a}}B_0 \setminus 2^{k\bar{a}}B_0} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} \\ &\lesssim \left\| \mathbf{1}_{2^{\bar{a}}B_0} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} + \sum_{k \in \mathbb{N}} 2^{-\nu\lambda k \underline{p}} \left\| \mathbf{1}_{2^{(k+1)\bar{a}}B_0 \setminus 2^{k\bar{a}}B_0} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} \\ &\lesssim \left\| \mathbf{1}_{B(\vec{0}_n, 1)} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} + \sum_{k \in \mathbb{N}} 2^{-\nu\lambda k \underline{p}} \left\| \mathbf{1}_{B(\vec{0}_n, 2^{ka_+})} \right\|_{L^{\bar{p}}(\mathbb{R}^n)}^{\underline{p}} \\ &\lesssim \sum_{k \in \mathbb{Z}_+} 2^{-\nu\lambda k \underline{p}} 2^{vka_+ \underline{p} / p_-} < \infty, \end{aligned}$$

where, for any $r \in (0, \infty)$, $B(\vec{0}_n, r) := \{y \in \mathbb{R}^n : |y| < r\}$. Thus, $M_\varphi^{(K,L)}(f) \in L^{\bar{p}}(\mathbb{R}^n)$.

In addition, by Lemmas 4.13 and 4.12, we conclude that, for any given $L \in (0, \infty)$, there exist some $I \in \mathbb{N}$ and a positive constant C_5 such that, for any $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\left\| M_I^{0(K,L)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \leq C_5 \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)}.$$

For any fixed $K \in \mathbb{Z}_+$, let

$$G_K := \left\{ x \in \mathbb{R}^n : M_I^{0(K,L)}(f)(x) \leq C_6 M_\varphi^{(K,L)}(f)(x) \right\},$$

where $C_6 := 2C_5$. Then

$$\left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\bar{p}}(\mathbb{R}^n)} \lesssim \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\bar{p}}(G_K)}, \quad (4.60)$$

due to the fact that

$$\left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\vec{p}}(G_K^c)} \leq C_6^{-1} \left\| M_I^{0(K,L)}(f) \right\|_{L^{\vec{p}}(G_K^c)} \leq C_5/C_6 \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

For any given $L \in (0, \infty)$, repeating the proof of [62, (4.17)] with p therein replaced by p_- and A as in (4.7), we find that, for any $r \in (0, p_-)$, $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in G_K$,

$$\left[M_\varphi^{(K,L)}(f)(x) \right]^r \lesssim M_{\text{HL}}^{\vec{a}} \left(\left[M_\varphi^{0(K,L)}(f) \right]^r \right)(x).$$

From this, (4.60) and Lemmas 4.10 and 4.3, we further deduce that, for any $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)}^r &\lesssim \left\| M_\varphi^{(K,L)}(f) \right\|_{L^{\vec{p}}(G_K)}^r \sim \left\| \left[M_\varphi^{(K,L)}(f) \right]^r \right\|_{L^{\vec{p}/r}(G_K)} \\ &\lesssim \left\| M_{\text{HL}}^{\vec{a}} \left(\left[M_\varphi^{0(K,L)}(f) \right]^r \right) \right\|_{L^{\vec{p}/r}(\mathbb{R}^n)} \\ &\lesssim \left\| \left[M_\varphi^{0(K,L)}(f) \right]^r \right\|_{L^{\vec{p}/r}(\mathbb{R}^n)} \sim \left\| M_\varphi^{0(K,L)}(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)}^r. \end{aligned} \tag{4.61}$$

Letting $K \rightarrow \infty$ in (4.61), by Lemma 4.11, we have

$$\left\| M_\varphi(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \lesssim \left\| M_\varphi^0(f) \right\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

This shows that (iii) implies (ii) and hence finishes the proof of Theorem 4.9. □

4.3 Dual spaces of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$

In this subsection, we mainly discuss the dual spaces of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$. Indeed, the dual spaces of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ were asked by Cleanthous *et al.* in [23] and part of them were obtained by Huang *et al.* in [46]. To present this, we first introduce the notion of the anisotropic mixed-norm Campanato space $\mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$ given in [46].

Definition 4.10. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $q \in [1, \infty]$ and $s \in \mathbb{Z}_+$. The *anisotropic mixed-norm Campanato space* $\mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$ is defined to be the set of all locally $L^q(\mathbb{R}^n)$ functions g such that, when $q \in [1, \infty)$,

$$\|g\|_{\mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)} := \sup_{B \in \mathfrak{B}} \inf_{P \in \mathbb{P}_s(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}} \left[\frac{1}{|B|} \int_B |g(x) - P(x)|^q dx \right]^{1/q} < \infty$$

and

$$\|g\|_{\mathcal{L}_{\vec{p},\infty,s}^{\vec{a}}(\mathbb{R}^n)} := \sup_{B \in \mathfrak{B}} \inf_{P \in \mathbb{P}_s(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}} \|g - P\|_{L^\infty(B)} < \infty,$$

where \mathfrak{B} is as in (4.1).

Remark 4.7. (i) Obviously, $\|\cdot\|_{\mathcal{L}_{\vec{p},q,s}^{\vec{a}}}(\mathbb{R}^n)$ is a seminorm and $\mathbb{P}_s(\mathbb{R}^n) \subset \mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$. Indeed,

$$\|g\|_{\mathcal{L}_{\vec{p},q,s}^{\vec{a}}}(\mathbb{R}^n) = 0 \quad \text{if and only if} \quad g \in \mathbb{P}_s(\mathbb{R}^n).$$

Thus, if we identify g_1 with g_2 when $g_1 - g_2 \in \mathbb{P}_s(\mathbb{R}^n)$, then $\mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$ becomes a Banach space. In what follows, we always identify $g \in \mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$ with $\{g + P : P \in \mathbb{P}_s(\mathbb{R}^n)\}$.

(ii) When $\vec{a} := (\overbrace{1, \dots, 1}^{n \text{ times}})$ and $\vec{p} := (\overbrace{p, \dots, p}^{n \text{ times}})$ with some $p \in (0, 1]$, for any $B \in \mathfrak{B}$, we have $\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)} = |B|^{1/p}$. Then the space $\mathcal{L}_{\vec{p},q,s}^{\vec{a}}(\mathbb{R}^n)$ is just the classical Campanato space $L_{\frac{1}{p}-1,q,s}(\mathbb{R}^n)$ introduced by Campanato in [18], which includes the classical space $\text{BMO}(\mathbb{R}^n)$, introduced by John and Nirenberg in [50], as a special case.

Via Theorems 4.3 and 4.4, the following duality theorem was established in [46, Theorem 3.10].

Theorem 4.10. Let $\vec{p} \in (0, 1]^n$ and \vec{a} , r and s be as in Definition 4.5. Then the dual space of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, denoted by $(H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n))^*$, is $\mathcal{L}_{\vec{p},r',s}^{\vec{a}}(\mathbb{R}^n)$ with $1/r + 1/r' = 1$ in the following sense:

(i) Suppose that $g \in \mathcal{L}_{\vec{p},r',s}^{\vec{a}}(\mathbb{R}^n)$. Then the linear functional

$$L_g : f \longmapsto L_g(f) := \int_{\mathbb{R}^n} f(x)g(x) dx, \tag{4.62}$$

initially defined for any $f \in H_{\vec{a},\text{fin}}^{\vec{p},r,s}(\mathbb{R}^n)$, has a bounded extension to $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$.

(ii) Conversely, any continuous linear functional on $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ arises as in (4.62) with a unique $g \in \mathcal{L}_{\vec{p},r',s}^{\vec{a}}(\mathbb{R}^n)$.

Moreover, $\|g\|_{\mathcal{L}_{\vec{p},r',s}^{\vec{a}}}(\mathbb{R}^n) \sim \|L_g\|_{(H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n))^*}$, where the positive equivalence constants are independent of g .

Remark 4.8. (i) When $\vec{a} := (\overbrace{1, \dots, 1}^{n \text{ times}})$ and $\vec{p} := (\overbrace{p, \dots, p}^{n \text{ times}})$ with some $p \in (0, 1]$, the two spaces $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ and $\mathcal{L}_{\vec{p},r',s}^{\vec{a}}(\mathbb{R}^n)$ become, respectively, the classical Hardy space $H^p(\mathbb{R}^n)$ and the classical Campanato space $L_{\frac{1}{p}-1,r',s}(\mathbb{R}^n)$. In this case, Theorem 4.2 was obtained by Taibleson and Weiss [82], which includes the famous duality result, obtained by Fefferman and Stein in [31], $(H^1(\mathbb{R}^n))^* = \text{BMO}(\mathbb{R}^n)$, as a special case.

(ii) Note that, when $\vec{a} := (\overbrace{1, \dots, 1}^{n \text{ times}})$, the space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is just the isotropic mixed-norm Hardy space. We point out that, even in this case, Theorem 4.10 is also new.

- (iii) When $\vec{p} \in (1, \infty)^n$, by Proposition 4.1, we know that $H_a^{\vec{p}}(\mathbb{R}^n) = L^{\vec{p}}(\mathbb{R}^n)$ with equivalent norms. This, together with Theorem 2.2, further implies that, for any $\vec{p} \in (1, \infty)^n$, $L^{\vec{p}'}(\mathbb{R}^n)$ is the dual space of $H_a^{\vec{p}}(\mathbb{R}^n)$. However, when $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$ with $p_{i_0} \in (0, 1]$ and $p_{j_0} \in (1, \infty)$ for some $i_0, j_0 \in \{1, \dots, n\}$, the dual space of $H_a^{\vec{p}}(\mathbb{R}^n)$ is still unknown so far.

4.4 Applications to boundedness of sublinear operators

This subsection is devoted to displaying some applications of Hardy spaces $H_a^{\vec{p}}(\mathbb{R}^n)$, which were obtained in [45]. More precisely, in this subsection, we first recall a criterion on the boundedness of sublinear operators from $H_a^{\vec{p}}(\mathbb{R}^n)$ into a quasi-Banach space. Then, applying this criterion, the boundedness of anisotropic convolutional δ -type and non-convolutional β -order Calderón–Zygmund operators from $H_a^{\vec{p}}(\mathbb{R}^n)$ to itself [or to $L^{\vec{p}}(\mathbb{R}^n)$] was obtained. In addition, we improve [45, Theorems 6.8 and 6.9].

Recall that a complete vector space is called a *quasi-Banach space* \mathcal{B} if its quasi-norm $\|\cdot\|_{\mathcal{B}}$ satisfies

- (i) $\|\psi\|_{\mathcal{B}} = 0$ if and only if ψ is the zero element of \mathcal{B} ;
- (ii) there exists a positive constant $K \in [1, \infty)$ such that, for any $\psi, \phi \in \mathcal{B}$,

$$\|\psi + \phi\|_{\mathcal{B}} \leq K(\|\psi\|_{\mathcal{B}} + \|\phi\|_{\mathcal{B}}).$$

Note that, when $K=1$, a quasi-Banach space \mathcal{B} is just a Banach space. In addition, for any given $\gamma \in (0, 1]$, a γ -quasi-Banach space \mathcal{B}_{γ} is defined to be a quasi-Banach space equipped with a quasi-norm $\|\cdot\|_{\mathcal{B}_{\gamma}}$ satisfying that there exists a constant $C \in [1, \infty)$ such that, for any $\kappa \in \mathbb{N}$ and $\{\psi_i\}_{i=1}^{\kappa} \subset \mathcal{B}_{\gamma}$,

$$\left\| \sum_{i=1}^{\kappa} \psi_i \right\|_{\mathcal{B}_{\gamma}}^{\gamma} \leq C \sum_{i=1}^{\kappa} \|\psi_i\|_{\mathcal{B}_{\gamma}}^{\gamma}$$

holds true (see [45, 60, 86–88]).

Let \mathcal{B}_{γ} be a γ -quasi-Banach space with $\gamma \in (0, 1]$, and \mathcal{Y} a linear space. An operator T from \mathcal{Y} to \mathcal{B}_{γ} is said to be \mathcal{B}_{γ} -sublinear if there exists a positive constant C such that, for any $\kappa \in \mathbb{N}$, $\{\mu_i\}_{i=1}^{\kappa} \subset \mathbb{C}$ and $\{\psi_i\}_{i=1}^{\kappa} \subset \mathcal{Y}$,

$$\left\| T \left(\sum_{i=1}^{\kappa} \mu_i \psi_i \right) \right\|_{\mathcal{B}_{\gamma}}^{\gamma} \leq C \sum_{i=1}^{\kappa} |\mu_i|^{\gamma} \|T(\psi_i)\|_{\mathcal{B}_{\gamma}}^{\gamma}$$

and, for any $\psi, \phi \in \mathcal{Y}$, $\|T(\psi) - T(\phi)\|_{\mathcal{B}_{\gamma}} \leq C\|T(\psi - \phi)\|_{\mathcal{B}_{\gamma}}$ (see [45, 60, 86–88]). Obviously, for any $\gamma \in (0, 1]$, the linearity of T implies its \mathcal{B}_{γ} -sublinearity.

We first state the following criterion on the boundedness of sublinear operators from $H_a^{\vec{p}}(\mathbb{R}^n)$ into a quasi-Banach space \mathcal{B}_{γ} , which was proved by Huang *et al.* in [45, Theorem 6.2] via Theorem 4.4.

Theorem 4.11. Assume that $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $r \in (\max\{p_+, 1\}, \infty]$ with p_+ as in (4.4), $\gamma \in (0, 1]$, s is as in (4.8) and \mathcal{B}_γ a γ -quasi-Banach space. If either of the following two statements holds true:

- (i) $r \in (\max\{p_+, 1\}, \infty)$ and $T: H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n) \rightarrow \mathcal{B}_\gamma$ is a \mathcal{B}_γ -sublinear operator satisfying that there exists a positive constant C such that, for any $f \in H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C \|f\|_{H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)};$$

- (ii) $T: H_{\vec{a}, \text{fin}}^{\vec{p}, \infty, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \rightarrow \mathcal{B}_\gamma$ is a \mathcal{B}_γ -sublinear operator satisfying that there exists a positive constant C such that, for any $f \in H_{\vec{a}, \text{fin}}^{\vec{p}, \infty, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C \|f\|_{H_{\vec{a}, \text{fin}}^{\vec{p}, \infty, s}(\mathbb{R}^n)},$$

then T uniquely extends to a bounded \mathcal{B}_γ -sublinear operator from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ into \mathcal{B}_γ . Moreover, there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Using Theorem 4.11, the following useful conclusion was also shown in [45, Corollary 6.3].

Corollary 4.2. Let $\vec{p} \in (0, 1]^n$ and \vec{a} , r , γ , s and \mathcal{B}_γ be as in Theorem 4.11. If either of the following two statements holds true:

- (i) $r \in (1, \infty)$ and T is a \mathcal{B}_γ -sublinear operator from $H_{\vec{a}, \text{fin}}^{\vec{p}, r, s}(\mathbb{R}^n)$ to \mathcal{B}_γ satisfying

$$\sup \left\{ \|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any } (\vec{p}, r, s)\text{-atom} \right\} < \infty;$$

- (ii) T is a \mathcal{B}_γ -sublinear operator defined on all continuous (\vec{p}, ∞, s) -atoms satisfying

$$\sup \left\{ \|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any continuous } (\vec{p}, \infty, s)\text{-atom} \right\} < \infty,$$

then T uniquely extends to a bounded \mathcal{B}_γ -sublinear operator from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ into \mathcal{B}_γ .

We now recall a class of anisotropic convolutional δ -type Calderón–Zygmund operators and anisotropic non-convolutional β -order Calderón–Zygmund operators in [45] as follows. In what follows, for any $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and $m \in \mathbb{Z}_+$, denote by $C^m(E)$ the set of all functions on E whose derivatives with order not greater than m are continuous.

Definition 4.11. For any $\delta \in (0, a_+]$, a linear operator T is called an *anisotropic convolutional δ -type Calderón–Zygmund operator* if T is bounded on $L^2(\mathbb{R}^n)$ with kernel $k \in \mathcal{S}'(\mathbb{R}^n)$ coinciding with a locally integrable function on $\mathbb{R}^n \setminus \{\vec{0}_n\}$, and satisfying that there exists a positive constant C such that, for any $x, y \in \mathbb{R}^n$ with $|x|_{\vec{a}} > 2|y|_{\vec{a}} \neq 0$,

$$|k(x-y) - k(x)| \leq C \frac{|y|_{\vec{a}}^\delta}{|x|_{\vec{a}}^{v+\delta}}$$

and, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $T(f)(x) := \text{p.v. } k * f(x)$.

Definition 4.12. Let $\vec{a} \in [1, \infty)^n$. For any given $\beta \in (0, \infty) \setminus \mathbb{N}$, a linear operator T is called an *anisotropic non-convolutional β -order Calderón–Zygmund operator* if T is bounded on $L^2(\mathbb{R}^n)$ and its kernel

$$\mathcal{K}: \Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\} \rightarrow \mathbb{C}$$

satisfies that $\mathcal{K} \in C^{[\beta]}(\Omega)$ and there exists a positive constant C such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = [\beta]$, and $x, y, z \in \mathbb{R}^n$ with $|x-y|_{\vec{a}} > 2|y-z|_{\vec{a}} \neq 0$,

$$|[\partial^\alpha \mathcal{K}(x, \cdot)](y) - [\partial^\alpha \mathcal{K}(x, \cdot)](z)| \leq C \frac{|y-z|_{\vec{a}}^{\beta a_+}}{|x-y|_{\vec{a}}^{v+\beta a_+}} \min \left\{ |y-z|_{\vec{a}}^{-[\beta] a_+}, |y-z|_{\vec{a}}^{-[\beta] a_-} \right\}$$

and, for any $f \in L^2(\mathbb{R}^n)$ with compact support, and $x \notin \text{supp } f$,

$$T(f)(x) = \int_{\text{supp } f} \mathcal{K}(x, y) f(y) dy.$$

For any $l \in \mathbb{N}$, an operator T is said to have the *vanishing moments up to order l* if, for any $a \in L^2(\mathbb{R}^n)$ having compact support and satisfying that, for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq l$, $\int_{\mathbb{R}^n} x^\gamma a(x) dx = 0$, then

$$\int_{\mathbb{R}^n} x^\gamma T(a)(x) dx = 0.$$

In [45, Theorems 6.4 and 6.5], Huang *et al.* established the following boundedness of anisotropic convolutional δ -type Calderón–Zygmund operators from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ to itself or to $L^{\vec{p}}(\mathbb{R}^n)$.

Theorem 4.12. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, \infty)^n$, $\delta \in (0, a_+]$ and $p_- \in (\frac{v}{v+\delta}, \infty)$ with p_- as in (4.4). Let T be an anisotropic convolutional δ -type Calderón–Zygmund operator. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Theorem 4.13. Let \vec{a} , \vec{p} , δ , p_- and T be as in Theorem 4.12. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Moreover, the following boundedness of anisotropic β -order Calderón–Zygmund operators T from $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ to itself or to $L^{\vec{p}}(\mathbb{R}^n)$ was also established in [45, Theorems 6.8 and 6.9].

Theorem 4.14. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, 2)^n$, $\beta \in (0, \infty) \setminus \mathbb{N}$,

$$p_- \in \left(\frac{\nu}{\nu + \beta a_-}, \frac{\nu}{\nu + \lfloor \beta \rfloor a_-} \right]$$

with p_- as in (4.4) and a_- as in (4.5), and T be an anisotropic β -order Calderón–Zygmund operator having the vanishing moments up to order $\lfloor \beta \rfloor$. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Theorem 4.15. Let \vec{a} , \vec{p} , β and p_- be the same as in Theorem 4.14 and T an anisotropic β -order Calderón–Zygmund operator. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Finally, we point out that, in Definition 4.12, the range of $\beta \in (0, \infty) \setminus \mathbb{N}$ can be revised as $\beta \in (0, \infty)$, and then we can improve the restriction of β in Theorems 4.14 and 4.15 into $\beta \in (0, \infty)$. To be precise, we have the following improved versions of Definition 4.12 and Theorems 4.14 and 4.15.

Definition 4.13. Let $\vec{a} \in [1, \infty)^n$. For any given $\beta \in (0, \infty)$, a linear operator T is called an anisotropic non-convolutional β -order Calderón–Zygmund operator if T is bounded on $L^2(\mathbb{R}^n)$ and its kernel

$$\mathcal{K}: \Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\} \rightarrow \mathbb{C}$$

satisfies that $\mathcal{K} \in C^{\lceil \beta \rceil - 1}(\Omega)$ and there exists a positive constant C such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = \lceil \beta \rceil - 1$, and $x, y, z \in \mathbb{R}^n$ with $|x - y|_{\vec{a}} > 2|y - z|_{\vec{a}} \neq 0$,

$$\begin{aligned} & |[\partial^\alpha \mathcal{K}(x, \cdot)](y) - [\partial^\alpha \mathcal{K}(x, \cdot)](z)| \\ & \leq C \frac{|y - z|_{\vec{a}}^{\beta a_+}}{|x - y|_{\vec{a}}^{\nu + \beta a_+}} \min \left\{ |y - z|_{\vec{a}}^{-\lceil \beta \rceil a_+}, |y - z|_{\vec{a}}^{-\lceil \beta \rceil a_-} \right\} \end{aligned}$$

and, for any $f \in L^2(\mathbb{R}^n)$ with compact support, and $x \notin \text{supp } f$,

$$T(f)(x) = \int_{\text{supp } f} \mathcal{K}(x, y) f(y) dy.$$

Theorem 4.16. Let $\vec{a} \in [1, \infty)^n$, $\vec{p} \in (0, 2)^n$, $\beta \in (0, \infty)$,

$$p_- \in \left(\frac{\nu}{\nu + \beta a_-}, \frac{\nu}{\nu + (\lceil \beta \rceil - 1)a_-} \right]$$

with p_- as in (4.4) and a_- as in (4.5), and T be an anisotropic β -order Calderón–Zygmund operator having the vanishing moments up to order $\lceil \beta \rceil - 1$. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

Theorem 4.17. Let \vec{a} , \vec{p} , β and p_- be the same as in Theorem 4.16 and T an anisotropic β -order Calderón–Zygmund operator. Then there exists a positive constant C such that, for any $f \in H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|f\|_{H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)}.$$

To prove Theorems 4.16 and 4.17, we only need to replace $\lfloor \beta \rfloor$ by $\lceil \beta \rceil - 1$ in the proofs of [45, Theorems 6.8 and 6.9], respectively. Thus, the details are omitted.

At the end of this section, we point out that all the results about the mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ associated to a vector $\vec{a} \in [1, \infty)^n$ can be extended to a more general anisotropic setting, namely, the mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ associated to an expansive matrix A . We refer the reader to [47] for the details.

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