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A Note on Gaussian BV Function and its Heat Semigroup Characterization

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Abstract. In this note, we investigate the properties of Gaussian BV functions and give a heat semigroup characterization of BV functions in Gauss space. In particular, the latter is the nontrivial generalization of classical De Giorgi's heat kernel characterization of function of bounded variation on Euclidean space to the case of Gauss space.

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1 Introduction

There are various definitions of variational functions, and the related class of bounded variational functions (or BV functions for short), is meaningful in different contexts and equivalent in general. On the Euclidean space, the variation of $f\in L^1(\mathbb{R}^n)$ with the Lebesgue measure can be defined as

$$\|Df\|(\mathbb{R}^n)=\sup\left\{\int_{\mathbb{R}^n} f\operatorname{div}\varphi dx:\varphi\in C_c^1(\mathbb{R}^n,\,\mathbb{R}^n)\quad\text{with}\quad \|\varphi\|_{L^\infty}\le 1\right\},$$

where $\operatorname{div}\varphi(x):=\sum_{i=1}^n\frac{\partial\varphi_i}{\partial x_i}$. In fact, the original definition of the variation of a function was given by De Giorgi through a thermonuclear regularization process (see [3,4]). He also proved that

$$\|Df\|(\mathbb{R}^n)=\lim_{t\to 0}\int_{\mathbb{R}^n}|\nabla T_t f|dx, \tag{1.1}$$

where ∇ denotes the gradient of the function f, and

$$T_t f(x)=\int_{\mathbb{R}^n} h(t,x-y)f(y)dy$$

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is the heat semigroup with the Gauss-Weierstrass kernel $h(t,x)=(4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}$. Later, Miranda, Pallara, Paronetto and Preunkert in [6] proved the equality (1.1) on the Riemannian manifold M

$$|Du|(M)=\lim_{t\to 0}\int_M |dP_t u|dV$$

with two geometric assumptions: the Ricci curvature is bounded from below and the volume of geodesic balls of fixed radius has a positive lower bound which does not depend on the center, where $\{P_t\}_{t\geq 0}$ is the heat semigroup generated by the Laplace-Beltrami operator on M. After that, Carbonaro and Mauceri proved the equality (1.1) based on properties of heat semigroups with the only restriction that the Ricci curvature is bounded from below in [1]. [2] implies that the equality (1.1) holds in a weaker sense and the authors provide two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups.

In order to state our main result, we recall some basic facts for the n dimensional Gauss space $\mathbb{G}^n$. This space is equipped with the following measure

$$\gamma(x)=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}},\quad \forall x\in\mathbb{R}^n,$$

the Gaussian volume element $dV_\gamma=\gamma dx$ and the γ-divergence $\operatorname{div}_\gamma\varphi=\operatorname{div}\varphi-x\cdot\varphi$, $\varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$. Next we also recall the Gaussian BV functions and its properties. For any open subset $\Omega\subseteq\mathbb{R}^n$, the γ-total variation of $f\in L^1(\Omega)$ is defined by

$$\|Df\|(\Omega;\mathbb{G}^n)=\sup\left\{\int_\Omega f\operatorname{div}_\gamma\varphi dV_\gamma:\varphi\in C_c^1(\Omega,\mathbb{R}^n)\ \text{ with }\ \|\varphi\|_{L^\infty}\leq 1\right\},$$

where $\|\varphi\|_{L^\infty}=\operatorname*{esssup}_{x\in\Omega}\,(|\varphi_1|^2+\ldots+|\varphi_n|^2)^{1/2}$. Particularly, if $\Omega=\mathbb{R}^n$, we denote $\|Df\|(\Omega;\mathbb{G}^n)$ by $\|Df\|(\mathbb{G}^n)$. The function $f\in L^1(\Omega)$ is of the Gaussian bounded variation on Ω and denoted by $f\in BV(\Omega;\mathbb{G}^n)$ if

$$\|Df\|(\Omega)<\infty.$$

When $\Omega=\mathbb{R}^n$, we denote $BV(\Omega;\mathbb{G}^n)$ by $BV(\mathbb{G}^n)$. The space $BV_{\text{loc}}(\mathbb{G}^n)$ is said to be of locally Gaussian bounded variation in $\mathbb{R}^n$, if

$$\|Df\|(N;\mathbb{G}^n)<\infty,$$

for every open set $N\subseteq\mathbb{R}^n$ and $\overline{N}$ is compact. For a set $E\subseteq\mathbb{R}^n$, $P_\gamma(E):=\|D\chi_E\|(\mathbb{G}^n)$ be the Gaussian perimeter of E. Refer to [5] for some properties of $P_\gamma(E)$. In particular, from [5] we know that the Gauss-Green formula is valid:

$$\int_E \operatorname{div}_\gamma v dV_\gamma=\int_{\partial^* E} v\cdot v_E dP_\gamma,\quad \forall v\in C_c^1(\mathbb{R}^n,\mathbb{R}^n),$$

where v_E is the outer unit normal to ∂E of a set E with the Gaussian perimeter $P_\gamma(E)<\infty$, and $\partial^* E$ stands for the reduced boundary of set E, and the Gaussian perimeter element $dP_\gamma=\gamma dP$ accompanied by the $(n-1)$ dimensional area element dP with the weight γ .

Moreover, if $f\in Lip(\mathbb{R}^n)$, then by the Rademacher theorem, we have

$$\|Df\|(\mathbb{G}^n)=\int_{\mathbb{R}^n}|\nabla f|dV_\gamma<\infty.$$

Finally, the Gaussian co-area formula given in [5] is also valid, that is, if $f\in BV(\mathbb{G}^n)$, then

$$\|Df\|(\mathbb{G}^n)=\int_{-\infty}^{\infty}P_\gamma(\{x\in\mathbb{R}^n:f(x)>t\})dt.$$

Now we consider a situation after scaling transformation and generalize the above facts to this situation. At this time, the Gauss space is equipped with the following measure

$$\gamma_n^B(x)dx=\frac{(\det B)^{1/2}}{\pi^{n/2}}e^{-Bx\cdot x}dx,\quad \forall x\in\mathbb{R}^n,$$

where the diagonal matrix is

$$B=\begin{pmatrix} d_1 & 0 & \cdots & 0\\ 0 & d_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & d_n, \end{pmatrix},\quad d_i>0,\ 1\le i\le n.$$

Following [9], we know that the Gaussian volume element is denoted by $dV_B=\gamma_n^B(x)dx$ and the B-divergence is denoted by

$$\mathrm{div}_B\varphi=\mathrm{div}\varphi-2Bx\cdot\varphi,\quad \varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n).$$

If $d_i=\frac{1}{2}$, $i=1,\dots n$, in diagonal matrix B, this is the same situation as γ. Similarly, we also introduce the Gaussian BV functions and its properties. For any open subset $\Omega\subseteq\mathbb{R}^n$, the B-total variation of $f\in L^1(\Omega)$ is defined by

$$\|Df\|_B(\Omega;\mathbb{G}^n)=\sup\left\{\int_\Omega f\mathrm{div}_B\varphi dV_B:\varphi\in C_c^1(\Omega,\mathbb{R}^n)\ \text{ with }\ \|\varphi\|_{L^\infty}\le 1\right\},$$

where $\|\varphi\|_{L^\infty}=\operatorname*{esssup}_{x\in\Omega}(|\varphi_1|^2+\ldots+|\varphi_n|^2)^{1/2}$. In the same way, if $\Omega=\mathbb{R}^n$, we denote $\|Df\|_B(\Omega;\mathbb{G}^n)$ by $\|Df\|_B(\mathbb{G}^n)$. The function $f\in L^1(\Omega)$ is of the Gaussian bounded variation on Ω and denoted by $f\in BV_B(\Omega;\mathbb{G}^n)$ if

$$\|Df\|_B(\Omega)<\infty.$$

When $\Omega=\mathbb{R}^n$, we denote $BV_B(\Omega;\mathbb{G}^n)$ by $BV_B(\mathbb{G}^n)$. The space $BV_{B,\mathrm{loc}}(\mathbb{G}^n)$ is said to be of locally Gaussian bounded variation in $\mathbb{R}^n$, if

$$\|Df\|_B(N;\mathbb{G}^n)<\infty,$$

where set $N\subseteq\mathbb{R}^n$ and $\overline{N}$ is compact.

In Section 2 of this paper, we investigate the Gaussian BV functions and Gaussian perimeter and study their properties. In Section 3, as the continuation of the classical De Giorgi's heat kernel characterization of function of bounded variation on Euclidean space, we investigate the heat semigroup of Gaussian BV functions. Our proof mainly applies the basic properties of heat semigroups in [5], precisely,

$$\|Df\|_B(\mathbb{G}^n)=\lim_{t\to 0}\|\nabla P_t f\|_{L^1}, \quad \forall f\in L^1(\mathbb{G}^n),$$

where P_t is defined in (3.1) and it is also called Ornstein-Uhlenbeck semigroup (cf. [5] or [9]).

2 Gaussian BV functions and Gaussian perimeters

For a set $E\subseteq\mathbb{R}^n$, the Gaussian perimeter of E is

$$P_B(E):=\|D\chi_E\|_B(\mathbb{G}^n),$$

where χ_E is the characteristic function of set E. And the Gaussian perimeter element $dP_B=\gamma_n^B dP$ accompanied by the $(n-1)$ dimensional area element dP with the weight γ_n^B.

The following Lemmas can be obtained by the method in [5]. We omit the details of the proofs.

Lemma 2.1. *If $f,g\in L^1(\mathbb{G}^n)$, then*

$$\|D\max\{f,g\}\|_B(\mathbb{G}^n)+\|D\min\{f,g\}\|_B(\mathbb{G}^n)\le\|Df\|_B(\mathbb{G}^n)+\|Dg\|_B(\mathbb{G}^n).$$

In particular, for sets $U,V\in\mathbb{R}^n$, if $f=\chi_U,\ g=\chi_V$, we have that

$$P_B(U\cup V)+P_B(U\cap V)\le P_B(U)+P_B(V).$$

Lemma 2.2. *For a set $E\subseteq\mathbb{R}^n$, we have*

$$\int_E \mathrm{div}_B\varphi dV_B=\int_{\partial^* E}\varphi\cdot v_E dP_B,\ \ \forall\varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n),$$

where v_E is the outer unit normal to ∂E of the set E with the Gaussian perimeter $P_B(E)<\infty$, and $\partial^* E$ stands for the reduced boundary of set E.

Lemma 2.3. *If $f\in Lip(\mathbb{R}^n)$, we have*

$$\|Df\|_B(\mathbb{G}^n)=\int_{\mathbb{R}^n}|\nabla f|dV_B<\infty.$$

The following two theorem can be proved by following the proof of [10, 5.2.1] and [10, 5.2.2] respectively.

Theorem 2.1. *For any open subset $\Omega\subseteq\mathbb{R}^n$, suppose $f_k\in BV_B(\Omega),(k=1,2,\dots)$ and $f_k\to f$ in $L^1_{loc}(\Omega)$, then*

$$\|Df\|_B(\Omega)\le\lim_{k\to\infty}\inf\|Df_k\|_B(\Omega).$$

Theorem 2.2. *For any open subset $\Omega\subseteq\mathbb{R}^n$, if $f\in BV_B(\Omega)$, there exist functions $\{f_k\}_{k=1}^{\infty}\subset BV_B(\Omega)\cap C^\infty(\Omega)$ such that*

(i) $f_k\to f$ *in* $L^1(\Omega)$.

(ii) $\|Df_k\|_B(\Omega)\to\|Df\|_B(\Omega)$ *as* $k\to\infty$.

Lemma 2.4. *For any $f\in BV_B(\mathbb{G}^n)$, $\varphi\in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n}f\mathrm{div_B}\varphi dV_B=-\int_{\mathbb{R}^n}\nabla f\cdot\varphi dV_B.$$

Proof. Via the definition of the gradient and $dV_B=\gamma_n^B(x)dx=\frac{(\det B)^{1/2}}{\pi^{n/2}}e^{-Bx\cdot x}dx$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}f\mathrm{div_B}\varphi dV_B=\int_{\mathbb{R}^n}f(\mathrm{div}\varphi-2Bx\cdot\varphi)dV_B\\
=&\int_{\mathbb{R}^n}f(\frac{\partial\varphi_1}{\partial x_1}+\cdots+\frac{\partial\varphi_n}{\partial x_n}-2d_1x_1\varphi_1-\cdots-2d_nx_n\varphi_n)dV_B\\
=&\int_{\mathbb{R}^n}f(\frac{\partial\varphi_1}{\partial x_1}+\cdots+\frac{\partial\varphi_n}{\partial x_n}-2d_1x_1\varphi_1-\cdots-2d_nx_n\varphi_n)\frac{(\det B)^{1/2}}{\pi^{n/2}}e^{-Bx\cdot x}dx\\
=&\int_{\mathbb{R}^n}-\nabla(f\gamma_n^B)\cdot\varphi-f\gamma_n^B(2d_1x_1\varphi_1+\cdots+2d_nx_n\varphi_n)dx\\
=&\int_{\mathbb{R}^n}-\gamma_n^B\nabla f\cdot\varphi-f\nabla\gamma_n^B\cdot\varphi-f\gamma_n^B(2d_1x_1\varphi_1+\cdots+2d_nx_n\varphi_n)dx.
\end{aligned}$$

Next, we check the fact:

$$-f\nabla\gamma_n^B\cdot\varphi-f\gamma_n^B(2d_1x_1\varphi_1+\cdots+2d_nx_n\varphi_n)=0.$$

In fact,

$$\begin{aligned}
&-f\nabla\gamma_n^B\cdot\varphi-f\gamma_n^B(2d_1x_1\varphi_1+\cdots+2d_nx_n\varphi_n)\\
=&-f\cdot\varphi_1\gamma_n^B(-2d_1x_1)-\cdots-f\cdot\varphi_n\gamma_n^B(-2d_nx_n)-f\gamma_n^B(2d_1x_1\varphi_1+\cdots+2d_nx_n\varphi_n)\\
=&0.
\end{aligned}$$

Then we get

$$\int_{\mathbb{R}^n} f\cdot \mathrm{div_B}\varphi dV_B = -\int_{\mathbb{R}^n}\gamma_n^B \nabla f\cdot\varphi dx = -\int_{\mathbb{R}^n}\nabla f\cdot\varphi dV_B.$$

So we proved the theorem.

□

Lemma 2.5. *If* $f\in BV_B(\mathbb{G}^n)$, *then*

$$\|Df\|_B(\mathbb{G}^n)=\int_{-\infty}^{\infty}P_B(\{x\in\mathbb{R}^n : f(x)>t\})dt.$$

Proof. Assume that $f\in BV_B(\mathbb{G}^n)\cap C^\infty(\mathbb{G}^n)$. For $t\in\mathbb{R}$, define

$$E_t=\{x\in\mathbb{R}^n : f(x)>t\}.$$

It is not hard to verify that

$$\int_{\mathbb{R}^n} f\mathrm{div_B}\varphi dV_B=\int_{-\infty}^{+\infty}\left(\int_{E_t}\mathrm{div_B}\varphi dV_B\right)dt,$$

where $\varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$ and $\|\varphi\|_\infty\le 1$. Hence, the inequality

$$\|Df\|_B(\mathbb{G}^n)\le\int_{-\infty}^{\infty}P_B(\{x\in\mathbb{R}^n : f(x)>t\})dt,$$

holds true. And then we prove the opposite inequality. Let

$$m(t)=\int_{\mathbb{R}^n\setminus E_t}|\nabla f|dV_B=\int_{\{f\le t\}}|\nabla f|dV_B.$$

Then the function of m is nondecreasing, and m' exists L^1 a.e., with

$$\int_{-\infty}^{+\infty}m'(t)dx\le\int_{\mathbb{R}^n}|\nabla f|dV_B.$$

Next, for any $-\infty<t<\infty$, $r>0$, define function

$$\eta(s)=\begin{cases} 0 & s\le t\\ \frac{s-t}{r} & t\le s\le t+r\\ 1 & s\ge t+r\end{cases},$$

then

$$\eta'(s)=\begin{cases}\frac{1}{r} & t<s<t+r\\ 0 & else\end{cases}.$$

Hence, for all $\varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$,

$$-\int_{\mathbb{R}^n}\eta(f(x))div_B\varphi dV_B=\int_{\mathbb{R}^n}\eta'(f(x))\nabla f\cdot\varphi dV_B=\frac{1}{r}\int_{E_t\setminus E_{t+r}}\nabla f\cdot\varphi dV_B.$$

Moreover,

$$\begin{aligned}&\frac{m(t+r)-m(t)}{r}\\=&\frac{1}{r}\left[\int_{\mathbb{R}^n\setminus E_{t+r}}|\nabla f|dV_B-\int_{\mathbb{R}^n\setminus E_t}|\nabla f|dV_B\right]\\=&\frac{1}{r}\int_{E_t\setminus E_{t+r}}|\nabla f|dV_B\geq\frac{1}{r}\int_{E_t\setminus E_{t+r}}\nabla f\cdot\varphi dV_B\\=&-\int_{\mathbb{R}^n}\eta(f(x))\mathrm{div}_\mathrm{B}\varphi dV_B.\end{aligned}$$

For those t such that $m'(t)$ exists, we let $r\to 0$:

$$m'(t)\geq -\int_{E_t}\mathrm{div}_\mathrm{B}\varphi dV_B.$$

Taking the supremum over all φ as above implies

$$P_B(\{x\in\mathbb{R}^n: f(x)>t\})\leq m'(t),$$

and

$$\int_{-\infty}^{+\infty}P_B(\{x\in\mathbb{R}^n: f(x)>t\})dt\leq\int_{\mathbb{R}^n}|\nabla f|dV_B=\|Df\|_B(\mathbb{G}^n).$$

In fact, the equation holds for any function $f\in BV_B(\mathbb{G}^n)$. Fixing $f\in BV_B(\mathbb{G}^n)$ and choosing $\{f_k\}_{k=1}^{\infty}$ as in Theorem 2.5, then we have $f_k\to f$ in $L^1(\mathbb{G}^n)$ as $k\to\infty$. Define

$$E_t^k=\{x\in\mathbb{R}^n, f_k(x)>t\}.$$

Now

$$\int_{-\infty}^{+\infty}\left|\chi_{E_t^k}(x)-\chi_{E_t}(x)\right|dt=\int_{\min\{f,f_k\}}^{\max\{f,f_k\}}dt=|f_k-f|.$$

Thus

$$\int_{\mathbb{R}^n}|f_k-f|dV_B=\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}^n}\left|\chi_{E_t^k}(x)-\chi_{E_t}(x)\right|dV_B\right)dt.$$

Since $f_k\to f$ in $L^1(\mathbb{G}^n)$, there exists a subsequence which upon reindexing by k if needs be, satisfies

$$\chi_{E_t^k}\to\chi_{E_t}\ \ in\ \ L^1(\mathbb{G}^n)\ \ as\ \ k\to\infty.$$

Then by the lower Semicontinuity Theorem and Fatou's Lemma we have

$$\begin{aligned}&\int_{-\infty}^{+\infty}P_B(\{x\in\mathbb{R}^n: f(x)>t\})dt\\\leq&\liminf_{k\to\infty}\int_{-\infty}^{+\infty}P_B(\{x\in\mathbb{R}^n: f_k(x)>t\})dt\\=&\lim_{k\to\infty}\|Df_k\|_B(\mathbb{G}^n)=\|Df\|_B(\mathbb{G}^n),\end{aligned}$$

which completes the proof. □

3 Heat semigroups characterization of $BV_B(\mathbb{G}^n)$

At first, we consider the operator L_B on $\mathbb{G}^n$ which is defined as following: for any $f \in C_c^2(\mathbb{R}^n)$,

$$L_B f(x) := \frac{1}{2}\Delta f(x) - Bx\cdot\nabla f(x) = \frac{1}{2}\mathrm{div}_B(\nabla f),$$

and the operator L_B is selfadjoint on $L^2(\mathbb{G}^n)$ based on the result of [9]. Let $t>0$, for any $f\in L^2(\mathbb{R}^n)$, then the semigroup associate with the operator L_B is defined as

$$P_t f(x) = \int_{\mathbb{R}^n} k_B(t,x,y) f(y)dy, \tag{3.1}$$

where

$$k_B(t,x,y) = \prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}} \exp\left(-\frac{(e^{-d_i t}x_i - y_i)^2 d_i}{1-e^{-2d_i t}}\right),\ t>0, x\in\mathbb{R}^n.$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^n}(P_t f) g dV_B = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_B(t,x,y) f(y) dy g(x) dV_B\\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_B(t,x,y) f(y) g(x) \frac{(\det B)^{1/2}}{\pi^{n/2}} e^{-Bx\cdot x} dxdy\\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}} \exp\left(-\frac{(e^{-d_i t}x_i - y_i)^2 d_i}{1-e^{-2d_i t}}\right)\right] f(y) g(x)\cdot \frac{(\det B)^{1/2}}{\pi^{n/2}} e^{-Bx\cdot x} dxdy\\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\prod_{i=1}^n \frac{d_i}{\pi(1-e^{-2d_i t})^{1/2}} \exp\left(-\frac{(x_i{}^2 + y_i{}^2 - 2e^{-d_i t}x_i y_i) d_i}{1-e^{-2d_i t}}\right)\right] f(y) g(x) dxdy\\
=&\int_{\mathbb{R}^n}(P_t g) f dV_B.
\end{aligned}$$

Hence, the semigroup $\{P_t\}_{t\ge 0}$ is symmetric in $L^2(\mathbb{G}^n)$.

Lemma 3.1. *For every $f\in L^1(\mathbb{G}^n)$,*

$$\lim_{t\to 0} P_t f = f \ \text{ in } \ L^1.$$

Proof. By calculation, it is obvious that

$$\int_{\mathbb{R}^n}\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}} \exp\left(-\frac{y_i{}^2 d_i}{1-e^{-2d_i t}}\right) dy = 1.$$

Following the definition of $P_t f$ and the above equality, we get

$$\begin{aligned}
&P_t f(x)-f(x)\\
=&\int_{\mathbb{R}^n}\left[\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}}\exp\left(-\frac{(e^{-d_i t}x_i-y_i)^2 d_i}{1-e^{-2d_i t}}\right)\right]f(y)dy\\
&\quad -f(x)\int_{\mathbb{R}^n}\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}}\exp(-\frac{{y_i}^2 d_i}{1-e^{-2d_i t}})dy\\
=&\int_{\mathbb{R}^n}\left[\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}}\exp(-\frac{{y_i}^2 d_i}{1-e^{-2d_i t}})\right][f(e^{-Bt}x-y)-f(x)]dy\\
=&\int_{\mathbb{R}^n}\left[\prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1-e^{-2d_i t})^{1/2}}\exp(-{y_i}^2 d_i)\right][f(e^{-Bt}x-\sqrt{1-e^{-2Bt}}y)-f(x)]dy.
\end{aligned}$$

Letting $t\to 0$, via the dominated convergence theorem we conclude that

$$\lim_{t\to 0}\|P_t f-f\|_{L^1}=0. \qquad \square$$

Lemma 3.2. *The semigroup $\{P_t\}_{t\in[0,+\infty)}$ satisfies the following properties:*

(i) *$t\mapsto P_t f$ is continuous from $[0,\infty)$ to $L^2(\mathbb{G}^n)$.*

(ii) $|\nabla P_t f(x)|\le \max\{e^{-d_i t}\}|P_t(\nabla f)(x)|,\ i=1,\dots n.$

(iii) *$\|P_t f\|_\infty \le \|f\|_\infty, \forall f\in C_b^0(\mathbb{R}^n)$, where $C_b^0(\mathbb{R}^n)$ consists of the bounded and continuous functions on $\mathbb{R}^n$.*

Proof. The property (i) is obviously available. Next we prove (ii), via the definition of $P_t f(x)$ and the property of the gradient we have

$$\begin{aligned}
\nabla P_t f(x)=&\nabla\int_{\mathbb{R}^n}k_B(t,x,y)f(y)dy=\int_{\mathbb{R}^n}\nabla k_B(t,x,y)f(y)dy\\
=&\int_{\mathbb{R}^n}\left(k_B\frac{-2(e^{-d_1 t}x_1-y_1)d_1}{1-e^{-2d_1 t}}e^{-d_1 t},\cdots,k_B\frac{-2(e^{-d_n t}x_n-y_n)d_n}{1-e^{-2d_n t}}e^{-d_n t}\right)f(y)dy.
\end{aligned}$$

Then integration by part implies

$$\begin{aligned}
P_t(\nabla f)(x)=&\int_{\mathbb{R}^n}k_B(t,x,y)\nabla f(y)dy=-\int_{\mathbb{R}^n}\nabla_y k_B f(y)dy\\
=&-\int_{\mathbb{R}^n}\left(k_B\frac{2(e^{-d_1 t}x_1-y_1)d_1}{1-e^{-2d_1 t}},\cdots,k_B\frac{2(e^{-d_n t}x_n-y_n)d_n}{1-e^{-2d_n t}}\right)f(y)dy.
\end{aligned}$$

Finally, we can obtain the result by taking the absolute value of $\nabla P_t f(x)$ and $P_t(\nabla f)(x)$,

$$|\nabla P_t f(x)|\le \max\{e^{-d_i t}\}|P_t(\nabla f)(x)|, i=1,\dots n.$$

For (iii), it is easy to see that

$$|P_tf(x)|=|\int_{\mathbb{R}^n}k_B(t,x,y)f(y)dy|\leq\int_{\mathbb{R}^n}k_B(t,x,y)dy\|f\|_\infty,$$

and then we take the infinite norm on both sides

$$\|P_tf(x)\|_\infty\leq\left\|\int_{\mathbb{R}^n}k_B(t,x,y)dy\right\|_\infty\|f\|_\infty\leq\|f\|_\infty.$$

□

Theorem 3.1. *Denote by $C^1_{bd}(\mathbb{R}^n,\mathbb{R}^n)$ the space of vector-valued functions with continuous partial derivatives of first order and bounded B-divergence. Then for every f in $L^1(\mathbb{G}^n)$, it holds*

$$\|Df\|_B(\mathbb{G}^n)=\sup\left\{\int_{\mathbb{R}^n}f\mathrm{div}_B\varphi dV_B:\varphi\in C^1_{bd}(\mathbb{R}^n,\mathbb{R}^n)\quad\text{with}\quad\|\varphi\|_\infty\leq 1\right\}.$$

Proof. Clearly,

$$\|Df\|_B(\mathbb{G}^n)\leq\sup\left\{\int_{\mathbb{R}^n}f\mathrm{div}_B\varphi dV_B:\varphi\in C^1_{bd}(\mathbb{R}^n,\mathbb{R}^n)\quad\text{with}\quad\|\varphi\|_\infty\leq 1\right\}.$$

In order to prove the opposite inequality, we choose a sequence of functions in such that

(a) $0\leq\phi_k\leq 1$ for all $x\in\mathbb{R}^n$ and $k\in\mathbb{N}$.

(b) for every compact set $K\subset\mathbb{R}^n$ there exists n_K such that $\phi_k=1$ on $\mathbb{R}^n$ if $k\geq n_K$.

(c) $\|\nabla\phi_k\|_\infty\to 0$ as $k\to\infty$.
If $\varphi\in C^1_{bd}(\mathbb{R}^n;\mathbb{R}^n)$, we have $\|\phi_n\varphi\|_\infty\leq\|\varphi\|_\infty$ and

$$\begin{aligned}\mathrm{div}_B(\varphi\phi_k)&=\mathrm{div}(\varphi\phi_k)-2Bx\cdot\varphi\phi_k=\phi_k\mathrm{div}\varphi-2Bx\cdot\varphi\phi_k+\varphi\cdot\nabla\phi_k\\&=\phi_k\mathrm{div}_B\varphi+\varphi\cdot\nabla\phi_k.\end{aligned}$$

Therefore, if $\varphi\in C^1_{bd}(\mathbb{R}^n,\mathbb{R}^n)$ and $\|\varphi\|_\infty\leq 1$, then using the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^n}f\mathrm{div}_B\varphi dV_B=\lim_{k\to\infty}\int_{\mathbb{R}^n}f\mathrm{div}_B(\phi_k\varphi)dV_B\leq\|Df\|_B(\mathbb{G}^n).$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *For every $f\in L^1(\mathbb{G}^n)$, we have*

$$\|Df\|_B(\mathbb{G}^n)=\lim_{t\to 0}\|\nabla P_tf\|_{L^1}.$$

Proof. At first, for any functions $f\in BV_B(\mathbb{G}^n)$ and $\varphi\in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$, via Lemma 2.4 we have

$$\int_{\mathbb{R}^n}\nabla f\cdot\varphi dV_B=-\int_{\mathbb{R}^n} f\mathrm{div}_B\varphi dV_B.$$

Via the definition of $\|Df\|_B(\mathbb{G}^n)$, Lemma 3.1 and Lemma 3.2, we get

$$\int_{\mathbb{R}^n} f\mathrm{div}_B\varphi dV_B=\lim_{t\to 0}\int_{\mathbb{R}^n} P_tf\mathrm{div}_B\varphi dV_B=-\lim_{t\to 0}\int_{\mathbb{R}^n}\nabla(P_tf)\cdot\varphi dV_B\le\lim_{t\to 0}\|\nabla P_tf\|_{L^1}.$$

Then taking the supremum over φ implies that

$$\|Df\|_B(\mathbb{G}^n)\le\lim_{t\to 0}\|\nabla P_tf\|_{L^1}. \tag{3.2}$$

Next, we prove the opposite inequality

$$\|Df\|_B(\mathbb{G}^n)\ge\lim_{t\to 0}\|\nabla P_tf\|_{L^1}. \tag{3.3}$$

Let φ be a form in $C_c^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $\|\varphi\|_\infty\le 1$. We claim that $P_t\varphi$ is first-order continuous differentiable according to the definition of P_t and $\|P_t\varphi\|_\infty\le 1$ which is based on (iii) of Lemma 3.2. Since(ii) of Lemma 3.2 implies

$$|\mathrm{div}P_t\varphi(x)|\le\max\{e^{-d_it}\}|P_t(\mathrm{div}\varphi)(x)|,\quad i=1,\ldots n,$$

then we have

$$\begin{aligned}\|\mathrm{div_B}(P_t\varphi)\|_\infty&=\|\mathrm{div}(P_t\varphi)-2Bx\cdot(P_t\varphi)\|_\infty\\&\le\|\mathrm{div}(P_t\varphi)\|_\infty+\|2Bx\cdot(P_t\varphi)\|_\infty\\&\le\|P_t\mathrm{div}\varphi\|_\infty+\|2Bx\cdot\varphi\|\\&\le\|\mathrm{div}\varphi\|_\infty+\|2Bx\cdot\varphi\|<\infty.\end{aligned}$$

Therefore,

$$P_t\varphi\in C_{bd}^1(\mathbb{R}^n;\mathbb{R}^n).$$

In (ii) of Lemma 3.2, we assume that e^{-d_it} can be maximized when $i=i_0$, and by Theorem 3.1, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\nabla P_tf\cdot\varphi dV_B\right|\\\le&\left|\int_{\mathbb{R}^n}\max\{e^{-d_it}\}P_t(\nabla f)\cdot\varphi dV_B\right|\le\left|\sum_{i=1}^n e^{-d_{i_0}t}\int_{\mathbb{R}^n}P_t\left(\frac{\partial f}{\partial x_i}\right)\varphi_i dV_B\right|\\\le&\left|\sum_{i=1}^n e^{-d_{i_0}t}\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}P_t(\varphi_i)dV_B\right|=\left|e^{-d_{i_0}t}\int_{\mathbb{R}^n}(\nabla f)\cdot P_t\varphi dV_B\right|\\=&\left|e^{-d_{i_0}t}\int_{\mathbb{R}^n} f\mathrm{div_B}(P_t\varphi)dV_B\right|\le e^{-d_{i_0}t}\|Df\|_B(\mathbb{G}^n),\end{aligned}$$

where we have used the property that semigroup P_t is symmetric in $L^2(\mathbb{G}^n)$. Thus, taking the supremum with respect to all $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^n)$ and $\|\varphi\|_\infty \leq 1$, we have

$$\|\nabla P_t f\|_{L^1} \leq e^{-d_{i_0}t}\|Df\|_B(\mathbb{G}^n).$$

Hence, we can obtain (3.3) by passing the limit as tends to 0. Finally, we conclude the proof by combining (3.2) with (3.3). □

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References

[1] Andrea C, Giancarlo M. A note on bounded variation and heat semigroup on Riemannian manifolds. Bull Austral Math Soc, 2007, 76: 155-160.
[2] Bramanti M, Miranda Jr M, Pallara D. Two characterization of BV functions on Carnot groups via the heat semigroup. Int Math Res Not, 2012, 17: 3845-3876.
[3] De Giorgi E. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. Ann Mate Pura Appl, 1954, 36: 191-213.
[4] De Giorgi E. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. Ricerche di Mat, 1955, 4: 95-113.
[5] Liu L, Xiao J, Yang D, et al. Gaussian Capacity Analysis. LNM 2225 Springer, 2018.
[6] Michele Jr M, Pallara D, Fabio P, et al. Heat semigroup and functions of bounded variation on Riemannian manifolds. J Reine Angew Math, 2007, 613: 99-119.
[7] Sjögren P. Operators associated with the hermite semigroup-A survey. J Fourier Anal Appl, 1997, 3: 813-823.
[8] Xiao J. Gaussian BV-Capacity. Adv Cal Var, 2019, 9: 187-200.
[9] Gutiérrez C E, Segovia C, Torrea J L. On higher Riesz transforms for Gaussian measures. J Fourier Anal Appl, 1996, 2 : 583-596.
[10] Lawrence E C, Ronald G F. Measure theory and fine properties of functions. Stud Adv Math, 1992.