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On 2-microlocal Herz Type Besov and Triebel-Lizorkin Spaces with Variable Exponents

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Abstract. In this paper, 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents are introduced for the first time. Then, we give characterizations of these spaces by so-called Peetre's maximal functions. Further, the atomic and molecular decompositions of these spaces are obtained. Finally, using the characterizations of the spaces by local means and molecular decomposition we obtain the wavelet characterizations.

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1 Introduction

The theory of function spaces with variable exponents has developed rapidly in recent years (see, e.g., [2–3, 5–6, 8–9, 16–17, 19–21, 23–28]). It is worth noting that variable exponent Lebesgue spaces first appeared in [16] by Orlicz in 1931. In 2009, Izuki [7] defined Herz spaces with variable exponent $K^{\alpha,q}_{p(\cdot)}$ and obtained wavelet characterization of those spaces by virtue of the result on $L^{p(\cdot)}(\mathbb{R}^n)$ [6, 13]. Moreover, Izuki [8-10] proved the boundedness of some sublinear operators and commutators on $K^{\alpha,q}_{p(\cdot)}$.

As a continuation of the work for Herz spaces with variable exponent, Shi and Xu [21] introduced Herz type Besov and Triebel-Lizorkin spaces with variable exponent, $K^{\alpha,q}_{p(\cdot)}B^s_\beta$ and $K^{\alpha,q}_{p(\cdot)}F^s_\beta$, and obtained their equivalent quasi-norms. Subsequently, Dong and Xu gave characterizations of $K^{\alpha(\cdot),q}_{p(\cdot)}B^s_\beta$ and $K^{\alpha(\cdot),q}_{p(\cdot)}F^s_\beta$ by Peetre's maximal functions in [4].

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On the other hand, the concept of 2-microlocal spaces has aroused the interest of some scholars, and this concept initially appeared in the book of Peetre [18]. 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability were studied first by Kempka in [11–12]. With even q variable in the Besov case, Almeida and Caetano [1–2] study various key properties for 2-microlocal Besov and Triebel-Lizorkin spaces with all exponents variable, including Sobolev type embeddings, atomic and molecular representations.

In this paper, combining its of Herz type spaces and 2-microlocal spaces, the authors introduce 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents (see Section 2), which is the extension of 2-microlocal Besov and Triebel-Lizorkin spaces and Herz type Besov and Triebel-Lizorkin spaces with variable exponents. In Section 3, we give a simple proof for the characterizations of these spaces by Peetre's maximal functions and the local means. Another of the main results demonstrates an embedding theorem about two sequence spaces and clarify some convergence issues. Then applying the convergence and the characterization of Peetre's maximal functions, it is asserted that the atomic, molecular and wavelet characterizations of these spaces in Sections 4 and 5.

2 Preliminaries and definitions

In this section, we introduce the basic notation in the theory of 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents.

Definition 2.1. Let $p:\mathbb{R}^n \to [1,\infty)$ be a measurable function. $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \right\}$$

and

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Denote $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions p on $\mathbb{R}^n$ with range in $[1,\infty)$ such that

$$1 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$

Moreover, we define $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of all measurable functions p on $\mathbb{R}^n$ with range in $(0,\infty)$ such that

$$0 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$

Given $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0}\in L^{q(\cdot)}(\mathbb{R}^n)$, where $0<p_0<p_-$, and $q(\cdot)=\frac{p(\cdot)}{p_0}\in\mathcal{P}(\mathbb{R}^n)$. Then one can define a quasi-norm on this space by

$$\|f\|_{L^{p(\cdot)}}=\||f|^{p_0}\|_{L^{q(\cdot)}}^{1/p_0}.$$

Definition 2.2. Let $f\in L^1_{loc}(\mathbb{R}^n)$, the standard Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}f(x)=\sup_{r>0}r^{-n}\int_{B_r(x)}|f(y)|dy,$$

where $L^1_{loc}(\mathbb{R}^n)=\{f:f\in L^1(\mathbb{K})$ for all compact subsets $\mathbb{K}\subset\mathbb{R}^n\}$, $B_r(x)=B(x,r)=\{y\in\mathbb{R}^n: |x-y|<r\}$. The key tool we need is the boundedness of the Hardy-Littlewood maximal operator on variable exponent function spaces. Denote $(\mathcal{M}(|f|^t))^{1/t}$ by $\mathcal{M}_tf$ for $0<t<\infty$. There exist some sufficient conditions on $p(\cdot)$ such that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$; see, for example, [15, 19]. $\mathcal{B}(\mathbb{R}^n)$ be the set of all $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.3. (i) A continuous function $g:\mathbb{R}^n\to\mathbb{R}$ is called locally log-Hölder continuous, abbreviated $g\in\mathcal{C}^{\log}_{loc}(\mathbb{R}^n)$, if there exists $c_{\log}>0$ such that for all $x,y\in\mathbb{R}^n$,

$$|g(x)-g(y)|\le\frac{c_{\log}}{\log(e+1/|x-y|)}.$$

(ii) A function g is called globally log-Hölder continuous, abbreviated $g\in\mathcal{C}^{\log}(\mathbb{R}^n)$, if g is locally log-Hölder continuous and there exists $g_\infty\in\mathbb{R}$ and $C_{\log}>0$ such that for all $x\in\mathbb{R}^n$,

$$|g(x)-g_\infty|\le\frac{C_{\log}}{\log(e+|x|)}.$$

If $q\in\mathcal{C}^{\log}(\mathbb{R}^n)$, then for every $q_0<q^-$ we have $q(\cdot)/q_0\in\mathcal{B}(\mathbb{R}^n)$, see [11, Theorem 3.6]. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for those variable exponents $p\in\mathcal{P}^0(\mathbb{R}^n)$ with $\frac{1}{p}\in\mathcal{C}^{\log}$. If $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, then we have for every $p_0<p^-$ that $\mathcal{M}$ is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$ or, equivalently, that $\mathcal{M}_t$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, where $t=\min(1,p_0)$.

For a set A, χ_A denotes its characteristic function. Let $B_k=\{x\in\mathbb{R}:|x|<2^k\}$, $E_k=B_k\backslash B_{k-1}$ and $\chi_k=\chi_{E_k}$ for $k\in\mathbb{Z}$. $L^{p(\cdot)}_{loc}(\mathbb{R}^n\backslash\{0\})=\{f:f\in L^{p(\cdot)}(\mathbb{K})$ for all compact subsets $\mathbb{K}\subset\mathbb{R}^n\backslash\{0\}\}$.

Definition 2.4. Let $0<q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous Herz space $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{p(\cdot)}_{loc}(\mathbb{R}^n\backslash\{0\}):\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{k=-\infty}^{\infty}\left\|2^{\alpha(\cdot)k}f\chi_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q}.$$

(ii) The nonhomogeneous Herz space $K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{p(\cdot)}_{loc}(\mathbb{R}^n\backslash\{0\}):\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{m=0}^{\infty}\left\|2^{\alpha(\cdot)m}f\chi_m\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q}.$$

In this article, we only discuss the nonhomogeneous Herz space $K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.5. Let $\alpha_0,\alpha_1,\alpha_2\in\mathbb{R}$ with $\alpha_0\geq 0$ and $\alpha_1\leq\alpha_2$. We say that a sequence of positive measurable functions $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$ if

(i) there exists $c>0$ such that

$$0<\omega_j(x)\leq c\,\omega_j(y)(1+2^j|x-y|)^{\alpha_0}$$

for all $j\in\mathbb{N}_0$ and $x,y\in\mathbb{R}^n$;

(ii) there holds

$$2^{\alpha_1}\omega_j(x)\leq\omega_{j+1}(x)\leq 2^{\alpha_2}\omega_j(x)$$

for all $j\in\mathbb{N}_0$ and $x\in\mathbb{R}^n$. Such a sequence will be called an admissible weight sequence.

We now recall the Fourier analytical approach to function spaces of Herz type Besov and Triebel-Lizorkin. The set $\mathcal{S}(\mathbb{R}^n)$ denotes the usual Schwartz class of infinitely differentiable rapidly decreasing complex-valued functions. Its topology is generated by the semi norms

$$\|\varphi\|_{k,l}=\sup_{x\in\mathbb{R}^n}\langle x\rangle^k\sum_{|\beta|\leq l}|D^\beta\varphi(x)|,\quad\text{where }\langle x\rangle^k=(1+|x|^2)^{k/2}.$$

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the dual space of $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a tempered distribution f is denoted by $\hat{f}$ while its inverse transform is denoted by $\check{f}$. Let $\varphi_0\in\mathcal{S}(\mathbb{R}^n)$ with $\varphi_0\geq 0$ and satisfy the following conditions:

$$\varphi_0(x)=\begin{cases}1, & |x|\leq 1,\\ 0, & |x|\geq 2.\end{cases}$$

Set $\varphi(x)=\varphi_0(x)-\varphi_0(2x)$ with $x\in\mathbb{R}^n$. For $j\in\mathbb{Z}$, we also put $\varphi_j(x)=\varphi(2^{-j}x)$ and $\Phi_j=\check{\varphi}_j$, then we call $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, it follows that

$$\sum_{j=0}^{\infty}\varphi_j(x)=1.$$

Remark 2.1. Such a resolution of unity can easily be constructed. Consider the following example. Let $\varphi_0\in\mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x)=1$ for $|x|\le 1$ and supp $\varphi_0\subseteq\{x\in\mathbb{R}^n:|x|\le 2\}$. For $j\ge 1$ we define

$$\varphi_j(x)=\varphi_0(2^{-j}x)-\varphi_0(2^{-j+1}x).$$

It is obvious that $\varphi=\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity.

Now we introduce 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents as follows.

Definition 2.6. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$.

(i) The set

$$\left\{f\in\mathcal{S}'(\mathbb{R}^n)(\mathbb{R}^n):\ \left(\sum_{j=0}^{\infty}\left\|\omega_j(\varphi_j*\widehat{f})^\vee\right\|^{\beta}_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta}<\infty\right\},$$

is named to the 2-microlocal Herz type Besov space with variable exponents and denoted by $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$. The quasi-norm of f in this space is denoted by

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)}=\left(\sum_{j=0}^{\infty}\left\|\omega_j(\varphi_j*\widehat{f})^\vee\right\|^{\beta}_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta}.$$

(ii) The set

$$\left\{f\in\mathcal{S}'(\mathbb{R}^n):\ \left\|\left(\sum_{j=0}^{\infty}|\omega_j(\varphi_j*\widehat{f})^\vee|^{\beta}\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\right\},$$

is named to the 2-microlocal Herz type Triebel-Lizorkin space with variable exponents and denoted by $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$. The quasi-norm of f in this space is denoted by

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)}=\left\|\left(\sum_{j=0}^{\infty}|\omega_j(\varphi_j*\widehat{f})^\vee|^{\beta}\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}.$$

In the next section, we will show that Definition 2.6 is independent of the choice of the function φ.

3 Characterization by Peetre's maximal functions and local mean

Now, we will characterize these spaces in terms of Peetre's maximal functions and local mean. To this aim we need following definition and lemmas. Let us recall the classical Peetre maximal operator, introduced in [17].

Definition 3.1. Given a sequence of functions $\{\Psi_j\}_j \in \mathcal{S}(\mathbb{R}^n)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and a positive number $a>0$, the Peetre's maximal functions are defined as

$$(\Psi_j^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Psi_j * f(y)|}{1+|2^j(x-y)|^a}, \quad x\in\mathbb{R}^n, \ j\in\mathbb{N}_0.$$

Lemma 3.1 ([3]). *If $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ and $1<\beta\le\infty$, then there exists a positive constant C such that for all sequences $\{f_j\}_{j=0}^\infty$ of locally integrable functions,*

$$\Big\|\{\mathcal{M}f_j\}_{j=0}^\infty\Big\|_{L^{p(\cdot)}(l_\beta)} \le C\Big\|\{f_j\}_{j=0}^\infty\Big\|_{L^{p(\cdot)}(l_\beta)}.$$

Lemma 3.2 ([4], Lemma 2.2). *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exist $0<\delta_1,\delta_2<1$ depending only on $p(\cdot)$ and n such that for balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2}. \tag{3.1}$$

Lemma 3.3 ([4], Theorem 2.8). *Let $0<q,\beta<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ and $\alpha(\cdot)$ be log-Hölder continuous, both at the origin and at infinity, such that $-n\delta_1<\alpha(0)\le\alpha_\infty<n\delta_2$, where $0<\delta_1,\delta_2<1$ are constants satisfying (3.1), then there is a positive constant C independent of sequences $\{f_j\}_{j=1}^\infty$ of locally integrable functions on $\mathbb{R}^n$ such that*

$$\Big\|\Big(\sum_{j=0}^\infty |\mathcal{M}f_j|^\beta\Big)^{1/\beta}\Big\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)} \le C\Big\|\Big(\sum_{j=0}^\infty |f_j|^\beta\Big)^{1/\beta}\Big\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}.$$

The next lemma is a discrete convolution inequality which we will need later on.

Lemma 3.4 ([4], Lemma 3.5). *Let $0<\beta,q\le\infty$ and $\delta>0$, and $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. For any sequence $\{g_j\}_{j=0}^\infty$ of nonnegative measurable functions on $\mathbb{R}^n$, let $G_j=\sum_{k=0}^\infty 2^{-|k-j|\delta}g_k$. Then*

$$\Big\|\{G_j\}_{j=0}^\infty\Big\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)} \le C_1\Big\|\{g_j\}_{j=0}^\infty\Big\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)}$$

and

$$\Big\|\{G_j\}_{j=0}^\infty\Big\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})} \le C_1\Big\|\{g_j\}_{j=0}^\infty\Big\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})},$$

where $C_1=C_1(q,\delta)$ and $C_2=C_2(p(\cdot),q,\delta)$ are positive constants.

The following two theorems are the main results of this section. Moreover, Theorem 3.1 shows that the definition of 2-microlocal Herz type Besov and Triebel-Lizorkin spaces of variable integrability is independent of the resolution of unity.

Theorem 3.1. *Let* $\omega=(\omega_k)_{k\in\mathbb{N}_0}\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<q,\delta<\infty$, $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ *with* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ *and* $\alpha(\cdot)$ *be log-Hölder continuous, both at the origin and at infinity, such that* $-n\delta_1<\alpha(0)\le\alpha_\infty<n\delta_2$, *where* $0<\delta_1,\delta_2<1$ *are constants satisfying* (3.1). *Let* $a\in\mathbb{R}$, $R\in\mathbb{N}_0$ *with* $R>\alpha_2$. *Further, let* ψ_0,ψ_1 *belong to* $\mathcal{S}(\mathbb{R}^n)$ *with*

$$D^\beta\psi_1(0)=0,\quad \text{for}\ \ 0\le|\beta|<R,$$

and

$$|\psi_0(x)|>0\qquad \text{on}\ \ \{x\in\mathbb{R}^n:|x|<\varepsilon\},$$
$$|\psi_1(x)|>0\qquad \text{on}\ \ \{x\in\mathbb{R}^n:\varepsilon/2<|x|<2\varepsilon\}$$

for some $\varepsilon>0$.

(i) *If there exists* $0<p_0<p^-$ *with* $p(\cdot)/p_0\in\mathcal{B}(\mathbb{R}^n)$, *then for* $a>\frac{n}{p_0}+\alpha_0$

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta(\mathbb{R}^n)}\sim\|(\Psi^*_kf)_a\omega_k\|_{l_\beta(K^{\alpha(\cdot),q}_{p(\cdot)})}\sim\|(\Psi_k*f)\omega_k\|_{l_\beta(K^{\alpha(\cdot),q}_{p(\cdot)})}$$

holds for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

(ii) *If there exists* $p_0<\min(p^-,\beta)$ *with* $p(\cdot)/p_0\in\mathcal{B}(\mathbb{R}^n)$, *then for* $a>\frac{n}{p_0}+\alpha_0$

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)}\sim\|(\Psi^*_kf)_a\omega_k\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\sim\|(\Psi_k*f)\omega_k\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(l_\beta)}$$

holds for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

Proof. The idea of the proof is from Rychkov in [20]. The whole proof is divided to two steps, which together give the proof of Theorem 3.2.

We start with two given functions $\psi_0,\psi_1\in\mathcal{S}(\mathbb{R}^n)$ and define

$$\psi_j(x)=\psi_1(2^{-j+1}x),\ \text{ for } x\in\mathbb{R}^n \text{ and } j\in\mathbb{N}.$$

Furthermore, for all $j\in\mathbb{N}_0$, we write $\Psi_j=\hat{\psi}_j$ and in an analogous manner we define Φ_j from two starting functions $\phi_0,\phi_1\in\mathcal{S}(\mathbb{R}^n)$.

Step 1. Firstly, proceeding as in the proof of Theorem 3.6 in [11], it can easily be seen that for $\delta>0$,

$$(\Phi^*_\nu f)_a(x)\omega_\nu(x)\le c\sum_{k=0}^{\infty}2^{-|k-\nu|\delta}(\Phi^*_kf)_a(x)\omega_k(x),\quad x\in\mathbb{R}^n.$$

Thus, according to Lemma 3.4, we prove that for all $f\in\mathcal{S}'(\mathbb{R}^n)$ the following estimates are true:

$$\|(\Psi_k^* f)_a\omega_k\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})}\le c\|(\Phi_k^* f)_a\omega_k\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})}$$

and

$$\|(\Psi_k^* f)_a\omega_k\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)}\le c\|(\Phi_k^* f)_a\omega_k\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)}.$$

Step 2. According to the proof of Theorem 3.8 in [11] and Lemma 3.4, replace $l_{\beta/r}(L_{q(\cdot)/r})$ and $L_{q(\cdot)/r}(l_{\beta/r})$ by $l_{\beta/r}(K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r})$ and $K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r}(l_{\beta/r})$, we have

$$\left\|\left((\Psi_\nu^* f)_a\omega_\nu\right)^r\right\|_{l_{\beta/r}(K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r})}\le c\left\|\mathcal{M}\left(|\Psi_k * f|^r(\omega_k)^r\right)\right\|_{l_{\beta/r}(K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r})}$$

and

$$\left\|\left((\Psi_\nu^* f)_a\omega_\nu\right)^r\right\|_{K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r}(l_{\beta/r})}\le c\left\|\mathcal{M}\left(|\Psi_k * f|^r(\omega_k)^r\right)(x)\right\|_{K_{p(\cdot)/r}^{\alpha(\cdot)r,q/r}(l_{\beta/r})}.$$

Now we choose $\frac{n}{a-\alpha}<r<p_0$, clearly, $p(\cdot)/r\in\mathcal{B}(\mathbb{R}^n)$. By Lemma 3.1 and Lemma 3.3, we have

$$\|(\Psi_k^* f)_a\omega_k\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})}\le c\|(\Psi_k * f)\omega_k\|_{l_\beta(K_{p(\cdot)}^{\alpha(\cdot),q})}$$

and

$$\|(\Psi_k^* f)_a\omega_k\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)}\le c\|(\Psi_k * f)\omega_k\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(l_\beta)}$$

holds for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

This finishes the proof of Theorem 3.1. □

Now, we discuss usual local mean characterization of 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents. Let $B_1(0)$ be the unit ball and $k_0,k^0\in\mathcal{S}(\mathbb{R}^n)$ with support in $B_1(0)$, and $(\hat{k_0})(0)\neq 0,(\hat{k^0})(0)\neq 0$. For $N\in\mathbb{N}_0$ we define the iterated Laplacian

$$k(y):=\triangle^N k^0(y)=\Big(\sum_{j=1}^n\frac{\partial^2}{\partial y_j^2}\Big)^N k^0(y),\quad y\in\mathbb{R}^n.$$

For a distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ the corresponding local mean is defined for $x\in\mathbb{R}^n$ and $t>0$ by (at least formally)

$$k(t,f)(x)=\int_{\mathbb{R}^n}k(y)f(x+ty)dy=t^{-n}\int_{\mathbb{R}^n}k(\frac{y-x}{t})f(y)dy.$$

Theorem 3.2. *Let* $\omega=(\omega_k)_{k\in\mathbb{N}_0}\in\mathcal{W}_{\alpha_1,\alpha_2}^{\alpha_0}$, $0<q,\delta<\infty$, $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ *with* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ *and* $\alpha(\cdot)$ *be log-Hölder continuous, both at the origin and at infinity, such that* $-n\delta_1<\alpha(0)\le\alpha_\infty<n\delta_2$, *where* $0<\delta_1,\delta_2<1$ *are constants satisfying* (3.1). *Further more, let*

$N\in\mathbb{N}_0$ with $2N>\alpha_2$ and let $k_0,k^0\in\mathcal{S}(\mathbb{R}^n)$ and the function k be defined as above.

(i) *If there exists a $q_0<q^-$ with $q(\cdot)/q_0\in\mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)\omega_0\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}+\Big(\sum_{j=1}^{\infty}\|k(2^{-j},f)\omega_j\|^{\beta}_{K^{\alpha(\cdot),q}_{p(\cdot)}}\Big)^{1/\beta}$$

is an equivalent norm on $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

(ii) *If $q^+<\infty$ and if there exists a $q_0\le\min(q^-,\beta)$ with $q(\cdot)/q_0\in\mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)\omega_0\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}+\Big\|\Big(\sum_{j=1}^{\infty}|k(2^{-j},f)(\cdot)\omega_j(\cdot)|^{\beta}\Big)^{1/\beta}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}$$

is an equivalent norm on $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$ for all $f\in\mathcal{S}'(\mathbb{R}^n)$.

Combining the proof of Theorem 2.4 in [11] and Theorem 3.1, it is evident to see that Theorem 3.2 holds.

4 Atomic and molecular decompositions

The main goal of this section is to obtain atomic and molecular decompositions for $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$, correspondingly. To do so, we give the definitions of $K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}(\mathbb{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}(\mathbb{R}^n)$ as follows.

Definition 4.1. Let $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. Then for all complex valued sequences $\lambda=\{\lambda_{\nu m}\in\mathbb{C}:\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n\}$ we define

$$K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}(\mathbb{R}^n)=\Big\{\lambda:\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}<\infty\Big\},$$

where

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}=\left(\sum_{\nu=0}^{\infty}\Big\|\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot)\Big\|^{\beta}_{K^{\alpha(\cdot),q}_{p(\cdot)}}\right)^{1/\beta}.$$

Furthermore, for $p^+<\infty$ we define

$$K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}(\mathbb{R}^n)=\Big\{\lambda:\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}<\infty\Big\},$$

where

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}=\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^{\beta}\omega^{\beta}_{\nu}(2^{-\nu}m)\chi_{\nu m}(\cdot)\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}.$$

We use the notation Q_{jm} with $j\in\mathbb{N}_0$ and $m\in\mathbb{Z}^n$, for the closed cube with sides parallel to the coordinate axes, centered at $2^{-j}m$ and with side length 2^{-j}. By χ_{jm} we denote the corresponding characteristic function. The notation dQ_{jm}, $d>0$ will stand for the closed cube concentric with Q_{jm} and of side length $d2^{-j}$. Then, we recall the definitions of atoms and molecules as follows.

Definition 4.2. Let K,L be a nonnegative integer. A function a in $C^K(\mathbb{R}^n)$ is called a (K,L)-atom for a cube Q_{vm} if it satisfies the following conditions:

$$\text{supp } a\subseteq \gamma Q_{vm}, \tag{4.1}$$

$$|D^\beta a(x)|\le 2^{|\beta|\nu}, \qquad \text{for } 0\le|\beta|\le K, \tag{4.2}$$

$$\int_{\mathbb{R}^n} x^\beta a(x)dx=0 \qquad \text{for } 0\le|\beta|\le L \text{ and } \nu\ge 1. \tag{4.3}$$

If the atom a is located at $Q_{\nu m}$, that means it fulfills (4.1), then we will denote it by $a_{\nu m}$. For $\nu=0$ or $L=0$ there are no moment conditions (4.3) required.

Definition 4.3. Let $K,L\in\mathbb{N}_0$ and $M>0$. A function μ in $C^K(\mathbb{R}^n)$ is called a (K,L,M)-molecule for a cube $Q_{\nu m}$, if for some $\nu\in\mathbb{N}_0$ and $m\in\mathbb{Z}^n$

$$|D^\beta \mu(x)|\le 2^{|\beta|\nu}(1+2^\nu|x-2^{-\nu}m|)^{-M}, \quad \text{for } 0\le|\beta|\le K, \tag{4.4}$$

$$\int_{\mathbb{R}^n} x^\beta \mu(x)dx=0 \quad \text{if } 0\le|\beta|\le L \text{ and } \nu\ge 1. \tag{4.5}$$

If a molecule is concentrated in $Q_{\nu m}$, that means it satisfies (4.4), then it is denoted by $\mu_{\nu m}$. For $\nu=0$ or $L=0$ there are no moment conditions (4.5) required. If $a_{\nu m}$ is a (K,L)-atom, then it is a (K,L,M)-molecule for every $M>0$.

Before passing to the main atomic and molecular representation statements, we would like to take the opportunity to clarify some convergence issues that typically are not so clearly mentioned in the literature. Therefore, the next task in this subsection is to clarify the convergence of the sum

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}, \tag{4.6}$$

where $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules and $\{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ belongs to some sequence space from Definition 4.1. At least, we have to show the convergence of (4.6) in $\mathcal{S}'(\mathbb{R}^n)$. To prove the convergence, we need to prove the following embedding theorem.

Theorem 4.1. *Let $\omega=(\omega_\nu)_\nu\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, then*

$$K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\min(p^-,q^-,\beta)}(\mathbb{R}^n)\hookrightarrow K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}(\mathbb{R}^n)\hookrightarrow K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\max(p^+,q^+,\beta)}(\mathbb{R}^n). \tag{4.7}$$

Proof. Let $f_\nu(x)=\sum\limits_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(x)$, $r=\min(p^-,q^-,\beta)$ and $s=\max(p^+,q^+,\beta)$.

First we prove the left side, we know

$$\Big\|\|f_\nu\|_{l_\beta}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\le\Big\|\|f_\nu\|_{l_r}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}=\Big\|(\sum_{\nu=0}^\infty f_\nu^r)^{1/r}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}=\Big\|\sum_{\nu=0}^\infty f_\nu^r\Big\|^{1/r}_{K^{\alpha(\cdot)r,q/r}_{p(\cdot)/r}(\mathbb{R}^n)}.$$

Using the triangle inequality, combining with the above inequality we obtain

$$\begin{aligned}&\Big\|\|f_\nu\|_{l_\beta}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\le\Big\|\sum_{\nu=0}^\infty f_\nu^r\Big\|^{1/r}_{K^{\alpha(\cdot)r,q/r}_{p(\cdot)/r}(\mathbb{R}^n)}\\ \le&\Big(\sum_{\nu=0}^\infty\|f_\nu^r\|_{K^{\alpha(\cdot)r,q/r}_{p(\cdot)/r}(\mathbb{R}^n)}\Big)^{1/r}=\Big(\sum_{\nu=0}^\infty\|f_\nu\|^r_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/r}.\end{aligned}\tag{4.8}$$

On the right side, using the triangle inequality which holds since $\frac{p(\cdot)}{s}\le1$ and $\frac{q(\cdot)}{s}\le1$, we obtain

$$\begin{aligned}&\Big\|\|f_\nu\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\Big\|_{l_s}=\Big(\sum_{\nu=0}^\infty\|f_\nu^s\|^{\frac1s\cdot s}_{K^{\alpha(\cdot)s,q/s}_{p(\cdot)/s}(\mathbb{R}^n)}\Big)^{1/s}\\ \le&\Big\|\sum_{\nu=0}^\infty f_\nu^s\Big\|^{1/s}_{K^{\alpha(\cdot)s,q/s}_{p(\cdot)/s}(\mathbb{R}^n)}=\Big\|(\sum_{\nu=0}^\infty f_\nu^s)^{1/s}\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{4.9}$$

Thus, Theorem 4.1 is proved. □

Now, we can state the theorem on the convergence of (4.6) in $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 4.2. *Let $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. Assume*

$$L>-\alpha_1,\quad K \text{ arbitrary and } M \text{ large enough.}$$

If $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta(\mathbb{R}^n)$ or $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta(\mathbb{R}^n)$ and $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules concentrated in $Q_{\nu m}$, then the sum

$$\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(x)\quad\text{converges in } \mathcal{S}'(\mathbb{R}^n).\tag{4.10}$$

Proof. For $\omega_\nu\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$ and $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta(\mathbb{R}^n)$ or $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta(\mathbb{R}^n)$. By the proof of Lemma 3.11 in [12], we choose $t<\min(p^-,q^-,\beta,1)$.

Denote $\langle y\rangle^k=(1+|y|^2)^{k/2}$, we have $\langle y\rangle^k\le\langle y-2^{-\nu}m\rangle^k\langle\xi\rangle^k$. Let $M>0,k>0$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Using the moment conditions of $\mu_{\nu m}$, the properties of the weight sequence and the boundedness of the maximal operator, we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy\Big|\\
\leq &\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}||\mu_{\nu m}(y)|\omega_\nu(y)\Big|\varphi(y)-\sum_{|\gamma|<L}\frac{D^\varphi\varphi(2^{-\nu}m)}{\gamma!}(y-2^{-\nu}m)^\gamma\Big|\omega_\nu^{-1}(y)\frac{\langle y\rangle^k}{\langle y\rangle^k}dy\\
\leq &\, c\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}2^{-\nu(L+\alpha_1)}(1+2^\nu|y-2^{-\nu}m|)^{L+k-M}|\lambda_{\nu m}|\omega_\nu(y)\langle y\rangle^{\gamma-k}\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^k\sum_{|\eta|=L}\frac{D^\eta\varphi(\xi)}{\eta!}dy\\
\leq &\, c\|\varphi\|_{k,L}\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}2^{-\nu(L+\alpha_1)}(1+2^\nu|y-2^{-\nu}m|)^{L+\gamma+k-M}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\langle y\rangle^{\gamma-k}dy\\
\leq &\, c2^{-\nu(L+\alpha_1)}\|\varphi\|_{k,L}\Big\|\sum_{m\in\mathbb{Z}^n}(1+2^\nu|y-2^{-\nu}m|)^{L+\gamma+k-M}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\\
\leq &\, c2^{-\nu(L+\alpha_1)}\|\varphi\|_{k,L}\Big\|\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot)\Big)\Big\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\\
\leq &\, c2^{-\nu(L+\alpha_1)}\|\varphi\|_{k,L}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot)\Big)^t\Big\|^{1/t}_{K^{\alpha(\cdot)t,q/t}_{p(\cdot)/t}(\mathbb{R}^n)}\\
\leq &\, c2^{-\nu(L+\alpha_1)}\|\lambda^t\|^{1/t}_{K^{\alpha(\cdot)t,q/t}_{p(\cdot)/t}b^\omega_\infty(\mathbb{R}^n)}.
\end{aligned}
\tag{4.11}
$$

According to Theorem 4.1, let $s=\max(p^+,q^+,\beta)$ and $f_\nu(x)=\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(x)$, we get the following facts

$$
\Big\|\|f_\nu^t\|_{K^{\alpha(\cdot)t,q/t}_{p(\cdot)/t}(\mathbb{R}^n)}\Big\|^{1/t}_{l_\infty}\leq\Big\|\|f_\nu^t\|_{K^{\alpha(\cdot)t,q/t}_{p(\cdot)/t}(\mathbb{R}^n)}\Big\|^{1/t}_{l_{\beta/t}}=\left(\sum_{j=0}^\infty\|f_\nu\|^{t\cdot\frac{\beta}{t}}_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{t}{\beta}\cdot\frac{1}{t}}=\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta}
$$

and

$$
\begin{aligned}
\Big\|\|f_\nu^t\|_{K^{\alpha(\cdot)t,q/t}_{p(\cdot)/t}(\mathbb{R}^n)}\Big\|^{1/t}_{l_\infty}&=\Big\|\|f_\nu\|^t_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\Big\|^{1/t}_{l_\infty}\\
\leq\Big\|\|f_\nu\|^t_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\Big\|^{1/t}_{l_{s/t}}&=\|\lambda\|^{t\cdot\frac{1}{t}}_{K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_s}\leq\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}.
\end{aligned}
$$

Thus, combining the above estimates, we obtain

$$
\Big|\Big\langle\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m},\varphi\Big\rangle\Big|\leq c2^{-\nu(L+\alpha_1)}\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta},\tag{4.12a}
$$

$$
\Big|\Big\langle\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m},\varphi\Big\rangle\Big|\leq c2^{-\nu(L+\alpha_1)}\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}.\tag{4.12b}
$$

Since $L>-\alpha_1$ and $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}$ or $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}$, (4.10) is proved. □

Now we come to the atomic and molecular decompositions. In 2010, Kempka [12] established the atomic and molecular decompositions for 2-microlocal Besov and Triebel-Lizorkin spaces with variable exponents. Using the argument in [12], we can obtain the atomic and molecular decompositions of 2-microlocal Herz type Besov and Triebel-Lizorkin spaces with variable exponents. We leave the details of their proofs.

Corollary 4.1. Let $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^{\infty}(\mathbb{R}^n)$

(i) If

$$K>\alpha_2 \quad and \quad L>\sigma_q-\alpha_1,$$

then for each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}$ there exist $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}$ and $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ such that

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m} \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{4.13}$$

holds. Moreover

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}}, \tag{4.14}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}$.

(ii) If

$$K>\alpha_2 \quad \text{and} \quad L>\sigma_{q,\beta}-\alpha_1,$$

then for each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}$ there exist $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}$ and $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ such that

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m} \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n) \tag{4.15}$$

holds. Moreover

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}}, \tag{4.16}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}$.

To get the reverse direction of the atomic decomposition theory we use the more general molecules. Afterwards the atomic decomposition theory follows easily.

Corollary 4.2. Let $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. Further, let $K,L\in\mathbb{N}_0$.

(i) Let

$$K>\alpha_2,\quad L>\sigma_p-\alpha_1$$

and $M>0$ large enough. If $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta$ and $\{\mu_{vm}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules, then for each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta$ such that

$$f=\sum_{v=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{vm}\mu_{vm}\quad\text{converging in }\mathcal{S}'(\mathbb{R}^n) \tag{4.17}$$

holds. Moreover

$$|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta(\mathbb{R}^n)}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta}, \tag{4.18}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta(\mathbb{R}^n)$.

(ii) If

$$K>\alpha_2,\quad L>\sigma_{p,q}-\alpha_1$$

and $M>0$ large enough. If $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta$ and $\{\mu_{vm}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules, then for each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta$ such that

$$f=\sum_{v=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{vm}\mu_{vm}\quad\text{converging in }\mathcal{S}'(\mathbb{R}^n) \tag{4.19}$$

holds. Moreover

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}, \tag{4.20}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)$.

Since every $[K,L]$-atoms is a $[K,L,M]$-molecule for every $M>0$, we obtain the following corollary.

Corollary 4.3. Let $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. Further, let $K,L\in\mathbb{N}_0$.

(i) Let

$$K>\alpha_2\quad\text{and}\quad L>\sigma_p-\alpha_1.$$

(a) If $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta$ and $\{a_{vm}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L]$-atoms, then

$$f=\sum_{v=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{vm}\mu_{vm}\quad\text{converging in }\mathcal{S}'(\mathbb{R}^n), \tag{4.21}$$

belongs to the spaces $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ and there exists a constant $c>0$ with

$$|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}, \tag{4.22}$$

where the constant $c>0$ is universal for all λ and $a_{\nu m}$.

(b) For each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$, there exist $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}$ and $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ such that there exists a representation

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\quad\text{converging in } \mathcal{S}'(\mathbb{R}^n) \tag{4.23}$$

holds. Moreover

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)}, \tag{4.24}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$.

(ii) Let

$$K>\alpha_2\quad\text{and}\quad L>\sigma_{p,q}-\alpha_1.$$

(a) If $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}$ and $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L]$-atoms, then for each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$ such that

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}\quad\text{converging in } \mathcal{S}'(\mathbb{R}^n) \tag{4.25}$$

holds. Moreover

$$|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}, \tag{4.26}$$

where the constant $c>0$ is universal for all λ and $a_{\nu m}$.

(b) For each $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$ there exist $\lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}$ and $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ such that

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\quad\text{converging in } \mathcal{S}'(\mathbb{R}^n) \tag{4.27}$$

holds. Moreover

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)}, \tag{4.28}$$

where the constant $c>0$ is universal for all $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$.

5 Wavelet decompositions

In this section we will present wavelets characterization for $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}$ $(\mathbb{R}^n)$. First, we recall some results from wavelet theory. The proposition of the following is taken over from [22, Theorem 1.61].

Proposition 5.1. (i) There are a real scaling function $\psi_F \in \mathcal{S}(\mathbb{R})$ and a real associated wavelet $\psi_M \in \mathcal{S}(\mathbb{R})$ such that their Fourier transforms have compact supports, $\hat{\psi}_F(0) = (2\pi)^{-1/2}$ and

$$\text{supp}\,\{\hat{\psi}_M\} \subseteq \left[-\frac{8}{3}\pi,-\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi,\frac{8}{3}\pi\right].$$

(ii) For any $K \in \mathbb{N}$ there exist a real compactly supported scaling function $\psi_F \in C^k(\mathbb{R})$ and a real compactly supported associated wavelet $\psi_M \in C^k(\mathbb{R})$ such that $\{\hat{\psi}_F\}(0) = (2\pi)^{-1/2}$ and

$$\int_{\mathbb{R}} x^m \psi_M(x)dx = 0 \quad \text{for all } m \in \{0,\cdots,k-1\}.$$

The wavelets in the first part (i) are called Meyer wavelets, the wavelets from the second part (ii) are called Daubechies wavelets.

Let ψ_M,ψ_F be the Meyer or Daubechies wavelets described above. Define

$$G^0 = \{F,M\}^n \;\text{ and }\; G^\nu = \{F,M\}^{n*} \;\; if\ \nu \geq 1,$$

where the $*$ indicates, that at least one G_i of $G = (G_1,\cdots,G_n) \in \{F,M\}^{n*}$ must be an M. The cardinal number of $\{F,M\}^{n*}$ is $2^n - 1$. For $x \in \mathbb{R}^n$, let

$$\Psi^\nu_{Gm}(x) = 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r),$$

where $G \in G^\nu$, $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $\{\Psi^\nu_{Gm} : \nu \in \mathbb{N}_0, G \in G^\nu, m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$. We have to adapt our sequence spaces $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{f}^{\omega}_{\beta}(\mathbb{R}^n)$ to the new situation as follows.

Definition 5.1. Let $\omega = (\omega_j)_j \in \mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0 < \beta, q \leq \infty$, $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. Further, let $K, L \in \mathbb{N}_0$. Then

$$K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n) = \left\{\lambda = \{\lambda^\nu_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n} \subset \mathbb{C} : \|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n)} < \infty \right\}, \tag{5.1a}$$

where

$$\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n)} = \left(\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\Big\|\sum_{m\in\mathbb{Z}^n}|\lambda^\nu_{Gm}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big\|^{\beta}_{K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta}. \tag{5.1b}$$

Furthermore, we define

$$K_{p(\cdot)}^{\alpha(\cdot),q}\widetilde{f}_{\beta}^{\omega}(\mathbb{R}^n)=\left\{\lambda=\{\lambda_{Gm}^{\nu}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}\subset\mathbb{C}:\|\lambda\|_{K_{p(\cdot)}^{\alpha(\cdot),q}\widetilde{f}_{\beta}^{\omega}(\mathbb{R}^n)}<\infty\right\}, \tag{5.2a}$$

where

$$\|\lambda\|_{K_{p(\cdot)}^{\alpha(\cdot),q}\widetilde{f}_{\beta}^{\omega}(\mathbb{R}^n)}=\left\|\left(\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n}|\lambda_{Gm}^{\nu}|^{\beta}\omega_{\nu}^{\beta}(2^{-\nu}m)\chi_{\nu m}\right)^{1/\beta}\right\|_{K_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}. \tag{5.2b}$$

We recall the local means with kernel k

$$k(t,f)(x)=\int_{\mathbb{R}^n}k(y)f(x+ty)dy=t^{-n}\int_{\mathbb{R}^n}k\left(\frac{y-x}{t}\right)f(y)dy.$$

Let

$$k(2^{-j},f)(2^{-j}l)=2^{jn}\int_{\mathbb{R}^n}k(2^jf-l)f(y)dy=\int_{\mathbb{R}^n}k_{jl}(y)f(y)dy=k_{jl}(f).$$

where $t=2^{-j}$, $x=2^{-j}l$, $j\in\mathbb{N}_0$ and $l\in\mathbb{Z}^n$. Now the usual properties on k get shifted to the kernels k_{jl}.

Definition 5.2. Let $A,B\in\mathbb{N}_0$ and $C>0$. Further, let $k_{jl}\in C^A(\mathbb{R}^n)$ with $j\in\mathbb{N}_0$ and $l\in\mathbb{Z}^n$ be functions in $\mathbb{R}^n$ with

$$|D^{\beta}k_{jl}(x)|\le c2^{j|\beta|+jn}(1+2^j|x-2^{-j}l|)^{-C},\quad |\beta|\le A \tag{5.3}$$

for all $x\in\mathbb{R}^n$, $j\in\mathbb{N}_0$, $l\in\mathbb{Z}^n$, and

$$\int_{\mathbb{R}^n}x^{\beta}k_{jl}(x)dx=0,\quad |\beta|\le B$$

for $j\ge 1$ and $l\in\mathbb{Z}^n$.

From the above definition it is clear that $\{2^{-jn}k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ are $[A,B,C]$ molecules.

We assume that $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$ molecules and that the $\{k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ are the above given functions from Definition 5.2. Before coming to the theorem we recall two fundamental lemmas. First, we have to give estimates of the quantity $|\langle\mu_{\nu m},k_{jl}\rangle|$.

Lemma 5.1 ([5], Appendix B)**.** (i) *Let $M>a+n$, $L\ge A$ and $\nu\ge j$, then*

$$|\langle\mu_{\nu m},k_{jl}\rangle|\le c2^{-(\nu-j)(A+n)}(1+2^j|2^{-\nu}m-2^{-j}l|)^{-\min\{M-A-n,\ C\}}. \tag{5.4}$$

(ii) *Let $C>K+n$, $B\ge K$ and $\nu\le j$, then*

$$|\langle\mu_{\nu m},k_{jl}\rangle|\le c2^{-(j-\nu)K}(1+2^{\nu}|2^{-\nu}m-2^{-j}l|)^{-\min\{M,\ C-K-n\}}. \tag{5.5}$$

Lemma 5.2. *Let* $0<t<1$ *and* $R>\frac{n}{t}$. *For any* $j,\nu\in\mathbb{N}_0$, *any* $l\in\mathbb{Z}^n$ *and any sequence* $\{h_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *of complex numbers, we have with* $x\in Q_{jl}$

$$\sum_{m\in\mathbb{Z}^n}|h_{\nu m}|(1+2^j|2^{-j}l-2^{-\nu}m|)^{-R} \\ \leq c\max(2^{(\nu-j)\frac{n}{t}},1)\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|h_{\nu m}\chi_{\nu m}|\Big)(x). \tag{5.6}$$

Now, we are ready to state the following theorem, which gives us one direction of the wavelet decomposition. We define $k(f)=\{k_{jl}(f):j\in\mathbb{N}_0,l\in\mathbb{Z}^n\}$.

Theorem 5.1. *Let* $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\leq\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ *and* $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ *and* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. *Further, let* $K,L\in\mathbb{N}_0$. *Further, let* $\{k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ *be as in Definition 5.2 with* $C>0$ *large enough and* $A,B\in\mathbb{N}_0$.

(i)

$$A>\sigma_p-\alpha_1 \ \ \text{and} \ \ B>\alpha_2,$$

then for some $c>0$ *and all* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta(\mathbb{R}^n)$,

$$\|k(f)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^\omega_\beta}\leq c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^\omega_\beta(\mathbb{R}^n)}. \tag{5.7}$$

(ii) *If*

$$A>\sigma_{p,q}-\alpha_1 \ \ \text{and} \ \ B>\alpha_2,$$

then for some $c>0$ *and all* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)$,

$$\|k(f)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}\leq c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)}. \tag{5.8}$$

Proof. we only prove the theorem for the Triebel-Lizorkin spaces. Proof for the Besov spaces follows the same line of arguments. We apply the decomposition by atoms to $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^\omega_\beta(\mathbb{R}^n)$ and decompose the sum in

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}=\sum_{\nu=0}^{j}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}+\sum_{\nu=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}=f_j+f^j,$$

where $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L]$-atoms with $K=B>\alpha_2$ and $L=A>\sigma_{p,q}-\alpha_1$. By Corollary 4.3 it is sufficient to find a $c>0$ with

$$\|k(f)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}\leq c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}$$

and derive

$$k_{jl}(f)=\int_{\mathbb{R}^n}k_{jl}(y)f_j(y)dy+\int_{\mathbb{R}^n}k_{jl}(y)f^j(y)dy=k_{jl}(f_j)+k_{jl}(f^j).$$

For $\nu\le j$ and $t<\min(1,q^-,\beta)$, we obtain

$$\omega_j(x)|k_{jl}(f_j)|\le c\sum_{\nu=0}^{j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\langle k_{jl},a_{\nu m}\rangle|\omega_j(x)$$

$$\le c\sum_{\nu=0}^{j}2^{-(j-\nu)(K-\alpha_2)}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\Big(1+2^j|2^{-\nu}m-2^{-j}l|\Big)^{-C+k+n+\alpha}$$

$$\le c\sum_{\nu=0}^{j}2^{-(j-\nu)(K-\alpha_2)}\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\right)(x). \tag{5.9}$$

Thus, we obtain

$$\|k_{jl}(f_j)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}\le c\left\|\left(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|k_{jl}(f_j)\omega_j(\cdot)\chi_{jl}(\cdot)|^\beta\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}$$

$$\le c\left\|\left(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left[\sum_{\nu=0}^{j}2^{-(j-\nu)}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\right]^\beta\chi_{jl}(\cdot)\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}$$

$$\le c\left\|\left(\sum_{\nu=0}^{\infty}\left[\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\right]^\beta\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}$$

$$\le c\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^\beta\omega_\nu^\beta(2^{-\nu}m)\chi_{\nu m}(\cdot)\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}$$

$$=c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^\omega_\beta}. \tag{5.10}$$

For $\nu>j$, we have

$$\omega_j(x)|k_{jl}(f^j)|\le c\sum_{\nu=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\langle k_{jl},a_{\nu m}\rangle|\omega_j(x)$$

$$\le c\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A+n+\alpha_1)}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)(1+2^j|2^{-\nu}m-2^{-j}l|)^{-C+\alpha}$$

$$\le c\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A+n+\alpha_1-n/t)}\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\right)(x). \tag{5.11}$$

Set $\delta=A-\sigma_{p,q}+\alpha_1>0$, we obtain

$$
\begin{aligned}
\|k_{jl}(f^j)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}&=c\left\|\left(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|k_{jl}(f^j)\omega_j(\cdot)\chi_{jl}(\cdot)|^{\beta}\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\\
&\le c\left\|\left(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left[\sum_{\nu=j+1}^{\infty}2^{-(\nu-j)\delta}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\right]^{\beta}\chi_{jl}(\cdot)\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\\
&\le c\left\|\left(\sum_{\nu=j+1}^{\infty}\left[\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\right]^{\beta}\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\\
&\le c\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^{\beta}\omega_\nu^{\beta}(2^{-\nu}m)\chi_{\nu m}(\cdot)\right)^{1/\beta}\right\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\\
&=c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}.
\end{aligned}\tag{5.12}
$$

Finally, we have

$$\|k_{jl}(f)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}f^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)}. \tag{5.13}$$

Similarly, we can obtain

$$\|k_{jl}(f)\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}\le c\|\lambda\|_{K^{\alpha(\cdot),q}_{p(\cdot)}b^{\omega}_{\beta}}\le c\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)}, \tag{5.14}$$

which prove both parts of the theorem. ☐

Theorem 5.2. *Let* $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}$ *be the Daubechies wavelets according to Proposition 5.1. Let* $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ *and* $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ *and* $\alpha(\cdot)\in L^{\infty}(\mathbb{R}^n)$.

(i) *For* $f\in\mathcal{S}'(\mathbb{R}^n)$ *and* $k>\max(\sigma_p-\alpha_1,\alpha_2)$. *Then* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$, *if and only if, it can be represented as*

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n}\lambda^{\nu}_{Gm}2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\quad with\ \lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n) \tag{5.15}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\zeta}_{\beta}(\mathbb{R}^n)$ *with* $\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|x|\to\infty$ *and all* ν *and also* $\sup_{x\in\mathbb{R}^n}\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|\nu|\to\infty$. *The representation* (5.15) *is unique, we have*

$$
\begin{aligned}
&\lambda^{\nu}_{Gm}=\lambda^{\nu}_{Gm}(f)=2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle,\\
&I:\ f\to\{2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle\}
\end{aligned}
$$

is an isomorphic map from $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ *onto* $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n)$. *Moreover, if in addition* $\max\{\beta,p^+\}<\infty$, *then* $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}$ *is an unconditional basis in* $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$.

(ii) *Let* $f\in\mathcal{S}'(\mathbb{R}^n)$ *and* $k>\max(\sigma_{p,q}-\alpha_1,\alpha_2)$. *Then* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$, *if and only if, it can be represented as*

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n}\lambda^{\nu}_{Gm}2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\quad with\ \lambda\in\widetilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.16}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\zeta}_{\beta}(\mathbb{R}^n)$ *with* $\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|x|\to\infty$ *and all* ν *and also* $\sup_{x\in\mathbb{R}^n}\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|\nu|\to\infty$. *The representation* (5.2) *is unique, we have*

$$\lambda^{\nu}_{Gm}=\lambda^{\nu}_{Gm}(f)=2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle,$$
$$I:\ f\to\{2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle\}$$

is an isomorphic map from $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$ *onto* $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{f}^{\omega}_{\beta}(\mathbb{R}^n)$. *Moreover, if in addition* $\max\{\beta,p^+\}<\infty$, *then* $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}$ *is an unconditional basis in* $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$.

By the proof of Theorem 4.8 in [12], using Theorem 5.1, we easy to find the proof is not difficult and so is omitted.

Now, we present a wavelet decomposition theorem with the help of Meyer wavelets, described in Proposition 5.1.

Theorem 5.3. *Let* $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}$ *be the Meyer wavelets according to Proposition* 5.1. *Let* $\omega=(\omega_j)_j\in\mathcal{W}^{\alpha_0}_{\alpha_1,\alpha_2}$, $0<\beta,q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ *and* $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ *and* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$.

(i) *For* $f\in\mathcal{S}'(\mathbb{R}^n)$ *and* $k>\max(\sigma_p-\alpha_1,\alpha_2)$. *Then* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$, *if and only if, it can be represented as*

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n}\lambda^{\nu}_{Gm}2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\quad with\ \lambda\in K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n) \tag{5.17}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\zeta}_{\beta}(\mathbb{R}^n)$ *with* $\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|x|\to\infty$ *and all* ν *and also* $\sup_{x\in\mathbb{R}^n}\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|\nu|\to\infty$. *The representation* (5.17) *is unique, we have*

$$\lambda^{\nu}_{Gm}=\lambda^{\nu}_{Gm}(f)=2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle,$$
$$I:\ f\to\{2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle\}$$

is an isomorphic map from $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$ *onto* $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{b}^{\omega}_{\beta}(\mathbb{R}^n)$. *Moreover, if in addition* $\max\{\beta,p^+\}<\infty$, *then* $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}$ *is an unconditional basis in* $K^{\alpha(\cdot),q}_{p(\cdot)}B^{\omega}_{\beta}(\mathbb{R}^n)$.

(ii) *Let* $f\in\mathcal{S}'(\mathbb{R}^n)$ *and* $k>\max(\sigma_{p,q}-\alpha_1,\alpha_2)$. *Then* $f\in K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$, *if and only if, it can be represented as*

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}\lambda^\nu_{Gm}2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm}\quad \text{with } \lambda\in\widetilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.18}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\zeta}_{\beta}(\mathbb{R}^n)$ *with* $\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|x|\to\infty$ *and all* ν *and also* $\sup_{x\in\mathbb{R}^n}\zeta_\nu(x)/\omega_\nu(x)\to 0$ *for* $|\nu|\to\infty$. *The representation* (5.18) *is unique, we have*

$$\lambda^\nu_{Gm}=\lambda^\nu_{Gm}(f)=2^{\nu\frac{n}{2}}\langle f,\Psi^\nu_{Gm}\rangle,$$
$$I:\ f\to\{2^{\nu\frac{n}{2}}\langle f,\Psi^\nu_{Gm}\rangle\}$$

is an isomorphic map from $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$ *onto* $K^{\alpha(\cdot),q}_{p(\cdot)}\widetilde{f}^{\omega}_{\beta}(\mathbb{R}^n)$. *Moreover, if in addition* $\max\{\beta,p^+\}<\infty$, *then* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ *is an unconditional basis in* $K^{\alpha(\cdot),q}_{p(\cdot)}F^{\omega}_{\beta}(\mathbb{R}^n)$.

The proof of Theorem 5.3 is the same as in Theorem 5.2, so it will be omitted.

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