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Some Generalized Clifford-Jacobi Polynomials and Associated Spheroidal Wavelets

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Abstract. In the present paper, by extending some fractional calculus to the framework of Clifford analysis, new classes of wavelet functions are presented. Firstly, some classes of monogenic polynomials are provided based on 2-parameters weight functions which extend the classical Jacobi ones in the context of Clifford analysis. The discovered polynomial sets are next applied to introduce new wavelet functions. Reconstruction formula as well as Fourier-Plancherel rules have been proved. The main tool reposes on the extension of fractional derivatives, fractional integrals and fractional Fourier transforms to Clifford analysis.

Key Words: Continuous wavelet transform, Clifford analysis, Clifford Fourier transform, Fourier-plancherel, fractional Fourier transform, fractional derivatives, fractional integrals, fractional Clifford Fourier transform, Monogenic functions.

AMS Subject Classifications: 26A33, 42A38, 42B10, 44A15, 30G35

1 Introduction

Clifford Algebra is characterized by additional concepts as it provides a simpler model of mathematical objects compared to vector algebra. It permits a simplification in the notations of mathematical expressions such as plane and volume segments in two, three and higher dimensions by using a coordinate-free representation. Such representation is characterized by an important feature resumed in the fact that the motion of an object may be described with respect to a coordinate frame defined on the object itself. This means that it permits to use a self-coordinate system related to the object in hand.

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In the present work, one aim is to provide a rigorous development of wavelets adapted to the sphere based on Clifford calculus. The frame is somehow natural as wavelets are characterized by scale invariance of approximation spaces. Clifford algebra is one mathematical object that owns this characteristic. Recall that multiplication of real numbers scales their magnitudes according to their position in or out from the origin. However, multiplication of the imaginary part of a complex number performs a rotation, it is a multiplication that goes round and round instead of in and out. So, a multiplication of spherical elements by each other results in an element of the sphere. Again, repeated multiplication of the imaginary part results in orthogonal components. Thus, we need a coordinates system that results always in the object, a concept that we will see again and again in the Algebra. In other words, Clifford algebra generalizes to higher dimensions by the same exact principles applied at lower dimensions, by providing an algebraic entity for scalars, vectors, bivectors, trivectors, and there is no limit to the number of dimensions it can be extended to. More details on Clifford analysis, Clifford calculus, origins, history, developments may be found in [1,13,15,26,29].

In the present work, we propose to develop new wavelet analysis constructed in the framework of Clifford analysis by adopting monogenic functions which may be described as solutions of the Dirac operator and are direct higher dimensional generalizations of holomorphic functions in the complex plane. We apply such extension to some well adapted Clifford weights to construct new spheroidal wavelets. Recall that wavelets are widespread in the last decades. They become an interesting and useful tool in many fields such as mathematics, quantum physics, electrical engineering and seismic geology and they have proved to meet a need in signal processing that Fourier transform was not the best answer. Classical Fourier analysis provides a global description of signals and did not provide a time localization.

Wavelet analysis starts by convoluting the analyzed function with copies $\psi_{a,b}$, $a > 0$ and $b \in \mathbb{R}$ (called wavelets) issued from a source (mother wavelet) ψ by dilation $a > 0$ and translation $b \in \mathbb{R}$,

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\Big(\frac{x-b}{a}\Big). \tag{1.1}$$

Generally, the source ψ has to satisfy the so-called admissibility condition

$$\mathcal{A}_\psi = \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(u)|^2}{|u|} du < +\infty, \tag{1.2}$$

where $\widehat{\psi}$ is the Fourier transform of ψ.

Wavelet analysis of functions starts by computing the Continuous Wavelet Transform (CWT) of the analyzed function f

$$C_{a,b}(f) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\overline{\psi\Big(\frac{x-b}{a}\Big)}dx, \tag{1.3}$$

which in turns permits to re-construct in some sense the analyzed function f via an inverse transform based on the admissibility assumption (1.2) as

$$f(x) = \frac{1}{\mathcal{A}_\psi} \int_{\mathbb{R}} \int_0^{+\infty} C_{a,b}(f) \psi_{a,b}(x) \frac{da}{a^2} db. \tag{1.4}$$

In the present paper we propose to construct some special spheroidal wavelets in the context of Clifford analysis with the help of fractional calculus. Recall that fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) orders. Such topic is not new but recently a coming back to its application has taken place in various areas of engineering, science, finance, applied mathematics and bio-engineering.

The wavelets constructed here are general copies of the Gegenbauer-Jacobi one developed in [5, 6, 8, 9] and [10], where the authors tried to point out a wavelet analysis in homogenous Euclidean spaces. The main idea applied there was by considering a Clifford Heaviside function to decompose the weight $(1-x)^\alpha(1+x)^\beta$ into semi-radial ones and thus to apply the radial bases of Euclidean spaces by assuming that $\alpha \pm \beta = 1$. The basic idea applied there is resumed in the fact that for this case the weight function may be decomposed into a sum of tow parts: one part is the well known radial classical weight $(1+|x|^2)^\alpha$ and a second semi-radial one which leads with the well known rule of derivation of integrals with parameters to the first part.

This was the main difference and motivation with our case where such a decomposition is not possible. We instead come back to fractional calculus to overcome the problem of the non radially symmetric weight applied here and did not restrict to the previous case. We will prove instead that fractional calculus may be a good tool to overcome the difficulties crossed in the new context.

The organization of this paper is as follows: in Section 2, a revision of fractional calculus such as fractional derivation and fractional Fourier transform is provided. Section 3 is devoted to Clifford calculus. Basic operations, Clifford fractional derivation, Clifford Fourier transform and Fractional Clifford Fourier transform are reviewed. Next, our idea of the generalization of Clifford-Jacobi polynomials is developed in Section 4. Section 5 is concerned with the development of new wavelets in the Clifford context associated to the polynomial class developed previously. Relative continuous wavelet is also provided and reconstruction rule is proved in the new framework. We concluded afterward.

2 Fractional calculus

We propose in this part to review the basic definitions of fractional differentiation as well as fractional Fourier transforms to be applied next. For backgrounds on this part, the readers may refer to [3, 12, 14, 22, 24, 25, 28, 30–33, 35, 36, 38, 39, 41].

2.1 Fractional derivation

There are in fact many ideas to introduce the fractional derivative(s) of functions. One well known method reposes on Riemman-Liouville fractional integral which consists in a natural extension of the Cauchy formula given by

$$J^n f(t) = f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau, \quad t>0, \quad n\in\mathbb{N}. \tag{2.1}$$

This formulation is extended to fractional orders and constitutes the Riemman-Liouville fractional integral expressed for $\alpha \in \mathbb{R}+$ as

$$_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad t>a, \tag{2.2}$$

where Γ is the Euler Gamma function.

A second formulation of fractional differentiation is based on the so-called Hadamard fractional integral as

$$_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log\frac{t}{\tau}\Big)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}. \tag{2.3}$$

Based on these formulations and analogous the fractional derivatives of arbitrary order $\alpha>0$ becomes a natural requirement. We seek a formulation that remains valid when applied for ordinary integer orders. One formulation is introduced as follows. Let $\alpha>0$ and $m\in\mathbb{N}$ be such that $m-1<\alpha\le m$. The α-derivative of a function f is

$$D^\alpha f(t) = D^m J^{m-\alpha} f(t). \tag{2.4}$$

Otherwise,

$$D^\alpha f(t) = \begin{cases} \dfrac{d^m}{dt^m}\Big[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{f(t)}{(t-\tau)^{\alpha+1-m}}d\tau\Big] & \text{for } m-1<\alpha<m, \\ \dfrac{d^m}{dt^m} f(t) & \text{for } \alpha=m. \end{cases} \tag{2.5}$$

The following alternative definition of fractional derivative is originally introduced by Riemann-Liouville and Caputo and formulated using Lagrange's rule for differential operators. Let $\alpha>0$ and $n=[\alpha]$. The α-derivative is

$$_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n}\, {}_aD_t^{-(n-\alpha)} f(t). \tag{2.6}$$

Otherwise, a different alternative has been already formulated by Caputo and states that

$$^c_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}. \tag{2.7}$$

There are in literature many alternatives and ideas that introduced the fractional derivatives. We refer to the book of Kilbas et al. [25] and Samko et al. [32] for more details and applications. In this context, we have the following result.

Lemma 2.1. *The following assertions hold.*

1. *Whenever p and q in $\mathbb{R}_+$ we have for all $\underline{b} \in \mathbb{R}^m$ fixed,*

$$D^q_{\underline{x}}(\underline{b}-\underline{x}) = m^q \frac{\Gamma(p)}{\Gamma(p-q)}(\underline{b}-\underline{x})^{p-q-1}\underline{x}.$$

2. *Whenever r and s in $\mathbb{R}_+$ we have for all $\underline{a} \in \mathbb{R}^m$ fixed,*

$$D^r_{\underline{x}}(\underline{a}+\underline{x})^{s-1} = e^{ir\pi} m^r \frac{\Gamma(s)}{\Gamma(s-r)}(\underline{a}+\underline{x})^{s-r-1}.$$

2.2 Fractional Fourier transform

Let f be in $L^1(\mathbb{R}^m)$. Its Fourier transform denoted usually $\widehat{f}$ or $\mathcal{F}(f)$ is given by the integral transform

$$\widehat{f}(\eta) = \mathcal{F}(f)(\eta) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-ix.\eta} f(x)dx,$$

where dx is the Lebesgue measure on $\mathbb{R}^m$ and $x.\eta$ is the standard inner product of x and η in $\mathbb{R}^m$.

The Fractional Fourier Transform has been intensively studied during the last decade. In this section, we will be concerned with the definition and some of its properties. It is based on the well-known Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ defined on the real line by

$$H_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

It is well-known that these polynomials form an orthonormal basis for $L^2(\mathbb{R}^m)$ and are eigenfunctions of the Fourier transform satisfying $\mathcal{F}H_n = (-i)^n H_n$. Consequently, given $f \in L^2(\mathbb{R}^m)$, its Fourier transform may be written by means of Hermite polynomials as

$$\mathcal{F}(f) = \sum_{n=0}^{\infty} \langle f, H_n \rangle (-i)^n H_n = \sum_{n=0}^{\infty} \langle f, H_n \rangle e^{-in\frac{\pi}{2}} H_n.$$

The idea of fractional Fourier transform consists in replacing the fraction $\frac{\pi}{2}$ by a real number a. Hence, we obtain

$$\mathcal{F}_a(f) = \sum_{n=0}^{\infty} \langle f, H_n \rangle e^{-ina} H_n, \quad (a \in \mathbb{R}).$$

Coming back to the Hermite operator

$$\mathcal{H} = \frac{1}{2}\Big(-\frac{d^2}{dx^2} + x^2 - 1\Big),$$

which has the Hermite polynomials as eigenfunctions, the fractional Fourier transform may be written by means of an exponential operator

$$\mathcal{F} = e^{-i\frac{\pi}{2}\mathcal{H}}$$

and the fractional Fourier transform may be similarly be written as

$$\mathcal{F}_a = e^{-ia\mathcal{H}}.$$

Backgrounds on fractional Fourier analysis may be found in [11].

3 Clifford analysis

This section is devoted to a brief review of basic concepts of Clifford analysis and fractional calculus as well as the extension of fractional Fourier transforms for the Clifford case. The readers may refer to [1,2,9–11,13,16,17,20,34].

3.1 Clifford calculus

Clifford analysis appeared as a generalization of the complex analysis and Hamiltonians. It extended complex calculus to some type of finite-dimensional associative algebra known as Clifford algebra endowed with suitable operations as well as inner products and norms. It is now applied widely in a variety of fields including geometry and theoretical physics.

Clifford analysis offers a functional theory extending the one of holomorphic functions of complex variable. Starting from the real space $\mathbb{R}^m$, $(m > 1)$ (or the complex space $\mathbb{C}^m$) endowed with an orthonormal basis $(e_1, \cdots, e_m)$, the Clifford algebra $\mathbb{R}_m$ (or its complexification $\mathbb{C}_m$) starts by introducing a suitable interior product, let

$$\begin{aligned}
&e_j^2 = -1, && j = 1, \cdots, m,\\
&e_j e_k + e_k e_j = 0, && j \neq k, \quad j,k = 1, \cdots, m.
\end{aligned}$$

It is straightforward that it is a non-commutative multiplication. Two anti-involutions on the Clifford algebra are important. The conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j, \quad j = 1, \cdots, m,$$

with the additional rule in the complex case,

$$\bar{i} = -i.$$

The inversion is defined as the anti-involution for which

$$e_j^+ = e_j, \quad j = 1, \cdots, m.$$

This yields a basis of the Clifford algebra ($e_A : A \subset \{1, \cdots, m\}$), where $e_\emptyset = 1$ is the identity element. As these rules are defined, the Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying the vector $x = (x_1, \cdots, x_m)$ with the vector $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

The product of two vectors is given by

$$\underline{x}\,\underline{y} = \underline{x}.\underline{y} + \underline{x} \wedge \underline{y},$$

where

$$\underline{x}.\underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j,$$

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j),$$

is the wedge product. In particular,

$$\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2.$$

Finally, we recall that whenever we manipulate elements in the complex Clifford algebra $\mathbb{C}_m$, we need the Hermitian conjugation defined for $\lambda = \sum_A \lambda_A e_A$, $\lambda_A \in \mathbb{C}$ by

$$\lambda^\dagger = \sum_A \lambda_A^c \overline{e_A},$$

where λ_A^c is the complex conjugate of λ_A and where

$$\overline{e_A} = (-1)^{\frac{|A|(|A|+1)}{2}} e_A.$$

3.2 Clifford monogenic functions

Let Ω be an open subset of $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$ and $f : \Omega \to \mathbb{A}$, where $\mathbb{A}$ is the reel Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. f may be written on the form

$$f = \sum_A f_A e_A,$$

where the functions f_A are $\mathbb{R}$-valued or $\mathbb{C}$-valued and $(e_A)_A$ is a suitable basis of $\mathbb{A}$. Despite the fact that Clifford analysis generalizes the most important features of classical complex analysis, monogenic functions do not enjoy all properties of holomorphic functions of complex variable.

For instance, due to the non-commutativity of the Clifford algebras, the product of two monogenic functions is in general not monogenic. It is therefore natural to look for specific techniques to construct monogenic functions.

In the literature, there are several techniques available to generate monogenic functions on $\mathbb{R}^{m+1}$ such as the Cauchy-Kowalevski extension (CK-extension) which consists in finding a monogenic extension g^* of an analytic function g defined on a given subset of $\mathbb{R}^{m+1}$ of positive codimension. For an analytic function g on the hyperplane $\{(x_0,\underline{x})\in\mathbb{R}^{m+1},\ x_0=0\}$ the problem consists of finding a function g^* defined on $\mathbb{R}^{m+1}$ satisfying $\partial_{x_0}g^*=-\partial_{\underline{x}}g^*$ on $\mathbb{R}^{m+1}$ and $g^*(0,\underline{x})=g(\underline{x})$ on $\mathbb{R}^m$. A formal solution is

$$g^*(x_0,\underline{x})=\exp(-x_0\partial_{\underline{x}})g(\underline{x})=\sum_{k=0}^{\infty}\frac{(-x_0)^k}{k!}\partial_{\underline{x}}^k g(\underline{x}). \tag{3.1}$$

It may be proved that (3.1) is a monogenic extension of the function g in $\mathbb{R}^{m+1}$. Moreover, by the uniqueness theorem for monogenic functions this extension is also unique. See [4, 13, 29, 37, 40] and the references therein.

An $\mathbb{R}_m$ or $\mathbb{C}_m$-valued function $F(x_1,\cdots,x_m)$, respectively $F(x_0,x_1,\cdots,x_m)$ is called right monogenic in an open region of $\mathbb{R}^m$, respectively, or $\mathbb{R}^{m+1}$, if in that region

$$F\partial_{\underline{x}}=0,\quad \text{respectively}\ \ F(\partial_{x_0}+\partial_{\underline{x}})=0.$$

Here $\partial_{\underline{x}}$ is the Dirac operator defined on $\mathbb{R}^m$ by

$$\partial_{\underline{x}}=\sum_{j=1}^{m}e_j\partial_{x_j},$$

which splits the Laplacian in $\mathbb{R}^m$ as

$$\Delta_m=-\partial_{\underline{x}}^2,$$

whereas $\partial_{x_0}+\partial_{\underline{x}}$ is the Cauchy-Riemann operator in $\mathbb{R}^{m+1}$, for which

$$\Delta_{m+1}=(\partial_{x_0}+\partial_{\underline{x}})(\partial_{x_0}+\overline{\partial_{\underline{x}}}).$$

Denoting S^{m-1} the unit sphere in $\mathbb{R}^m$ and introducing spherical coordinates in $\mathbb{R}^m$ by

$$\underline{x}=r\underline{\omega},\quad r=|\underline{x}|\in[0,+\infty],\quad \underline{\omega}\in S^{m-1},$$

the Dirac operator takes the form

$$\partial_{\underline{x}}=\underline{\omega}\left(\partial_r+\frac{1}{r}\Gamma_{\underline{\omega}}\right),$$

where

$$\Gamma_{\underline{\omega}}=-\sum_{i<j}e_ie_j(x_i\partial_{x_j}-x_j\partial_{x_i})$$

is the so-called spherical Dirac operator which depends only on the angular coordinates.

3.3 Clifford Fourier transform

As for the Euclidean case, Fourier analysis is extended to Clifford Fourier analysis [6, 10, 11]. The idea behind the definition of the Clifford Fourier transform originates from the exponential operator representation of the classical Fourier transform by means of Hermite operators. Recall that

$$\mathcal{F}(f) = \exp\Big(- i\frac{\pi}{2}\mathcal{H}_m\Big)(f) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n\, \mathcal{H}_m^n(f)$$

with $\mathcal{H}_m$ the the classical m-dimensional Hermite operator given by

$$\mathcal{H}_m = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - m).$$

To introduce the Clifford analysis character in the Fourier transform, the exponential operator and $\mathcal{H}_m$ have to be replaced by Clifford algebra-valued ones. The starting step consists in factorizing the operator $\mathcal{H}_m$ making use of the factorization of the Laplace operator by the Dirac Γ defined by

$$\Gamma_{\underline{x}} = -\frac{1}{2}(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} - m).$$

Next, two Clifford-Hermite operators $H_m^{\pm}$ are introduced,

$$\mathcal{H}_m^{\pm} = \mathcal{H}_m \pm \Big(\Gamma_{\underline{x}} + \frac{m}{2}\Big).$$

Definition 3.1. *The Clifford-Fourier transform is the pair of transformations*

$$\mathcal{F}_m^{+} = \exp\Big(- i\frac{\pi}{2}\mathcal{H}_m^{+}\Big) \quad \textit{and} \quad \mathcal{F}_m^{-} = \exp\Big(- i\frac{\pi}{2}\mathcal{H}_m^{-}\Big)$$

Since $\mathcal{H}_m$ commutes with Γ, we have

$$\mathcal{F}_m^{+}\mathcal{F}_m^{-} = \exp(-i\pi\mathcal{H}_m).$$

Note that

$$\mathcal{F}_m^{\pm} = \exp\Big(\pm i\Big(\frac{\pi}{2}\Big)\Big(\Gamma_{\underline{x}} + \frac{m}{2}\Big)\Big)\, \mathcal{F}_m,$$

where $\mathcal{F}_m$ is the classical m-dimensional Fourier transform which acts by integration against the scalar-valued kernel

$$K_m(\underline{x},\underline{y}) = (2\pi)^{\frac{-m}{2}}\, e^{-i\langle \underline{x},\underline{y}\rangle}.$$

As a consequence, we obtain an integral representation for the Clifford-Fourier transform

$$\mathcal{F}_m^{\pm}(f)(\underline{x}) = \int_{\mathbb{R}^m} \exp\Big(\pm i\frac{\pi}{2}\Big)\Big(\Gamma_{\underline{x}} + \frac{m}{2}\Big)\, K_m(\underline{x},\underline{y})\, f(\underline{y})\, dV(\underline{y}),$$

from which we see that $\mathcal{F}_m^{\pm}$ acts by integration against the Clifford-valued kernel

$$C_m^{\pm}(\underline{x},\underline{y}) = \exp\Big(\pm i\frac{\pi}{2}\Big)\Big(\Gamma_{\underline{x}} + \frac{m}{2}\Big)\, K_m(\underline{x},\underline{y}).$$

$dV(\underline{y})$ is the Lebesgue measure on $\mathbb{R}_m$.

Definition 3.2. *The two fractional Clifford-Fourier transform is defined by*

$$\mathcal{F}_{m,a}^{\pm} = \exp(-ia\mathcal{H}_m^{\pm}).$$

Since we have

$$\mathcal{F}_{m,a}^{+}\mathcal{F}_{m,a}^{-} = \mathcal{F}_{m,2a},$$

the fractional Clifford-Fourier transforms act by integration against the kernels

$$C_{m,a}^{\pm}\Big(\underline{x},\underline{y}\Big) = \exp\Big(\pm ia\Big(\Gamma_{\underline{x}} + \frac{m}{2}\Big)\Big)\, K_{m,a}(\underline{x},\underline{y}),$$

where $K_{m,a}$ is the classical m-dimensional fractional Fourier transform.

Throughout this article the Clifford-Fourier transform of f is given by

$$\mathcal{F}(f)(\underline{y}) = \int_{\mathbb{R}^m} e^{-i\langle \underline{x},\underline{y}\rangle} f(\underline{x})dV(\underline{x}).$$

We also denote the inner product of functions in the framework of Clifford analysis by

$$\langle f,g\rangle = \int_{\mathbb{R}^m} f(\underline{x})\overline{g(\underline{x})}dV(\underline{x}).$$

In the present work, we propose to apply such topics to output some generalizations of multidimensional Continuous Wavelet Transform in the context of Clifford analysis.

4 Generalized Clifford-Jacobi polynomials

A special case has been developed in [9] where the authors tried to point out a wavelet analysis in homogenous Euclidean spaces. The main idea applied there was by considering a Clifford Heaviside function to decompose the weight $(1-x)^{\alpha}(1+x)^{\beta}$ into semi-radial ones and thus to apply the radial bases of Euclidean spaces by assuming that $\alpha \pm \beta = 1$. The basic idea may be resumed in the fact that for this case the weight function may be decomposed into a sum of tow parts: one part is the well known radial classical weight $(1+|x|^2)^{\alpha}$ and a second semi-radial one which leads with the well known rule of derivation of integrals with parameters to the first part. This was the main difference and motivation with our case where such a decomposition is not possible.

In the present paper we instead come back to fractional calculus to overcome the problem of the non radially symmetric weight applied here and did not restrict to the previous case. We will prove instead that fractional calculus may be a good tool to overcome the difficulties crossed in the new context.

In this section we consider the generalization to the Clifford analysis of the classical Jacobi polynomials. Consider the Clifford algebra-valued weight function

$$\omega_{\alpha,\beta}(\underline{x}) = (1-\underline{x})^{\alpha}(1+\underline{x})^{\beta},$$

with $\alpha,\beta \in \mathbb{R}$ and the monogenic function

$$F^*(t,\underline{x}) = \sum_{\ell=0}^{\infty} \frac{t^{\ell}}{\ell!} G_{\ell,m}^{\alpha,\beta}(\underline{x})\omega_{\alpha-\ell,\beta-\ell}(\underline{x}).$$

We will evaluate now $\partial_{\underline{x}}(\omega_{\alpha,\beta}(\underline{x}))$. The idea is inspired from the entire series development of the powers $(1+x)^{\alpha}$ on $\mathbb{R}$. So, assume that $|\underline{x}|<1$ (The case where $|\underline{x}|>1$ may be inspired by the same way and gives a Laurent series analogue). We have

$$(1-\underline{x})^{\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}(-1)^n \underline{x}^n \quad \text{and} \quad (1+\underline{x})^{\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!}\underline{x}^n,$$

where for $s \in \mathbb{R}$,

$$(s)_n = \frac{\Gamma(s+1)}{\Gamma(s-n)}.$$

Thus, we obtain

$$(1-\underline{x})^{\alpha}(1+\underline{x})^{\beta} = \sum_{n=0}^{\infty} a_n(\alpha,\beta)\underline{x}^n$$

with

$$a_n(\alpha,\beta) = \sum_{k=0}^{n} \frac{(\beta)_k}{k!}\frac{(\alpha)_{n-k}}{(n-k)!}(-1)^{n-k}.$$

Next, we use the following technical lemma.

Lemma 4.1. *For $n \in \mathbb{N}$, we have*

$$\partial_{\underline{x}}(\underline{x}^n) = \gamma_{n,m}\underline{x}^{n-1},$$

where

$$\gamma_{n,m} = \begin{cases} -n, & \text{if } n \text{ is even}, \\ -(m+n-1), & \text{if } n \text{ is odd}. \end{cases}$$

Consequently,

$$\partial_{\underline{x}}(\omega_{\alpha,\beta}(\underline{x})) = \sum_{n=0}^{\infty} a_n(\alpha,\beta)\partial_{\underline{x}}(\underline{x}^n) = \sum_{n=0}^{\infty} a_n(\alpha,\beta)\gamma_{n,m}\underline{x}^{n-1}.$$

Otherwise,

$$\begin{aligned}\partial_{\underline{x}}(\omega_{\alpha,\beta}(\underline{x})) &= -\sum_{p=0}^{\infty} a_{2p}(2p)\underline{x}^{2p-1} - \sum_{p=0}^{\infty} a_{2p+1}(m+2p+1-1)\underline{x}^{2p} \\ &= -\sum_{n=0}^{\infty} a_{n+1}(n+1)\underline{x}^{n} - \frac{m-1}{\underline{x}}\sum_{n=0}^{\infty} a_{2n+1}\underline{x}^{2n+1}.\end{aligned}$$

Denote

$$\Gamma_{\alpha,\beta}(\underline{x}) = \sum_{n=0}^{\infty} a_{n+1}(n+1)\underline{x}^{n}.$$

We now use the next lemma.

Lemma 4.2. *For $n \in N$, we have*

$$(n+1)a_{n+1}(\alpha,\beta) = \beta a_n(\alpha,\beta-1) - \alpha a_n(\alpha-1,\beta).$$

Hence, we obtain

$$\Gamma_{\alpha,\beta}(\underline{x}) = \alpha\sum_{n=0}^{\infty} a_n(\alpha-1,\beta)\underline{x}^{n} - \beta\sum_{n=0}^{\infty} a_n(\alpha,\beta-1)\underline{x}^{n} = \alpha\omega_{\alpha-1,\beta}(\underline{x}) - \beta\omega_{\alpha,\beta-1}(\underline{x}).$$

As a result,

$$\partial_{\underline{x}}(\omega_{\alpha,\beta}(\underline{x})) = \alpha\omega_{\alpha-1,\beta}(\underline{x}) - \beta\omega_{\alpha,\beta-1}(\underline{x}) - \frac{m-1}{2\underline{x}}[\omega_{\alpha,\beta}(\underline{x}) - \omega_{\alpha,\beta}(\underline{-x})].$$

As previously, we will investigate the fact that $F^*(t,\underline{x})$ is monogenic. Observe that

$$\begin{aligned}\partial_t F^*(t,\underline{x}) &= \sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}G^{\alpha,\beta}_{\ell+1,m}(\underline{x})\,\omega_{\alpha-\ell-1,\beta-\ell-1}(\underline{x}), \\ \partial_{\underline{x}} F^*(t,\underline{x}) &= \partial_{\underline{x}} G^{\alpha,\beta}_{\ell,m}(\underline{x})\omega_{\alpha-\ell,\beta-\ell}(\underline{x}) + G^{\alpha,\beta}_{\ell,m}(\underline{x})\partial_{\underline{x}}\left[\omega_{\alpha-\ell,\beta-\ell}(\underline{x})\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\partial_{\underline{x}}(\omega_{\alpha-\ell,\beta-\ell}(\underline{x})) =& (\alpha-\ell)\omega_{\alpha-\ell-1,\beta-\ell}(\underline{x}) - (\beta-\ell)\omega_{\alpha-\ell,\beta-\ell-1}(\underline{x}) \\ &- \frac{m-1}{2\underline{x}}[\omega_{\alpha-\ell,\beta-\ell}(\underline{x}) - \omega_{\alpha-\ell,\beta-\ell}(\underline{-x})].\end{aligned}$$

Hence,

$$\begin{aligned}\partial_{\underline{x}} F^*(t,\underline{x}) =& \partial_{\underline{x}} G^{\alpha,\beta}_{\ell,m}(\underline{x})\omega_{\alpha-\ell,\beta-\ell}(\underline{x}) \\ &+ G^{\alpha,\beta}_{\ell,m}(\underline{x})\Big[(\alpha-\ell)\omega_{\alpha-\ell-1,\beta-\ell}(\underline{x}) - (\beta-\ell)\omega_{\alpha-\ell,\beta-\ell-1}(\underline{x}) \\ &- \frac{m-1}{2\underline{x}}[\omega_{\alpha-\ell,\beta-\ell}(\underline{x}) - \omega_{\alpha-\ell,\beta-\ell}(\underline{-x})]\Big].\end{aligned}$$

Then, writing

$$
\begin{aligned}
(\partial_t+\partial_{\underline{x}})F^*(t,\underline{x}) =& G^{\alpha,\beta}_{\ell+1,m}(\underline{x})\omega_{\alpha-\ell-1,\beta-\ell-1}+\omega_{\alpha-\ell,\beta-\ell}(\underline{x})\partial_{\underline{x}}G^{\alpha,\beta}_{\ell,m}(\underline{x})\\
&+G^{\alpha,\beta}_{\ell,m}(\underline{x})\Big[(\alpha-\ell)\omega_{\alpha-\ell-1,\beta-\ell}(\underline{x})-(\beta-\ell)\omega_{\alpha-\ell,\beta-\ell-1}(\underline{x})\\
&-\frac{m-1}{2\underline{x}}[\omega_{\alpha-\ell,\beta-\ell}(\underline{x})-\omega_{\alpha-\ell,\beta-\ell}(\underline{-x})]\Big]=0,
\end{aligned}
$$

we get

$$
G^{\alpha,\beta}_{\ell+1,m}(\underline{x})=-\left[\alpha-\beta+(\alpha+\beta-2\ell)\underline{x}\right]G^{\alpha,\beta}_{\ell,m}(\underline{x})-(1-\underline{x}^2)\partial_{\underline{x}}G^{\alpha,\beta}_{\ell,m}(\underline{x}). \tag{4.1}
$$

This allows to compute $G^{\alpha,\beta}_{\ell,m}(\underline{x})$ recursively. Starting from $G^{\alpha,\beta}_{0,m}(\underline{x})=1$, we deduce that

$$
G^{\alpha,\beta}_{1,m}(\underline{x})=-\left[(\alpha-\beta)+(\alpha+\beta)\underline{x}\right].
$$

Next, For $\ell=1$,

$$
G^{\alpha,\beta}_{2,m}(\underline{x})=\left[(\alpha+\beta)(\alpha+\beta-2+m)\right]\underline{x}^2+\left[(\alpha-\beta)(2\alpha+2\beta-2)\right]\underline{x}+(\alpha-\beta)^2-m(\alpha+\beta).
$$

For $\ell=2$, we get

$$
\begin{aligned}
G^{\alpha,\beta}_{3,m}(\underline{x})=&-\left[(\alpha+\beta)(\alpha+\beta-2+m)+(\alpha+\beta-2)\right]\underline{x}^3\\
&-(\alpha-\beta)[(\alpha+\beta)(\alpha+\beta-2+m)+(2\alpha+2\beta-2)(\alpha+\beta-4-m)]\underline{x}^2\\
&-[(\alpha-\beta)^2(2\alpha+2\beta-2)+(\alpha+\beta)[(\alpha-\beta)^2-m(\alpha+\beta)-2(\alpha+\beta-2+m)]\underline{x}\\
&-(\alpha-\beta)[(\alpha+\beta)^2-m(\alpha+\beta)-m(2\alpha+2\beta-2)].
\end{aligned}
$$

Proposition 4.1. *The generalized Clifford-Jacobi polynomials can be written as*

$$
G^{\alpha,\beta}_{\ell,m}(\underline{x})=(-1)^{\ell}\omega_{\ell-\alpha,\ell-\beta}(\underline{x})\partial^{\ell}_{\underline{x}}\omega_{\alpha,\beta}(\underline{x}).
$$

Proof. The proof is based on the recurrence principle. We will sketch it just for the first two steps. The rest is left to the reader. For $\ell=1$, we have

$$
\begin{aligned}
\partial_{\underline{x}}\omega_{\alpha,\beta}(\underline{x}) =&\alpha\,\omega_{\alpha-1,\beta}(\underline{x})-\beta\omega_{\alpha,\beta-1}(\underline{x})-\frac{m-1}{2\underline{x}}[\omega_{\alpha,\beta}(\underline{x})-\omega_{\alpha,\beta}(-\underline{x})]\\
=&(-1)\omega_{\alpha-1,\beta-1}(\underline{x})\left[(\alpha-\beta)+(\alpha+\beta)\underline{x}\right]\\
=&(-1)\omega_{\alpha-1,\beta-1}(\underline{x})G^{\alpha,\beta}_{1,m}(\underline{x}).
\end{aligned}
$$

Which means that

$$
G^{\alpha,\beta}_{1,m}=(-1)\,\omega_{1-\alpha,1-\beta}(\underline{x})\partial_{\underline{x}}\,\omega_{\alpha,\beta}(\underline{x}).
$$

For $\ell = 2$, we shall get

$$G_{2,m}^{\alpha,\beta} = (-1)^2 \omega_{2-\alpha,2-\beta}(\underline{x}) \partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}). \tag{4.2}$$

Indeed,

$$\begin{aligned}
\partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}) =& \partial_{\underline{x}} \left(\alpha\, \omega_{\alpha-1,\beta}(\underline{x}) - \beta \omega_{\alpha,\beta-1}(\underline{x}) - \frac{m-1}{2\underline{x}} [\omega_{\alpha,\beta}(\underline{x}) - \omega_{\alpha,\beta}(-\underline{x})] \right) \\
=& \partial_{\underline{x}} \left(\alpha\, \omega_{\alpha-1,\beta}(\underline{x}) - \beta \omega_{\alpha,\beta-1}(\underline{x}) - (m-1) \sum_{n=0}^{\infty} a_{2n+1} \underline{x}^{2n} \right) \\
=& \alpha\, \partial_{\underline{x}}(\omega_{\alpha-1,\beta}(\underline{x})) - \beta \partial_{\underline{x}}(\omega_{\alpha,\beta-1}(\underline{x})) - (m-1) \partial_{\underline{x}} \left(\sum_{n=0}^{\infty} a_{2n+1} \underline{x}^{2n} \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}) =& \alpha \left[(\alpha-1) \omega_{\alpha-2,\beta}(\underline{x}) - \beta \omega_{\alpha-1,\beta-1}(\underline{x}) - \frac{m-1}{2\underline{x}} (\omega_{\alpha-1,\beta}(\underline{x}) - \omega_{\alpha-1,\beta}(-\underline{x})) \right] \\
& - \beta \left[\alpha \omega_{\alpha-1,\beta-1}(\underline{x}) - (\beta-1) \omega_{\alpha,\beta-2}(\underline{x}) - \frac{m-1}{2\underline{x}} (\omega_{\alpha,\beta-1}(\underline{x}) - \omega_{\alpha,\beta-1}(-\underline{x})) \right] \\
& + \frac{m-1}{\underline{x}^2} (\omega_{\alpha,\beta}(\underline{x}) - \omega_{\alpha,\beta}(\underline{-x})).
\end{aligned}$$

Otherwise,

$$\begin{aligned}
\partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}) =& (-1)^2 \omega_{\alpha-2,\beta-2}(\underline{x}) \Big[\alpha(\alpha-1)(1+\underline{x})^2 - \alpha\beta(1-\underline{x}^2) \\
& - \alpha \frac{m-1}{2\underline{x}} [(1-\underline{x})(1+\underline{x})^2 - (1-\underline{x})^2(1+\underline{x})] \\
& - \alpha\beta(1-\underline{x}^2) + \beta(\beta-1)(1-\underline{x})^2 \\
& + \beta \frac{m-1}{2\underline{x}} [(1-\underline{x})^2(1+\underline{x}) - (1-\underline{x})(1+\underline{x})^2] \\
& + \frac{m-1}{\underline{x}^2} [(1-\underline{x})^2(1+\underline{x})^2 - (1-\underline{x})^2(1+\underline{x})^2] \Big].
\end{aligned}$$

So, finally,

$$\begin{aligned}
\partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}) =& (-1)^2 \omega_{\alpha-2,\beta-2}(\underline{x}) [\alpha(\alpha-1) + 2\alpha(\alpha-1)\underline{x} \\
& + \alpha(\alpha-1)\underline{x}^2 - 2\alpha\beta + 2\alpha\beta\underline{x}^2 - \alpha(m-1) \\
& + \alpha(m-1)\underline{x}^2 + \beta(\beta-1) - 2\beta(\beta-1)\underline{x} \\
& + \beta(\beta-1)\underline{x}^2 + \beta(m-1)\underline{x}^2 - \beta(m-1)].
\end{aligned}$$

By regrouping the quantities with $\underline{x}^2$ and $\underline{x}$, we get

$$\partial_{\underline{x}}^{(2)} \omega_{\alpha,\beta}(\underline{x}) = (-1)^2 \omega_{\alpha-2,\beta-2}(\underline{x})[G_{2,m}^{\alpha,\beta}(\underline{x})],$$

where $G_{2,m}^{\alpha,\beta}(\underline{x})$ is as in (4.2). □

Proposition 4.2. *Let the integral*

$$I_{\ell,t}^{\alpha,\beta} = \int_{\mathbb{R}^m} \underline{x}^\ell G_{t,m}^{\alpha+t,\beta+t}(\underline{x})\, \omega_{\alpha,\beta}(\underline{x})\, dV(\underline{x}).$$

Then, whenever $2t < 1-\alpha-\beta-m$ *and* $\ell < t$, *we have the orthogonality relation*

$$I_{\ell,t,t}^{\alpha,\beta} = 0.$$

Proof. Denote,

$$I_{\ell,t} = \int_{\mathbb{R}^m} \underline{x}^\ell \partial_{\underline{x}}^t \omega_{\alpha+t,\beta+t}(\underline{x}) dV(\underline{x}).$$

Using Stokes's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} \underline{x}^\ell\, G_{t,m}^{\alpha+t,\beta+t}(\underline{x}) \omega_{\alpha,\beta}(\underline{x}) dV(\underline{x}) \\
=&(-1)^t \int_{\mathbb{R}^m} \underline{x}^\ell \omega_{t-\alpha-t,t-\beta-t}(\underline{x}) [\partial_{\underline{x}}^t \omega_{\alpha+t,\beta+t}(\underline{x})] \omega_{\alpha,\beta}(\underline{x}) dV(\underline{x}) \\
=&(-1)^t \int_{\mathbb{R}^m} \underline{x}^\ell \partial_{\underline{x}}^t \omega_{\alpha+t,\beta+t}(\underline{x}) dV(\underline{x}) \\
=&(-1)^t \int_{\mathbb{R}^m} \underline{x}^\ell \partial_{\underline{x}} \left[\partial_{\underline{x}}^{t-1} \omega_{\alpha+t,\beta+t}(\underline{x})\right] dV(\underline{x}) \\
=&(-1)^t \left[\int_{\partial\mathbb{R}^m} \underline{x}^\ell \partial_{\underline{x}}^{t-1} \omega_{\alpha+t,\beta+t}(\underline{x}) \partial\Gamma(\underline{x}) - \int_{\mathbb{R}^m} \partial_{\underline{x}}(\underline{x}^\ell) \partial_{\underline{x}}^{t-1} \omega_{\alpha+t,\beta+t}(\underline{x}) dV(\underline{x})\right].
\end{aligned}$$

Denote now

$$\begin{aligned}
I &= \int_{\partial\mathbb{R}^m} \underline{x}^\ell \partial_{\underline{x}}^{t-1} \omega_{\alpha+t,\beta+t}(\underline{x}) \partial\Gamma(\underline{x}), \\
II &= \int_{\mathbb{R}^m} \partial_{\underline{x}}(\underline{x}^\ell) \partial_{\underline{x}}^{t-1} \omega_{\alpha+t,\beta+t}(\underline{x}) dV(\underline{x}).
\end{aligned}$$

The integral I vanishes due to the assumption $2t < 1-\alpha-\beta-m$. Now, using Lemma 4.1, we evaluate II. Indeed,

$$II = \gamma_{l,m} \int_{\mathbb{R}^m} \underline{x}^{\ell-1} \partial_{\underline{x}}^{t-1}\, \omega_{\alpha+t,\beta+t}(\underline{x})\, dV(\underline{x}) = \gamma_{l,m} I_{\ell-1,t-1}.$$

Hence we obtain

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^\ell G_{t,m}^{\alpha+t,\beta+t}(\underline{x})\omega_{\alpha,\beta}(\underline{x})dV(\underline{x}) = (-1)^{t+1}\gamma_{l,m} I_{\ell-1,t-1}\\
=&(-1)^{t+1}\gamma_{l,m}[(-1)^t\gamma_{l-1,m}I_{\ell-2,t-2}]\\
=&(-1)^{2t+1}\gamma_{l,m}\gamma_{l-1,m}I_{\ell-2,t-2}\cdots\\
=&C(m,\ell,t)I_0\\
=&0.\end{aligned}$$

The constant $C(m,\ell,t)$ above is given by

$$C(m,\ell,t) = (-1)^{ml+1}\prod_{k=0}^{m}\gamma_{k,m}.$$

We complete the proof. $\square$

One of the important concepts in wavelet theory is the orthogonality property of the dilation-translation copies of the wavelet mother. In Clifford extension this orthogonality may be inherited from the Clifford polynomials introduced as a post step yielding next the Clifford wavelets. However, it seems that this property of orthogonality does not hold for the case of Clifford Jacobi polynomials namely. The so-called Cauchy-Kowalewskaia extension of the real Jacobi weight function to the monogenic extension in the Clifford framework does not yield unfortunately to orthogonal polynomials although a recurrence relation and Rodrigues formula are already obtained. However, the Rodrigues formula together with Stokes's theorem did not lead to an orthogonality relation for the generalized Clifford-polynomials.

This led the authors in [9] to come back to a restricted case $\alpha = \beta + 1$ to overcome the problem. In this case the weight $\omega_{\alpha,\beta}(\underline{x})$ is splitted using the so-called Clifford-Heaviside functions

$$P^+ = \frac{1}{2}\Big(1 + i\frac{\underline{x}}{|x|}\Big), \quad P^- = \frac{1}{2}\Big(1 - i\frac{\underline{x}}{|x|}\Big),$$

to generate polynomials $J_{l,\beta}(\underline{x})$ satisfying an orthogonality relation of the type

$$\int_{\mathbb{R}^m} J_{l,\beta+l}(\underline{x})J_{k,\beta+k}^{\dagger}(\underline{x})(1+\underline{x})(1+|x|^2)^{\beta}\, dV(\underline{x}) = 0.$$

Besides, in [7] the authors developed a different technique that did not use the CK extension already used in quasi all the Clifford extensions of orthogonal polynomials to construct a set of Clifford-Jacobi orthogonal polynomials on the unit ball $B(0,1)$ of $\mathbb{R}^m$. The main idea consists in applying the so-called Clifford-Heaviside functions $P^{\pm}$ above to transform the orthogonality relation on the open unit ball to an orthogonality relation on the real line.

In the present case, ze did not need neither the restriction nor the decomposition applied there. Besides we may prove an orthogonality relation using Proposition 4.2 directly. We obtain immediately the following result.

Lemma 4.3.

$$\int_{\mathbb{R}^m} G_{l,m}^{\alpha+l,\beta+l}(\underline{x})(G_{k,m}^{\alpha+k,\beta+k}(\underline{x}))^{\dagger}\omega_{\alpha,\beta}(\underline{x})\, dV(\underline{x}) = 0,$$

whenever $2k < 1-\alpha-\beta-m$ *and* $\ell < k$.

5 The Generalized Clifford-Jacobi continuous wavelet transform

Now we are able to introduce a new class of wavelets relative to the generalized Clifford-Jacobi polynomials developed previously. Recall that for $0 < t < \frac{1-\alpha-\beta-m}{2}$ we have

$$\int_{\mathbb{R}^m} G_{t,m}^{\alpha+t,\beta+t}(\underline{x})\omega_{\alpha,\beta}(\underline{x})dV(\underline{x}) = 0.$$

Consequently, we get the following definition.

Definition 5.1. *The generalized Clifford-Jacobi wavelet mother is defined by*

$$\psi_{\ell,m}^{\alpha,\beta}(\underline{x}) = G_{\ell,m}^{\alpha+\ell,\beta+\ell}(\underline{x})\omega_{\alpha,\beta}(\underline{x}).$$

The wavelet $\psi_{\ell,m}^{\alpha,\beta}(\underline{x})$ have vanishing moments as is shown in the next proposition.

Proposition 5.1. *The following assertions hold.*

1. *The wavelet* $\psi_{\ell,m}^{\alpha,\beta}$ *have vanishing moments if the condition* $\forall k,\ 0 \le k \le -(m+\ell+\alpha+\beta)$ *and* $0 \le k \le \ell$ *is fulfilled. More precisely,*

$$\int_{\mathbb{R}^m} \underline{x}^k \psi_{\ell,m}^{\alpha,\beta}(\underline{x})dV(\underline{x}) = 0 \quad \textit{for}\ \ 0 \le\ k \le -(m+\ell+\alpha+\beta) \quad \textit{and} \quad 0 \le k \le \ell. \tag{5.1}$$

2. *The Clifford-Fourier transform of* $\psi_{\ell,m}^{\alpha,\beta}$ *is given by*

$$\widehat{\psi_{\ell,m}^{\alpha,\beta}}(\underline{u}) = (-i\underline{u})^{\ell} C_{p,q} e^{-ir\pi} C_{r,s} \sum_{k=0}^{N_\alpha}\sum_{n=0}^{N_\beta} C_{N_\alpha}^k C_{N_\beta}^n (-1)^k \widetilde{C_{k,q}}\widetilde{C_{n,r}} K_{k,n}^{q,r}(\underline{u}), \tag{5.2}$$

where

$$C_{p,q} = m^{-q}\frac{\Gamma(p-q)}{\Gamma(p)}, \quad \widetilde{C_{k,q}} = \frac{k!m^{k-q}}{\Gamma(k+1-q)}, \quad N_\alpha = N_{\alpha,l}^q = \alpha+\ell+q,$$

$$K_{k,n}^{q,r}(\underline{u}) = e^{i\dfrac{\pi}{2}(k+n-q-r)}\frac{(2\pi)^{\frac{m}{2}}K_m}{\Gamma(m)}\Gamma(k+n-q-r+m-1)|\underline{u}|^{q+r-k-n-m}.$$

Proof. We first observe

$$\begin{aligned}
\widehat{\psi_{\ell,m}^{\alpha,\beta}}(\underline{u}) &= \int_{\mathbb{R}^m} \psi_{\ell,m}^{\alpha,\beta}(\underline{x}) e^{-i\underline{x}.\underline{u}}\, d\underline{x} \\
&=(-1)^{\ell} \int_{\mathbb{R}^m} \partial_{\underline{x}}^{\ell} \left(\omega_{\alpha+\ell,\beta+\ell}(\underline{x})\right) e^{-i\underline{x}.\underline{u}}\, d\underline{x} \\
&=(-1)^{\ell} \int_{\mathbb{R}^m} \omega_{\alpha+\ell,\beta+\ell}(\underline{x}) e^{-i\underline{x}.\underline{u}} (i\underline{u})^{\ell}\, d\underline{x} \\
&=(-1)^{\ell}\, (i\underline{u})^{\ell} \int_{\mathbb{R}^m} \omega_{\alpha+\ell,\beta+\ell}(\underline{x})\, e^{-i\underline{x}.\underline{u}}\, d\underline{x} \\
&=(-1)^{\ell}\, (i\underline{u})^{\ell} \int_{\mathbb{R}^m} (1-\underline{x})^{\alpha+\ell}(1+\underline{x})^{\beta+\ell}\, e^{-i\underline{x}.\underline{u}}\, d\underline{x} \\
&=(-1)^{\ell} (i\underline{u})^{\ell} \widehat{\omega_{\alpha+\ell,\beta+\ell}}(\underline{u}).
\end{aligned}$$

Applying Lemma 2.1, for $b=1$, $p-q-1=\alpha+\ell$, we get

$$D_{\underline{x}}^{q}(1-\underline{x})^{p-1} = m^{q} \frac{\Gamma(p)}{\Gamma(p-q)} (1-\underline{x})^{\alpha+\ell}.$$

Therefore,

$$(1-\underline{x})^{\alpha+\ell} = C_{p,q} D_{\underline{x}}^{q}(1-\underline{x})^{p-1}.$$

Consequently,

$$\begin{aligned}
\widehat{\psi_{\ell,m}^{\alpha,\beta}}(\underline{u}) &=(-i\underline{u})^{\ell} C_{p,q} \int_{\mathbb{R}^m} D_{\underline{x}}^{q}(1-\underline{x})^{p-1}(1+\underline{x})^{\beta+\ell} e^{-i\underline{x}.\underline{u}} d\underline{x} \\
&=(-i\underline{u})^{\ell} C_{p,q} \int_{\mathbb{R}^m} D_{\underline{x}}^{q}(1-\underline{x})^{\alpha+\ell+q}(1+\underline{x})^{\beta+\ell} e^{-i\underline{x}.\underline{u}} d\underline{x}.
\end{aligned}$$

For $q=[\alpha]+1-\alpha$, it holds that $\alpha+\ell+q\in\mathbb{N}$. Hence,

$$\begin{aligned}
\widehat{\psi_{\ell,m}^{\alpha,\beta}}(\underline{u}) &=(-i\underline{u})^{\ell} C_{p,q} \sum_{k=0}^{\alpha+\ell+q} C_{\alpha+\ell+q}^{k} (-1)^{k} \int_{\mathbb{R}^m} D_{\underline{x}}^{q} \underline{x}^{k} (1+\underline{x})^{\beta+\ell} e^{-i\underline{x}.\underline{u}} d\underline{x} \\
&=(-i\underline{u})^{\ell} C_{p,q} \sum_{k=0}^{\alpha+\ell+q} C_{\alpha+\ell+q}^{k} (-1)^{k} H_{k,q} \int_{\mathbb{R}^m} \underline{x}^{k-q} (1+\underline{x})^{\beta+\ell} e^{-i\underline{x}.\underline{u}} d\underline{x} \\
&=(-i\underline{u})^{\ell} C_{p,q} \sum_{k=0}^{\alpha+\ell+q} C_{\alpha+\ell+q}^{k} (-1)^{k} H_{k,q} I_{q,\beta}^{l,k}(\underline{u}),
\end{aligned}$$

because of the fact that

$$D_{\underline{x}}^{q}(\underline{x}^{k}) = H_{k,q} x^{k-q}, \quad \text{with } \; H_{k,q} = \prod_{r=0}^{q-1} \gamma_{k-r,m}$$

and where we set

$$I_{q,\beta}^{l,k}(\underline{u}) = \int_{\mathbb{R}^m} \underline{x}^{k-q}(1+\underline{x})^{\beta+\ell} e^{-i\underline{x}.\underline{u}} d\underline{x}.$$

We now evaluate this last integral. To do it, we need again Lemma 2.1. For $s-r-1=\beta+\ell$, we get

$$(1+\underline{x})^{\beta+\ell}=e^{-ir\pi}C_{r,s}D_{\underline{x}}^{r}(1+\underline{x})^{s-1},$$

where $s-1=\beta+\ell+r$. For $r=[\beta]+1-\beta$, we get $s-1=\ell+[\beta]+1\in\mathbb{N}$. Consequently,

$$\begin{aligned}(1+\underline{x})^{\beta+\ell}=&e^{-ir\pi}C_{r,s}D_{\underline{x}}^{r}\sum_{n=0}^{\beta+\ell+r}C_{\beta+\ell+r}^{n}\,\underline{x}^{n}\\ =&e^{-ir\pi}C_{r,s}\sum_{n=0}^{\beta+\ell+r}C_{\beta+\ell+r}^{n}D_{\underline{x}}^{r}\,\underline{x}^{n}\\ =&e^{-ir\pi}C_{r,s}\sum_{n=0}^{\beta+\ell+r}C_{\beta+\ell+r}^{n}H_{n,r}\,\underline{x}^{n-r}.\end{aligned}$$

As a result,

$$I_{q,\beta}^{l,k}(\underline{u})=e^{-ir\pi}C_{r,s}\sum_{n=0}^{\beta+\ell+r}C_{\beta+\ell+r}^{n}H_{n,r}\int_{\mathbb{R}^m}\underline{x}^{k+n-q-r}e^{-i\underline{x}\underline{u}}d\underline{x}.$$

Denote now

$$K_{k,n}^{q,r}(\underline{u})=\int_{\mathbb{R}^m}\underline{x}^{k+n-q-r}e^{-i\underline{x}\underline{u}}d\underline{x}.$$

It is straightforward that

$$K_{k,n}^{q,r}(\underline{u})=(-i)^{q+r-k-n}D_{\underline{u}}^{k+n-q-r}\int_{\mathbb{R}^m}e^{-i\underline{x}\underline{u}}d\underline{x}.$$

Therefore,

$$\widehat{\psi_{\ell,m}^{\alpha,\beta}}(\underline{u})=(-i\underline{u})^{\ell}C_{p,q}e^{-ir\pi}C_{r,s}\sum_{k=0}^{N_\alpha}\sum_{n=0}^{N_\beta}C_{N_\alpha}^{k}C_{N_\beta}^{n}(-1)^{k}H_{k,q}H_{n,r}K_{k,n}^{q,r}(\underline{u}).$$

We complete the proof. □

We now evaluate the quantity $K_{k,n}^{q,r}(\underline{u})$. So denote

$$K_{k,n}^{q,r}(\underline{u})=(-i)^{q+r-k-n}D_{\underline{u}}^{k+n-q-r}\left(\int_{\mathbb{R}^m}e^{-i\underline{x}\underline{u}}d\underline{x}\right),\quad J(\underline{u})=\int_{\mathbb{R}^m}e^{-i\underline{x}\underline{u}}d\underline{x}.$$

We have the following technical lemma.

Lemma 5.1.

$$\int_{S^{m-1}}e^{-i\langle r\underline{\omega},\rho\underline{\xi}\rangle}d\underline{\xi}=\frac{(2\pi)^{\frac{m}{2}}J_{\frac{m}{2}-1}(r\rho)}{(r\rho)^{\frac{m}{2}-1}},$$

where $J_{\frac{m}{2}-1}$ is the Bessel function of first kind with order $\frac{m}{2}-1$.

By applying the spherical coordinates

$$\underline{x}=t\underline{\omega},\quad \underline{u}=\rho\underline{\xi},\quad t=|\underline{x}|,\quad \rho=|\underline{u}|,\quad \omega,\xi\in S^{m-1},$$

(S^{m-1} being the unit sphere in $\mathbb{R}^m$), we get

$$\begin{aligned} J(\underline{u}) &= \int_0^{+\infty} t^{m-1}dt\int_{S^{m-1}} e^{-it\rho\underline{\omega}\underline{\xi}}d\tau(\underline{\omega}) \\ &= \int_0^{+\infty} t^{m-1}(2\pi)^{\frac{m}{2}}\frac{J_{\frac{m}{2}-1}(t\rho)}{t^{\frac{m}{2}-1}\rho^{\frac{m}{2}-1}}dt \\ &=(2\pi)^{\frac{m}{2}}\int_0^{+\infty}\Big(\frac{t}{\rho}\Big)^{\frac{m}{2}-1}J_{\frac{m}{2}-1}(t\rho)tdt \\ &=(2\pi)^{\frac{m}{2}}\rho^{-m}\int_0^{+\infty}x^{\frac{m}{2}}J_{\frac{m}{2}-1}(x)dx. \end{aligned}$$

Denoting now

$$K_m=\int_0^{+\infty}x^{\frac{m}{2}}J_{\frac{m}{2}-1}(x)dx,$$

we obtain

$$K_{k,n}^{q,r}(\underline{u})=(-i)^{q+r-k-n}(2\pi)^{\frac{m}{2}}K_mD_\rho^{k+n-q-r}(\rho^{-m}).$$

Now, as

$$D_\rho^\nu(\rho^{-m})=\frac{\Gamma(m+\nu-1)}{\Gamma(m)}e^{i\nu\pi}\rho^{-m-\nu},$$

we get

$$K_{k,n}^{q,r}(\underline{u})=\Gamma_{q,r}^{m,n}(k)|\underline{u}|^{q+r-k-n-m},$$

where

$$\Gamma_{q,r}^{m,n}(k)=e^{i\dfrac{\pi}{2}(k+n-q-r)}\frac{(2\pi)^{\frac{m}{2}}K_m}{\Gamma(m)}\Gamma(k+n-q-r+m-1).$$

Now, we introduce the generalized Clifford-Jacobi continuous wavelet transform.

Definition 5.2. *For $a>0$ and $\underline{b}\in\mathbb{R}_m$, the $(a,\underline{b})$-copy of the wavelet mother $\psi_{\ell,m}^{\alpha,\beta}$ is defined by*

$${}_{\underline{b}}^{a}\psi_{\ell,m}^{\alpha,\beta}(\underline{x})=a^{-\frac{m}{2}}\psi_{\ell,m}^{\alpha,\beta}\Big(\frac{\underline{x}-\underline{b}}{a}\Big). \tag{5.3}$$

The generalized Clifford-Jacobi CWT of a function $f\in L_2(\mathbb{R}^m)$ is defined by

$$C_{a,\underline{b}}(f)=\langle{}_{\underline{b}}^{a}\psi_{\ell,m}^{\alpha,\beta},f\rangle.$$

Lemma 5.2. *Consider the inner product*

$$\langle C_{a,\underline{b}}(f), C_{a,\underline{b}}(g)\rangle = \frac{1}{\mathcal{A}^{\alpha,\beta}_{\ell,m}} \int_{\mathbb{R}^m}\int_0^{+\infty} C_{a,\underline{b}}(f)\overline{C_{a,\underline{b}}(g)}\frac{da}{a^{m+1}}dV(\underline{b}).$$

Then, the following Parseval formula hold

$$\langle C_{a,\underline{b}}(f), C_{a,\underline{b}}(g)\rangle = \langle f, g\rangle.$$

The following theorem guaranties the construction of the analyzed function $f \in L_2(\mathbb{R}^m)$ from its wavelet transform.

Theorem 5.1. *Let $\psi^{\alpha,\beta}_{\ell,m}$ be an analyzing wavelet as in Definition 5.1. The following assertions hold.*

1. *$\psi^{\alpha,\beta}_{\ell,m}$ is admissible in the sense that*
$$\mathcal{A}^{\alpha,\beta}_{\ell,m} = \frac{1}{\omega_m}\int_{\mathbb{R}^m}\left|\widehat{\psi^{\alpha,\beta}_{\ell,m}}(\underline{x})\right|^2 \frac{dV(\underline{x})}{|\underline{x}|^m} < +\infty,$$
where ω_m is the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.

2. *The analyzed function $f \in L_2(\mathbb{R}^m)$ may be reconstructed in the L_2-sense by*
$$f(x) = \frac{1}{\mathcal{A}^{\alpha,\beta}_{\ell,m}}\int_{a>0}\int_{b\in\mathbb{R}^m} C_{a,\underline{b}}(f)\psi^{\alpha,\beta}_{\ell,m}\left(\frac{\underline{x}-\underline{b}}{a}\right)\frac{da\, dV(\underline{b})}{a^{m+1}}.$$

Proof. Assertion 1 is based on the asymptotic behaviour of Bessel functions (analogue result is already checked in [5]). For Assertion 2, using the Clifford Fourier transform, we observe that

$$C_{a,\underline{b}}(f)(\underline{b}) = a^{\frac{m}{2}}\widetilde{\widehat{\widehat{f}(\underline{.})\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{.})}}(\underline{b}),$$

where $\widetilde{h}(\underline{u}) = h(-\underline{u})$, $\forall h$. Thus,

$$C_{a,\underline{b}}(f)\overline{C_{a,\underline{b}}(g)} = \widehat{\left(\widehat{f}(\underline{.})a^{\frac{m}{2}}\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{.})\right)}(-\underline{b})\overline{\widehat{\left(\widehat{g}(\underline{.})a^{\frac{m}{2}}\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{.})\right)}(-\underline{b})}.$$

Consequently,

$$\begin{aligned}
\langle C_{a,\underline{b}}(f), C_{a,\underline{b}}(g)\rangle &= \int_{a>0}\int_{\mathbb{R}^m}\widehat{f}(\underline{.})a^{\frac{m}{2}}\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{.})\,\overline{\widehat{g}(\underline{.})a^{\frac{m}{2}}\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{.})}\frac{da\, dV(\underline{b})}{a^{m+1}}\\
&= \int_{a>0}\int_{\mathbb{R}^m}\widehat{f}(\underline{b})\overline{\widehat{g}(\underline{b})}\frac{a^m|\widehat{\psi^{\alpha,\beta}_{\ell,m}}(a\underline{b})|^2}{a^{m+1}}\,da\, dV(\underline{b})\\
&=\mathcal{A}^{\alpha,\beta}_{\ell,m}\int_{\omega\in\mathbb{R}^m}\widehat{g}(b)\overline{\widehat{g}(\underline{b})}dV(\underline{b})\\
&=\mathcal{A}^{\alpha,\beta}_{\ell,m}\langle\widehat{f},\widehat{g}\rangle\\
&=\langle f, g\rangle.
\end{aligned}$$

Thus, we complete the proof. □

6 Conclusions

In the present work, a new class of Clifford-Jacobi polynomials has been developed relatively to a general Jacobi weight. Next, new associated wavelets in the Clifford context have been introduced using fractional calculus such as derivatives and fractional Fourier transform on Clifford algebras. The new classes of polynomials as well as wavelets extend the works of Brackx and his collaborators in [5,6,8,9] and [10].

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