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Weighted Norm Inequalities for Marcinkiewicz Integrals with Non-Smooth Kernels on Spaces of Homogeneous Type

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Abstract. In this article, we obtain some weighted estimates for Marcinkiewicz integrals with non-smooth kernels on spaces of homogeneous type. The weight w considered here belongs to the Muckenhoupt's class A_∞. Moreover, weighted estimates for commutators of BMO functions and Marcinkiewicz integrals are also given.

Key Words: Commutators, Muckenhoupt weights, Marcinkiewicz integrals, Singular integrals, Sharp maximal functions, BMO functions, Young functions, Luxemburg norm, Spaces of homogeneous type.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, endowed with a metric distance d on $\mathcal{X} \times \mathcal{X}$ satisfying

$$d(x,z) \le \kappa \left(d(x,y) + d(y,z) \right) \text{ for some fixed constant } \kappa \ge 1 \text{ and for all } x,y,z \in \mathcal{X}, \tag{1.1}$$

and a regular Borel measure μ on $\mathcal{X}$ such that the doubling property

$$\mu(B(x;2r)) \le C\mu(B(x;r)) < \infty \tag{1.2}$$

holds for some fixed constant $C \ge 1$, for all $x \in \mathcal{X}$ and for all $r > 0$, where $B(x;r) = \{y \in \mathcal{X} : d(x,y) < r\}$. The above property implies that there exist some fixed constants $C \ge 1$, $n > 0$ such that

$$\mu(B(x;\lambda r)) \le C\lambda^n \mu(B(x;r)), \tag{1.3}$$

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uniformly for all $\lambda \geq 1$, $x \in \mathcal{X}$, and $r > 0$. The parameter n measures the "dimension" of the space $\mathcal{X}$. There also exist constants C, N ($C \geq 1, 0 \leq N \leq n$) such that

$$\mu(B(y;r)) \leq C \left(1 + \frac{d(x,y)}{r}\right)^N \mu(B(x;r)) \tag{1.4}$$

uniformly for all $x, y \in \mathcal{X}$ and all $r > 0$. The reader can find more information on this subject in [2,3].

Let T be a bounded linear operator on $L^2(\mathcal{X})$ with an associated kernel $K(x,y)$ in the sense that

$$Tf(x) = \int_{\mathcal{X}} K(x,y) f(y) d\mu(y), \tag{1.5}$$

where f is a continuous function with compact support, $x \notin \text{supp} f$; and $K(x,y)$ is a measurable function defined on $(\mathcal{X} \times \mathcal{X}) \setminus \Delta$ with $\Delta = \{(x,x) : x \in \mathcal{X}\}$.

The authors in [4,6] assumed that there exists a class of operators A_t $(t > 0)$ which can be represented by the kernels $a_t(x,y)$ in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y) u(y) d\mu(y) \quad \text{for every function } \ u \in L^1(\mathcal{X}) \cap L^2(\mathcal{X}).$$

Moreover, the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)| \leq h_t(x,y) \quad \text{for all } \ x, y \in \mathcal{X}, \tag{1.6a}$$

$$\text{where } h_t(x,y) = (\mu(B(x;t^{1/m})))^{-1} s((d(x,y))^m t^{-1}) \text{ for some positive constant } m. \tag{1.6b}$$

Here s is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\sigma} s(r^m) = 0 \tag{1.7}$$

for some $\sigma > N$, where n and N appear in (1.3) and (1.4) respectively.

Remark 1.1. The functions h_t above satisfy the following properties (see [5,6]):

1) There exist positive constants C_1 and C_2 such that

$$C_1 \leq \int_{\mathcal{X}} h_t(x,y) d\mu(x) \leq C_2 \ \text{ uniformly in } \ t \ \text{ and } \ dy.$$

2) There exists a positive constant C such that

$$\int_{\mathcal{X}} h_t(x,y) |f(x)| d\mu(x) \leq C \mathcal{M} f(y) \quad \text{and} \quad \int_{\mathcal{X}} h_t(x,y) |f(y)| d\mu(y) \leq C \mathcal{M} f(x).$$

Here $\mathcal{M} f(x)$, the Hardy-Littlewood maximal function, is defined by

$$\mathcal{M} f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \right\},$$

where the supremum is taken over all balls B containing x.

The class of operators A_t plays the role of approximation to the identity. The existence of such a class of operators A_t was verified in [4]. Now let $\{A_t\}_{t\in(0,\infty)}$ and $\{B_t\}_{t\in(0,\infty)}$ be two classes of operators which satisfy (1.6a)–(1.7). Denote by $K(x,y)-K_t(x,y)$ the kernels of the operators $(T-TB_t)$, and $K(x,y)-K^t(x,y)$ as the kernels of $(T-A_tT)$ respectively. We state below a list of assumptions which lead to interesting results:

(i) T is a bounded linear operator from $L^2(\mathcal{X})$ to $L^2(\mathcal{X})$;

(ii) There exist positive constants c_1 and C_A such that

$$\int_{d(x,y)\geq c_1 t^{1/m}} |K(x,y)-K_t(x,y)|d\mu(x) \leq C_A \quad \text{for all } \ y\in\mathcal{X}.$$

(iii) There exist positive constants c_2 and C_A such that

$$\int_{d(x,y)\geq c_2 t^{1/m}} |K(x,y)-K^t(x,y)|d\mu(y) \leq C_A \quad \text{for all } \ x\in\mathcal{X}.$$

(iv) There exist positive constants c_2, c_4 and β such that

$$|K(x,y)-K^t(x,y)| \leq \frac{c_4}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{[d(x,y)]^\beta}, \quad \text{whenever } \ d(x,y)\geq c_2 t^{1/m}.$$

Using Assumptions (i), (ii) and (iii), Duong and McIntosh [4] obtained the L^p- boundedness of the singular integral operator T. Afterward, Martell [13] extended their results to weighted spaces with weights $w\in A_p$, under hypotheses (i), (ii) and (iv). For further interesting results about the singular integral and its commutators, the reader may view [6,10,14,17] among many other excellent references.

The purpose of this paper is to obtain weighted L^p- estimates ($1\leq p<\infty$) for the Marcinkiewicz integral

$$\nu(f)(x)=\left\{\int_0^\infty \left|\int_{d(x,y)<\tau} K(x,y)f(y)d\mu(y)\right|^2 \frac{d\tau}{\tau^3}\right\}^{1/2}, \tag{1.8}$$

where the weight $w\in A_\infty$ and the kernel $K(x,y)$ satisfies similar conditions as (i), (ii) and (iv) above. In addition, we also investigate weighted L^p- estimates ($1<p<\infty$) for the commutator of BMO function and Marcinkiewicz integral

$$\nu_b(f)(x)=\left\{\int_0^\infty \left|\int_{d(x,y)<\tau} (b(x)-b(y))K(x,y)f(y)d\mu(y)\right|^2 \frac{d\tau}{\tau^3}\right\}^{1/2}. \tag{1.9}$$

For background information about Marcinkiewicz integral, please see [12,19]. Our main theorems are given in Sections 3 and 4. In Section 5, we obtain the weighted estimates ($w\in A_\infty$) of $\nu(f)$ and $\nu_b(f)$ for a special case when $\mathcal{X}=\mathbb{R}^n$ and K is a convolution kernel, which satisfies some classical assumptions. For the rest of this paper, the letter C denotes a positive constant which may vary at each occurrence; however, it is independent of any essential variable.

2 Background

2.1 Approximation of the identity

Denote by $L_b^\infty(\mathcal{X})$ the set of all functions in $L^\infty(\mathcal{X})$ with bounded support. Note that $L_b^\infty(\mathcal{X})$ is dense in $L^p(\mathcal{X})$ for $p \in (0,\infty)$ (see [8,9] for example). For $f \in L_b^\infty(\mathcal{X})$, define the linear operator F by

$$F(f)(x,\tau) = \int_{d(x,y)<\tau} K(x,y) f(y) d\mu(y),$$

where $K(x,y)$ is a measurable function defined on $(\mathcal{X}\times\mathcal{X})\diagdown\Delta$ with $\Delta = \{(x,x) : x \in \mathcal{X}\}$. Define the Marcinkiewicz integral $\nu(f)$ by

$$\nu(f)(x) = \left\{ \int_0^\infty |F(f)(x,\tau)|^2 \frac{d\tau}{\tau^3} \right\}^{1/2}.$$

Denote

$$\mathcal{B}_1 = \mathbb{C} \quad \text{and} \quad \mathcal{B}_2 = L^2\Big(\mathbb{R}^+; \frac{d\tau}{\tau^3}\Big),$$

where $\mathbb{R}^+ = (0,\infty)$. Then

$$\nu(f)(x) = ||F(f)(x,\cdot)||_{\mathcal{B}_2},$$

so that

$$||\nu(f)||_{L^p(\mathcal{X})} = ||\|F(f)\|_{\mathcal{B}_2}||_{L^p(\mathcal{X})} := ||F(f)||_{L^p(\mathcal{X};\mathcal{B}_2)}.$$

Note also that $L^p(\mathcal{X};\mathcal{B}_1) = L^p(\mathcal{X})$.

In the sequel, we assume the existence of two classes of operators $\{A_t\}_{t\in(0,\infty)}$ and $\{B_t\}_{t\in(0,\infty)}$, both of which can be represented by kernels $a_t(x,y)$ and $b_t(x,y)$ respectively in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y) u(y) d\mu(y) \quad \text{for } u \in L^1(\mathcal{X}) \cap L^r(\mathcal{X}) \quad \text{and for some } r > 1,$$

and similar definition for B_t. Moreover, both kernels $a_t(x,y)$ and $b_t(x,y)$ are assumed to satisfy inequalities (1.6a)–(1.7). Let $K_t(x,y)$ and $K^t(x,y)$ represent the kernels of the operators FB_t and A_tF $(t>0)$ respectively. We may assume that both FB_t and A_tF have integral forms

$$(FB_t)(f)(x,\tau) = \int_{d(x,y)<\tau} K_t(x,y) f(y) d\mu(y), \tag{2.1a}$$

$$(A_tF)(f)(x,\tau) = \int_{d(x,y)<\tau} K^t(x,y) f(y) d\mu(y). \tag{2.1b}$$

To see this, consider the kernel b_t of the operator B_t defined in [4] by

$$b_t(y,z) = \chi_{B(z;t^{1/m})}(y)[\mu(B(z;t^{1/m}))]^{-1}.$$

Now, let $B_{t,\tau}$ be the operator whose kernel $b_{t,\tau}(y,z)$ is defined by

$$b_{t,\tau}(y,z)=\chi_{B(z;\tau)}(y)b_t(y,z).$$

Then $|b_{t,\tau}(y,z)|\le|b_t(y,z)|\le h_t(y,z)$ for all $t,\tau>0$ and $y,z\in\mathcal{X}$. Moreover,

$$\begin{aligned}\|(FB_{t,\tau})(f)(x,\cdot)\|_{\mathcal{B}_2}=&\left\|\int_{d(x,z)<2\kappa\tau}K_t(x,z)f(z)d\mu(z)\right\|_{\mathcal{B}_2}\\ =&2\kappa\left\|\int_{d(x,z)<\tau}K_t(x,z)f(z)d\mu(z)\right\|_{\mathcal{B}_2},\end{aligned}$$

where κ appears in (1.1). Similarly, if we let $A_{t,\tau}$ be the operator whose kernel $a_{t,\tau}(x,y)$ is given by

$$a_{t,\tau}(x,y)=\chi_{B(x;\tau)}(y)a_t(x,y)=\chi_{B(x;\tau)}(y)\chi_{B(x;t^{1/m})}(y)[\mu(B(x;t^{1/m}))]^{-1},$$

then the above equation also holds for $\|(A_{t,\tau}F)(f)(x,\cdot)\|_{\mathcal{B}_2}$ (with $K^t(x,z)$ instead of $K_t(x,z)$). Therefore, for simplicity, we will assume that (2.1) holds true; and we will work with the operators A_t and B_t for the rest of this paper. Moreover, we will need the following assumptions for our theorems:

(a) ν is a bounded operator from $L^r(\mathcal{X})$ to $L^r(\mathcal{X})$ with the bound C_r for some $r>1$;

(b) there exist positive constants c_1 and C_A such that

$$\int_{d(x,y)\ge c_1t^{1/m}}\frac{|K(x,y)-K_t(x,y)|}{d(x,y)}d\mu(x)\le C_A\quad\text{for all }\ y\in\mathcal{X};$$

($\bar{\text{b}}$) there exist positive constants c_1 and c_3 such that

$$\frac{|K(x,y)-K_t(x,y)|}{d(x,y)}\le\frac{c_3}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^{\beta}}\quad\text{whenever }\ d(x,y)\ge c_1t^{1/m}.$$

(c) there exist positive constants c_2 and c_4 such that

$$\frac{|K(x,y)-K^t(x,y)|}{d(x,y)}\le\frac{c_4}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^{\beta}}\quad\text{whenever }\ d(x,y)\ge c_2t^{1/m}.$$

Remark 2.1. Note that hypothesis ($\bar{\text{b}}$) implies hypothesis (b). Assumption ($\bar{\text{b}}$) will be used to prove weak type $(1,1)$ inequality for the Marcinkiewicz integral. The above assumptions may not look natural in comparison with the classical case. In the last section, we will give an example about a specific kernel K which satisfies all of these assumptions.

2.2 Luxemburg norm

A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex, increasing and satisfying $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Its complementary function $\psi : [0,\infty) \to [0,\infty)$ is defined by

$$\psi(y) = \sup\{xy - \varphi(x) : x \geq 0\}, \quad y \in [0,\infty).$$

Observe that the pair (φ,ψ) satisfies Young's inequality

$$xy \leq \varphi(x) + \psi(y), \quad x,y \in [0,\infty). \tag{2.2}$$

We define the $\varphi-$average of a function f over a ball B by means of the Luxemburg norm

$$\|f\|_{\varphi,B} = \inf\left\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \varphi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

An application of Young's inequality (2.2) yields the following generalized Hölder inequality

$$\frac{1}{\mu(B)}\int_B |f(x)g(x)| d\mu(x) \leq 2\|f\|_{\varphi,B}\|g\|_{\psi,B}. \tag{2.3}$$

See [18] for more information on this topic. The Young functions considered in this article are $\varphi(t) = t\left(\log(e+t)\right)^k := t\log^k(e+t)$, $k \in \mathbb{N}$. Note that the $\varphi-$average of a function g over a ball B is

$$\|g\|_{\varphi,B} := \|g\|_{L(\log L)^k,B} = \inf\left\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \frac{|g(x)|}{\lambda}\log^k\left(e + \frac{|g(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

The maximal operators $M_\varphi := M_{L(\log L)^k}$ associated to φ are defined by

$$M_{L(\log L)^k}(g)(x) = \sup_{B\ni x}\left\{\|g\|_{L(\log L)^k,B}\right\}.$$

The complementary Young functions are $\psi(t) \cong \exp(t^{1/k})$ (see [15]). We define the $\psi-$average of a function f over a ball B to be

$$\|f\|_{\psi,B} := \|f\|_{\exp(L^{1/k}),B} = \inf\left\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \exp\left(\left(\frac{|f(x)|}{\lambda}\right)^{1/k}\right) d\mu(x) \leq 2\right\}.$$

Again, by Young's inequality (2.2), we see that

$$\frac{1}{\mu(B)}\int_B |f(x)g(x)| d\mu(x) \leq 3\|f\|_{\exp(L^{1/k}),B}\|g\|_{L(\log L)^k,B}, \quad (k \in \mathbb{N}). \tag{2.4}$$

It follows from [15,16] that there exists a constant $C_k > 1$ such that, for any non-negative integer k,

$$C_k^{-1}M_{L(\log L)^k}f(x) \leq \mathcal{M}^{k+1}f(x) \leq C_k M_{L(\log L)^k}f(x), \tag{2.5}$$

where $\mathcal{M}^k$ stands for the Hardy-Littlewood maximal operator iterated k times. Recall that the definition of the Hardy-Littlewood maximal operator $\mathcal{M}$ was given in Remark 1.1.

2.3 Sharp maximal operator

For $k \in \mathbb{N} \cup \{0\}$, we define the sharp maximal operator $M^{\sharp}_{A,L(\log L)^k}$ by

$$M^{\sharp}_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) = \sup_{B \ni x} \left\{ ||||(g - A_{t_B} g)(\cdot)||_{\mathcal{B}_2}||_{L(\log L)^k, B} \right\}.$$

Observe that when $k = 0$,

$$M_{L(\log L)^k}(f)(x) = \mathcal{M}(f)(x) \quad \text{and} \quad M^{\sharp}_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) = M^{\sharp}_{A}(||g(\cdot)||_{\mathcal{B}_2})(x),$$

where

$$M^{\sharp}_{A}(||g(\cdot)||_{\mathcal{B}_2})(x) := \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B ||(g - A_{t_B} g)(y, \cdot)||_{\mathcal{B}_2} d\mu(y) \right\}.$$

2.4 Muckenhoupt weights

Recall that a weight w belongs to $A_\infty(\mathcal{X})$ if there exist positive constants C_w and δ_w such that, for any ball $B \subset \mathcal{X}$ and any measurable subset $E \subset B$,

$$\frac{w(E)}{w(B)} \le C_w \left(\frac{\mu(E)}{\mu(B)} \right)^{\delta_w}. \tag{2.6}$$

For more information on topics of weights, the reader may view [7,20].

2.5 Preliminary lemmas and theorem

The following lemmas and theorem are necessary for the proof of our theorems.

Lemma 2.1 ([1], Covering Lemma). *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. Let $\mathfrak{B} = \{B_\alpha : \alpha \in \Gamma\}$ be a family of balls such that the set $E = \bigcup_{\alpha \in \Gamma} B_\alpha$ is measurable and $\mu(E) < \infty$. Then there exists a disjoint sequence $\{B_i\} = \{B(x_i; r_i)\} \subset \mathfrak{B}$ such that $E \subset \bigcup_i B(x_i; c_8 r_i)$ with a constant c_8 depending only on κ (which appears in (1.1)). Moreover, every $B \in \mathfrak{B}$ is contained in some $B(x_i; c_8 r_i)$.*

Lemma 2.2 ([10]). *Let $k \in \mathbb{N} \cup \{0\}$. Let S be a sub-linear operator which is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$ and from $L^{p_o}(\mathcal{X})$ to $L^{p_o,\infty}(\mathcal{X})$ for some $p_o \in (1, \infty)$. Given any two measurable sets V_1 and V_2 with $\mu(V_1) \le \mu(V_2) < \infty$, and function f supported on V_1, there exists a constant $C_k > 0$ such that*

$$||Sf||_{L(\log L)^k, V_2} \le C_k ||f||_{L(\log L)^{k+1}, V_1}.$$

Lemma 2.3 ([10,13]). *Let $g \in L^\infty_b(\mathcal{X}; \mathcal{B}_2)$, where $\mathcal{B}_2 = L^2(\mathbb{R}^+; \frac{d\tau}{\tau^3})$. Given $\lambda > 0$ and a ball B_0 such that there exists $x_0 \in B_0$ with $\mathcal{M}_{L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x_0) \le \lambda$. Then for every $\eta \in (0,1)$, there exists $\gamma > 0$ (independent of λ, B_0, g, x_0) such that*

$$\mu\left(\left\{ x \in B_0 : \mathcal{M}_{L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) > A\lambda, M^{\sharp}_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) \le \gamma\lambda \right\} \right)$$
$$\le \eta \mu(B_0),$$

where $A>1$ is a fixed constant which only depends on the space $\mathcal{X}$ and the approximation of the identity $\{A_t, t>0\}$; and γ only depends on η.

Moreover, if $w \in A_\infty$, then by (2.6)

$$w\left(\left\{x \in B_0 : \mathcal{M}_{L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) > A\lambda, M^\sharp_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})(x) \le \gamma\lambda\right\}\right) \le C_w \eta^{\delta_w} w(B_0).$$

Lemma 2.4 ([10,13])**.** *Let $k \in \mathbb{N}\cup\{0\}$, $0<p<\infty$, $w \in A_\infty$, and $g \in L^{p_o}(\mathcal{X};\mathcal{B}_2)$ for some $p_o \in (1,\infty)$. Assume that $\mathcal{M}(||g||_{\mathcal{B}_2}) \in L^p(\mathcal{X},w)$. Then there exists a positive constant C only depending on p, C_w and δ_w such that*

$$\left|\left|\mathcal{M}_{L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})\right|\right|_{L^p(\mathcal{X},w)} \le C\left|\left|M^\sharp_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})\right|\right|_{L^p(\mathcal{X},w)}, \qquad \text{if } \mu(\mathcal{X})=\infty,$$

$$\begin{aligned}&\left|\left|\mathcal{M}_{L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})\right|\right|_{L^p(\mathcal{X},w)}\\ \le\, & C\left|\left|M^\sharp_{A,L(\log L)^k}(||g(\cdot)||_{\mathcal{B}_2})\right|\right|_{L^p(\mathcal{X},w)} + C[w(\mathcal{X})]^{1/p}\, ||||g(\cdot)||_{\mathcal{B}_2}||_{L(\log L)^k,\mathcal{X}}, \quad \text{if } \mu(\mathcal{X})<\infty.\end{aligned}$$

Theorem 2.1 ([11])**.** *Assume that hypotheses* (a), (b) *and* (c) *(as stated in previous sub-section) hold true. Then the operator ν has well-defined extensions on $L^p(\mathcal{X})$ for $1\le p<\infty$. Moreover, there exist positive constants $C_{\mathcal{X}}$ and C (where C depends on p, $C_{\mathcal{X}}$, C_A and C_r) such that*

$$\begin{aligned}&||\nu(f)||_{L^{1,\infty}(\mathcal{X})} \le C_{\mathcal{X}}(C_A + C_r)||f||_{L^1(\mathcal{X})},\\ &||\nu(f)||_{L^p(\mathcal{X})} \le C||f||_{L^p(\mathcal{X})} \quad \text{for } 1<p<\infty.\end{aligned}$$

Lemmas 2.3 and 2.4 are modified versions of Lemma 2.6 [10] and Theorem 2.1 [10] respectively (see also [13]). The proofs of these lemmas are essentially similar to the proofs of those in [10]. We therefore omit the proofs of these lemmas. Note that Lemmas 2.3 and 2.4 still hold true even if $\mathcal{B}_2$ is replaced by arbitrary complex Banach space $\mathcal{B}$, provided that the following inequality holds true:

$$\begin{aligned}||A_t g(x)||_{\mathcal{B}} &= \left|\left|\int_{\mathcal{X}} a_t(x,y)g(y)d\mu(y)\right|\right|_{\mathcal{B}}\\ &\le \int_{\mathcal{X}} ||a_t(x,y)||_{\mathcal{B}\to\mathcal{B}}||g(y)||_{\mathcal{B}}d\mu(y)\\ &\le \int_{\mathcal{X}} h_t(x,y)||g(y)||_{\mathcal{B}}d\mu(y).\end{aligned}$$

3 Marcinkiewicz Integrals

Theorem 3.1. *Let $k\in\mathbb{N}$, $p\in(0,\infty)$ and $w\in A_\infty$. Assume that hypotheses* (a), (b) *and* (c) *hold true. Then there exists a positive constant C depending on k, p, C_w and δ_w such that, for any $f \in L^\infty_b(\mathcal{X})$,*

$$\int_{\mathcal{X}}\left(\mathcal{M}^k(\nu f)(x)\right)^p w(x)d\mu(x) \le C\int_{\mathcal{X}}\left(\mathcal{M}^{k+1}(f)(x)\right)^p w(x)d\mu(x).$$

In particular, if $w \in A_p$ for some $p \in (1,\infty)$, then

$$||\nu f||_{L^p(\mathcal{X},w)} \le C||f||_{L^p(\mathcal{X},w)}.$$

In order to prove the theorem, we need the following lemma.

Lemma 3.1. *Let k be a non-negative integer. Assume that hypotheses* (a), (b) *and* (c) *hold true. Then there exists a constant $C > 0$ such that*

$$M^{\sharp}_{A,L(\log L)^k}(||F(f)(\cdot)||_{\mathcal{B}_2})(x) \le C\mathcal{M}_{L(\log L)^{k+1}} f(x), \quad \forall x \in \mathcal{X}.$$

Proof. Recall that $\nu(f)(x) = ||F(f)(x,\cdot)||_{\mathcal{B}_2}$. Take a ball B which contains $x \in \mathcal{X}$. Let $t_B = r_B^m$, where r_B^m is the radius of the ball B. We write $f = f\chi_{c_7B} + f\chi_{(c_7B)^c} =: f_1 + f_2$, where $c_7 = \kappa(c_2+3)$ (c_2 appears in hypothesis (c)). By Theorem 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\Big\| ||F(f_1)||_{\mathcal{B}_2} \Big\|_{L(\log L)^k,B} = ||\nu(f_1)||_{L(\log L)^k,B} \\
\le & C\,||\nu(f_1)||_{L(\log L)^k,c_7B} \le C\,||f_1||_{L(\log L)^{k+1},c_7B}\,.
\end{aligned} \tag{3.1}$$

For $y \in B$,

$$\begin{aligned}
&||A_{t_B}F(f_1)(y,\cdot)||_{\mathcal{B}_2} \le \int_{\mathcal{X}} h_{t_B}(y,z)||F(f_1)(z,\cdot)||_{\mathcal{B}_2}d\mu(z) \\
\le & \frac{C}{\mu(c_7B)}\int_{c_7B}||F(f_1)(z,\cdot)||_{\mathcal{B}_2}d\mu(z) \\
&+C\sum_{j=0}^{\infty}2^{(j+1)n}s(2^{jm})\frac{1}{\mu(2^{j+1}c_7B)}\int_{2^{j+1}c_7B\setminus 2^jc_7B}||F(f_1)(z,\cdot)||_{\mathcal{B}_2}d\mu(z) \\
\le & C||\nu(f_1)||_{L(\log L)^k,c_7B}+C\sum_{j=0}^{\infty}2^{(j+1)n}s(2^{jm})||\nu(f_1)||_{L(\log L)^k,2^{j+1}c_7B} \\
\le & C||f_1||_{L(\log L)^{k+1},c_7B}+C\sum_{j=0}^{\infty}2^{(j+1)n}s(2^{jm})||f_1||_{L(\log L)^{k+1},c_7B} \\
\le & C||f_1||_{L(\log L)^{k+1},c_7B}.
\end{aligned} \tag{3.2}$$

The second inequality holds true since

$$h_{t_B}(y,z) \le \frac{C}{\mu(B)} \le \frac{C}{\mu(c_7B)},$$

when $y \in B$ and $z \in c_7B$; and

$$h_{t_B}(y,z) \le \frac{C2^{(j+1)n}s(2^{jm})}{\mu(2^{j+1}c_7B)},$$

when $y \in B$ and $z \in 2^{j+1}c_7B \setminus 2^jc_7B$. The fourth inequality follows from Theorem 2.1 and Lemma 2.2. Inequality (3.2) implies that

$$\Big\| ||A_{t_B}F(f_1)(y,\cdot)||_{\mathcal{B}_2} \Big\|_{L(\log L)^k,B} \le C||f_1||_{L(\log L)^{k+1},c_7B}. \tag{3.3}$$

Combining the estimates in (3.1) and (3.3), and taking the supremum over all balls B containing x yields

$$\begin{aligned} &M^\sharp_{A,L(\log L)^k}(||F(f_1)(\cdot)||_{\mathcal{B}_2})(x) \\ =&\sup_{B\ni x}\Big\{ ||||(F-A_{t_B}F)(f_1)||_{\mathcal{B}_2}||_{L(\log L)^k,B}\Big\} \\ \le& CM_{L(\log L)^{k+1}}f(x). \end{aligned} \tag{3.4}$$

It remains to estimate $M^\sharp_{A,L(\log L)^k}(||F(f_2)(\cdot)||_{\mathcal{B}_2})(x)$. By hypothesis (c), we have

$$\begin{aligned} &||(F-A_{t_B}F)(f_2)(y,\cdot)||_{\mathcal{B}_2} \\ =&\left(\int_0^\infty \left|\int_{d(y,z)<\tau}[K(y,z)-K^{t_B}(y,z)]f_2(z)d\mu(z)\right|^2 \frac{d\tau}{\tau^3}\right)^{1/2} \\ \le& \int_{(c_7B)^c} \frac{|K(y,z)-K^{t_B}(y,z)|}{d(y,z)}|f_2(z)|d\mu(z) \\ =&\sum_{j=0}^\infty \int_{2^{j+1}c_7B\setminus 2^jc_7B} \frac{|K(y,z)-K^{t_B}(y,z)|}{d(y,z)}|f_2(z)|d\mu(z) \\ \le& C\sum_{j=0}^\infty \frac{2^{-j\beta}}{\mu(B(y;2^jc_7r_B))}\int_{2^{j+1}c_7B}|f_2(z)|d\mu(z) \\ \le& C\sum_{j=0}^\infty 2^{-j\beta}\frac{1}{\mu(2^{j+1}c_7B)}\int_{2^{j+1}c_7B}|f_2(z)|d\mu(z) \\ \le& C\sum_{j=0}^\infty 2^{-j\beta}\mathcal{M}f_2(x) \le C\mathcal{M}f(x), \end{aligned}$$

which implies that

$$M^\sharp_{A,L(\log L)^k}(||F(f_2)(\cdot)||_{\mathcal{B}_2})(x) \le C\mathcal{M}f(x) \le CM_{L(\log L)^{k+1}}f(x). \tag{3.5}$$

By inequalities (3.4) and (3.5), we obtain

$$\begin{aligned} &M^\sharp_{A,L(\log L)^k}(||F(f)(\cdot)||_{\mathcal{B}_2})(x) \\ \le& M^\sharp_{A,L(\log L)^k}(||F(f_1)(\cdot)||_{\mathcal{B}_2})(x) + M^\sharp_{A,L(\log L)^k}(||F(f_2)(\cdot)||_{\mathcal{B}_2})(x) \\ \le& CM_{L(\log L)^{k+1}}f(x). \end{aligned}$$

Thus, we complete the proof. $\square$

We now prove the theorem.

Proof of Theorem 3.1. Observe that if $\mu(\mathcal{X}) < \infty$, then it follows from Theorem 2.1 and Lemma 2.2 that

$$\begin{aligned}
&w(\mathcal{X})\Big\| ||F(f)||_{\mathcal{B}_2} \Big\|^p_{L(\log L)^{k-1},\mathcal{X}} \\
=&w(\mathcal{X})\, ||\nu(f)||^p_{L(\log L)^{k-1},\mathcal{X}} \le Cw(\mathcal{X})\, ||f||^p_{L(\log L)^k,\mathcal{X}} \\
\le& Cw(\mathcal{X}) \left(\inf_{x\in\mathcal{X}} \left\{ M_{L(\log L)^k} f(x) \right\} \right)^p \\
\le& C \int_{\mathcal{X}} [M_{L(\log L)^k} f(x)]^p w(x) d\mu(x).
\end{aligned} \tag{3.6}$$

Now set

$$\delta_{\mathcal{X}} = \begin{cases} 0, & \text{if } \mu(\mathcal{X}) = \infty, \quad \text{otherwise}, \\ w(\mathcal{X}), & \text{if } \mu(\mathcal{X}) < \infty. \end{cases}$$

By Lemmas 2.4, 3.1 and inequalities (2.5), (3.6), we infer that

$$\begin{aligned}
&\int_{\mathcal{X}} [\mathcal{M}^k(\nu f)(x)]^p w(x) d\mu(x) \\
\le& C_k \int_{\mathcal{X}} [M_{L(\log L)^{k-1}}(\nu(f))(x)]^p w(x) d\mu(x) \\
=& C_k \int_{\mathcal{X}} [M_{L(\log L)^{k-1}}(||F(f)(\cdot)||_{\mathcal{B}_2})(x)]^p w(x) d\mu(x) \\
\le& C \int_{\mathcal{X}} [M^{\sharp}_{A,L(\log L)^{k-1}}(||F(f)(\cdot)||_{\mathcal{B}_2})(x)]^p w(x) d\mu(x) \\
&\quad + C\delta_{\mathcal{X}} w(\mathcal{X})\, ||||F(f)||_{\mathcal{B}_2}||^p_{L(\log L)^{k-1},\mathcal{X}} \\
\le& C \int_{\mathcal{X}} [M_{L(\log L)^k} f(x)]^p w(x) d\mu(x) \\
\le& C \int_{\mathcal{X}} [\mathcal{M}^{k+1} f(x)]^p w(x) d\mu(x).
\end{aligned}$$

The last statement of the theorem follows from the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^p(\mathcal{X}, w)$, where $w \in A_p$ for some $p \in (1, \infty)$. □

By applying Theorem 3.1 and Theorem 5.1 in [17], we obtain the following corollary:

Corollary 3.1. *Let $p \in (1, \infty)$ and $w \in A_\infty$. Assume that hypotheses (a), (b) and (c) hold true. Then there exists a positive constant C depending on p, C_w and δ_w such that*

$$\int_{\mathcal{X}} [\nu f(x)]^p w(x) d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^p \mathcal{M}w(x) d\mu(x).$$

Theorem 3.2. *Let $f \in L_b^\infty(\mathcal{X})$, $w \in A_\infty$. Assume that hypotheses* (a), $(\bar{b})$ *and* (c) *hold true. Then there exists a positive constant C depending on C_w and δ_w such that, for all $\lambda > 0$,*

$$\int_{\{x\in\mathcal{X}:|\nu f(x)|>\lambda\}} w(x)d\mu(x) \le \frac{C}{\lambda}\int_{\mathcal{X}} |f(x)|\mathcal{M}w(x)d\mu(x).$$

In particular, if $w \in A_1$, then

$$w\left(\{x \in \mathcal{X} : |\nu f(x)| > \lambda\}\right) \le \frac{C}{\lambda}||f||_{L^1(\mathcal{X},w)}.$$

Proof. We may assume that $\mathcal{M}w(x)$ is finite almost everywhere. Let $f \in L_b^\infty(\mathcal{X})$. Set

$$\lambda_{\mathcal{X}} = \begin{cases} 0, & \text{if } \mu(\mathcal{X}) = \infty, \text{ otherwise,} \\ ||f||_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}, & \text{if } \mu(\mathcal{X}) < \infty. \end{cases}$$

For $\lambda > \lambda_{\mathcal{X}}$, we apply Lemma 2.1 to f at height λ to obtain a disjoint sequence of balls $\{B_j\}_{j\in\mathbb{N}}$ and a constant $c_8 \ge 1$ such that for any $x \in \mathcal{X} \setminus \bigcup_{j\in\mathbb{N}} c_8 B_j$ and any ball B centered at x,

$$\frac{1}{\mu(B)}\int_B |f(y)|d\mu(y) \le \lambda, \tag{3.7}$$

and for all $j \in \mathbb{N}$,

$$\frac{1}{\mu(c_8B_j)}\int_{c_8B_j} |f(y)|d\mu(y) \le \lambda \le \frac{1}{\mu(B_j)}\int_{B_j} |f(y)|d\mu(y). \tag{3.8}$$

Following [1], we set

$$V_1 = c_8B_1 \setminus \bigcup_{l\le 2} B_l,$$

and for $j \ge 2$,

$$V_j = c_8B_j \setminus \left\{\bigcup_{l=1}^{j-1}\left(V_l \cup (\cup_{i\ge j+1}B_i)\right)\right\}.$$

Then $B_j \subset V_j \subset c_8B_j$, and

$$\bigcup_{j\in\mathbb{N}} V_j = \bigcup_{j\in\mathbb{N}} c_8B_j.$$

Moreover, $\{V_j\}$ are mutually disjoint; and $f(x) = g(x) + b(x)$, where

$$g(x) = f(x)\chi_{(\cup_j V_j)^c}(x) + \sum_j \left(\frac{1}{\mu(V_j)}\int_{V_j} f(y)d\mu(y)\right)\chi_{V_j}(x),$$

$$b(x) = \sum_j b_j(x) \quad \text{with } b_j(x) = \left(f(x) - \frac{1}{\mu(V_j)}\int_{V_j} f(y)d\mu(y)\right)\chi_{V_j}(x).$$

Note that

$$g(x) \leq C\lambda \quad \text{for almost every } \; x \in \mathcal{X}. \tag{3.9}$$

Now let $E_\lambda = \bigcup_{j\in\mathbb{N}} c_9 B_j$, where $c_9 = c_8(c_1+3)\kappa$ (c_1 appears in hypothesis $(\bar{\mathrm{b}})$). Then

$$\begin{aligned}
&w(\{x \in \mathcal{X} : \nu(f)(x) > \lambda\})\\
\leq &w(E_\lambda) + w(\{x \in E_\lambda^c : \nu(g)(x) > \lambda/2\}) + w(\{x \in E_\lambda^c : \nu(b)(x) > \lambda/2\})\\
\equiv &Z_1 + Z_2 + Z_3.
\end{aligned}$$

Observe that for any $x \in B_j$,

$$\frac{w(c_9B_j)}{\mu(c_9B_j)} = \frac{1}{\mu(c_9B_j)} \int_{c_9B_j} w(y)d\mu(y) \leq \mathcal{M}w(x),$$

which implies that

$$\frac{w(c_9B_j)}{\mu(c_9B_j)} \leq \inf_{x\in B_j} \{\mathcal{M}w(x)\}.$$

The above inequality, together with the doubling property of μ and (3.8), imply that

$$\begin{aligned}
Z_1 =&w(E_\lambda) \leq \sum_j \frac{w(c_9B_j)}{\mu(c_9B_j)} \mu(c_9B_j) \leq C \sum_j \inf_{x\in B_j} \{\mathcal{M}w(x)\}\, \mu(B_j)\\
\leq &\frac{C}{\lambda} \sum_j \int_{B_j} |f(y)|\mathcal{M}w(y)d\mu(y) \leq \frac{C}{\lambda} \int_{\mathcal{X}} |f(y)|\mathcal{M}w(y)d\mu(y).
\end{aligned}$$

Recall that $w \in A_\infty \Rightarrow w \in A_p$ for some $p > 1$. By Theorem 3.1 and by inequality (3.9), we have

$$\begin{aligned}
Z_2 \leq &\left(\frac{2}{\lambda}\right)^p \int_{\mathcal{X}} |\nu g(x)|^p w(x) \chi_{E_\lambda^c}(x) d\mu(x)\\
\leq &\frac{C}{\lambda^p} \int_{\mathcal{X}} |g(x)|^p w(x) d\mu(x) \leq \frac{C}{\lambda} \int_{\mathcal{X}} |g(x)| w(x) d\mu(x)\\
= &\frac{C}{\lambda} \int_{\mathcal{X}} |g(x)|(w\chi_{E_\lambda})(x)d\mu(x) + \frac{C}{\lambda} \int_{\mathcal{X}} |g(x)|(w\chi_{E_\lambda^c})(x) d\mu(x)\\
=: &S_1 + S_2.
\end{aligned}$$

Again, by inequality (3.9) and from the estimate of Z_1, we obtain

$$S_1 \leq Cw(E_\lambda) = CZ_1 \leq \frac{C}{\lambda} \int_{\mathcal{X}} |f(y)|\mathcal{M}w(y)d\mu(y).$$

On the other hand,

$$\begin{aligned}
S_2 \leq &\frac{C}{\lambda} \int_{\mathcal{X}} |g(x)|\mathcal{M}(w\chi_{E_\lambda^c})(x)d\mu(x)\\
\leq &\frac{C}{\lambda} \int_{\mathcal{X}\setminus\cup_j V_j} |f(x)|\mathcal{M}(w\chi_{E_\lambda^c})(x)d\mu(x) + \frac{C}{\lambda} \sum_j \int_{V_j} |g(x)|\mathcal{M}(w\chi_{E_\lambda^c})(x)d\mu(x).
\end{aligned}$$

Observe that

$$\mathcal{M}(w\chi_{E_\lambda^c})(x) \leq C \inf_{y\in V_j}\left\{\mathcal{M}(w\chi_{E_\lambda^c})(y)\right\},$$

since $x,y \in V_j$ and $w\chi_{E_\lambda^c}$ lives far away from V_j (see [7], p. 160). Therefore, for each $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{V_j} |g(x)|\mathcal{M}(w\chi_{E_\lambda^c})(x)d\mu(x)\\
\leq &C \inf_{y\in V_j}\left\{\mathcal{M}(w\chi_{E_\lambda^c})(y)\right\}\int_{V_j}|g(x)|d\mu(x)\\
\leq &C\lambda\mu(c_8B_j)\inf_{y\in V_j}\left\{\mathcal{M}(w\chi_{E_\lambda^c})(y)\right\}\\
\leq &C\lambda\mu(B_j)\inf_{y\in B_j}\left\{\mathcal{M}(w\chi_{E_\lambda^c})(y)\right\}\\
\leq &C \inf_{y\in B_j}\left\{\mathcal{M}(w\chi_{E_\lambda^c})(y)\right\}\int_{B_j}|f(x)|d\mu(x)\\
\leq &C\int_{B_j}|f(x)|\mathcal{M}w(x)d\mu(x).
\end{aligned}$$

The second and fourth inequalities follow from (3.9) and (3.8) respectively. Therefore,

$$S_2 \leq \frac{C}{\lambda}\int_{\mathcal{X}}|f(x)|\mathcal{M}w(x)d\mu(x).$$

Summing the estimates of S_1 and S_2 yields that

$$Z_2 \leq \frac{C}{\lambda}\int_{\mathcal{X}}|f(x)|\mathcal{M}w(x)d\mu(x).$$

For the estimate of Z_3, let $t_j = r_j^m$, where r_j is the radius of the ball B_j. We write

$$\begin{aligned}
Z_3 =&w\left(\{x\in E_\lambda^c : ||F(b)(x,\cdot)||_{\mathcal{B}_2} > \lambda/2\}\right)\\
=&w\left(\left\{x\in E_\lambda^c : \left|\left|\sum_j F(b_j)(x,\cdot)\right|\right|_{\mathcal{B}_2} > \lambda/2\right\}\right)\\
\leq &w\left(\left\{x\in E_\lambda^c : \sum_j ||(F-FB_{t_j})(b_j)(x,\cdot)||_{\mathcal{B}_2} > \lambda/4\right\}\right)\\
&\quad + w\left(\left\{x\in E_\lambda^c : \left|\left|\sum_j FB_{t_j}(b_j)(x,\cdot)\right|\right|_{\mathcal{B}_2} > \lambda/4\right\}\right)\\
=:& Z_4 + Z_5.
\end{aligned}$$

Observe that if $y \in B_j$ and $x \notin E_\lambda = \bigcup_j c_9 B_j$, then $d(x,y) \geq c_1 r_j = c_1 t_j^{1/m}$. Note also that

$$
\begin{aligned}
&||(F - FB_{t_j})(b_j)(x,\cdot)||_{\mathcal{B}_2} \\
=&\left\{ \int_0^\infty \left| \int_{d(x,y)<\tau} \Big(K(x,y) - K_{t_j}(x,y) \Big) b_j(y) d\mu(y) \right|^2 \frac{d\tau}{\tau^3} \right\}^{1/2} \\
\leq& \int_{V_j} \frac{|K(x,y) - K_{t_j}(x,y)|}{d(x,y)} |b_j(y)| d\mu(y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Z_4 \leq& \frac{4}{\lambda} \int_{\mathcal{X}\setminus E_\lambda} \sum_j ||(F - FB_{t_j})(b_j)(x,\cdot)||_{\mathcal{B}_2} w(x) \chi_{E_\lambda^c}(x) d\mu(x) \\
\leq& \frac{C}{\lambda} \int_{\mathcal{X}\setminus E_\lambda} \sum_j \left(\int_{V_j} \frac{|K(x,y) - K_{t_j}(x,y)|}{d(x,y)} |b_j(y)| d\mu(y) \right) w(x) \chi_{E_\lambda^c}(x) d\mu(x) \\
\leq& \frac{C}{\lambda} \sum_j \int_{V_j} \left(\int_{d(x,y) \geq c_1 t_j^{1/m}} \frac{|K(x,y) - K_{t_j}(x,y)|}{d(x,y)} w(x) \chi_{E_\lambda^c}(x) d\mu(x) \right) |b_j(y)| d\mu(y).
\end{aligned}
$$

It follows from hypothesis $(\bar{\mathrm{b}})$ that

$$
\begin{aligned}
&\int_{d(x,y) \geq c_1 t_j^{1/m}} \frac{|K(x,y) - K_{t_j}(x,y)|}{d(x,y)} w(x) \chi_{E_\lambda^c}(x) d\mu(x) \\
=& \sum_{i=0}^\infty \int_{d(x,y) \cong 2^i c_1 t_j^{1/m}} \frac{|K(x,y) - K_{t_j}(x,y)|}{d(x,y)} w(x) \chi_{E_\lambda^c}(x) d\mu(x) \\
\leq& C \sum_{i=0}^\infty 2^{-i\beta} \frac{1}{\mu(B(y; 2^{i+1} c_1 t_j^{1/m}))} \int_{B(y; 2^{i+1} c_1 t_j^{1/m})} w(x) \chi_{E_\lambda^c}(x) d\mu(x) \\
\leq& C \mathcal{M}(w \chi_{E_\lambda^c})(y),
\end{aligned}
$$

where $d(x,y) \cong 2^i c_1 t_j^{1/m}$ means $2^i c_1 t_j^{1/m} \leq d(x,y) \leq 2^{i+1} c_1 t_j^{1/m}$. Therefore,

$$
\begin{aligned}
Z_4 \leq& \frac{C}{\lambda} \sum_j \int_{V_j} |b_j(y)| \mathcal{M}(w \chi_{E_\lambda^c})(y) d\mu(y) \\
\leq& \frac{C}{\lambda} \sum_j \int_{V_j} |f(y)| \mathcal{M}(w \chi_{E_\lambda^c})(y) d\mu(y) \\
&+ \frac{C}{\lambda} \sum_j \left(\frac{1}{\mu(V_j)} \int_{V_j} |f(x)| d\mu(x) \right) \left(\int_{V_j} \mathcal{M}(w \chi_{E_\lambda^c})(y) d\mu(y) \right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{\lambda}\sum_j \left\{ \int_{V_j} |f(y)|\mathcal{M}(w\chi_{E_\lambda^c})(y)d\mu(y) + \left(\inf_{x\in V_j}\mathcal{M}(w\chi_{E_\lambda^c})(x)\right)\int_{V_j}|f(x)|d\mu(x)\right\}\\
&\le \frac{C}{\lambda}\sum_j \left\{ \int_{V_j} |f(y)|\mathcal{M}(w\chi_{E_\lambda^c})(y)d\mu(y) + \int_{V_j}|f(x)|\mathcal{M}(w\chi_{E_\lambda^c})(x)d\mu(x)\right\}\\
&\le \frac{C}{\lambda}\int_{\mathcal{X}} |f(x)|\mathcal{M}w(x)d\mu(x).
\end{aligned}
$$

It remains to estimate Z_5. By inequality (3.8) and Proposition 2.5 in [5] (or see Lemma 1 in [4]), we have

$$
\begin{aligned}
|B_{t_j}b_j(x)| &\le \int_{\mathcal{X}} h_{t_j}(x,y)|b_j(y)|d\mu(y) \le \left(\sup_{y\in V_j} h_{t_j}(x,y)\right)\int_{V_j}|b_j(y)|d\mu(y)\\
&\le 2\left(\sup_{y\in V_j} h_{t_j}(x,y)\right)\int_{c_8B_j}|f(y)|d\mu(y) \le C\lambda\mu(c_8B_j)\left(\sup_{y\in V_j} h_{t_j}(x,y)\right)\\
&\le C\lambda\mu(V_j)\left(\sup_{y\in V_j} h_{t_j}(x,y)\right) \le C\lambda\mu(V_j)\left(\inf_{y\in V_j} h_{\theta t_j}(x,y)\right)\\
&\le C\lambda\int_{V_j} h_{\theta t_j}(x,y)\chi_{V_j}(y)d\mu(y), \qquad (3.10)
\end{aligned}
$$

where $\theta \ge 1$ is a positive constant (see Lemma 1 in [4]). Now $w \in A_\infty \Rightarrow w \in A_r$ for some $r > 1$, which implies that $w^{1-r'} \in A_{r'}$. Take a function $u \ge 0$, $u \in L^{r'}(\mathcal{X}, w^{1-r'})$ with $||u||_{L^{r'}(\mathcal{X},w^{1-r'})} \le 1$. From inequality (3.10) and Remark 1.1-ii, we infer that

$$
\begin{aligned}
&\left|\left\langle \sum_j |B_{t_j}b_j|, u\right\rangle\right| \le \sum_j \left|\langle |B_{t_j}b_j|, u\rangle\right|\\
&\le C\lambda\sum_j\int_{\mathcal{X}}\left(\int_{\mathcal{X}} h_{\theta t_j}(x,y)u(x)d\mu(x)\right)\chi_{V_j}(y)d\mu(y)\\
&\le C\lambda\sum_j\int_{\mathcal{X}}\mathcal{M}u(y)\chi_{V_j}(y)d\mu(y) \le C\lambda\int_{\cup_j V_j}\mathcal{M}u(y)d\mu(y)\\
&\le C\lambda\left\{\int_{\mathcal{X}}|\mathcal{M}u(y)|^{r'}[w(y)]^{1-r'}d\mu(y)\right\}^{1/r'}\left\{\int_{\cup_j V_j} w(y)d\mu(y)\right\}^{1/r}\\
&\le C\lambda||u||_{L^{r'}(\mathcal{X},w^{1-r'})}[w(\cup_j V_j)]^{1/r} \le C\lambda[w(E_\lambda)]^{1/r}\\
&\le C\lambda^{1/r'}\left(\int_{\mathcal{X}}|f(x)|\mathcal{M}w(x)d\mu(x)\right)^{1/r},
\end{aligned}
$$

where the last inequality follows from the estimate of Z_1. Hence,

$$\begin{aligned}\left|\left|\sum_j |B_{t_j}b_j|\right|\right|_{L^r(\mathcal{X},w)} =& \sup_{||u||_{L^{r'}(\mathcal{X},w^{1-r'})\le 1}} \left\{\left|\left\langle \sum_j |B_{t_j}b_j|, u\right\rangle\right|\right\}\\ \le& C\lambda^{1/r'}\left(\int_{\mathcal{X}} |f(x)|\mathcal{M}w(x)d\mu(x)\right)^{1/r}.\end{aligned}$$

The above equality follows from the fact that for any $f \in L^r(\mathcal{X}, w)$,

$$\begin{aligned}&||f||_{L^r(\mathcal{X},w)} = ||\bar{f}||_{L^r(\mathcal{X},w^{1-r'})}\\ =& \sup_{||g||_{L^{r'}(\mathcal{X},w^{1-r'})}\le 1} \left|\int_{\mathcal{X}} \bar{f}gw^{1-r'}d\mu\right| = \sup_{||g||_{L^{r'}(\mathcal{X},w^{1-r'})}\le 1} \left|\int_{\mathcal{X}} fgd\mu\right|,\end{aligned}$$

where $\bar{f} = fw^{r'-1}$, and $\frac{1}{r} + \frac{1}{r'} = 1$. By Theorem 3.1 and from the estimate above, we see that

$$\begin{aligned}Z_5 \le& \left(\frac{4}{\lambda}\right)^r \int_{\mathcal{X}} \left|\left|F\left(\sum_j B_{t_j}b_j\right)(x,\cdot)\right|\right|_{\mathcal{B}_2}^r w(x)d\mu(x)\\ =& \left(\frac{4}{\lambda}\right)^r \int_{\mathcal{X}} \left|\nu\left(\sum_j B_{t_j}b_j\right)(x)\right|^r w(x)d\mu(x)\\ \le& \frac{C}{\lambda^r}\int_{\mathcal{X}} \left|\sum_j B_{t_j}b_j(x)\right|^r w(x)d\mu(x)\\ \le& \frac{C}{\lambda^r}\int_{\mathcal{X}} \left(\sum_j |B_{t_j}b_j(x)|\right)^r w(x)d\mu(x)\\ \le& \frac{C}{\lambda}\int_{\mathcal{X}} |f(x)|\mathcal{M}w(x)d\mu(x).\end{aligned}$$

Summing over all of the estimates $Z_1, \cdots, Z_5$ yields that, for all $\lambda > \lambda_{\mathcal{X}}$,

$$\int_{\{x\in\mathcal{X}:|\nu f(x)|>\lambda\}} w(x)d\mu(x) \le \frac{C}{\lambda}\int_{\mathcal{X}} |f(x)|\mathcal{M}w(x)d\mu(x).$$

If $\mathcal{X}$ is unbounded, then we are done. If $\mathcal{X}$ is bounded, then for $0 < \lambda \le \lambda_{\mathcal{X}} = ||f||_{L^1(\mathcal{X})}[\mu(\mathcal{X})]^{-1}$, we have

$$\begin{aligned}&\int_{\{x\in\mathcal{X}:\nu f(x)>\lambda\}} w(x)d\mu(x)\\ \le& w(\mathcal{X}) \le \frac{w(\mathcal{X})}{\mu(\mathcal{X})}\frac{||f||_{L^1(\mathcal{X})}}{\lambda}\\ \le& \frac{1}{\lambda}\int_{\mathcal{X}} |f(x)|\left(\inf_{y\in\mathcal{X}}\{\mathcal{M}w(y)\}\right)d\mu(x)\end{aligned}$$

$$\leq \frac{1}{\lambda}\int_{\mathcal{X}}|f(x)|\mathcal{M}w(x)d\mu(x),$$

where the third inequality follows from the fact that $\frac{w(\mathcal{X})}{\mu(\mathcal{X})} \leq \mathcal{M}w(x)$ for any $x \in \mathcal{X}$. Consequently, for all $\lambda > 0$,

$$\int_{\{x\in\mathcal{X}:|\nu f(x)|>\lambda\}} w(x)d\mu(x) \leq \frac{C}{\lambda}\int_{\mathcal{X}}|f(x)|\mathcal{M}w(x)d\mu(x).$$

The last statement of the theorem is obvious since $\mathcal{M}w(x) \leq Cw(x)$ for almost every $x \in \mathcal{X}$, when $w \in A_1$. □

Remark 3.1. In the proof of Theorem 1.6 in [10], the authors [10] obtained an estimate of a quantity (which is similar to Z_2 in the proof above) by applying Theorem 1.5 in [10]. Theorem 1.5 in [10], in turn, was derived from Theorem 1.2 in [10] by duality. Observe that their techniques cannot be applied to our case, since the Marcinkiewicz integral is sub-linear. Our estimate of Z_2 relies essentially on the good behavior of the function $g(x)$ (see inequality (3.9)). Note also that our technique should work for their case.

4 Commutators

Let b be a BMO function whose BMO semi-norm is denoted by $||b||_*$. For a ball $B \subset \mathcal{X}$, let

$$b_B = \frac{1}{\mu(B)}\int_B b(x)d\mu(x).$$

For $f \in L_b^\infty(\mathcal{X})$, define the linear operator $F_b(f)$ by

$$F_b(f)(x,\tau) = \int_{d(x,y)<\tau}\Big(b(x)-b(y)\Big)K(x,y)f(y)d\mu(y),$$

where $K(x,y)$ is a measurable function defined on $(\mathcal{X}\times\mathcal{X})\diagdown\Delta$ with $\Delta = \{(x,x) : x \in \mathcal{X}\}$. Define the commutator of Marcinkiewicz integral and BMO function, $\nu_b(f)$, by

$$\nu_b f(x) = ||F_b(f)(x,\cdot)||_{\mathcal{B}_2} = \left\{\int_0^\infty |F_b(f)(x,\tau)|^2\frac{d\tau}{\tau^3}\right\}^{1/2}.$$

We have the following theorem.

Theorem 4.1. *Let $f \in L_b^\infty(\mathcal{X})$, $w \in A_\infty(\mathcal{X})$, $b \in BMO(\mathcal{X})$, and $\mu(\mathcal{X}) = \infty$. Suppose that hypotheses (a), (b) and (c) hold true. Then there exists a positive constant C, depending on p, C_w and δ_w such that*

$$\int_{\mathcal{X}}|\nu_b f(x)|^p w(x)d\mu(x) \leq C||b||_*^p\int_{\mathcal{X}}(\mathcal{M}^3 f(x))^p w(x)d\mu(x) \quad \textit{for } 1<p<\infty.$$

We need the following lemmas.

Lemma 4.1 ([7]). *Let $b \in BMO(\mathcal{X})$, and denote its BMO semi-norm by $||b||_*$. Then*
(i) there exists a constant $C > 0$ such that whenever $M \geq 2$, $|b_B - b_{MB}| \leq C(\log M)||b||_*$ for every ball $B \subset \mathcal{X}$;
(ii) there exists a positive constant c_p $(0 < p < \infty)$ such that

$$\left(\frac{1}{\mu(B)}\int_B |b(y) - b_B|^p d\mu(y)\right)^{1/p} \leq c_p||b||_*$$

for all balls $B \subset \mathcal{X}$;
(iii) There exist constants $C_1, C_2 > 1$ such that

$$\frac{1}{\mu(B)}\int_B \exp\left(\frac{|b(y) - b_B|}{C_2||b||_*}\right) d\mu(y) \leq C_1$$

for all balls $B \subset \mathcal{X}$.

Lemma 4.2. *For $j \in \mathbb{N} \cup \{0\}$ and any ball B containing $x \in \mathcal{X}$,*

$$\begin{aligned} I_1(x) :=& \frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B} |b(y) - b_B|||F(f)(y,\cdot)||_{\mathcal{B}_2} d\mu(y) \\ \leq& C(j+1)||b||_* M_{L\log L}(\nu f)(x). \end{aligned}$$

Proof. By inequalities (2.4), (2.5) and Lemma 4.1, we have

$$\begin{aligned} I_1(x) \leq& \frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B} (|b(y) - b_{2^{j+1}B}| + |b_B - b_{2^{j+1}B}|)||F(f)(y,\cdot)||_{\mathcal{B}_2} d\mu(y) \\ \leq& C(j+1)||b||_*\mathcal{M}(\nu f)(x) + 3||b - b_{2^{j+1}B}||_{\exp(L),2^{j+1}B}||\nu f||_{L\log L,2^{j+1}B} \\ \leq& C(j+1)||b||_* M_{L\log L}(\nu f)(x). \end{aligned}$$

This completes the proof the lemma. □

Lemma 4.3. *Let B be any ball which contains $x \in \mathcal{X}$. Let $t_B = r_B^m$, where r_B is the radius of the ball B, and m appears in* (1.6b)–(1.7)*. Then*

$$\begin{aligned} I_2(x) :=& \frac{1}{\mu(B)}\int_B ||A_{t_B}\left((b - b_B)F(f)\right)(y,\cdot)||_{\mathcal{B}_2} d\mu(y) \\ \leq& C||b||_* M_{L\log L}(\nu f)(x). \end{aligned}$$

Proof. Recall that $c_7 = \kappa(c_2+3)$. Observe that

$$\begin{aligned}
&||A_{t_B}\left((b-b_B)F(f)\right)(y,\cdot)||_{\mathcal{B}_2}\\
\leq &\int_{\mathcal{X}} h_{t_B}(y,z)|b(z)-b_B|||F(f)(z,\cdot)||_{\mathcal{B}_2}d\mu(z)\\
\leq &\frac{C}{\mu(c_7B)}\int_{c_7B}|b(z)-b_B|||F(f)(z,\cdot)||_{\mathcal{B}_2}d\mu(z)\\
&\quad + C\sum_{j=0}^{\infty}\frac{2^{(j+1)n}s(2^{jm})}{\mu(2^{j+1}c_7B)}\int_{2^{j+1}c_7B}|b(z)-b_B|||F(f)(z,\cdot)||_{\mathcal{B}_2}d\mu(z)\\
\leq &C\left(1+\sum_{j=0}^{\infty}(j+1)2^{(j+1)n}s(2^{jm})\right)||b||_*M_{L\log L}(\nu f)(x).\\
\leq &C||b||_*M_{L\log L}(\nu f)(x),
\end{aligned}$$

where the third inequality follows from Lemma 4.2. Therefore,

$$\begin{aligned}
I_2(x) =&\frac{1}{\mu(B)}\int_B ||A_{t_B}\left((b-b_B)F(f)\right)(y,\cdot)||_{\mathcal{B}_2}d\mu(y)\\
\leq &C||b||_*M_{L\log L}(\nu f)(x).
\end{aligned}$$

We complete the proof. □

Lemma 4.4. *Let $k\in\mathbb{N}\cup\{0\}$ and B be a ball with finite measure. Then there exists a constant $C_k>0$ such that*

$$||fg||_{L(\log L)^k,B}\leq C_k||f||_{\exp(L),B}||g||_{L(\log L)^{k+1},B}.$$

Remark 4.1. Lemma 4.4 appears in [10] without its proof. For clarity and for reader's convenience, we show the proof of this lemma below.

Proof. If $k=0$, then the result follows immediately from inequality (2.4). Now suppose $k\in\mathbb{N}$. Denote

$$C_k = 2^k(k+6),\quad \alpha_1 := ||f||_{L\log L,B}\quad \text{and}\quad \lambda_{k+1} := ||g||_{L(\log L)^{k+1},B}.$$

Observe that

$$\begin{aligned}
&\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\log^k\left(e+\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\right)\\
\leq &\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\log^k\left(e+\frac{|f(x)g(x)|}{\alpha_1\lambda_{k+1}}\right)\\
\leq &\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\left\{\log\left(1+\frac{|f(x)|}{\alpha_1}\right)+\log\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)\right\}^k
\end{aligned}$$

$$\begin{aligned}&\le 2^k\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\left\{\log^k\left(1+\frac{|f(x)|}{\alpha_1}\right)+\log^k\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)\right\}\\&\le\frac{2^k}{C_k}\left(\frac{|f(x)|}{\alpha_1}\right)^{k+1}\frac{|g(x)|}{\lambda_{k+1}}+\frac{2^k}{C_k}\frac{|f(x)|}{\alpha_1}\frac{|g(x)|}{\lambda_{k+1}}\log^k\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)\\&=:F(x)+G(x).\end{aligned}\tag{4.1}$$

By Young's inequality (2.2),

$$\begin{aligned}\left(\frac{|f(x)|}{\alpha_1}\right)^{k+1}\frac{|g(x)|}{\lambda_{k+1}}&\le\exp\left(\left(\left(\frac{|f(x)|}{\alpha_1}\right)^{k+1}\right)^{\frac{1}{k+1}}\right)+\frac{|g(x)|}{\lambda_{k+1}}\log^{k+1}\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)\\&\le\exp\left(\frac{|f(x)|}{\alpha_1}\right)+\frac{|g(x)|}{\lambda_{k+1}}\log^{k+1}\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B\left(\frac{|f(x)|}{\alpha_1}\right)^{k+1}\frac{|g(x)|}{\lambda_{k+1}}d\mu(x)\\&\le\frac{1}{\mu(B)}\int_B\exp\left(\frac{|f(x)|}{\alpha_1}\right)d\mu(x)+\frac{1}{\mu(B)}\int_B\frac{|g(x)|}{\lambda_{k+1}}\log^{k+1}\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)d\mu(x)\\&\le 2+1=3.\end{aligned}\tag{4.2}$$

Similarly,

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B\frac{|f(x)|}{\alpha_1}\frac{|g(x)|}{\lambda_{k+1}}\log^k\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)d\mu(x)\\&\le\frac{1}{\mu(B)}\int_B\exp\left(\frac{|f(x)|}{\alpha_1}\right)d\mu(x)\\&\quad+\frac{1}{\mu(B)}\int_B\frac{|g(x)|}{\lambda_{k+1}}\log^k\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)\log\left(e+\frac{|g(x)|}{\lambda_{k+1}}\log^k(e+\frac{|g(x)|}{\lambda_{k+1}})\right)d\mu(x)\\&\le 2+(k+1)\int_B\frac{|g(x)|}{\lambda_{k+1}}\log^{k+1}\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)d\mu(x)\\&\le k+3,\end{aligned}\tag{4.3}$$

where the second inequality follows from the fact that

$$\begin{aligned}\log\left(e+\frac{|g(x)|}{\lambda_{k+1}}\log^k(e+\frac{|g(x)|}{\lambda_{k+1}})\right)&\le\log\left(e+\left(\frac{|g(x)|}{\lambda_{k+1}}\right)^{k+1}\right)\\&\le\log\left(e+\frac{|g(x)|}{\lambda_{k+1}}\right)^{k+1}\end{aligned}$$

$$=(k+1)\log\left(e+(\frac{|g(x)|}{\lambda_{k+1}})\right).$$

Combining inequalities (4.1)–(4.3), we obtain

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\log^k\left(e+\frac{|f(x)g(x)|}{C_k\alpha_1\lambda_{k+1}}\right)d\mu(x)\\ \le&\frac{1}{\mu(B)}\int_B\left(F(x)+G(x)\right)d\mu(x)\\ \le&\frac{2^k(k+6)}{C_k}=1,\end{aligned}$$

which implies that

$$||fg||_{L(\log L)^k,B}\le C_k\alpha_1\lambda_{k+1}=C_k||f||_{\exp(L),B}||g||_{L(\log L)^{k+1},B}.$$

Thus, we complete the proof. □

Lemma 4.5.

$$\begin{aligned}M_A^\sharp(||F_b(f)||_{\mathcal{B}_2})(x):=&\sup_{B\ni x}\left\{\frac{1}{\mu(B)}\int_B||\left(F_b-A_{t_B}F_b\right)(f)(y,\cdot)||_{\mathcal{B}_2}d\mu(y)\right\}\\ \le&C||b||_*\left(\mathcal{M}^2(\nu f)(x)+\mathcal{M}^3f(x)\right).\end{aligned}$$

Proof. For an arbitrary $x\in\mathcal{X}$, choose a ball B which contains x. We write $f=f\chi_{c_7B}+f\chi_{(c_7B)^c}\equiv f_1+f_2$, and

$$\begin{aligned}&||\left(F_b-A_{t_B}F_b\right)(f)(y,\cdot)||_{\mathcal{B}_2}\\ \le&||\left(b(y)-b_B\right)F(f)(y,\cdot)||_{\mathcal{B}_2}+||A_{t_B}\left((b-b_B)F(f)\right)(y,\cdot)||_{\mathcal{B}_2}\\ &+||F\left((b-b_B)f_1\right)(y,\cdot)||_{\mathcal{B}_2}+||A_{t_B}F\left((b-b_B)f_1\right)(y,\cdot)||_{\mathcal{B}_2}\\ &+||(F-A_{t_B}F)\left((b-b_B)f_2\right)(y,\cdot)||_{\mathcal{B}_2}\\ =:&\sum_{i=1}^5K_i(y).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B||\left(F_b(f)-A_{t_B}F_b(f)\right)(y,\cdot)||_{\mathcal{B}_2}d\mu(y)\\ \le&\sum_{i=1}^5\frac{1}{\mu(B)}\int_BK_i(y)d\mu(y)=:\sum_{i=1}^5J_i(x).\end{aligned}$$

By Lemmas 4.2 and 4.3 respectively, we see that

$$\max\left\{J_1(x),J_2(x)\right\}\le C||b||_*M_{L\log L}(\nu f)(x)\le C||b||_*\mathcal{M}^2(\nu f)(x).$$

On the other hand, it follows from Lemmas 2.2, 4.4 and 4.1 that

$$\begin{aligned} J_3(x) =&\frac{1}{\mu(B)}\int_B \nu\left((b-b_B)f_1\right)(y)d\mu(y) \leq C||(b-b_B)f_1||_{L\log L,B} \\ \leq& C||b-b_B||_{\exp(L),B}||f||_{L(\log L)^2,B} \leq C||b||_* M_{L(\log L)^2}f(x) \\ \leq& C||b||_*\mathcal{M}^3 f(x). \end{aligned}$$

Again, by Lemmas 2.2, 4.4 and 4.1, we obtain

$$\begin{aligned} K_4(y) =&||A_{t_B}F\left((b-b_B)f_1\right)(y,\cdot)||_{\mathcal{B}_2} \\ \leq& C\int_{\mathcal{X}} h_{t_B}(y,z)||F\left((b-b_B)f_1\right)(z,\cdot)||_{\mathcal{B}_2}d\mu(z) \\ \leq& \frac{C}{\mu(c_7B)}\int_{c_7B}\nu\left((b-b_B)f_1\right)(z)d\mu(z) \\ &+C\sum_{j=0}^{\infty}2^{(j+1)n}s(2^{jm})\frac{1}{\mu(2^{j+1}c_7B)}\int_{2^{j+1}c_7B}\nu\left((b-b_B)f_1\right)(z)d\mu(z) \\ \leq& C\left(1+\sum_{j=0}^{\infty}2^{(j+1)n}s(2^{jm})\right)||(b-b_B)f_1||_{L\log L,c_7B} \\ \leq& C||b-b_B||_{\exp(L),c_7B}||f_1||_{L(\log L)^2,c_7B} \\ \leq& C||b||_*M_{L(\log L)^2}f(x) \leq C||b||_*\mathcal{M}^3 f(x). \end{aligned}$$

Therefore,

$$J_4(x) = \frac{1}{\mu(B)}\int_B K_4(y)d\mu(y) \leq C||b||_*\mathcal{M}^3 f(x).$$

It remains to estimate $J_5(x)$. As in the proof of Lemma 3.1, we have

$$\begin{aligned} K_5(y) =&||(F-A_{t_B}F)\left((b-b_B)f_2\right)(y,\cdot)||_{\mathcal{B}_2} \\ \leq& C\sum_{j=0}^{\infty}2^{-j\beta}\frac{1}{\mu(2^{j+1}c_7B)}\int_{2^{j+1}c_7B}|b(z)-b_B||f_2(z)|d\mu(z) \\ \leq& C||b||_*\sum_{j=0}^{\infty}(j+1)2^{-j\beta}M_{L\log L}f(x) \\ \leq& C||b||_*M_{L\log L}f(x), \end{aligned}$$

where the second inequality follows from Lemma 4.2. Hence,

$$J_5(x) = \frac{1}{\mu(B)}\int_B K_5(y)d\mu(y) \leq C||b||_*\mathcal{M}^3 f(x).$$

Summing all of the estimates $J_1,\cdots,J_5$, and taking the supremum over all balls B containing x yield the desired conclusion. □

We now prove the theorem.

Proof of Theorem 4.1. By Lemma 2.4, Lemma 4.5 and Theorem 3.1, we infer that

$$\begin{aligned}||\nu_b f||_{L^p(\mathcal{X},w)} \leq& ||\mathcal{M}(\nu_b f)||_{L^p(\mathcal{X},w)} = ||\mathcal{M}(||F_b(f)||_{\mathcal{B}_2})||_{L^p(\mathcal{X},w)} \\ \leq& C\left|\left|M^{\sharp}_{A}(||F_b(f)||_{\mathcal{B}_2})\right|\right|_{L^p(\mathcal{X},w)} \\ \leq& C||b||_* \Big(||\mathcal{M}^2(\nu f)||_{L^p(\mathcal{X},w)} + ||\mathcal{M}^3(f)||_{L^p(\mathcal{X},w)}\Big) \\ \leq& C||b||_*||\mathcal{M}^3(f)||_{L^p(\mathcal{X},w)}.\end{aligned}$$

This completes the proof. □

Remark 4.2. Consider the commutator

$$\nu_{\vec{b}}(f)(x) = \left\{\int_0^\infty \left|\int_{d(x,y)<\tau}\left(\prod_{i=1}^{k}(b_i(x)-b_i(y))\right)K(x,y)f(y)d\mu(y)\right|^2 \frac{d\tau}{\tau^3}\right\}^{1/2},$$

where $b_i \in \mathrm{BMO}(\mathcal{X})$, $1 \leq i \leq k$. It would be interesting to know if Theorem 4.1 can also be extended to the case $\nu_{\vec{b}}(f)$ with weight $w \in A_\infty(\mathcal{X})$.

5 Application

We will need the following propositions for the next theorem.

Proposition 5.1. *Suppose that there exist constants $\delta_1 > 1$, $C_5 > 0$ and $\beta > 0$ such that*

$$\frac{|K(x,y)-K(x,z)|}{d(x,y)} \leq \frac{C_5}{\mu(B(x;d(x,y)))}\left\{\frac{d(y,z)}{d(x,y)}\right\}^\beta \quad whenever\ \ d(x,y) \geq \delta_1 d(y,z). \tag{5.1}$$

Then there exist positive constants C_6 and δ_2 such that

$$\frac{|K(x,y)-K_t(x,y)|}{d(x,y)} \leq \frac{C_6}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^\beta} \quad whenever\ \ d(x,y) \geq \delta_2 t^{1/m}.$$

Proposition 5.2. *Suppose that there exist constants $\delta_3 > 1$, $C_7 > 0$ and $\beta > 0$ such that*

$$\frac{|K(x,y)-K(z,y)|}{d(x,y)} \leq \frac{C_7}{\mu(B(x;d(x,y)))}\left\{\frac{d(x,z)}{d(x,y)}\right\}^\beta \quad whenever\ \ d(x,y) \geq \delta_3 d(x,z). \tag{5.2}$$

Then there exist positive constants C_8 and δ_4 such that

$$\frac{|K(x,y)-K^t(x,y)|}{d(x,y)} \leq \frac{C_8}{\mu(B(x;d(x,y)))}\frac{t^{\beta/m}}{(d(x,y))^\beta} \quad whenever\ \ d(x,y) \geq \delta_4 t^{1/m}.$$

We omit the proofs of these propositions since they are identically similar to the proof of Proposition 2 in [4].

Now let $\mathcal{X} = \mathbb{R}^n$, $n \geq 2$. Let Ω be homogeneous of degree zero, $\Omega(x) = \Omega(\frac{x}{|x|})$ for every nonzero $x \in \mathbb{R}^n$, and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

Assume further that Ω satisfies the $\beta-$ Hölder's continuity:

There exist constants $C > 0$ and $\beta \in (0,1]$ such that for all nonzero $x, y \in \mathbb{R}^n$,

$$|\Omega(x) - \Omega(y)| \leq C \left| \frac{x}{|x|} - \frac{y}{|y|} \right|^{\beta}.$$

Now let K be the convolution kernel defined by

$$K(x) = \frac{\Omega(x)}{|x|^{n-1}}, \quad n \geq 2.$$

Let

$$\begin{aligned} F_{\Omega}(f)(x,t) &= \int_{|x-y|<t} K(x-y)f(y)dy, \\ F_{\Omega,b}(f)(x,t) &= \int_{|x-y|<t} (b(x)-b(y))\, K(x-y)f(y)dy, \end{aligned}$$

where b is a BMO function. Recall that the Marcinkiewicz integral and its commutator are given by

$$\begin{aligned} \nu_{\Omega}(f)(x) &= \left(\int_0^{\infty} |F_{\Omega}(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2}, \\ \nu_{\Omega,b}(f)(x) &= \left(\int_0^{\infty} |F_{\Omega,b}(f)(x,t)|^2 \frac{dt}{t^3} \right)^{1/2}. \end{aligned}$$

We have the following theorem for such $\nu_{\Omega}(f)(x)$ and $\nu_{\Omega,b}(f)(x)$.

Theorem 5.1. *The results of Theorem* 3.1, *Corollary* 3.1 *and Theorem* 3.2 *hold true for* $\nu_{\Omega}(f)(x)$ *and the result of Theorem* 4.1 *also holds true for* $\nu_{\Omega,b}(f)(x)$.

Proof. It suffices to verify that hypotheses (a), ($\bar{\text{b}}$) and (c) are satisfied. For assumptions ($\bar{\text{b}}$) and (c), it is enough to show that the kernel K satisfies the estimates (5.1) and (5.2) in Propositions 5.1-5.2 respectively. We begin to show that

$$\frac{|K(x-y) - K(x-z)|}{|x-y|} \leq \frac{C}{|x-y|^n} \left\{ \frac{|y-z|}{|x-y|} \right\}^{\beta} \quad \text{whenever } |x-y| \geq 2|y-z|. \tag{5.3}$$

We have

$$
\begin{aligned}
&|K(x-y)-K(x-z)| = \left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-z)}{|x-z|^{n-1}}\right| \\
\leq &\left|\frac{\Omega(x-y)-\Omega(x-z)}{|x-y|^{n-1}}\right| + |\Omega(x-z)| \cdot \left|\frac{1}{|x-y|^{n-1}} - \frac{1}{|x-z|^{n-1}}\right| \\
=: &\, A + B.
\end{aligned}
$$

For the estimate of A, observe that

$$
\left|\frac{\Omega(x-y)-\Omega(x-z)}{|x-y|^{n-1}}\right| \leq \frac{C}{|x-y|^{n-1}} \left|\frac{x-y}{|x-y|} - \frac{x-z}{|x-z|}\right|^{\beta} \leq \frac{C}{|x-y|^{n-1}} \frac{|y-z|^{\beta}}{|x-y|^{\beta}}.
$$

For the estimate of B, note that Ω is bounded, and thus

$$
|B| \leq \frac{C}{|x-y|^{n-1}} \frac{|y-z|}{|x-y|} \leq \frac{C}{|x-y|^{n-1}} \left(\frac{|y-z|}{|x-y|}\right)^{\beta},
$$

where the last inequality follows since $0 < \beta \leq 1$ and $|x-y| \geq 2|y-z|$. Inequality (5.3) is proved. By similar calculations as above, we also obtain

$$
\frac{|K(x-y)-K(z-y)|}{|x-y|} \leq \frac{C}{|x-y|^{n}} \left\{\frac{|x-z|}{|x-y|}\right\}^{\beta} \quad \text{whenever} \quad |x-y| \geq 2|x-z|. \tag{5.4}
$$

It remains to check assumption (a). We will show that ν_Ω is a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Define the measures σ_t by

$$
(\sigma_t * f)(x) = \frac{1}{t}\int_{|y|<t} K(y) f(x-y) dy = \frac{1}{t}\int_{|y|<t} \frac{\Omega(y')}{|y|^{n-1}} f(x-y) dy \left(y' = \frac{y}{|y|}\right).
$$

Then

$$
|\sigma_t * f(x)| = \left|\frac{1}{t} F_\Omega(f)(x,t)\right| \quad \text{and} \quad \nu_\Omega(f)(x) = \left(\int_0^\infty |\sigma_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}.
$$

In terms of Fourier transform,

$$
\hat{\sigma}_t(\xi) = \frac{1}{t}\int_{|y|<t} \frac{\Omega(y')}{|y|^{n-1}} e^{i\xi\cdot y} dy.
$$

We have the following lemma for $\hat{\sigma}_t(\xi)$.

Lemma 5.1. $|\hat{\sigma}_t(\xi)| \leq C \min\left\{|\xi t|, |\xi t|^{-1/2}\right\}$.

Proof. By the cancellation condition of Ω, we have

$$|\hat{\sigma}_t(\xi)| = \left|\frac{1}{t}\int_{|y|<t}\frac{\Omega(y')}{|y|^{n-1}}\left(e^{i\xi\cdot y}-1\right)dy\right| \le \frac{1}{2}||\Omega||_{L^1(S^{n-1})}|\xi t| \le C|\xi t|.$$

Moreover,

$$\hat{\sigma}_t(\xi) = \frac{1}{t}\int_{S^{n-1}}\Omega(y')\left(\int_0^t e^{i|\xi|r(\xi'\cdot y')}dr\right)d\sigma(y') \equiv \int_{S^{n-1}}\Omega(y')I(y')d\sigma(y'). \tag{5.5}$$

It is clear that

$$|I(y')| = \left|\frac{1}{t}\int_0^t e^{i|\xi|r(\xi'\cdot y')}dr\right| \le 1 \quad \text{for all } t>0. \tag{5.6}$$

On the other hand, a direct integration gives

$$|I(y')| \le \frac{2}{|\xi t||\xi'\cdot y'|},$$

which together with inequality (5.6) imply that

$$|I(y')| \le C|\xi t|^{-1/2}|\xi'\cdot y'|^{-1/2}. \tag{5.7}$$

Substituting (5.7) into (5.5) and using the fact that Ω is bounded, we obtain

$$|\hat{\sigma}_t(\xi)| \le C|\xi t|^{-1/2},$$

finishing the proof of this lemma. □

It follows from the above lemma that for any $\xi \neq 0$,

$$\int_0^\infty |\hat{\sigma}_t(\xi)|^2\frac{dt}{t} = \int_0^{|\xi|^{-1}}|\hat{\sigma}_t(\xi)|^2\frac{dt}{t} + \int_{|\xi|^{-1}}^\infty |\hat{\sigma}_t(\xi)|^2\frac{dt}{t} \le C < \infty, \tag{5.8}$$

where the constant C is independent of $|\xi|$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n}|\nu_\Omega(f)(x)|^2dx &= \int_{\mathbb{R}^n}\int_0^\infty|\sigma_t * f(x)|^2\frac{dt}{t}dx = \int_0^\infty\left(\int_{\mathbb{R}^n}|\sigma_t * f(x)|^2dx\right)\frac{dt}{t}\\ &= \int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\left(\int_0^\infty|\hat{\sigma}_t(\xi)|^2\frac{dt}{t}\right)d\xi \le C\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2d\xi\\ &= C\int_{\mathbb{R}^n}|f(x)|^2dx,\end{aligned}$$

where the last inequality follows from (5.8). The proof of this theorem is completed. □

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