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Harmonic Analysis Associated with the Heckman-Opdam-Jacobi Operator on $\mathbb{R}^{d+1}$

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Abstract. In this paper we consider the Heckman-Opdam-Jacobi operator Δ_{HJ} on $\mathbb{R}^{d+1}$. We define the Heckman-Opdam-Jacobi intertwining operator V_{HJ}, which turn out to be the transmutation operator between Δ_{HJ} and the Laplacian Δ_{d+1}. Next we construct ${}^tV_{HJ}$ the dual of this intertwining operator. We exploit these operators to develop a new harmonic analysis corresponding to Δ_{HJ}.

Key Words: Heckman-Opdam-Jacobi operator, generalized intertwining operator and its dual, generalized Fourier transform, generalized translation operators.

AMS Subject Classifications: 33E30, 42B10, 44A15, 35K05

1 Introduction

Recently, Mejjaoli and Trimèche in [15], have considered and studied the Dunkl-Bessel-Laplace operator on $\mathbb{R}^d \times \mathbb{R}_+$ defined by

$$\Delta_{k,\beta} = \Delta_{k,x'} + L_{\beta,x_{d+1}}, \qquad x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}_+,$$

where Δ_k is the Dunkl Laplacian on $\mathbb{R}^d$ and L_β is the Bessel operator on $\mathbb{R}_+$ given by

$$L_\beta = \frac{d^2}{dx_{d+1}^2} + \frac{2\beta+1}{x_{d+1}} \frac{d}{dx_{d+1}}, \qquad \beta > -\frac{1}{2},$$

and have developed an harmonic analysis associated with it. Based on this paper, several works have been elaborated. We can cite, for example, the work of Hassini and Trimèche in [11]. They have studied the generalized wavelets and generalized windowed transforms associated to the Dunkl-Bessel operator.

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The Jacobi and the Cherednik Heckman-Opdam operators are differential and differential-difference operators defined respectively on $\mathbb{R}$ and $\mathbb{R}^d$, they are well studied in [1, 2, 4, 5, 12, 14, 17] and references there. These operators play a prominent role in the new harmonic analysis theory associated to a new class of differential-difference operators that we attempt to consider in this paper. This class of operators is given by

$$\Delta_{HJ} = \Delta_{k,x'} + (\Delta_{a,b} + \xi^2)_{x_{d+1}}, \qquad x = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R},$$

where Δ_k is the Heckman-Opdam Laplacian on $\mathbb{R}^d$ (see [3, 9, 16, 18, 20]) and $\Delta_{a,b}$ is the Jacobi operator on $\mathbb{R}$ (see [1, 4, 5]).

Throughout this article, we have overcome several difficulties in proving some tools of harmonic analysis associated with the differential-difference operator Δ_{HJ} on $\mathbb{R}^{d+1}$ such that the Plancherel formula which is not verified but at the end we came up with an analog of it. The importance of this new class of operators is derived from those of the two operators Δ_k and $\Delta_{a,b}$.

The outline of this paper is as follows. The second and third sections are devoted to some basic results of harmonic analysis associated respectively with the Jacobi operator on $\mathbb{R}$ and the Cherednik operator on $\mathbb{R}^d$. In the last section, we study the harmonic analysis associated to Δ_{HJ}. In a more specific way, we give some properties of the eigenfunction Λ of this operator equal to 1 at zero. We introduce the generalized intertwining operator and its dual, we define the generalized Fourier transform and we prove for this transform the Paley-Wiener theorem and the inversion formulas. We finish by the generalized translation operators and convolution product that give us the Plancherel type formula. In a latest paper and as an application, we have solved the generalized heat equation associated to Δ_{HJ} and we have shown that the heat semi group has a positive kernel.

2 Harmonic analysis associated to the Jacobi operator on $\mathbb{R}$

In this section we recall some basic results that constitute the harmonic analysis for the Jacobi operator on $\mathbb{R}$. For more details we refer to [4–8, 18].

2.1 The Jacobi operator and function

For $a \geq b \geq -\frac{1}{2}$, $a \neq -\frac{1}{2}$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the Jacobi function $\varphi_\lambda^{(a,b)}$ is defined by

$$\varphi_\lambda(x) = \varphi_\lambda^{(a,b)}(x) = {}_2F_1\left(\frac{\xi + i\lambda}{2}; \frac{\xi - i\lambda}{2}; a+1; -(\sinh x)^2\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\xi = a + b + 1. \tag{2.1}$$

The function $x \longmapsto \varphi_\lambda(x)$ is the unique solution on $\mathbb{R}$ of the differential equation

$$\begin{cases} \Delta_{a,b}u(x) = -(\lambda^2+\xi^2)u(x), \\ u(0)=1, \quad u'(0)=0, \end{cases}$$

where $\Delta_{a,b}$ is the Jacobi differential operator

$$\Delta_{a,b} = \frac{d^2}{dx^2} + [(2a+1)\coth x + (2b+1)\tanh x]\frac{d}{dx} = \frac{1}{A_{a,b}(x)}\frac{d}{dx}\left[A_{a,b}(x)\frac{d}{dx}\right], \tag{2.2a}$$

$$\text{with } A_{a,b}(x) = 2^{2\xi}(\sinh|x|)^{2a+1}(\cosh x)^{2b+1}. \tag{2.2b}$$

For twice differentiable compactly supported function f on $\mathbb{R}$ and twice differentiable function g on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} (\Delta_{a,b}+\xi^2)f(y)g(y)A_{a,b}(y)dy = \int_{\mathbb{R}} f(y)(\Delta_{a,b}+\xi^2)g(y)A_{a,b}(y)dy. \tag{2.3}$$

Proposition 2.1. *The Jacobi function satisfies the following properties:*

1. *For each λ in $\mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is an even infinitely differentiable function on $\mathbb{R}$ and for each x in $\mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$.*

2. *For each $\lambda \in \mathbb{C}$, φ_λ has the following integral representation (see [14])*

$$\forall x \in \mathbb{R}, \qquad \varphi_\lambda(x) = \int_{-|x|}^{|x|} K_{a,b}(x,y)e^{i\lambda y}dy,$$

where $K_{a,b}$ is a nonnegative kernel given by

$$\begin{aligned} K_{a,b}(x,y) =& \frac{2^{(-a+\frac{1}{2})}\Gamma(a+1)}{\sqrt{\pi}\Gamma(a+\frac{1}{2})(\sinh x)^{2a}(\cosh x)^{2b}}(\cosh 2x - \cosh 2y)^{a-\frac{1}{2}} \\ &\times {}_2F_1\left(a+b; a-b; a+\frac{1}{2}; \frac{\cosh x - \cosh y}{2\cosh x}\right). \end{aligned} \tag{2.4}$$

3. *Let $\lambda \in \mathbb{C}$ such that $|\Im(\lambda)| \le \xi$. For all $x \in \mathbb{R}_+$, we have*

   (a) $|\varphi_\lambda(x)| \le 1$,

   (b) $e^{-\xi x} \le \varphi_0(x) \le C(1+x)e^{-\xi x}$, *where C is a positive constant.*

4. *For all $n \in \mathbb{N}$, there exists a positive constant M_n such that*

$$\left|\frac{d^n}{dx^n}\varphi_\lambda(x)\right| \le M_n(1+|x|)(1+|\lambda|)^n e^{(|\Im(\lambda)|-\xi)|x|}, \quad \forall \lambda \in \mathbb{C}, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

5. *There exists a positive constant M such that for all $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| \le M(1+|x|)^{n+1}e^{(|\Im(\lambda)|-\xi)|x|}. \tag{2.6}$$

2.2 The Mehler transforms $\chi_{a,b}$ and ${}^t\chi_{a,b}$

Notation 2.1. We denote by

- $\mathcal{E}_*(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$.
- $\mathcal{D}_*(\mathbb{R})$ the subspace of $\mathcal{E}_*(\mathbb{R})$ consisting of compactly supported functions.
- $\mathcal{S}_*(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$ rapidly decreasing with all its derivatives equipped with the usual topology.
- $\mathcal{S}_{2,*}(\mathbb{R})$ the subspace of $\mathcal{E}_*(\mathbb{R})$ consisting of functions f such that for all $n,m \in \mathbb{N}$,

$$N_{n,m}(f) = \sup_{x\in\mathbb{R}}(1+|x|)^n(\varphi_0(x))^{-1}\left|f^{(m)}(x)\right| < +\infty.$$

  Its topology is defined by the semi-norms $(N_{l,m})_{n,m\in\mathbb{N}}$.

- $PW_{*,r}(\mathbb{C})$, $r>0$, the space of even entire rapidly decreasing functions ψ of exponential type r, that is, for all $m \in \mathbb{N}$,

$$Q_m(\psi) = \sup_{\lambda\in\mathbb{C}}(1+|\lambda|^2)^m e^{-r|\Im(\lambda)|}\,|\psi(\lambda)| < +\infty.$$

  The space

$$PW_*(\mathbb{C}) = \bigcup_{r>0} PW_{*,r}(\mathbb{C})$$

  is equipped with inductive limit topology.

Definition 2.1. *The dual Mehler transform denoted by $\chi_{a,b}$ is defined, for an even continuous and bounded function f on $\mathbb{R}$, by*

$$\forall x \in \mathbb{R}, \qquad \chi_{a,b}(f)(x) = \int_{-|x|}^{|x|} K_{a,b}(x,y)f(y)dy,$$

where $K_{a,b}$ is given by the relation (2.4).

Remark 2.1. We notice that

$$\forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \quad \chi_{a,b}(e^{i\langle\lambda,.\rangle})(x) = \varphi_\lambda(x). \tag{2.7}$$

Theorem 2.1. *$\chi_{(a,b)}$ is a topological isomorphism from $\mathcal{E}_*(\mathbb{R})$ onto itself such that*

$$\chi_{a,b}\frac{d^2}{dx^2} = (\Delta_{a,b}+\xi^2)\chi_{a,b}, \tag{2.8a}$$

$$\chi_{a,b}f(0) = f(0), \quad \forall f \in \mathcal{E}_*(\mathbb{R}). \tag{2.8b}$$

Definition 2.2. *1. The dual* ${}^t\chi_{a,b}$ *of* $\chi_{a,b}$ *is defined by the following relation*

$$\int_{\mathbb{R}} g(x)\, {}^t\chi_{a,b}(f)(x)dx = \int_{\mathbb{R}} \chi_{a,b}(g)(y)f(y)A_{a,b}(y)dy, \\ \forall f \in \mathcal{S}_{2,*}(\mathbb{R}), \quad \forall g \in \mathcal{E}_*(\mathbb{R}). \tag{2.9}$$

2. For $f \in \mathcal{D}_*(\mathbb{R})$ *(resp.* $\mathcal{S}_{2,*}(\mathbb{R})$*), this operator is given by the following integral*

$$\forall x \in \mathbb{R}, \qquad {}^t\chi_{a,b}(f)(x) = \int_{|y|\geq|x|} K_{a,b}(x,y)f(y)A_{a,b}(y)dy, \tag{2.10}$$

where $K_{a,b}$ *is given by the relation* (2.4)*.*

Theorem 2.2. *The operator* ${}^t\chi_{a,b}$ *is a linear topological isomorphism from*

1. $\mathcal{D}_*(\mathbb{R})$ *onto itself,*
2. $\mathcal{S}_{2,*}(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$*,*

satisfying

$${}^t\chi_{a,b}(\Delta_{a,b} + \xi^2)f = \frac{d^2}{dx^2}\, {}^t\chi_{a,b}f. \tag{2.11}$$

2.3 The generalized Fourier transform associated to the Jacobi operator

Definition 2.3. *The generalized Fourier transform associated to the Jacobi operator* $\mathcal{F}_{a,b}$ *is defined on* $\mathcal{D}_*(\mathbb{R})$ *(resp.* $\mathcal{S}_{2,*}(\mathbb{R})$*) by*

$$\forall \lambda \in \mathbb{C}, \quad \mathcal{F}_{a,b}(f)(\lambda) = \int_{\mathbb{R}} f(x)\varphi_\lambda(x)A_{a,b}(x)dx. \tag{2.12}$$

Remark 2.2. Using the relation (2.3) for all f in $\mathcal{D}_*(\mathbb{R})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R})$), we obtain

$$\forall \lambda \in \mathbb{C}, \quad \mathcal{F}_{a,b}((\Delta_{a,b} + \xi^2)f)(\lambda) = -\lambda^2 \mathcal{F}_{a,b}(f)(\lambda). \tag{2.13}$$

Theorem 2.3. *1. The generalized Fourier transform* $\mathcal{F}_{a,b}$ *is a topological isomorphism from*

- $\mathcal{D}_*(\mathbb{R})$ *onto* $PW_*(\mathbb{C})$*,*
- $\mathcal{S}_{2,*}(\mathbb{R})$ *onto* $\mathcal{S}_*(\mathbb{R})$*.*

2. (Inversion formula) For f *in* $\mathcal{S}_*(\mathbb{R})$*, the inverse* $\mathcal{F}^{-1}_{a,b}(f)$ *is given by*

$$\forall x \in \mathbb{R}, \quad \mathcal{F}^{-1}_{a,b}(f)(x) = \int_{\mathbb{R}} f(\lambda)\varphi_\lambda(x)d\sigma(\lambda), \tag{2.14}$$

where

$$d\sigma(\lambda) = |c_{a,b}(\lambda)|^{-2}\, d\lambda, \tag{2.15}$$

with $|c_{a,b}(\lambda)|^{-2}$ *is an even continuous function on* $\mathbb{R}$ *satisfying the estimate:*
There exist positive constants k*,* k_1 *and* k_2 *such that, for* $a > -1/2$*,*

$$\forall \lambda > k, \quad k_1\lambda^2 \leq |c_{a,b}(\lambda)|^{-2} \leq k_2\lambda^2.$$

3 Harmonic analysis associated to the Heckman-Opdam Laplacian on $\mathbb{R}^d$

In this section we give a brief reminder about the theory of the Cherednik operators (see, e.g., [2,3,10,12,16,17]). More precisely, we introduce the Fourier transform and the intertwining operators associated with these operators.

Notation 3.1. We denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$.
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ consisting of compactly supported functions.
- $\mathcal{S}(\mathbb{R}^d)$ the classical Schwartz space.
- $\mathcal{S}_2(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ consisting of functions f such that for all $l, n \in \mathbb{N}$,

$$p_{l,n}(f) = \sup_{|\nu|\le n,\ x\in\mathbb{R}^d} (1+\|x\|)^l \left(F_0(x)\right)^{-1} |D^\nu f(x)| < +\infty,$$

  where $\nu = (\nu_1, \cdots, \nu_d) \in \mathbb{N}^d$, $|\nu| = \sum_{i=1}^d \nu_i$, $D^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\cdots\partial x_d^{\nu_d}}$ and F_0 is the Heckman-Opdam hypergeometric function defined by

$$F_0(x) = \frac{1}{|W|}\sum_{w\in W} G_0(wx)$$

  with G_0 is the Opdam hypergeometric kernel given later. Its topology is defined by the semi-norms $p_{l,n}$ for $l, n \in \mathbb{N}$.

- $PW_a(\mathbb{C}^d)$, $a > 0$, the space of entire functions g on $\mathbb{C}^d$ satisfying, for $m \in \mathbb{N}$,

$$q_m(g) = \sup_{\lambda\in\mathbb{C}^d} (1+\|\lambda\|)^m e^{-a\|\Im(\lambda)\|} |g(\lambda)| < +\infty.$$

  The space

$$PW(\mathbb{C}^d) = \bigcup_{a>0} PW_a(\mathbb{C}^d)$$

  is called the Paley-Wiener space. It is equipped with inductive limit topology.

3.1 Root system, multiplicity function and the Cherednik operators

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle\cdot,\cdot\rangle$ and $\|x\| = \sqrt{\langle x, x\rangle}$. For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha$$

be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\mathcal{R} \cap \mathbb{R}^d$, $\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha(\mathcal{R}) = \mathcal{R}$ for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$, the reflections σ_α, $\alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$ called the reflection group associated with $\mathcal{R}$. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$. Then, for each $\alpha \in \mathcal{R}$, either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$. A root α is called non-multipliable if 2α is not a root.

A function $k : \mathbb{R}^d \longrightarrow [0, +\infty]$ on the root system $\mathcal{R}$ is called a multiplicity function if it is invariant under the action of the reflection group W.

Moreover, let A_k denotes the weight function

$$\forall x \in \mathbb{R}^d, \quad A_k(x) = \prod_{\alpha \in \mathcal{R}_+} \left| 2 \sinh \left\langle \frac{\alpha}{2}, x \right\rangle \right|^{2k(\alpha)}, \tag{3.1}$$

which is W-invariant. For any suitable function f on $\mathbb{R}^d$ and for $x \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$, the Cherednik operators $T_j, j = 1, 2, \cdots ., d$, associated with the reflection group W and the multiplicity function k, are defined by

$$\begin{aligned} T_j f(x) &= \partial_j f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{1 - e^{-\langle \alpha, x \rangle}} - \rho_j f(x), \\ \text{where} \quad \rho_j &= \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha_j, \quad \alpha_j = \langle \alpha, e_j \rangle . \end{aligned} \tag{3.2}$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the operators T_j, $j = 1, \cdots, d$, reduce to the corresponding partial derivatives. We suppose in the following that $k > 0$. The Cherednik operators form a commutative system of differential-difference operators. The Heckman-Opdam Laplacian is defined, for a regular function f on $\mathbb{R}^d$ and for $x \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$, by

$$\begin{aligned} \Delta_k f(x) =& \sum_{j=1}^{d} T_j^2 f(x) \\ =& \Delta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \coth \left[\frac{\langle \alpha, x \rangle}{2} \right] \langle \nabla f(x), \alpha \rangle \\ &+ \|\rho\|^2 f(x) - \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \frac{|\alpha|^2}{4 \sinh^2 \frac{\langle \alpha, x \rangle}{2}} \{ f(x) - f(\sigma_\alpha(x)) \}, \end{aligned}$$

where $\Delta = \sum_{j=1}^d \partial_j^2$ and $\nabla = (\partial_1, \cdots, \partial_d)$ are respectively the Laplacian and the gradient on $\mathbb{R}^d$.

Theorem 3.1. *For every twice differentiable compactly supported function f and twice differen-*

tiable function g on $\mathbb{R}^d$, we have for all $j=1,\cdots,d$,

$$\int_{\mathbb{R}^d} T_j f(x) g(-x) A_k(x) dx = \int_{\mathbb{R}^d} f(x) T_j g(-x) A_k(x) dx. \tag{3.3a}$$

$$\int_{\mathbb{R}^d} \Delta_k f(x) g(-x) A_k(x) dx = \int_{\mathbb{R}^d} f(x) \Delta_k g(-x) A_k(x) dx. \tag{3.3b}$$

Proof. In order to demonstrate the relation (3.3a), we recall the duality relation given in [19] by

$$\int_{\mathbb{R}^d} T_j f(x) g(x) A_k(x) dx = -\int_{\mathbb{R}^d} f(x)(T_j + S_j) g(x) A_k(x) dx,$$

where

$$S_j g(x) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j g(\sigma_\alpha(x)).$$

Using this relation above, we have

$$\begin{aligned}\int_{\mathbb{R}^d} T_j f(x) g(-x) A_k(x) dx &= \int_{\mathbb{R}^d} T_j f(x) \check{g}(x) A_k(x) dx \\ &= -\int_{\mathbb{R}^d} f(x)(T_j + S_j)\check{g}(x) A_k(x) dx.\end{aligned}$$

By the definition of the operators T_j and S_j, we obtain

$$\begin{aligned}&-T_j\check{g}(x) - S_j\check{g}(x)\\ =&\partial_j g(-x) - \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j \frac{g(-x)-g(\sigma_\alpha(-x))}{1-e^{-\langle\alpha,x\rangle}} + \rho_j g(-x) - \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j \check{g}(\sigma_\alpha(x)).\end{aligned}$$

Since

$$[T_j g](-x) = \partial_j g(-x) + \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j \frac{g(-x)-g(\sigma_\alpha(-x))}{1-e^{-\langle\alpha,-x\rangle}} - \rho_j g(-x),$$

and using the relation (3.2), we have

$$\begin{aligned}&-T_j\check{g}(x) - S_j\check{g}(x) - [T_j g](-x)\\ =&\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j [g(-x)-g(\sigma_\alpha(-x))]\left[1-\frac{1}{1-e^{-\langle\alpha,x\rangle}}-\frac{1}{1-e^{\langle\alpha,x\rangle}}\right]=0,\end{aligned}$$

which gives the desired result.

For the relation (3.3b), it follows immediately from the duality relation (3.3a). □

3.2 The Opdam hypergeometric kernel

The Opdam hypergeometric kernel G_λ, $\lambda\in\mathbb{C}^d$, is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda = \lambda_j G_\lambda, \quad j=1,\cdots,d,\\ G_\lambda(0)=1. \end{cases} \tag{3.4}$$

Proposition 3.1 ([17,20]). *The function G_λ possesses the following properties:*

1. *For all $\lambda \in \mathbb{C}^d$, the function $x \longmapsto G_\lambda(x)$ is infinitely differentiable on $\mathbb{R}^d$.*
2. *For all $x \in \mathbb{R}^d$, the function $\lambda \longmapsto G_\lambda(x)$ is entire on $\mathbb{C}^d$.*
3. *For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have (see [16])*
$$\overline{G_\lambda(x)} = G_{-\overline{\lambda}}(x), \quad |G_\lambda(x)| \le |W|^{1/2}\, e^{\max_{w\in W}\Re(\langle w\lambda, x\rangle)}. \tag{3.5}$$
In particular, for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$, we have $|G_{i\lambda}(x)| \le |W|^{1/2}$.
4. *Let p and q be polynomials of degree m and n respectively. Then, there exists a positive constant M such that, for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have*
$$\left| p\left(\frac{\partial}{\partial\lambda}\right) q\left(\frac{\partial}{\partial x}\right) G_\lambda(x)\right| \le M(1+\|x\|)^m(1+\|\lambda\|)^n F_0(x) e^{\max_{w\in W}\Re\langle w\lambda, x\rangle}. \tag{3.6}$$
5. *Let $\overline{a_+}$ be the closure of the positive Weyl chamber a_+ defined by*
$$a_+ = \{x \in \mathbb{R}^d; \forall \alpha \in \mathcal{R}_+, \langle \alpha, x\rangle > 0\}, \tag{3.7}$$
and $\mathcal{R}_+^0$ be the set of positive non-multipliable roots. Then for every x in $\overline{a_+}$, the function $G_0(x)$ satisfies the following estimate
$$G_0(x) \asymp \prod_{\alpha\in\mathcal{R}_+^0} (1+\langle \alpha, x\rangle) e^{-\langle \rho, x\rangle}, \tag{3.8}$$
where $\rho = (\rho_1, \cdots, \rho_d) \in \mathbb{R}^d$ with ρ_j is given by the relation (3.2).

3.3 The Cherednik transform

Definition 3.1. *The Cherednik transform $\mathcal{H}_k$ of a function f in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) is defined by*
$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}_k(f)(\lambda) = \int_{\mathbb{R}^d} f(x) G_{i\lambda}(-x) A_k(x)dx,$$
where A_k is given by the relation (3.1).

Proposition 3.2. *For every $\lambda \in \mathbb{R}^d$ and $f \in \mathcal{D}(\mathbb{R}^d)$, we have*
$$\mathcal{H}_k(T_j f)(\lambda) = i\lambda_j \mathcal{H}_k(f)(\lambda), \quad \forall j = 1, \cdots, d, \tag{3.9a}$$
$$\mathcal{H}_k(\Delta_k f)(\lambda) = -\|\lambda\|^2 \mathcal{H}_k(f)(\lambda). \tag{3.9b}$$

Proof. It follows immediately from the system (3.4) and the relations (3.3a) and (3.3b). □

Theorem 3.2 ([9,20]).

1. *The Cherednik transform $\mathcal{H}_k$ is a topological isomorphism from*
   - $\mathcal{D}(\mathbb{R}^d)$ *onto* $PW(\mathbb{C}^d)$,
   - $\mathcal{S}_2(\mathbb{R}^d)$ *onto* $\mathcal{S}(\mathbb{R}^d)$.
2. *(Inversion formula) For every* $f \in \mathcal{S}_2(\mathbb{R}^d)$, *we have*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}_k(f)(\lambda) G_{i\lambda}(x) C_k(\lambda) d\lambda, \quad \forall x \in \mathbb{R}^d, \tag{3.10}$$

where

$$\begin{aligned} C_k(\lambda) = & c_0 \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma\left(i\left\langle \lambda, \frac{2}{\|\alpha\|}\alpha \right\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha)\right)}{\Gamma\left(i\left\langle \lambda, \frac{2}{\|\alpha\|}\alpha \right\rangle + \frac{1}{2}k(\frac{\alpha}{2})\right)} \\ & \times \frac{\Gamma\left(-i\left\langle \lambda, \frac{2}{\|\alpha\|}\alpha \right\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha) + 1\right)}{\Gamma\left(-i\left\langle \lambda, \frac{2}{\|\alpha\|}\alpha \right\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + 1\right)}, \end{aligned} \tag{3.11}$$

and c_0 *is a positive constant.*

3.4 The transmutation operators associated to the Cherednik operators

Definition 3.2. *The dual Cherednik transmutation operator* ${}^tV_k$ *is defined on* $\mathcal{S}_2(\mathbb{R}^d)$ *by the following relation*

$${}^tV_k = \mathcal{F}^{-1} \circ \mathcal{H}_k, \tag{3.12}$$

where $\mathcal{F}$ *is the classical Fourier transform on* $\mathbb{R}^d$ *given by*

$$\forall x \in \mathbb{R}^d, \quad \mathcal{F}f(x) = \int_{\mathbb{R}^d} f(y) e^{-i\langle x,y \rangle} dy.$$

Theorem 3.3. *The operator* ${}^tV_k$ *is a linear topological isomorphism from*

- $\mathcal{D}(\mathbb{R}^d)$ *onto itself,*
- $\mathcal{S}_2(\mathbb{R}^d)$ *onto* $\mathcal{S}(\mathbb{R}^d)$.

Proof. It is deduced immediately from Theorem 3.2 and the relation (3.12). □

Proposition 3.3. *For all function* f *in* $\mathcal{S}_2(\mathbb{R}^d)$, *we have*

$$\forall j = 1, \cdots, d, \quad {}^tV_k(T_j f) = \partial_j({}^tV_k f), \tag{3.13a}$$

$${}^tV_k \Delta_k f = \Delta {}^tV_k f. \tag{3.13b}$$

Proof. The relations (3.9a) and (3.12) give the desired results. □

Definition 3.3. *We define the Cherednik transmutation operator V_k on $\mathcal{E}(\mathbb{R}^d)$ by the following duality relation*

$$\int_{\mathbb{R}^d} V_k(g)(x)f(x)A_k(x)dx = \int_{\mathbb{R}^d} g(x)^t V_k(f)(x)dx, \tag{3.14}$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) and $g \in \mathcal{E}(\mathbb{R}^d)$.

Proposition 3.4 ([19]). *For all $g \in \mathcal{E}(\mathbb{R}^d)$, we have*

$$\forall j = 1, \cdots, d, \quad T_j V_k(g) = V_k(\partial_j g), \tag{3.15a}$$

$$\Delta_k V_k(g) = V_k(\Delta g). \tag{3.15b}$$

Corollary 3.1. *For $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$,*

$$G_\lambda(x) = V_k(e^{\langle \lambda, \cdot \rangle})(x). \tag{3.16}$$

4 Harmonic analysis associated to the Heckman-Opdam-Jacobi operator on $\mathbb{R}^{d+1}$

Notation 4.1. We denote by

- $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$, $x = (x_1, \cdots, x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$.
- $C^2_*(\mathbb{R}^{d+1})$ the space of twice continuously differentiable functions on $\mathbb{R}^{d+1}$ even with respect to the last variable.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$ even with respect to the last variable.
- $\mathcal{D}_*(\mathbb{R}^{d+1})$ the subspace of $\mathcal{E}_*(\mathbb{R}^{d+1})$ consisting of compactly supported functions.
- $\mathcal{S}_*(\mathbb{R}^{d+1})$ the subspace of the classical Schwartz space $\mathcal{S}(\mathbb{R}^{d+1})$ consisting of functions which are even with respect to the last variable.
- $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ the subspace of $\mathcal{E}_*(\mathbb{R}^{d+1})$ such that for all $m, n \in \mathbb{N}$,

$$\tau_{m,n}(f) = \sup_{|\nu| \le n,\ x \in \mathbb{R}^{d+1}} (1 + \|x\|)^m \left(\varphi_0(x_{d+1})\right)^{-1} \left(F_0(x')\right)^{-1} |D^\nu f(x)| < +\infty,$$

  with $\nu = (\nu_1, \cdots, \nu_{d+1}) \in \mathbb{N}^{d+1}$, $|\nu| = \sum_{i=1}^{d+1} \nu_i$ and $D^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_{d+1}^{\nu_{d+1}}}$. Its topology is defined by the semi-norms $\tau_{m,n}$ for $m, n \in \mathbb{N}$.

4.1 The Heckman-Opdam-Jacobi operator

Let

$$\rho = (\rho_1, \cdots, \rho_d, \rho_{d+1}) = (\rho', \rho_{d+1}) \in \mathbb{R}^d \times \mathbb{R},$$

where ρ_j, $1 \leq j \leq d$, is defined by the relation (3.2) and ρ_{d+1} is given by the relation (2.1).

For all $f \in C^2_*(\mathbb{R}^{d+1})$, we define the Heckman-Opdam-Jacobi operator Δ_{HJ} as follows

$$\forall x \in \mathbb{R}^{d+1}, \quad \Delta_{HJ} f(x) = (\Delta_{a,b} + \rho^2_{d+1})_{x_{d+1}} f(x) + \Delta_{k,x'} f(x).$$

In the limit case $k = 0$ and $a = b = -\frac{1}{2}$, Δ_{HJ} reduces to the usual Laplacian. For every $x \in \mathbb{R}^{d+1}$ and $\lambda \in \mathbb{R}^{d+1}$, the Heckman-Opdam-Jacobi kernel Λ given by

$$\Lambda(x, \lambda) = G_{i\lambda'}(x') \varphi_{\lambda_{d+1}}(x_{d+1}),$$

is a solution of the system

$$\begin{cases} \Delta_{HJ} u(x, \lambda) = -\|\lambda\|^2 u(x, \lambda), \\ u(0, \lambda) = 1, \quad \dfrac{\partial u}{\partial x_{d+1}}((x', 0), \lambda) = 0. \end{cases}$$

Remark 4.1. Let $d = 1$ and consider the root system $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$ and $W = \mathbb{Z}_2$. We consider a positive multiplicity function k on W and we put $k_1 = k(\alpha) + k(2\alpha)$, $k_2 = k(2\alpha)$ so $\rho = k(\alpha) + 2k(2\alpha) = k_1 + k_2$. The Cherednik operator and the Heckman-Opdam Laplacian are given by

$$\begin{aligned} T_1 f(x) &= f'(x) + \left(\frac{2k(\alpha)}{1 - e^{-2x}} + \frac{4k(2\alpha)}{1 - e^{-4x}} \right) [f(x) - f(-x)] - \rho f(x) \\ &= f'(x) + (k_1 \coth x + k_2 \tanh x) [f(x) - f(-x)] - \rho f(-x), \\ \Delta_k f(x) &= \Delta_{a,b} f(x) - \Big[(k_1 \coth^2 x + k_2 \tanh^2 x - \rho) \Big] [f(x) - f(-x)] + \rho^2 f(x), \end{aligned}$$

where $c = k_1 - \frac{1}{2}$ and $e = k_2 - \frac{1}{2}$.

So, the operator Δ_{HJ} can be written as follow

$$\begin{aligned} \Delta_{HJ} f(x, y) = &\left(\Delta_{a,b} + \rho_2^2 \right)_y f(x, y) + (\Delta_{c,e} + \rho_1^2)_x f(x, y) \\ &+ \Big((k_1 - k_2) \tanh^2 x + (k_2 - 2k_1) \Big) (f(x, y) - f(-x, y)) \end{aligned}$$

and the function Λ becomes

$$\Lambda((x, y), (\lambda_1, \lambda_2)) = \varphi^{a,b}_{\lambda_2}(y) \left[\varphi^{c,e}_{\lambda_1}(x) + \frac{1}{i\lambda_1 - \rho_1} \frac{d}{dx} \varphi^{c,e}_{\lambda_1}(x) \right].$$

Furthermore, if we take $\lambda = \lambda_1 = \lambda_2$ and $(a,b) = (c,e)$, we obtain

$$\begin{aligned}\Lambda((x,y),(\lambda,\lambda)) =&\varphi_\lambda^{c,e}(x)\varphi_\lambda^{c,e}(y) + \frac{1}{i\lambda-\rho_1}\varphi_\lambda^{c,e}(y)\frac{d}{dx}\varphi_\lambda^{c,e}(x)\\ =&\tau^y\varphi_\lambda^{c,e}(x) + \frac{1}{i\lambda-\rho_1}\varphi_\lambda^{c,e}(y)\frac{d}{dx}\varphi_\lambda^{c,e}(x),\end{aligned}$$

where τ^x is the generalized translation associated to the Jacobi operator defined by

$$\forall f\in\mathcal{S}_{2,*}(\mathbb{R}),\quad \forall y\in\mathbb{R},\quad \mathcal{F}_{a,b}(\tau^x f)(y) = \varphi_\lambda^{a,b}(x)\mathcal{F}_{a,b}f(y),$$

and $\mathcal{F}_{a,b}$ is given by the relation (2.12).

Proposition 4.1. *The function Λ satisfies the following properties:*

1. *For $\lambda\in\mathbb{R}^{d+1}$, the function $x\longmapsto\Lambda(x,\lambda)$ is infinitely differentiable on $\mathbb{R}^{d+1}$ and for $x\in\mathbb{R}^{d+1}$, the function $\lambda\longmapsto\Lambda(x,\lambda)$ is infinitely differentiable on $\mathbb{R}^{d+1}$.*

2. • *For all $\nu=(\nu_1,\cdots,\nu_d,\nu_{d+1})\in\mathbb{N}^{d+1}$, there exists a positive constant M such that for all $\lambda\in\mathbb{R}^{d+1}$ and $x\in\mathbb{R}^{d+1}$,*

$$|D_x^\nu\Lambda(x,\lambda)|\le M(1+\|\lambda'\|)^{|\nu'|}(1+|\lambda_{d+1}|)^{\nu_{d+1}}(1+|x_{d+1}|)F_0(x')e^{-\rho_{d+1}|x_{d+1}|},$$

   where

$$|\nu'| = \sum_{i=1}^d \nu_i \quad\text{and}\quad |\nu| = \sum_{i=1}^{d+1}\nu_i.$$

   • *For all $\mu=(\mu_1,\cdots,\mu_{d+1})\in\mathbb{N}^{d+1}$, there exists a positive constant M such that for all $\lambda\in\mathbb{R}^{d+1}$ and $x\in\mathbb{R}^{d+1}$,*

$$\left|D_\lambda^\mu\Lambda(x,\lambda)\right|\le M(1+\|x'\|)^{|\mu'|}(1+|x_{d+1}|)^{\mu_{d+1}+1}F_0(x')e^{-\rho_{d+1}|x_{d+1}|},$$

   where

$$|\mu'| = \sum_{i=1}^d \mu_i \quad\text{and}\quad |\mu| = \sum_{i=1}^{d+1}\mu_i.$$

3. *For $x\in\mathbb{R}^{d+1}$ and $\lambda\in\mathbb{R}^{d+1}$, we have*

$$|\Lambda(x,\lambda)|\le|W|^{\frac{1}{2}}. \tag{4.1}$$

4. *For $x=(x',x_{d+1})$ in $\overline{\mathfrak{a}_+}\times\mathbb{R}_+$, there exist two positive constants c_1 and c_2 such that*

$$c_1e^{-\langle\rho,x\rangle}\prod_{\alpha\in\mathcal{R}_+^0}(1+\langle\alpha,x'\rangle)\le\Lambda(x,0)\le c_2(1+x_{d+1})e^{-\langle\rho,x\rangle}\prod_{\alpha\in\mathcal{R}_+^0}(1+\langle\alpha,x'\rangle),$$

   where $\overline{\mathfrak{a}_+}$ is the closure of the positive Weyl chamber given by the relation (3.7)*, $\mathcal{R}_+^0$ is the set of positive indivisible roots.*

Proof. 1. It is deduced from the properties of $\varphi_\lambda(x)$ given in Proposition 2.1 and the properties of $G_\lambda(x)$ given in Proposition 3.1.

2. By using the inequalities (2.5), (2.6) and (3.6), we obtain the estimations below.

3. We deduce the result by combining the property (3) in Proposition 2.1 with the inequality (3.5).

4. Using the inequalities in the property (3) and the estimate (3.8), we obtain the result.

Thus, we complete the proof. □

4.2 The generalized transmutation operator and its dual

Definition 4.1. *The generalized transmutation operator is defined on $\mathcal{E}_*(\mathbb{R}^{d+1})$ by*

$$\forall x\in\mathbb{R}^{d+1},\quad V_{HJ}f(x)=\begin{cases}\displaystyle\int_{-|x_{d+1}|}^{|x_{d+1}|}K_{a,b}(x_{d+1},y)V_{k,x'}f(x',y)dy, & \text{if } x_{d+1}\neq 0,\\ V_kf(x',0), & \text{if } x_{d+1}=0,\end{cases}\tag{4.2}$$

where $K_{a,b}$ is given by the relation (2.4).

Remark 4.2. The operator V_{HJ} can also be written in the form $V_{HJ}=\chi_{a,b}\otimes V_k$ and satisfies the following property:

$$\forall x,\lambda\in\mathbb{R}^{d+1},\quad V_{HJ}(e^{i\langle\lambda,\cdot\rangle})(x)=\Lambda(x,\lambda).\tag{4.3}$$

Indeed, by using the relations (2.7) and (3.16), we obtain

$$\begin{aligned}V_{HJ}(e^{i\langle\lambda,\cdot\rangle})(x)&=\left[\chi_{a,b,x_{d+1}}\otimes V_{k,x'}\right]e^{i\langle(\lambda',\lambda_{d+1}),\cdot\rangle}(x',x_{d+1})\\&=\left[\chi_{a,b,x_{d+1}}\otimes V_{k,x'}\right]e^{i\lambda_{d+1}.x_{d+1}}e^{i\langle\lambda',x'\rangle}\\&=\chi_{a,b,x_{d+1}}(e^{i\lambda_{d+1}.x_{d+1}}).V_{k,x'}(e^{i\langle\lambda',x'\rangle})\\&=\varphi_{\lambda_{d+1}}(x_{d+1})G_{i\lambda'}(x')=\Lambda(x,\lambda).\end{aligned}$$

Theorem 4.1. *For every function $f\in\mathcal{E}_*(\mathbb{R}^{d+1})$, $V_{HJ}f$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$ and*

$$\Delta_{HJ}V_{HJ}f=V_{HJ}\Delta_{d+1}f,$$

where $\Delta_{d+1}=\sum_{i=1}^{d+1}\partial_i^2$ is the classical Laplacian on $\mathbb{R}^{d+1}$.

Proof. Let $f\in\mathcal{E}_*(\mathbb{R}^{d+1})$. From the relation (4.2) and the properties of V_k and $K_{a,b}$, we deduce that $V_{HJ}f$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$. By using the relations (2.8) and (3.15b) and for every $x \in \mathbb{R}^{d+1}$, we obtain that

$$\begin{aligned}
&\Delta_{HJ} V_{HJ} f(x) = \Delta_{HJ} \left[\chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] f(x)\\
=&(\Delta_{a,b} + \rho_{d+1}^2)_{x_{d+1}} \left[\chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] f(x) + \Delta_{k,x'} \left[\chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] f(x)\\
=& \left[(\Delta_{a,b} + \rho_{d+1}^2)_{x_{d+1}} \chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] f(x) + \left[\chi_{a,b,x_{d+1}} \otimes \Delta_{k,x'} V_{k,x'}\right] f(x)\\
=& \left[\chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] \frac{\partial^2}{\partial x_{d+1}^2} f(x) + \left[\chi_{a,b,x_{d+1}} \otimes V_{k,x'}\right] \Delta_d f(x) = V_{HJ} \Delta_{d+1} f(x).
\end{aligned}$$

Thus, we complete the proof. □

Definition 4.2. *The generalized dual transmutation operator ${}^tV_{HJ}$ is defined by*

$${}^tV_{HJ} f(y) = \int_{|y_{d+1}| \le |t|} K_{a,b}(y_{d+1}, t)\, {}^tV_{k,y'} f(y', t) A_{a,b}(t) dt, \quad \forall f \in \mathcal{D}_*(\mathbb{R}^{d+1}), \quad \forall y \in \mathbb{R}^{d+1}.$$

Remark 4.3. We can write ${}^tV_{HJ}$ in the form ${}^tV_{HJ} = {}^t\chi_{a,b} \otimes {}^tV_k$, where ${}^t\chi_{a,b}$ is given by the relation (2.10) and ${}^tV_k$ is defined by the relation (3.12).

Theorem 4.2. *The operator ${}^tV_{HJ}$ is a topological isomorphism from*

- $\mathcal{D}_*(\mathbb{R}^{d+1})$ *onto itself,*
- $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ *onto* $\mathcal{S}_*(\mathbb{R}^{d+1})$,

satisfying ${}^tV_{HJ} \Delta_{HJ} f = \Delta_{d+1} {}^tV_{HJ} f$.

Proof. As ${}^tV_k$ is a topological isomorphism from $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) onto itself (resp. $\mathcal{S}(\mathbb{R}^d)$) and ${}^t\chi_{a,b}$ is a topological isomorphism from $\mathcal{D}_*(\mathbb{R})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R})$) onto itself (resp. $\mathcal{S}(\mathbb{R})$), we deduce that ${}^tV_{HJ}$ is a topological isomorphism from $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) onto itself (resp. $\mathcal{S}_*(\mathbb{R}^{d+1})$) (see [13, p. 40, Corollary 1]).

Let $y \in \mathbb{R}^{d+1}$. Using the relations (2.11) and (3.13b), we obtain

$$\begin{aligned}
&{}^tV_{HJ}(\Delta_{HJ} f)(y) = \left[{}^t\chi_{a,b} \otimes {}^tV_k\right] (\Delta_{HJ} f)(y)\\
=& \left[{}^t\chi_{a,b,y_{d+1}} \otimes {}^tV_{k,y'}\right] \left[(\Delta_{a,b} + \rho_{d+1}^2)_{y_{d+1}} + \Delta_{k,y'}\right] f(y)\\
=& \left[{}^t\chi_{a,b,y_{d+1}} (\Delta_{a,b} + \rho_{d+1}^2)_{y_{d+1}} \otimes {}^tV_{k,y'}\right] f(y) + \left[{}^t\chi_{a,b,y_{d+1}} \otimes {}^tV_{k,y'} \Delta_{k,y'}\right] f(y)\\
=& \frac{\partial^2}{\partial y_{d+1}^2} \left[{}^t\chi_{a,b,y_{d+1}} \otimes {}^tV_{k,y'}\right] f(y) + \Delta_d \left[{}^t\chi_{a,b,y_{d+1}} \otimes {}^tV_{k,y'}\right] f(y) = \Delta_{d+1} {}^tV_{HJ} f(y).
\end{aligned}$$

Thus, we complete the proof. □

Proposition 4.2. *The integral transforms V_{HJ} and ${}^tV_{HJ}$ are transposate, i.e., for all $f \in \mathcal{D}_*(\mathbb{R}^{d+1})$ and $g \in \mathcal{E}_*(\mathbb{R}^{d+1})$, we have the following duality relation:*

$$\int_{\mathbb{R}^{d+1}} {}^tV_{HJ} f(y) g(y) dy = \int_{\mathbb{R}^{d+1}} f(y) V_{HJ} g(y) A_{HJ}(y) dy, \tag{4.4}$$

where

$$A_{HJ}(y) = A_k(y')A_{a,b}(y_{d+1}), \tag{4.5}$$

with $A_{a,b}$ *and* A_k *are given respectively by the relations* (2.2b) *and* (3.1).

Proof. Let $f \in \mathcal{D}_*(\mathbb{R}^{d+1})$ and $g \in \mathcal{E}_*(\mathbb{R}^{d+1})$. By using the duality relations (2.9) and (3.14), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} {}^tV_{HJ}f(y)g(y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}} {}^tV_{HJ}f(y',y_{d+1})g(y',y_{d+1})dy_{d+1}dy' \\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{|y_{d+1}|\geq|t|} K_{a,b}(y_{d+1},t)\,{}^tV_{k,y'}f(y',t)A_{a,b}(t)dtg(y',y_{d+1})dy_{d+1}dy' \\
=&\int_{\mathbb{R}}\int_{|y_{d+1}|\geq|t|} K_{a,b}(y_{d+1},t)\left[\int_{\mathbb{R}^d} {}^tV_{k,y'}f(y',t)g(y',y_{d+1})dy'\right]A_{a,b}(t)dtdy_{d+1} \\
=&\int_{\mathbb{R}}\int_{|y_{d+1}|\geq|t|} K_{a,b}(y_{d+1},t)\left[\int_{\mathbb{R}^d} f(y',t)V_{k,y'}g(y',y_{d+1})A_k(y')dy'\right]A_{a,b}(t)dtdy_{d+1} \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left[\int_{|y_{d+1}|\geq|t|} K_{a,b}(y_{d+1},t)f(y',t)A_{a,b}(t)dt\right]V_{k,y'}g(y',y_{d+1})A_k(y')dy'dy_{d+1} \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}^d} {}^t\chi_{a,b,y_{d+1}}f(y',y_{d+1})V_{k,y'}g(y',y_{d+1})A_k(y')dy'dy_{d+1} \\
=&\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}} {}^t\chi_{a,b,y_{d+1}}f(y',y_{d+1})V_{k,y'}g(y',y_{d+1})dy_{d+1}\right]A_k(y')dy' \\
=&\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}} f(y',y_{d+1})\chi_{a,b,y_{d+1}}V_{k,y'}g(y',y_{d+1})A_{a,b}(y_{d+1})dy_{d+1}\right]A_k(y')dy' \\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}} f(y',y_{d+1})V_{HJ}g(y',y_{d+1})A_{a,b}(y_{d+1})dy_{d+1}A_k(y')dy' \\
=&\int_{\mathbb{R}^{d+1}} f(y)V_{HJ}g(y)A_{HJ}(y)dy.
\end{aligned}$$

Thus, we complete the proof. □

4.3 The generalized Fourier transform

Notation 4.2. We denote by

- $PW_{r,*}(\mathbb{C}^{d+1}), r>0$, the space of entire functions g on $\mathbb{C}^{d+1}$ even with respect to the last variable and satisfying, for $m \in \mathbb{N}$,

$$P_m(g) = \sup_{\lambda\in\mathbb{C}^{d+1}} (1+\|\lambda\|)^m e^{-r\|\Im(\lambda)\|}\,|g(\lambda)| < +\infty.$$

  The space

$$PW_*(\mathbb{C}^{d+1}) = \bigcup_{r>0} PW_{r,*}(\mathbb{C}^{d+1})$$

  is called the Paley-Wiener space. It is equipped with inductive limit topology.

- $L^p_{HJ,*}(\mathbb{R}^{d+1}), 1 \leq p \leq +\infty$, the space of measurable functions on $\mathbb{R}^{d+1}$ even with respect to the last variable satisfying

$$\begin{cases} \|f\|_{HJ,p} = \left(\int_{\mathbb{R}^{d+1}} |f(x)|^p A_{HJ}(x) dx \right)^{1/p} < +\infty & \text{for } 1 \leq p < +\infty, \\ \|f\|_{HJ,\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^{d+1}} |f(x)| < +\infty & \text{for } p = +\infty, \end{cases}$$

where A_{HJ} is given by the relation (4.5).

Definition 4.3. *For $f \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), the generalized Fourier transform is defined by*

$$\forall \lambda \in \mathbb{R}^{d+1}, \qquad \mathcal{F}_{HJ} f(\lambda) = \int_{\mathbb{R}^{d+1}} f(x) \Lambda(-x, \lambda) A_{HJ}(x) dx.$$

Proposition 4.3. *The generalized Fourier transform satisfies the following properties:*

1. *For all function f in $L^1_{HJ,*}(\mathbb{R}^{d+1})$, we have*

$$\|\mathcal{F}_{HJ}(f)\|_{HJ,\infty} \leq |W|^{\frac{1}{2}} \|f\|_{HJ,1} \, .$$

2. *For all function f in $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), we have*

$$\forall \lambda \in \mathbb{R}^{d+1}, \qquad \mathcal{F}_{HJ}(f)(\lambda) = \mathcal{F}_0^{-1} \circ {}^t V_{HJ}(\check{f})(\lambda), \tag{4.6}$$

where $\check{f}(x) = f(-x)$ and $\mathcal{F}_0$ the classical Fourier transform on $\mathbb{R}^{d+1}$ defined by

$$\mathcal{F}_0(f)(\lambda) = \int_{\mathbb{R}^{d+1}} f(x) e^{-i\langle x, \lambda \rangle} dx.$$

3. *For all function f in $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), we have*

$$\forall \lambda \in \mathbb{R}^{d+1}, \quad \mathcal{F}_{HJ}(\Delta_{HJ} f)(\lambda) = - \|\lambda\|^2 \mathcal{F}_{HJ} f(\lambda). \tag{4.7}$$

Proof. 1. Let $f \in L^1_{HJ,*}(\mathbb{R}^{d+1})$. We conclude the result by using the relation (4.1) and the following inequality:

$$\begin{aligned} \forall \lambda \in \mathbb{R}^{d+1}, \quad |\mathcal{F}_{HJ} f(\lambda)| &= \left| \int_{\mathbb{R}^{d+1}} f(x) \Lambda(-x, \lambda) A_{HJ}(x) dx \right| \\ &\leq |W|^{\frac{1}{2}} \int_{\mathbb{R}^{d+1}} |f(x)| \, A_{HJ}(x) dx = |W|^{\frac{1}{2}} \|f\|_{HJ,1} \, . \end{aligned}$$

2. Let $f \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) and $\lambda \in \mathbb{R}^{d+1}$. By using the relations (4.3) and (4.4), we obtain

$$\begin{aligned}\mathcal{F}_{HJ}(f)(\lambda) &= \int_{\mathbb{R}^{d+1}} f(x)\Lambda(-x,\lambda)A_{HJ}(x)dx\\ &= \int_{\mathbb{R}^{d+1}} f(x)V_{HJ}(e^{i\langle\lambda,.\rangle})(-x)A_{HJ}(x)dx\\ &= \int_{\mathbb{R}^{d+1}} \check{f}(x)V_{HJ}(e^{i\langle\lambda,\cdot\rangle})(x)A_{HJ}(x)dx\\ &= \int_{\mathbb{R}^{d+1}} {}^tV_{HJ}(\check{f})(x)e^{i\langle\lambda,x\rangle}dx = \mathcal{F}_0^{-1}\circ {}^tV_{HJ}(\check{f})(\lambda).\end{aligned}$$

3. Let $f \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) and $\lambda \in \mathbb{R}^{d+1}$. Using the relations (2.13) and (3.9b), we obtain

$$\begin{aligned}&\mathcal{F}_{HJ}(\Delta_{HJ}f)(\lambda) = \int_{\mathbb{R}^{d+1}} \Delta_{HJ}f(x)\Lambda(-x,\lambda)A_{HJ}(x)dx\\ =&\int_{\mathbb{R}^d}\int_{\mathbb{R}}(\Delta_{a,b}+\rho^2_{d+1})_{x_{d+1}}f(x',x_{d+1})\Lambda(-(x',x_{d+1}),(\lambda',\lambda_{d+1}))\\ &\quad\times A_{a,b}(x_{d+1})dx_{d+1}A_k(x')dx'\\ &\quad+\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Delta_{k,x'}f(x',x_{d+1})\Lambda(-(x',x_{d+1}),(\lambda',\lambda_{d+1}))A_k(x')dx'A_{a,b}(x_{d+1})dx_{d+1}\\ =&-\lambda^2_{d+1}\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}}f(x',x_{d+1})\varphi_{\lambda_{d+1}}(x_{d+1})A_{a,b}(x_{d+1})dx_{d+1}\right]G_{i\lambda'}(-x')A_k(x')dx'\\ &\quad-\|\lambda'\|^2\int_{\mathbb{R}}\left[\int_{\mathbb{R}^d}f(x',x_{d+1})G_{i\lambda'}(-x')A_k(x')dx'\right]\varphi_{\lambda_{d+1}}(x_{d+1})A_{a,b}(x_{d+1})dx_{d+1}\\ =&-\|\lambda\|^2\int_{\mathbb{R}^{d+1}}f(x)\Lambda(-x,\lambda)A_{HJ}(x)dx\\ =&-\|\lambda\|^2\,\mathcal{F}_{HJ}f(\lambda).\end{aligned}$$

Thus, we complete the proof. □

Remark 4.4. We notice that for every f in $\mathcal{D}_*(\mathbb{R}^{d+1})$ (*resp.* $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), we have

$$\forall\lambda=(\lambda',\lambda_{d+1})\in\mathbb{R}^{d+1},\quad \mathcal{F}_{HJ}f(\lambda)=(\mathcal{H}_k)_{\lambda'}\otimes(\mathcal{F}_{a,b})_{\lambda_{d+1}}\,f(\lambda). \tag{4.8}$$

Theorem 4.3. *The Fourier transform $\mathcal{F}_{HJ}$ is a topological isomorphism from*

1. $\mathcal{D}_*(\mathbb{R}^{d+1})$ *onto* $PW_*(\mathbb{C}^{d+1})$,
2. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ *onto* $\mathcal{S}_*(\mathbb{R}^{d+1})$.

Proof. The result follows by combining the relation (4.6), Theorem 4.2 and the Paley-Wiener theorem for the classical Fourier transform. □

Proposition 4.4. *The inverse formula is given, for a function f in $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$, by*

$$f(x)=\int_{\mathbb{R}^{d+1}}\mathcal{F}_{HJ}(f)(\lambda)\Lambda(x,\lambda)d\mu_k(\lambda),\quad \forall x\in\mathbb{R}^{d+1}, \tag{4.9}$$

where

$$d\mu_k(\lambda)=d\sigma(\lambda_{d+1})C_k(\lambda')d\lambda' \quad \text{for } \lambda\in\mathbb{R}^{d+1}$$

with $d\sigma$ and C_k are given respectively by the relations (2.15) *and* (3.11).

Proof. We deduce the result by combining the relations (2.14), (3.10) and (4.8). □

4.4 The generalized translation operators

Definition 4.4. *The generalized translation operator T^x_{HJ}, is defined for $f\in\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ and $x\in\mathbb{R}^{d+1}$, by*

$$\forall\lambda\in\mathbb{R}^{d+1},\quad \mathcal{F}_{HJ}(T^x_{HJ}f)(\lambda)=\Lambda(x,\lambda)\mathcal{F}_{HJ}f(\lambda). \tag{4.10}$$

Proposition 4.5. *For every x in $\mathbb{R}^{d+1}$, the generalized translation operator T^x_{HJ} satisfies the following properties:*

1. *For all $f\in\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ and $y\in\mathbb{R}^{d+1}$, we have*

$$T^x_{HJ}f(y)=\int_{\mathbb{R}^{d+1}}\Lambda(x,\lambda)\Lambda(y,\lambda)\mathcal{F}_{HJ}(f)(\lambda)d\mu_k(\lambda). \tag{4.11}$$

2. *For all $f\in\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$ and $y\in\mathbb{R}^{d+1}$, we have*

$$T^x_{HJ}f(y)=T^y_{HJ}f(x),\quad T^x_{HJ}f(0)=f(x)\quad \text{and}\quad T^x_{HJ}T^y_{HJ}f=T^y_{HJ}T^x_{HJ}f.$$

3. *For all $f\in\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$, we have $\Delta_{HJ}(T^x_{HJ}f)=T^x_{HJ}(\Delta_{HJ}f)$.*

Proof. 1. We obtain the result by combining the relations (4.9) and (4.10).

2. It is deduced from the relations (4.10) and (4.11).

3. By using the relations (4.7) and (4.10), we obtain

$$\begin{aligned}\mathcal{F}_{HJ}(\Delta_{HJ}(T^x_{HJ}f))(\lambda)=&-\|\lambda\|^2\,\mathcal{F}_{HJ}(T^x_{HJ}f)(\lambda)=-\|\lambda\|^2\,\Lambda(x,\lambda)\mathcal{F}_{HJ}f(\lambda)\\ =&\Lambda(x,\lambda)\mathcal{F}_{HJ}(\Delta_{HJ}f)(\lambda)=\mathcal{F}_{HJ}(T^x_{HJ}(\Delta_{HJ}f))(\lambda).\end{aligned}$$

Therefore,

$$\Delta_{HJ}(T^x_{HJ}f)=T^x_{HJ}(\Delta_{HJ}f).$$

Thus, we complete the proof. □

4.5 The generalized convolution product

Definition 4.5. *The generalized convolution product $f *_{(HJ)} g$ of the functions $f,g \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) is defined by*

$$\forall x \in \mathbb{R}^{d+1}, \quad f *_{(HJ)} g(x) = \int_{\mathbb{R}^{d+1}} f(y) T^x_{HJ} g(-y) A_{HJ}(y) dy. \tag{4.12}$$

Proposition 4.6. *For all $f,g \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), the function $f *_{(HJ)} g$ belongs to $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) and we have*

$$\forall \lambda \in \mathbb{R}^{d+1}, \quad \mathcal{F}_{HJ}(f *_{(HJ)} g)(\lambda) = \mathcal{F}_{HJ} f(\lambda) \cdot \mathcal{F}_{HJ} g(\lambda). \tag{4.13}$$

Proof. Let $f,g \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$). Using Fubini's theorem and the relation (4.12), we obtain

$$\begin{aligned} f *_{(HJ)} g(x) &= \int_{\mathbb{R}^{d+1}} f(y) T^x_{HJ} g(-y) A_{HJ}(y) dy \\ &= \int_{\mathbb{R}^{d+1}} f(y) \left(\int_{\mathbb{R}^{d+1}} \Lambda(x,\lambda)\Lambda(-y,\lambda) \mathcal{F}_{HJ}(g)(\lambda) d\mu_k(\lambda) \right) A_{HJ}(y) dy \\ &= \int_{\mathbb{R}^{d+1}} \Lambda(x,\lambda) \mathcal{F}_{HJ}(g)(\lambda) \left(\int_{\mathbb{R}^{d+1}} f(y) \Lambda(-y,\lambda) A_{HJ}(y) dy \right) d\mu_k(\lambda) \\ &= \int_{\mathbb{R}^{d+1}} \Lambda(x,\lambda) \mathcal{F}_{HJ}(f)(\lambda) \mathcal{F}_{HJ}(g)(\lambda) d\mu_k(\lambda). \end{aligned} \tag{4.14}$$

Since $\mathcal{F}_{HJ}(f)\mathcal{F}_{HJ}(g) \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_*(\mathbb{R}^{d+1})$), there exists $\varphi \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) such that

$$\mathcal{F}_{HJ}(\varphi) = \mathcal{F}_{HJ}(f) \mathcal{F}_{HJ}(g).$$

From the inversion formula (4.9), we obtain that $f *_{(HJ)} g = \varphi$.

Consequently, we deduce that $f *_{(HJ)} g$ belongs to $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$) and

$$\mathcal{F}_{HJ}(f *_{(HJ)} g) = \mathcal{F}_{HJ} f \cdot \mathcal{F}_{HJ} g.$$

The proof is completed. □

Theorem 4.4 (Plancherel type formula)**.** *For all f,g in $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), we have*

$$\int_{\mathbb{R}^{d+1}} f(x) g(-x) A_{HJ}(x) dx = \int_{\mathbb{R}^{d+1}} \mathcal{F}_{HJ} f(\lambda) \mathcal{F}_{HJ} g(\lambda) d\mu_k(\lambda). \tag{4.15}$$

Proof. For $f,g \in \mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), the relations (4.12) and (4.14) give

$$\begin{aligned} f *_{(HJ)} g(x) &= \int_{\mathbb{R}^{d+1}} \Lambda(x,\lambda) \mathcal{F}_{HJ} f(\lambda) \mathcal{F}_{HJ} g(\lambda) d\mu_k(\lambda) \\ &= \int_{\mathbb{R}^{d+1}} f(y) T^x_{HJ} g(-y) A_{HJ}(y) dy. \end{aligned}$$

Therefore, we obtain the result by taking $x=0$ and using the relations

$$\forall y \in \mathbb{R}^{d+1}, \qquad T^0_{HJ} g(-y) = g(-y) \quad \text{and} \quad \forall \lambda \in \mathbb{R}^{d+1}, \quad \Lambda(0,\lambda) = 1.$$

Thus, we complete the proof. □

Remark 4.5. For all f, g in $\mathcal{D}_*(\mathbb{R}^{d+1})$ (resp. $\mathcal{S}_{2,*}(\mathbb{R}^{d+1})$), we have

$$\int_{\mathbb{R}^{d+1}} T^x_{HJ} f(-y) g(y) A_{HJ}(y) dy = \int_{\mathbb{R}^{d+1}} f(y) T^x_{HJ} g(-y) A_{HJ}(y) dy.$$

In fact, by using the relation (4.10) and Plancherel type formula (4.15), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} T^x_{HJ} f(-y) g(y) A_{HJ}(y) dy \\
=&\int_{\mathbb{R}^{d+1}} \mathcal{F}_{HJ}(T^x_{HJ} f)(\lambda) \mathcal{F}_{HJ}(g)(\lambda) d\mu_k(\lambda) \\
=&\int_{\mathbb{R}^{d+1}} \Lambda(x,\lambda) \mathcal{F}_{HJ} f(\lambda) \mathcal{F}_{HJ} g(\lambda) d\mu_k(\lambda) \\
=&\int_{\mathbb{R}^{d+1}} \mathcal{F}_{HJ} f(\lambda) \mathcal{F}_{HJ}(T^x_{HJ} g)(\lambda) d\mu_k(\lambda) \\
=&\int_{\mathbb{R}^{d+1}} f(y) T^x_{HJ} g(-y) A_{HJ}(y) dy.
\end{aligned}$$

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