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Regularity of Viscosity Solutions of the Biased Infinity Laplacian Equation

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Abstract. In this paper, we are interested in the regularity estimates of the nonnegative viscosity super solution of the $\beta-$biased infinity Laplacian equation

$$\Delta_\infty^\beta u = 0,$$

where $\beta \in \mathbb{R}$ is a fixed constant and $\Delta_\infty^\beta u := \Delta_\infty^N u + \beta|Du|$, which arises from the random game named biased tug-of-war. By studying directly the $\beta-$biased infinity Laplacian equation, we construct the appropriate exponential cones as barrier functions to establish a key estimate. Based on this estimate, we obtain the Harnack inequality, Hopf boundary point lemma, Lipschitz estimate and the Liouville property etc.

Key Words: $\beta-$biased infinity Laplacian, viscosity solution, exponential cone, Harnack inequality, Lipschitz regularity.

AMS Subject Classifications: 35J62, 35J70, 35B53

1 Introduction

In this work, we devote to the regularity of the viscosity solution of the $\beta-$biased infinity Laplacian equation

$$\Delta_\infty^\beta u = 0, \tag{1.1}$$

where $\beta \in \mathbb{R}$ ($\beta \neq 0$) is a fixed constant and

$$\Delta_\infty^\beta u := \Delta_\infty^N u + \beta|Du|$$

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with

$$\Delta_\infty^N u := \frac{1}{|Du|^2}(D^2uDu)\cdot Du.$$

If $\beta=0$, it coincides with the unbiased operator.

The so-called $\beta-$biased infinity Laplacian operator Δ_∞^β is first introduced in [25] when modelling the biased tug-of-war. Let us briefly recall the two-player, random-turn, β-biased ε-tug-of-war game. Let F be a real final payoff function defined on $\partial\Omega$. The starting position is $x_0\in\Omega$. At the k-th step the two players toss a suitably biased coin (player I wins with odds of $\exp(\beta\varepsilon)$ to 1), and the winner chooses x_k with $d(x_k,x_{k-1})<\varepsilon$. The game ends when $x_k\in\partial\Omega$, and player II pays the amount $F(x_k)$ to player I. The existence and uniqueness of the viscosity solutions were obtained by game theory under the Dirichlet boundary condition. Furthermore, they also proved that viscosity solutions of (1.1) satisfies the comparison property with the exponential cones. For the general equations including (1.1), existence and uniqueness were proved by PDE methods under strong smoothness conditions on the boundary of the domain in [4]. For the inhomogeneous equation

$$\Delta_\infty^\beta u = f, \tag{1.2}$$

existence and uniqueness of Dirichlet or Dirichlet-Neumann mixed boundary problem were obtained by finite difference approximation [2]. In [20], the wellposedness and Lipschitz regularity of the initial-Dirichlet boundary problem related to the evolutionary equation

$$u_t-\Delta_\infty^\beta u=f,$$

were proven. Recently the infinity Laplacian equations arising from game theory have received a lot of attentions because they are not only degenerate from PDE point of view but also have many applications including the image processing, see for example [1, 10–12, 14, 18, 21, 22]. Notice that Δ_∞^β is 1-homogeneous and bounded when the gradient vanishes. This property of scaling invariance is very important in our proofs and applications including the image processing [10–12].

In the special case $\beta=0$, the unbiased operator is the well known normalized infinity Laplacian Δ_∞^N related to the absolutely minimizing Lipschitz extensions which has been extensively studied in the past two decades. See [3, 5, 7, 9, 23, 24, 26] and the references therein. Normalized infinity Laplacian equation was well studied by game theory named tug-of-war in [26] and by partial differential equation methods in [23] respectively. In [9], Crandall, Evans and Gariepy developed the method of comparison with cones for infinity harmonic functions. In [13], Harnack inequality was proven for the smooth nonnegative infinity harmonic functions. The Harnack inequality of the nonnegative infinity superharmonic functions(a supersolution to $\Delta_\infty^N u=0$) is obtained by $p-$approximation method in [19]. In [6], another proof was given using the distance functions as test functions.

In this paper, we are interested in the $\beta-$biased case and we focus on the $\beta-$biased operator itself. The trick is to construct suitable exponential cones as barrier functions and then we utilize these exponential cones to establish the key estimate (in Lemma 2.3)

$$u(x) \geq \frac{u(x_0)}{1-e^{-\beta r}} \cdot \left(1 - e^{\beta(|x-x_0|-r)}\right), \quad x \in B_r(x_0),$$

where u is a nonnegative viscosity supersolution of (1.1). Then by this estimate, we first establish the Harnack inequality. The idea comes from [6], but there are several new elements and difficulties that were not present in the unbiased case, partly because the connection to absolutely minimizing Lipschitz extensions is not there any more. Besides the Harnack inequality, we also establish Hopf boundary lemma, Lipschitz estimate and the Liouville property. As far as we know, these results are new to $\beta-$biased operator. Furthermore, among them, we find that the $\beta-$biased infinity Laplacian operator shows some interesting facts different from the unbiased one.

Notice that the biased operator Δ_∞^β is degenerate and singular when the gradient vanishes. In fact, if we let $\nu = \frac{Du}{|Du|}$ be the unit gradient vector, then the $\beta-$biased infinity Laplacian can be written as

$$\Delta_\infty^\beta u = D_\nu^2 u + \beta D_\nu u.$$

This means that the $\beta-$biased infinity Laplacian is not degenerate only in the direction of the gradient of u. The $\beta-$biased equation (1.1) is quite degenerate, and we do not know whether the solutions are differentiable. In fact, even for the unbiased case, the regularity of the infinity harmonic functions is an open problem. In dimension 2, due to the fine topological structure, infinity harmonic functions are proven to be in $C^{1,\alpha}$ for some $\alpha > 0$, see [15]. For $n \geq 3$, infinity harmonic functions in $\mathbb{R}^n$ are differentiable [16,17].

This paper is organized in the following order. In Section 2, we give the definition of viscosity solutions to Eq. (1.1) by semicontinuous extension and a key estimate used frequently in this paper. In Section 3, we give the main results including the Harnack inequality, Hopf boundary point lemma, Lipschitz estimate and the Liouville property.

2 Preliminaries

In this section, we first give the definition of viscosity solutions for Eq. (1.1). Normally, one can define viscosity solutions as in [8]. Due to the singularity of Eq. (1.1), we have to take care of the situation when the gradient vanishes to guarantee the viscosity solutions are well defined. We introduce some notations which will be used throughout this paper. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, $\partial\Omega$ be its boundary. $B_r(x_0)$ denotes the open ball in $\mathbb{R}^n$ of radius r and center x_0. We use $\lambda_{\max}(\mathcal{M})$ and $\lambda_{\min}(\mathcal{M})$ to denote the largest and the smallest eigenvalues of a symmetric matrix $\mathcal{M}$ respectively, namely

$$\lambda_{\max}(\mathcal{M}) = \max_{|\eta|=1} \left[(\mathcal{M}\eta)\cdot\eta\right] \quad \text{and} \quad \lambda_{\min}(\mathcal{M}) = \min_{|\eta|=1} \left[(\mathcal{M}\eta)\cdot\eta\right].$$

We now begin to give the definitions for viscosity solutions to (1.1). These definitions are similar to those in dealing with singularities in [2,20,23–26]. We include them here for completeness.

Definition 2.1. *Suppose that $u:\Omega\to\mathbb{R}$ is an upper semi-continuous function. If for every $x_0\in\Omega$ and test function $\varphi\in C^2(\Omega)$ such that $u(x_0)=\varphi(x_0)$ and $u(x)<\varphi(x)$ for all $x\in\Omega$ with $x\neq x_0$, there holds*

$$\begin{cases} \Delta_\infty^\beta\varphi(x_0)\geq 0, & \text{if } D\varphi(x_0)\neq 0,\\ \lambda_{\max}\left(D^2\varphi(x_0)\right)\geq 0, & \text{if } D\varphi(x_0)=0, \end{cases}$$

then we say u is a viscosity sub-solution of Eq. (1.1).

Suppose that $u:\Omega\to\mathbb{R}$ is a lower semi-continuous function. If for every $x_0\in\Omega$ and test function $\varphi\in C^2(\Omega)$ such that $u(x_0)=\varphi(x_0)$ and $u(x)>\varphi(x)$ for all $x\in\Omega$ with $x\neq x_0$, there holds

$$\begin{cases} \Delta_\infty^\beta\varphi(x_0)\leq 0, & \text{if } D\varphi(x_0)\neq 0,\\ \lambda_{\min}\left(D^2\varphi(x_0)\right)\leq 0, & \text{if } D\varphi(x_0)=0, \end{cases}$$

then we say u is a viscosity super-solution of Eq. (1.1).

If a continuous function u is both a viscosity subsolution and a viscosity super-solution of Eq. (1.1), *then we say u is a viscosity solution of Eq.* (1.1).

In the above definition, at the points where the gradient of u vanishes, we interpret Eq. (1.1) as the differential inclusion $\Delta_\infty^\beta u\in[\lambda_{\min}\left(D^2u\right),\lambda_{\max}\left(D^2u\right)]$.

For any constants $A>0$ and C, we define the exponential cones

$$\begin{aligned} h(x):=&A\left(1-e^{-\beta\operatorname{dist}(x,\,\partial B_r(x_0))}\right)\operatorname{sgn}(\beta)+C\\ =&A\left(1-e^{\beta(|x-x_0|-r)}\right)\operatorname{sgn}(\beta)+C,\quad x\in B_r(x_0). \end{aligned}\tag{2.1}$$

When $A=1$, $C=0$, we denote

$$H(x):=\left(1-e^{\beta(|x-x_0|-r)}\right)\operatorname{sgn}(\beta).$$

For $x\neq x_0$, since

$$\begin{aligned} Dh(x)&=-A|\beta|e^{\beta(|x-x_0|-r)}\cdot\frac{x-x_0}{|x-x_0|},\\ D^2h(x)&=-A|\beta|e^{\beta(|x-x_0|-r)}\bigg[\beta\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2}\\ &\qquad+\left(\frac{1}{|x-x_0|}I-\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^3}\right)\bigg], \end{aligned}$$

where $\otimes$ denotes the tensor product, we have

$$\Delta_\infty^\beta h(x)=0,\quad x\in B_r(x_0)\backslash\{x_0\}.$$

Lemma 2.1. *For any constants $A>0$ and C, the function h defined as in* (2.1) *is a viscosity solution of* (1.1) *in $B_r(x_0)\backslash\{x_0\}$.*

Proof. The fact that a classical solution is a viscosity solution follows easily from the definition of a viscosity solution. □

We should point out that $h(x)$ is only a viscosity supersolution of (1.1) in the whole ball $B_r(x_0)$. In fact, near $x = x_0$, the function looks like an exponential cone having vertex at the center of the ball, and the exponential conical shape prevents testing from below (hence automatically a super-solution), but allows test-functions with non-zero gradient and arbitrary Hessian from the other side.

Lemma 2.2. *Let* $0 \neq \beta \in \mathbb{R}$, $r > 0$, $x_0 \in \mathbb{R}^n$ *and* $\alpha > 1$. *The function* h *is defined as in* (2.1) *with* $C \equiv 0$. *Then for any* $x \in B_r(x_0)$ *and* $x \neq x_0$, *we have*

$$\Delta_\infty^\beta \left(h(x)^\alpha\right) = \alpha(\alpha-1)A^2\beta^2 h^{\alpha-2} e^{2\beta(|x-x_0|-r)} > 0. \tag{2.2}$$

Proof. By direct calculation, we have

$$\begin{aligned} D(h(x)^\alpha) &= \alpha h^{\alpha-1} Dh(x) = -\alpha A h^{\alpha-1}|\beta| e^{\beta(|x-x_0|-r)} \cdot \frac{x-x_0}{|x-x_0|}, \\ D^2(h(x)^\alpha) &= \alpha(\alpha-1)h^{\alpha-2} Dh \otimes Dh + \alpha h^{\alpha-1} D^2 h \\ &= \alpha(\alpha-1)A^2 h^{\alpha-2}\beta^2 e^{2\beta(|x-x_0|-r)} \cdot \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2} \\ &\quad - \alpha A h^{\alpha-1}\beta|\beta| e^{\beta(|x-x_0|-r)} \cdot \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2} \\ &\quad - \alpha A h^{\alpha-1}|\beta| e^{\beta(|x-x_0|-r)} \cdot \left(\frac{1}{|x-x_0|} I - \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^3}\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \Delta_\infty^\beta(h^\alpha) &= \left(D^2(h^\alpha)\frac{D(h^\alpha)}{|D(h^\alpha)|}\right) \cdot \left(\frac{D(h^\alpha)}{|D(h^\alpha)|}\right) + \beta|D(h^\alpha)| \\ &= \alpha(\alpha-1)A^2\beta^2 h^{\alpha-2} e^{2\beta(|x-x_0|-r)} > 0, \end{aligned} \tag{2.3}$$

where we have used $\alpha > 1$. □

Lemma 2.2 shows that for any $A > 0$, $C \equiv 0$ and $\alpha > 1$, the function $(h(x))^\alpha$ is a smooth strict subsolution of (1.1) in $B_r(x_0)\backslash\{x_0\}$.

Next, we take $h(x)$ as a universal test function for the $\beta-$biased operator to establish the key estimate (2.4). With this estimate in hand, we can establish a series of regularity results including the Harnack inequality. To the best of our knowledge, these results are new to $\beta-$biased operator. It is well known that one can get Lipschitz or Hölder continuity from the Harnack inequality for the linear uniformly elliptic equation. As a matter of fact, for the highly degenerate biased infinity Laplacian equation, we can also establish the local Lipschitz estimate from the Harnack inequality.

Now we choose $A=\frac{u(x_0)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)}$ and $C\equiv 0$ in (2.1) and denote

$$\begin{aligned}\theta(x) &:= \frac{u(x_0)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)}\cdot\left(1-e^{\beta(|x-x_0|-r)}\right)\mathrm{sgn}(\beta),\\ \Theta(x) &:= (\theta(x))^{\alpha}.\end{aligned}$$

Lemma 2.3. *Let $u\in C(\Omega)$ be a nonnegative function such that $\Delta_\infty^\beta u\le 0$ in the viscosity sense in a domain Ω. Suppose $x_0\in\Omega$, $u(x_0)>0$ and $0<r\le \mathrm{dist}(x_0,\partial\Omega)$. Then we have*

$$u(x)\ge\theta(x),\quad x\in B_r(x_0). \tag{2.4}$$

Proof. We argue by contradiction. Suppose that there exists a point $x_*\in\overline{B_r(x_0)}$ such that $u-\theta$ attains its negative minimum at the interior point x_*, i.e.,

$$u(x_*)-\theta(x_*)=\inf_{B_r(x_0)}(u-\theta)<0. \tag{2.5}$$

Since

$$\begin{aligned}&u(x)\ge\theta(x)\quad\text{on }\ \partial B_r(x_0),\\ &u(x_0)=\theta(x_0),\end{aligned}$$

we must have $x_*\neq x_0$. Due to (2.5), now we take $\alpha>1$ sufficiently close to 1 such that $u-\Theta$ attains its negative minimum at the interior point x_*^α and $x_*^\alpha\neq x_0$. Since Θ is smooth in a neighborhood of x_*^α, by the definition of viscosity supersolution of u, we get

$$\Delta_\infty^\beta\Theta(x_*^\alpha)\le 0. \tag{2.6}$$

Recalling that

$$A=\frac{u(x_0)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)},$$

by Lemma 2.2, we have

$$\Delta_\infty^\beta\Theta(x_*^\alpha)=\left(\frac{u(x_0)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)}\right)^2\cdot\alpha(\alpha-1)\beta^2\left(\theta(x_*^\alpha)\right)^{\alpha-2}e^{2\beta(|x_*^\alpha-x_0|-r)}>0,$$

which contradicts to (2.6). □

Remark 2.1. By (2.4), we can immediately get

$$u(x)-u(x_0)\ge u(x_0)\cdot\left(\frac{1-e^{\beta(|x-x_0|-r)}}{1-e^{-\beta r}}-1\right)=-u(x_0)\cdot\frac{e^{\beta|x-x_0|}-1}{e^{\beta r}-1}. \tag{2.7}$$

Remark 2.2. We can drop the nonnegative assumption condition in the above lemma. In fact, if u is a viscosity super-solution of (1.1), then

$$v := u - \inf_{\Omega} u$$

is a nonnegative viscosity super-solution. Therefore, we can apply Lemma 2.3 to the function v to obtain

$$u(x) - \inf_{\Omega} u \geq \frac{u(x_0) - \inf_{\Omega} u}{(1 - e^{-\beta r})\mathrm{sgn}(\beta)} \cdot \left(1 - e^{\beta(|x-x_0|-r)}\right) \mathrm{sgn}(\beta), \quad x \in B_r(x_0). \tag{2.8}$$

Remark 2.3. Let $x = x_0 + s\vec{e}$ and $y = x_0 + t\vec{e}$, where $0 < s < t < r$ and $\vec{e}$ is a unit vector in $\mathbb{R}^n$. It is easy to see that $y \in B_{r-s}(x) \subset B_r(x_0)$. Applying Lemma 2.3 to $B_{r-s}(x)$, we obtain

$$u(y) \geq \frac{u(x)\left(1 - e^{\beta(|x-y|-(r-s))}\right) \mathrm{sgn}(\beta)}{\left(1 - e^{-\beta(r-s)}\right) \mathrm{sgn}(\beta)} = \frac{u(x)H(y)}{H(x)}.$$

It yields

$$\frac{u(y)}{H(y)} \geq \frac{u(x)}{H(x)}.$$

This means that the function

$$f(t) := \frac{u(x_0 + t\vec{e})}{H(x_0 + t\vec{e})}$$

is nondecreasing along radial lines emanating from x_0.

Corollary 2.1. *Suppose that $u \in C(\Omega)$ is a nonnegative viscosity super-solution to* (1.1) *in* Ω. *Let $x_0 \in \Omega$, $u(x_0) > 0$ and $0 < r \leq \mathrm{dist}(x_0, \partial\Omega)$. Then we have*

$$u(x) \geq u(x_0) \cdot \frac{1 - e^{-\beta r/2}}{1 - e^{-\beta r}} \quad \textit{for all} \quad x \in B_{r/2}(x_0).$$

An immediate consequence of Corollary 2.1 is the following strict maximum principle.

Corollary 2.2. *Let $u \in C(\Omega)$ be a nonnegative viscosity supersolution to* (1.1) *in* Ω. *Suppose that $u(x_0) > 0$ for some $x_0 \in \Omega$. Then we have $u > 0$ in Ω.*

Proof. Denote

$$\mathcal{S} = \{x \in \Omega : u(x) > 0\}.$$

It is clear that $\mathcal{S}$ is open. Let y be a limit point of $\mathcal{S}$. Then either $y \in \partial\Omega$ or $y \in \Omega$. If $y \in \Omega$, then there exists a sufficiently small ball $B_r(y)$ which is compactly contained in Ω. Since y is a limit point of $\mathcal{S}$, there exists a point $z \in B_{r/4}(y) \cap \mathcal{S}$. Due to $u(z) > 0$, $y \in B_{r/4}(z) \subset B_{r/2}(y)$ and $B_{r/2}(z) \subset B_r(y)$, Corollary 2.1 implies that

$$u(y) \geq u(z) \cdot \frac{1 - e^{-\beta r/4}}{1 - e^{-\beta r/2}} > 0.$$

That is $y \in \mathcal{S}$. Hence $\mathcal{S}$ is both open and closed. Thus we have $\mathcal{S} = \Omega$ due to Ω being connected. $\square$

3 Main results

In this section, we use the crucial estimate in Section 2 to establish the Harnack inequality, Hopf boundary point lemma and Liouville property for the $\beta-$biased infinity Laplacian equation. In addition, we use the key estimate (2.4) to give a new and elementary proof of the local Lipschitz continuity which was given by finite difference approximation in [2]. The idea of such a statement, together with the strategy of the proof, comes from [6]; however, some of the details are quite different.

Now we first prove the Harnack inequality.

Theorem 3.1. *Suppose that u is a nonnegative viscosity super-solution of* (1.1) *in* Ω. *If* $x_0 \in \Omega$, $u(x_0) > 0$ *and* $0 < r \leq \operatorname{dist}(x_0, \partial\Omega)$, *then we have*

$$\inf_{B_{r/2}(x_0)} u(x) \geq \left(\sup_{B_{r/2}(x_0)} u(x) \right) \cdot \frac{1-e^{-\beta r/2}}{1-e^{-\beta r}} \cdot \left(\frac{1-e^{-\beta r/4}}{1-e^{-\beta r/2}} \right)^2.$$

Proof. The maximum principle Corollary 2.2 implies that $u > 0$ in Ω. We set

$$u(x_*) = \inf_{B_{r/2}(x_0)} u.$$

Hence Corollary 2.1 implies that

$$u(x_*) \geq u(x_0) \cdot \frac{1-e^{-\beta r/2}}{1-e^{-\beta r}}.$$

For any $x \in B_{r/2}(x_0)$, we denote $\bar{x} = \frac{x_0+x}{2}$ and $\rho = |x - x_0|$. Since $\rho \leq r/2$, $\bar{x} \in B_{r/4}(x)$ and $x_0 \in B_{r/4}(\bar{x})$, once again, Corollary 2.1 yields

$$u(\bar{x}) \geq u(x) \cdot \frac{1-e^{-\beta r/4}}{1-e^{-\beta r/2}},$$
$$u(x_0) \geq u(\bar{x}) \cdot \frac{1-e^{-\beta r/4}}{1-e^{-\beta r/2}}.$$

With these inequalities in hand, for any $x \in B_{r/2}(x_0)$, we get

$$u(x_*) \geq u(x) \cdot \frac{1-e^{-\beta r/2}}{1-e^{-\beta r}} \cdot \left(\frac{1-e^{-\beta r/4}}{1-e^{-\beta r/2}} \right)^2.$$

Clearly, this means that

$$u(x_*) \geq \left(\sup_{B_{r/2}(x_0)} u(x) \right) \cdot \frac{1-e^{-\beta r/2}}{1-e^{-\beta r}} \cdot \left(\frac{1-e^{-\beta r/4}}{1-e^{-\beta r/2}} \right)^2.$$

So, we complete the proof. □

The following is the Hopf boundary point lemma and one should notice that we do not need the nonnegativity of the viscosity super-solution.

Theorem 3.2. *Suppose that u satisfies $\Delta_\infty^\beta u \le 0$ in Ω in the viscosity sense. Let $y \in \partial\Omega$ be such that $u(y) = \inf_\Omega u$ and there exists a ball $B_r(x_0) \subset \Omega$ such that $y \in \partial B_r(x_0) \cap \partial\Omega$ and $u(x_0) > u(y)$. Then we have*

$$\liminf_{x\to y} \frac{u(x)-u(y)}{|x-y|} > 0,$$

where the limit is non-tangential, that is, taken over the set of x for which the angle between $y-x$ and the outer normal at y is less than $\pi/2-\alpha$ for some fixed $0<\alpha<\pi/2$.

Proof. We denote

$$w(x) := u(x)-u(y) \ge 0, x\in\Omega.$$

Then w is also a viscosity super-solution to (1.1) and $w(x_0)>0$. By Lemma 2.3, we have

$$w(x) \ge \frac{w(x_0)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)} \cdot H(x), \quad x\in B_r(x_0).$$

Then we have

$$\liminf_{x\to y} \frac{u(x)-u(y)}{H(x)} \ge \frac{u(x_0)-u(y)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)} > 0.$$

A direct calculation yields

$$\begin{aligned} 0 <& \frac{u(x_0)-u(y)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)} \\ \le& \liminf_{x\to y} \frac{u(x)-u(y)}{H(x)} \\ =& \liminf_{x\to y} \frac{u(x)-u(y)}{|x-y|} \cdot \frac{|x-y|}{\left(1-e^{\beta(|x-x_0|-r)}\right)\mathrm{sgn}(\beta)} \\ \le& \liminf_{x\to y} \frac{u(x)-u(y)}{|x-y|} \cdot \frac{|x-y|}{\left(1-e^{-\beta|x-y|}\right)\mathrm{sgn}(\beta)} \\ =& \liminf_{x\to y} \frac{u(x)-u(y)}{|x-y|} \cdot \frac{|x-y|}{\beta|x-y|\mathrm{sgn}(\beta)} \\ =& \liminf_{x\to y} \frac{u(x)-u(y)}{|x-y|} \cdot \frac{1}{|\beta|}. \end{aligned}$$

Thus, we complete the proof. □

Now we use Lemma 2.3 to establish the local Lipschitz estimate of a viscosity super-solution u. One should notice that the following result does not require the assumption of nonnegativity. For another finite difference approximation, one can see [2].

Theorem 3.3. *Assume that u is a viscosity super-solution of* (1.1) *in* Ω. *Let* $y \in \Omega$ *and* $r = \operatorname{dist}(y, \partial\Omega) > 0$. *Then for all* $x \in B_{r/4}(y)$, *we have*

$$|u(x) - u(y)| \leq \frac{(\operatorname{osc} u)(e^{\beta r} - 1)}{e^{\beta r/2}\left(e^{\beta r/2} - 1\right)^2} \cdot \left(e^{\beta|y-x|} - 1\right).$$

Proof. Set

$$w(x) := u(x) - \inf_{\Omega} u, \quad x \in \Omega.$$

It is clear that w is a nonnegative viscosity super-solution in Ω. For arbitrary $x \in B_{r/4}(y)$, by (2.7), one has

$$w(x) - w(y) \geq -w(y) \cdot \frac{e^{\beta|x-y|} - 1}{e^{\beta r} - 1}. \tag{3.1}$$

Due to $y \in B_{r/2}(x) \subset B_r(y)$, once again (2.7) yields

$$w(y) - w(x) \geq -w(x) \cdot \frac{e^{\beta|y-x|} - 1}{e^{\beta r/2} - 1}. \tag{3.2}$$

By (3.1) and (3.2), we get

$$-w(y) \cdot \frac{e^{\beta|x-y|} - 1}{e^{\beta r} - 1} \leq w(x) - w(y) \leq w(x) \cdot \frac{e^{\beta|y-x|} - 1}{e^{\beta r/2} - 1}. \tag{3.3}$$

Notice that Corollary 2.1 implies

$$w(y) \geq w(x) \cdot \frac{1 - e^{-\beta r/2}}{1 - e^{-\beta r}}. \tag{3.4}$$

Substituting (3.4) into (3.3), one has

$$-w(y) \cdot \frac{e^{\beta|x-y|} - 1}{e^{\beta r} - 1} \leq w(x) - w(y) \leq w(y) \cdot \frac{e^{\beta r} - 1}{e^{\beta r/2}\left(e^{\beta r/2} - 1\right)^2} \cdot \left(e^{\beta|y-x|} - 1\right).$$

Hence, we obtain

$$\begin{aligned}
|u(x) - u(y)| =& |w(x) - w(y)| \\
\leq& w(y) \cdot \frac{e^{\beta r} - 1}{e^{\beta r/2}(e^{\beta r/2} - 1)^2} \cdot \left(e^{\beta|x-y|} - 1\right) \\
\leq& \frac{M(e^{\beta r} - 1)}{e^{\beta r/2}(e^{\beta r/2} - 1)^2} \cdot \left(e^{\beta|x-y|} - 1\right),
\end{aligned}$$

where $M = \sup_\Omega w = \operatorname{osc} u$. $\square$

In the unbiased case, infinity harmonic functions in $\mathbb{R}^n$ are differentiable [16, 17]. In dimension 2, due to the fine topological structure, infinity harmonic functions are proven to be in $C^{1,\alpha}$ for some $\alpha > 0$ [15].

Next we give a Liouville type theorem stating that global super-solutions of $\Delta_\infty^\beta u = 0$ which are bounded below are constant. See [9] for the infinity subharmonic functions.

Theorem 3.4. *Suppose that u is bounded below and satisfies* $\Delta_\infty^\beta u \leq 0$ *in* $\mathbb{R}^n$ *in the viscosity sense. Then u is a constant function.*

Proof. Since u is bounded below, we can assume $u(x) \geq m$ for all x in $\mathbb{R}^n$, where m is a constant. Then we consider $U(x) := u(x) - m \geq 0$ for $x \in \mathbb{R}^n$. For arbitrary two different points x and y in $\mathbb{R}^n$, we apply Lemma 2.3 to the function U to get

$$U(y) \geq \frac{U(x)}{(1-e^{-\beta r})\mathrm{sgn}(\beta)} \cdot H(y) = \frac{U(x)}{(1-e^{-\beta r})} \cdot \left(1 - e^{\beta(|y-x|-r)}\right),$$

where $r > |y-x|$. Letting $r \to \infty$, we get $U(y) \geq U(x)$. By symmetry, we also have $U(x) \geq U(y)$. $\square$

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