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Blow-Up Phenomena for Some Pseudo-Parabolic Equations with Nonlocal Term

Gongwei Liu and Shuying Tian*

[1] *College of Science, Henan University of Technology, Zhengzhou, Henan 450001, China*
[2] *Department of Mathematics, School of Science, Wuhan University of Technology, Wuhan, Hubei 430070, China*

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Abstract. We investigate the initial boundary value problem of some semilinear pseudo-parabolic equations with Newtonian nonlocal term. We establish a lower bound for the blow-up time if blow-up does occur. Also both the upper bound for T and blow up rate of the solution are given when $J(u_0) < 0$. Moreover, we establish the blow up result for arbitrary initial energy and the upper bound for T. As a product, we refine the lifespan when $J(u_0) < 0$.

Key Words: Nonlocal pseudo-parabolic equations, blow-up, upper bound, lower bound.

AMS Subject Classifications: 35B44, 35K70

1 Introduction

In this paper, we are concerned with the following initial boundary value problem (IBVP) of some pseudo-parabolic equations with nonlocal term

$$\begin{cases} u_t - \Delta u_t - \Delta u + u = a\phi_u u + b|u|^{p-1}u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,t) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1,5)$ and ϕ_u is the Newtonian nonlocal term

$$\phi_u(x) = \int_\Omega \frac{u^2(y)}{4\pi|x-y|} dy, \quad x \in \mathbb{R}^3. \tag{1.2}$$

*Corresponding author. *Email addresses:* `gongweiliu@126.com` (G. Liu), `sytian@whut.edu.cn` (S. Tian)

Elliptic problems with the nonlocal term $\phi_u u$ have been extensively investigated in the field of physic and mathematics by many authors, for example, the Klein-Gordon-Maxwell equations [1, 2] and the Schrödinger-Poisson-Slater problem [3]. Recently, the local and global existence of solutions for the parabolic equations with the nonlocal term $\phi_u u$ were considered in [4]. More specifically, they considered the following three cases: (A) $a = b = 1$; (B) $a = -1, b = 1$; (C) $a = 1, b = 0$.

Since the appearance of $-\Delta u_t$, Eq. (1.1) is called the pseudo-parabolic equation, which describes many interesting physical and biological phenomena, for example, the non-stationary process in semiconductors in the presence of sources and $\Delta u_t - u_t$ stands for the free electron density rate (see [5]). Pseudo-equations have been extensively studied by many authors since the work of Ting [6,7], see for example [8–14] and references therein. Most of these papers were interested in the existence and the blow up of the solutions to the pseudo-equations with polynomial nonlinear source term (i.e., $a = 0$, $b = 1$). Recently, the existence and blow up of solutions for the pseudo-equation with nonlocal term $\phi_u u$ have been investigated, that is the IBVP(1.1) with Case (A) $a = b = 1$ [15] and the IBVP (1.1) with Case (B) $a = -1$, $b = 1$ [16]. The existence, asymptotic behavior and blow up results were obtained respectively in [15, 16] via energy equality together with the potential wells methods. By the way, the following nonlocal problem

$$u_t - \Delta u = \Big(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}} dy \Big) |u|^{p-2}u, \quad x \in \Omega \subset R^n, \quad t > 0,$$

has been investigated in [17–20].

In this paper, we will continue to study the blow up properties of the IBVP (1.1). For convenience, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the associated inner product on $L^2(\Omega)$. Moreover, we use $\|\cdot\|_p$ and $\|\cdot\|_{H_0^1}$ to denote the norms on $L^p(\Omega)$ and $H_0^1(\Omega)$) respectively. We also denote C as the different constant in different line and T as the maximal existence time.

First, we introduce the definition of the weak solution to the IBVP(1.1).

Definition 1.1 ([15,16]). *A function $u \in L^2_{loc}([0,T), H_0^1(\Omega))$ with $u_t \in L^2_{loc}([0,T), H_0^1(\Omega))$ is called a weak solution of IBVP* (1.1) *provided*

(i) for a.e. $t \in [0,T)$ it holds that

$$\begin{aligned} &(u_t(t), v) + (\nabla u_t(t), \nabla v) + (u(t), v) + (\nabla u(t), \nabla v) \\ =&(a\phi_u u + b|u|^{p-1}u, v), \quad v \in H_0^1(\Omega). \end{aligned}$$

(ii) $u(x,0) = u_0(x)$.

In order to compare with our work in this paper, For the completeness, we summarize the existence and blow-up results obtained in [15, 16] as following.

(**RES 1**) (see [15, 16]) Assume that $u_0 \in H_0^1(\Omega)$, then IBVP (1.1) admits a unique weak solution $u \in C^1([0,T), H_0^1(\Omega))$ and u can be represented in the following integral form

$$u(t) = u_0 + \int_0^t \big\{ (I-\Delta)^{-1}[a\phi_u(s)u(s) + b|u(s)|^{p-1}u(s)] - u(s) \big\} ds, \quad t \in [0,T),$$

or

$$u(t) = e^{-t}u_0 + \int_0^t e^{-(t-s)}(I-\Delta)^{-1}\big(a\phi_u(s)u(s) + b|u(s)|^{p-1}u(s)\big) ds, \quad t \in [0,T).$$

Moreover, if $T<\infty$, then

$$\limsup_{t\to T^-} \|u(t)\|_{H_0^1} = \infty.$$

We introduce the energy functional and Nehari functional on $H_0^1(\Omega)$ for the IBVP (1.1) by

$$J(u) = J_{ab}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{a}{4}\int_\Omega \phi_u u^2 - \frac{b}{p+1}\|u\|_{p+1}^{p+1}, \tag{1.3a}$$

$$I(u) = I_{ab}(u) = \|u\|_{H_0^1}^2 - a\int_\Omega \phi_u u^2 - b\|u\|_{p+1}^{p+1}. \tag{1.3b}$$

Let

$$d = d_{ab} = \inf_{u\in N} J(u) = \inf_{u\in H_0^1(\Omega)\ \{0\}} \max_{\lambda\ge 0} J(\lambda u),$$

where

$$N = N_{ab} = \{u \in H_0^1(\Omega)\{0\} : I_{ab}(u) = 0\}.$$

If there is no confusion, we will still use the symbols $J(u)$, $I(u)$, d and N to denote $J_{ab}(u)$, $I_{ab}(u)$, d_{ab} and N_{ab} respectively. Now we state the blow up results obtained in [15, 16] which are relevant to the work in this paper.

(**RES 2**) Let $u_0 \in H_0^1(\Omega)$ with $J(u_0) \le d$, and $u(t) = u(x,t;u_0)$ be the solutions of the IBVP (1.1) whose maximal existence time is T. If $I(u_0) < 0$, then $T<\infty$ and u blows up at T, that is

$$\lim_{t\to T^-} \|u(t)\|_{H_0^1} = \infty.$$

Remark 1.1. (I) Throughout this paper, we restrict the range of the exponent p as the same as that of [15, 16], that is: for Case (A) $a=b=1$, $p\in(1,5)$, see [15]; for Case (B) $a=-1, b=1$, $p \in [3,5)$, see [16].

(II) Through there are no results on the IBVP (1.1) with the Case (C) $a=1, b=0$, we can obtain the similar existence, asymptotic behavior and blow up results as those of [15] by the similar methods. Hence, (RES 1) and (RES 2) also hold for the Case (C).

In summary, when $J(u_0) \leq d, I(u_0) < 0$, the authors obtained the blow up results. To the best of our knowledge, there is little information on the bounds for blow up time to the IBVP (1.1). When blow up occurs, the blow up time T cannot usually be computed exactly. Hence, it is important to determine lower and upper bounds for T in practice. The main task of this paper is to give some results about this question.

This paper is organized as follows. In Section 2, we delicate a lower bound for T if blow up occurs. In Section 3, firstly, we give both the upper bound for T and blow up rate of the solution when $J(u_0) < 0$, then, we establish the blow up result for arbitrary initial energy and the upper bound for T.

2 Lower bound for the blow up time

In this section, we determine the lower bound for the blow up time to IBVP (1.1). First, we give some notations. If $u_0 \in H_0^1(\Omega)$, then it follows from the Sobolev embedding and Lemma 2.1 of [4] that

$$\phi_u \in L^6(\Omega) \quad \text{and} \quad \|\phi_u\|_6 \leq C\|u\|_{H_0^1}^2.$$

Moreover, $\phi_u u \in L^2(\Omega)$,

$$\|\phi_u u\|_2 \leq C\|u\|_{H_0^1}^3 \quad \text{and} \quad \int_\Omega \phi_u u^2 dx \leq C\|u\|_{H_0^1}^4.$$

Hence, we can obtain there exists a positive constant C^* such that

$$C^* = \sup_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega \phi_u u^2 dx}{\|u\|_{H_0^1}^4}.$$

We also denote positive constant

$$C_* = \sup_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|_{p+1}}{\|u\|_{H_0^1}}.$$

Theorem 2.1. *Let $u_0 \in H_0^1(\Omega)$and $u(t) = u(x,t;u_0)$ be the nontrivial solution of the IBVP* (1.1) *which blows up in finite time T in H_0^1-norm. Then T is bounded below as follows*

- *Case* (A) $a = b = 1$
$$T \geq \int_{\varphi(0)}^{+\infty} \frac{dy}{2C^* y^2 + 2C_*^{p+1} y^{\frac{p+1}{2}}}. \tag{2.1}$$

- *Case* (B) $a = -1, b = 1$
$$T \geq \frac{\|u_0\|_{H_0^1}^{-p+1}}{(p-1)C_*^{p+1}}. \tag{2.2}$$

- *Case* (C) $a=1, b=0$

$$T \geq \frac{\|u_0\|_{H_0^1}^{-2}}{2C_*}. \tag{2.3}$$

Proof. We define the auxiliary function

$$\varphi(t) = \|u(t)\|_{H_0^1}^2 = \int_\Omega u^2(t)dx + \int_\Omega |\nabla u(t)|^2 dx. \tag{2.4}$$

Differentiating (2.4) with respect to t, and using (1.1) and (1.3b), we have

$$\begin{aligned}\varphi'(t) =& 2\int_\Omega u u_t dx + 2\int_\Omega \nabla u \cdot \nabla u_t dx \\ =& -2\int_\Omega |\nabla u(t)|^2 dx - 2\int_\Omega u^2(t)dx + 2a\int_\Omega \phi_u u^2 dx + 2b\int_\Omega |u(t)|^{p+1}dx \\ =& -2I(u).\end{aligned} \tag{2.5}$$

Now, we are in position to consider the three different cases: (A), (B) and (C).

Case (A) $a=b=1$: It follows from (2.5) that

$$\begin{aligned}\varphi'(t) \leq& 2\int_\Omega \phi_u u^2 dx + 2\int_\Omega |u(t)|^{p+1}dx \\ \leq& 2C^*\|u(t)\|_{H_0^1}^4 + 2C_*^{p+1}\|u(t)\|_{H_0^1}^{p+1} \\ =& 2C^*\varphi^2(t) + 2C_*^{p+1}\varphi^{\frac{p+1}{2}}(t).\end{aligned} \tag{2.6}$$

Integrating the differential inequality (2.6) from 0 to t, we obtain

$$\int_{\varphi(0)}^{\varphi(t)} \frac{dy}{2C^*y^2 + 2C_*^{p+1}y^{\frac{p+1}{2}}} \leq t,$$

hence, if $u(t)$ blows up in H_0^1-norm as $t \to T^-$, we obtain the lower bound (2.1)

Case (B) $a=-1, b=1$: It follows from (2.5) that

$$\varphi'(t) \leq 2\int_\Omega |u(t)|^{p+1}dx \leq 2C_*^{p+1}\varphi^{\frac{p+1}{2}}(t),$$

which is equivalent to

$$\frac{\varphi'(t)}{\varphi^{\frac{p+1}{2}}(t)} \leq 2C_*^{p+1}.$$

Integrating the above inequality from 0 to t, we have

$$\varphi^{-\frac{p-1}{2}}(t) \geq \varphi^{-\frac{p-1}{2}}(0) - (p-1)C_*^{p+1}t,$$

hence, if $u(t)$ blows up in H_0^1-norm as $t \to T^-$, we obtain the lower bound (2.2).

Case (C) $a=1, b=0$: It follows from (2.5) that

$$\varphi'(t) \leq 2\int_\Omega \phi_u u^2 dx \leq 2C^*\varphi^2(t).$$

Then, we can deduce (2.3) as similar as the proof of the Case (B). □

3 Upper bound for the blow up time

In this section, we prove that the solution $u(t)$ of IBVP (1.1) blows up in finite time under some conditions. Firstly, we give both the upper bound for T and blow-up rate of the solution when $J(u_0)<0$.

Theorem 3.1. *Let $u_0\in H_0^1(\Omega)$, $J(u_0)<0$, $p_0=\min\{2,\frac{p+1}{2}\}$ and $u(t)=u(x,t;u_0)$ be the nontrivial solution of the IBVP* (1.1)*, then $u(t)$ blows up in finite time T in H_0^1-norm. Moreover, an upper bound for T is given by follows*

- *Case* (A) $a=b=1$

$$T\le\frac{\|u_0\|^2_{H_0^1}}{4p_0(1-p_0)J(u_0)}.\tag{3.1}$$

- *Case* (B) $a=-1, b=1$

$$T\le\frac{\|u_0\|^2_{H_0^1}}{(1-p^2)J(u_0)}.\tag{3.2}$$

- *Case* (C) $a=1, b=0$

$$T\le\frac{\|u_0\|^2_{H_0^1}}{-8J(u_0)}.\tag{3.3}$$

Proof. Differentiating (1.3b) and making use of (1.1), we obtain

$$\frac{d}{dt}J(u(t))=-\|u_t(t)\|_2^2-\|\nabla u_t(t)\|_2^2=-\|u_t(t)\|^2_{H_0^1}.\tag{3.4}$$

Now, we are in position to consider the three different cases: (A), (B) and (C).

Case (A) $a=b=1$. Suppose that $p_0=\min\{2,\frac{p+1}{2}\}$ for $p\in(1,5)$ as [15], then $p_0\in(1,2]$. Let

$$\psi(t)=-4p_0J(u(t))=-2p_0\|u(t)\|^2_{H_0^1}+p_0\int_\Omega\phi_u u^2dx+\frac{4p_0}{p+1}\|u(t)\|^{p+1}_{p+1}.$$

Then, (3.4) implies

$$\psi'(t)=4p_0(\|u_t(t)\|_2^2+\|\nabla u_t(t)\|_2^2)$$

Given $\varphi(t)$ be the function defined in (2.4), since $p_0\in(1,2]$, we obtain

$$\begin{aligned}\varphi'(t)=&-2\int_\Omega u^2(t)dx-2\int_\Omega|\nabla u(t)|^2dx+2\int_\Omega\phi_u u^2dx+2\int_\Omega|u(t)|^{p+1}dx\\ \ge&\psi(t).\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\varphi(t)\psi'(t) =&4p_0\Big(\int_\Omega u^2(t)dx+\int_\Omega|\nabla u(t)|^2dx\Big)\Big(\int_\Omega u_t^2(t)dx+\int_\Omega|\nabla u_t(t)|^2dx\Big)\\ \geq&4p_0\Big(\int_\Omega u(t)u_t(t)dx+\int_\Omega\nabla u(t)\cdot\nabla u_t(t)dx\Big)=p_0[\varphi'(t)]^2.\end{aligned}$$

It follows from (3.4) and the assumption $J(u_0)<0$ that $\psi(t)>0$ for all $t\geq 0$. Hence, we have

$$\varphi(t)\psi'(t)\geq p_0\varphi'(t)\psi(t),$$

which an be rewritten as

$$\frac{\psi'(t)}{\psi(t)}\geq p_0\frac{\varphi'(t)}{\varphi(t)}.$$

Integrating the above inequality from 0 to t, noticing $\varphi'(t)\geq\psi(t)$, we deduce

$$\frac{\psi(t)}{[\varphi(t)]^{p_0}}\geq\frac{\psi(0)}{[\varphi(0)]^{p_0}}$$

and

$$\frac{\varphi'(t)}{[\varphi(t)]^{p_0}}\geq\frac{\psi(0)}{[\varphi(0)]^{p_0}}. \tag{3.5}$$

Then, a further integration results in

$$\frac{1}{[\varphi(t)]^{p_0-1}}\leq\frac{1}{[\varphi(0)]^{p_0-1}}-(p_0-1)\frac{\psi(0)}{[\varphi(0)]^{p_0}}t. \tag{3.6}$$

It is obvious that (3.6) can not hold for all time t and u blows up in some finite T, i.e., $\lim_{t\to T^-}\varphi(t)=+\infty$, where

$$T\leq\frac{\varphi(0)}{(p_0-1)\psi(0)},$$

which implies the conclusion (3.1) of this theorem.

Case (B) $a=-1$, $b=1$. Let $\varphi(t)$ be the same as (2.4) and

$$\psi(t)=-2(p+1)J(u(t))=-(p+1)\|u(t)\|^2_{H_0^1}-\frac{p+1}{2}\int_\Omega\phi_u u^2dx+2\|u(t)\|^{p+1}_{p+1}.$$

Since $p\in[3,5)$, using Schwarz's inequality, by direct computation, we deduce $\varphi'(t)\geq\psi(t)$ and

$$\varphi(t)\psi'(t)\geq\frac{p+1}{2}[\varphi'(t)]^2.$$

Then by the similar computation as Case (A), see also [10], we can obtain the conclusion (3.2).

Case (C) $a=1$, $b=0$. Let $\varphi(t)$ be the same as (2.4) and

$$\psi(t)=-8J(u(t))=-4\|u(t)\|^2_{H_0^1}+2\int_\Omega\phi_u u^2dx.$$

Then, by the similar argument as the Case (A), we can obtain the conclusion (3.3). □

Remark 3.1. In this remark, we obtain the upper blow up rate when $J(u_0) < 0$ for all cases. For Case (A) $a = b = 1$, we integrate the inequality (3.5) from t to T, noticing $\lim_{t\to T^-} \varphi(t) = +\infty$, we obtain

$$\varphi(t) \leq \left[\frac{(p_0-1)\psi(0)}{[\varphi(0)]^{p_0}}\right]^{\frac{1}{1-p_0}} (T-t)^{-\frac{1}{p_0-1}},$$

then it follows from the definitions of $\varphi(t)$ and $\psi(t)$ that

$$\|u(t)\|_{H_0^1} \leq \left[\frac{4p_0(1-p_0)J(u_0)}{\|u_0\|_{H_0^1}^{2p_0}}\right]^{\frac{1}{2(1-p_0)}} (T-t)^{-\frac{1}{2(p_0-1)}}.$$

Similarly, we can obtain the upper blow up rate as

$$\|u(t)\|_{H_0^1} \leq \left[\frac{(1-p^2)J(u_0)}{\|u_0\|_{H_0^1}^{p+1}}\right]^{\frac{1}{1-p}} (T-t)^{-\frac{1}{p-1}} \qquad \text{for Case (B) } a=-1,\ b=1,$$

$$\|u(t)\|_{H_0^1} \leq \left[\frac{(-8J(u_0))}{\|u_0\|_{H_0^1}^{4}}\right]^{-\frac{1}{2}} (T-t)^{-\frac{1}{2}} \qquad \text{for Case (C) } a=1,\ b=0.$$

Now, we establish the blow-up results for arbitrary initial energy and the upper bounds for the maximal existence time T.

Theorem 3.2. *Let $u_0 \in H_0^1(\Omega)$, $p_0 = \min\{2, \frac{p+1}{2}\}$ and $u(t) = u(x,t;u_0)$ be the nontrivial solution of the IBVP* (1.1)*, then $u(t)$ blows up in finite time T in H_0^1-norm with the initial energy satisfies*

- *Case* (A) $a = b = 1$
$$J(u_0) < \frac{p_0-1}{2p_0}\|u_0\|_{H_0^1}^2, \tag{3.7}$$
then $u(t)$ blows up at some finite in H_0^1-norm. Moreover, the upper bound can be estimated by
$$T \leq \frac{2p_0\|u_0\|_{H_0^1}^2}{(p_0-1)^2[(p_0-1)\|u_0\|_{H_0^1}^2 - 2p_0J(u_0)]}. \tag{3.8}$$

- *Case* (B) $a = -1, b = 1$
$$J(u_0) < \frac{p-1}{2(p+1)}\|u_0\|_{H_0^1}^2, \tag{3.9}$$
then $u(t)$ blows up at some finite in L^2-norm. Moreover, the upper bound can be estimated by
$$T \leq \frac{8(p+1)\|u_0\|_{H_0^1}^2}{(p-1)^2[(p-1)\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)]}. \tag{3.10}$$

- *Case* (A) $a=1, b=0$

$$J(u_0)<\frac{1}{4}\|u_0\|^2_{H^1_0}, \tag{3.11}$$

then $u(t)$ blows up at some finite in L^2-norm. Moreover, the upper bound can be estimated by

$$T\leq\frac{4\|u_0\|^2_{H^1_0}}{\|u_0\|^2_{H^1_0}-4J(u_0)]}. \tag{3.12}$$

Proof. We mainly consider Case (A). Suppose $u(t)$ be the nontrivial solution of the problem (1.1) with the initial energy satisfying (3.7) for Case (A) $a=b=1$ (resp. (3.9) for Case (B) $a=-1, b=1$ and (3.11) for Case (C) $a=1, b=0$). We may assume $J(u(t))\geq 0$, otherwise, there must exist some $t_0\geq 0$ such that $J(u(t_0))<0$, then $u(t)$ will blow up at some finite time via Theorem 3.1. So in the following, we assume that $u(t)$ exists globally and $J(u(t))\geq 0$ for all $t\geq 0$.

Case (A) $a=b=1$. It follows from (1.3a) and (2.5) that

$$\begin{aligned}\varphi'(t)=&-2\int_\Omega u^2(t)dx-2\int_\Omega|\nabla u(t)|^2dx+2\int_\Omega\phi_u u^2dx+2\int_\Omega|u(t)|^{p+1}dx\\ =&(p-1)\|u(t)\|^2_{H^1_0}+\Big(2-\frac{p+1}{2}\Big)\int_\Omega\phi_u u^2dx-2(p+1)J(u(t)),\quad 1<p<3,\\ =&2\|u(t)\|^2_{H^1_0}+\Big(2-\frac{8}{p+1}\Big)\|u(t)\|^{p+1}_{p+1}-8J(u(t)),\qquad 3\leq p<5.\end{aligned}$$

Now let $p_0=\min\{2,\frac{p+1}{2}\}$, then

$$\varphi'(t)\geq 2(p_0-1)\|u(t)\|^2_{H^1_0}-4p_0J(u(t))=2(p_0-1)\Big(\|u(t)\|^2_{H^1_0}-\frac{2p_0}{p_0-1}J(u(t))\Big).$$

Let

$$H(t)=\|u(t)\|^2_{H^1_0}-\frac{2p_0}{p_0-1}J(u(t)).$$

Since $\frac{d}{dt}J(u(t))\leq 0$, by a simple calculation, we can obtain

$$\frac{d}{dt}H(t)\geq 2(p_0-1)H(t)$$

for all $t\geq 0$. By using Gronwall's inequality, we have

$$H(t)\geq e^{2(p_0-1)t}H(0).$$

Noticing that $H(0)>0$ via (3.7), and the assumption $J(u(t))\geq 0$ for $t\geq 0$, we have

$$\|u(t)\|_{H^1_0}\geq\sqrt{H(0)}e^{(p_0-1)t},\quad t\geq 0. \tag{3.13}$$

On other hand, since

$$\int_0^t \|u_s(s)\|_{H_0^1} ds \geq \|\int_0^t u_s(s)ds\|_{H_0^1} = \|u(t)-u_0\|_{H_0^1} \geq \|u(t)\|_{H_0^1} - \|u_0\|_{H_0^1}, \quad t\geq 0,$$

then combining the above inequality, Hölder's inequality and $J(u_0)\geq J(u(t))\geq 0$, we deduce

$$\begin{aligned}\|u(t)\|_{H_0^1} \leq& \|u_0\|_{H_0^1} + \int_0^t \|u_s(s)\|_{H_0^1} ds\\ \leq& \|u_0\|_{H_0^1} + t^{\frac{1}{2}}\Big[\int_0^t \|u_s(s)\|^2_{H_0^1} ds\Big]^{\frac{1}{2}}\\ =& \|u_0\|_{H_0^1} + t^{\frac{1}{2}}[J(u_0)-J(u(t))]^{\frac{1}{2}}\\ \leq& \|u_0\|_{H_0^1} + t^{\frac{1}{2}}(J(u_0))^{\frac{1}{2}}, \quad t\geq 0,\end{aligned}$$

which is a contradiction with (3.13) for sufficiently large t. Hence, $u(t)$ blows up at some finite time, that is $T<\infty$.

Next, we establish an upper bound estimate of T. Firstly, we claim that

$$I(u(t)) = \|u(t)\|^2_{H_0^1} - \int_\Omega \phi_u u^2 dx - \|u(t)\|^{p+1}_{p+1} < 0, \quad \forall t\in[0,T). \tag{3.14}$$

Indeed, in view of (1.3a) and (1.3b), after a simple calculation, we have for $1<p<3$

$$J(u(t)) = \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|u(t)\|^2_{H_0^1} + \Big(\frac{1}{p+1}-\frac{1}{4}\Big)\int_\Omega \phi_u u^2 dx + \frac{1}{p+1}I(u(t)), \tag{3.15a}$$

and for $3\leq p<5$,

$$J(u(t)) = \Big(\frac{1}{2}-\frac{1}{4}\Big)\|u(t)\|^2_{H_0^1} + \Big(\frac{1}{4}-\frac{1}{p+1}\Big)\int_\Omega \phi_u u^2 dx + \frac{1}{4}I(u(t)). \tag{3.15b}$$

Combining the above equation and the condition (3.7), we can deduce $I(u_0)<0$. Hence, if (3.14) doesn't hold, there must exist a $t_0\in(0,T)$ such that $I(u(t_0))=0$, $I(u(t))<0$, for $t\in[0,t_0)$. Then by (2.5), we obtain that $\|u(t)\|^2_{H_0^1}$ is strictly increasing on $[0,t_0)$. Then, it follows from (3.7) that

$$J(u_0) < \frac{p_0-1}{2p_0}\|u_0\|^2_{H_0^1} < \frac{p_0-1}{2p_0}\|u(t_0)\|^2_{H_0^1}. \tag{3.16}$$

On the other hand, combining (3.4) and (3.15), we have for $1<p<3$,

$$\begin{aligned}&J(u_0)\geq J(u(t_0))\\ =&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\|u(t_0)\|^2_{H_0^1} + \Big(\frac{1}{p+1}-\frac{1}{4}\Big)\int_\Omega \phi_{u(t_0)}u^2(t_0)dx + \frac{1}{p+1}I(u(t_0)\\ \geq&\frac{p-1}{2(p+1)}\|u(t_0)\|^2_{H_0^1},\end{aligned}$$

and for $p \geq 3$,

$$\begin{aligned} & J(u_0) \geq J(u(t_0)) \\ =& \Big(\frac{1}{2}-\frac{1}{4}\Big)\|u(t_0)\|^2_{H^1_0} + \Big(\frac{1}{4}-\frac{1}{p+1}\Big)\|u(t_0)\|^{p+1}_{p+1} + \frac{1}{4}I(u(t_0)) \\ \geq& \frac{1}{4}\|u(t_0)\|^2_{H^1_0}. \end{aligned}$$

The above two inequalities imply that

$$J(u_0) \geq \frac{p_0-1}{2p_0}\|u(t_0)\|^2_{H^1_0},$$

which contradicts (3.16). Hence, $I(u(t) < 0$ and $\|u(t)\|^2_{H^1_0}$ is strictly increasing on $[0,T)$.

For any $\tilde{T} \in (0,T)$, we define the functional

$$F(t) = \int_0^t \|u(s)\|^2_{H^1_0} ds + (T-t)\|u_0\|^2_{H^1_0} + \beta(t+\gamma)^2, \quad t \in [0,\tilde{T}],$$

with two positive constants β,γ to be determined later. Noticing $\|u(t)\|^2_{H^1_0}$ is strictly increasing, we have

$$\begin{aligned} F'(t) =& \|u(t)\|^2_{H^1_0} - \|u_0\|^2_{H^1_0} + 2\beta(t+\gamma) \\ =& \int_0^t \frac{d}{ds}\|u(s)\|^2_{H^1_0} ds + 2\beta(t+\gamma) > 0. \end{aligned} \tag{3.17}$$

It follows from (3.4) that

$$\begin{aligned} F''(t) =& \frac{d}{dt}\|u(t)\|^2_{H^1_0} + 2\beta \\ =& 2(p_0-1)\|u(t)\|^2_{H^1_0} - 4p_0 J(u(t)) + 2\beta \\ \geq& 2(p_0-1)\|u(t)\|^2_{H^1_0} - 4p_0 J(u_0) + 4p_0\int_0^t \|u_s\|^2_{H^1_0} ds. \end{aligned} \tag{3.18}$$

Since

$$F(0) = T\|u_0\|^2_{H^1_0} + \beta\gamma^2 > 0 \quad \text{and} \quad F'(0) = 2\beta\gamma > 0,$$

making use of Hölder's inequality, Young's inequality and the element algebraic inequality:

$$ab + cd \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2},$$

after a simple calculation, we have

$$\begin{aligned}\xi(t) \triangleq& \Big(\int_0^t \|u(s)\|_{H_0^1}^2 ds + \beta(t+\gamma)^2 \Big) \Big(\int_0^t \|u_s\|_{H_0^1}^2 ds + \beta \Big) \\ &- \Big(\int_0^t \frac{1}{2} \frac{d}{ds} \|u(s)\|_{H_0^1}^2 ds + \beta(t+\gamma) \Big)^2 \\ \geq& 0.\end{aligned}$$

Hence, by making use of the above inequality and (3.17), we have

$$\begin{aligned}-(F'(t))^2 =& -4\Big[\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2 ds + \beta(t+\gamma)\Big]^2 \\ =& 4\bigg(\xi(t) - \big(F(t) - (T-t)\|u_0\|_{H_0^1}^2\big)\bigg)\bigg(\int_0^t \|u_s(s)\|_{H_0^1}^2 ds + \beta)\bigg) \\ \geq& -4F(t)\bigg(\int_0^t \|u_s(s)\|_{H_0^1}^2 ds + \beta\bigg).\end{aligned}$$

Combining the above inequality and (3.18), for any $\theta > 0$, we obtain

$$\begin{aligned}& F(t)F''(t) - \theta(F'(t))^2 \\ \geq& F(t)\bigg(F''(t) - 4\theta\int_0^t \|u_s(s)\|_{H_0^1}^2 ds - 4\theta\beta\bigg) \\ \geq& F(t)\bigg(2(p_0-1)\|u(t)\|_{H_0^1}^2 - 4p_0 J(u_0) + 4(p_0-\theta)\int_0^t \|u_s\|_{H_0^1}^2 ds - 4\theta\beta\bigg) \\ =& F(t)\bigg[4p_0\bigg(\frac{p_0-1}{2p_0}\|u(t)\|_{H_0^1}^2 - J(u_0)\bigg) + 4(p_0-\theta)\int_0^t \|u_s\|_{H_0^1}^2 ds - 4\theta\beta\bigg] \\ \geq& F(t)\bigg[4p_0\bigg(\frac{p_0-1}{2p_0}\|u_0\|_{H_0^1}^2 - J(u_0)\bigg) + 4(p_0-\theta)\int_0^t \|u_s\|_{H_0^1}^2 ds - 4\theta\beta\bigg]\end{aligned}$$

for any $t \in [0, \tilde{T}]$, where we also use the fact that $\|u(t)\|_{H_0^1}^2$ is strictly increasing. In view of (3.7), taking $\theta = p_0$ and letting β sufficiently small such that

$$0 < \beta \leq \beta_0 \triangleq \frac{p_0-1}{2p_0}\|u_0\|_{H_0^1}^2 - J(u_0), \tag{3.19}$$

we have

$$F(t)F''(t) - p_0(F'(t))^2 \geq 0, \quad t \in [0, \tilde{T}].$$

Define $G(t) = F^{1-p_0}(t)$ for $t \in [0, \tilde{T}]$. After a simple calculation, since $F(t) > 0$, $F'(t) > 0$ and $p_0 > 1$, we have

$$\begin{aligned}G'(t) &= (1-p_0)F^{-p_0}(t)F'(t) < 0, \\ G''(t) &= (1-p_0)F^{-p_0-1}[F(t)F''(t) - p_0(F'(t))^2] \leq 0,\end{aligned}$$

holds for all $t \in [0, \tilde{T}]$, which means that $G(t)$ is concave on $[0, \tilde{T}]$. Hence, we have

$$G(\tilde{T}) - G(0) = \int_0^1 G'(\eta \tilde{T}) d\eta \tilde{T} \leq G'(0)\tilde{T}. \tag{3.20}$$

By the definition of $G(t)$ and (3.17), we have

$$G'(0) = (1 - p_0)F^{-p_0}(0)F'(0) = 2(1 - p_0)\beta\gamma F^{-p_0}(0) < 0.$$

Hence, it follows from the above inequality and (3.20) that

$$\tilde{T} \leq \frac{G(\tilde{T})}{G'(0)} - \frac{G(0)}{G'(0)} < -\frac{G(0)}{G'(0)}.$$

Substituting $G(0)$ and $G'(0)$ into the above inequality, we get

$$\tilde{T} < \frac{T\|u\|^2_{H^1_0} + \beta\gamma^2}{2(p_0 - 1)\beta\gamma} = \frac{\|u\|^2_{H^1_0}}{2(p_0 - 1)\beta\gamma} T + \frac{\gamma}{2(p_0 - 1)}, \quad \forall \tilde{T} \in [0, T).$$

Letting $\tilde{T} \to T^-$, we obtain

$$T \leq \frac{\|u\|^2_{H^1_0}}{2(p_0 - 1)\beta\gamma} T + \frac{\gamma}{2(p_0 - 1)}. \tag{3.21}$$

Fixing an arbitrary β satisfying (3.19), then take γ sufficiently large such that

$$\frac{\|u_0\|^2_{H^1_0}}{2(p_0 - 1)\beta} < \gamma < +\infty.$$

Then, (3.21) yields

$$T \leq \frac{\beta\gamma^2}{2(p_0 - 1)\beta\gamma - \|u_0\|^2_{H^1_0}}. \tag{3.22}$$

Define the function $T_\beta(\gamma)$ as

$$T_\beta(\gamma) = \frac{\beta\gamma^2}{2(p_0 - 1)\beta\gamma - \|u_0\|^2_{H^1_0}}, \quad \gamma \in \left(\frac{\|u_0\|^2_{H^1_0}}{2(p_0 - 1)\beta}, +\infty \right).$$

We can easily verify that the function $T_\beta(\gamma)$ has a unique minimum at

$$\gamma_\beta \triangleq \frac{\|u_0\|^2_{H^1_0}}{(p_0 - 1)\beta} \in \left(\frac{\|u_0\|^2_{H^1_0}}{2(p_0 - 1)\beta}, +\infty \right).$$

Then, it follows from (3.22) that

$$T \leq \inf_{\gamma \in \left(\frac{\|u_0\|^2_{H_0^1}}{2(p_0-1)\beta}, +\infty\right)} T_\beta(\gamma) = T_\beta(\gamma_\beta) = \frac{\|u_0\|^2_{H_0^1}}{(p_0-1)^2\beta}$$

holds for any β satisfying (3.19). Hence, we obtain

$$T \leq \inf_{\beta\in(0,\beta_0]} \frac{\|u_0\|^2_{H_0^1}}{(p_0-1)^2\beta} = \frac{\|u_0\|^2_{H_0^1}}{(p_0-1)^2\beta_0} = \frac{2p_0\|u_0\|^2_{H_0^1}}{(p_0-1)^2[(p_0-1)\|u_0\|^2_{H_0^1} - 2p_0 J(u_0)]},$$

i.e., (3.8) holds. The proof of Theorem 3.2 for Case (A) is completed.

By taking $p_0 = \frac{p+1}{2}$ and $p_0 = 2$ for Case (B) and Case (C), under the condition (3.9) and (3.11), we can obtain (3.10) and (3.12) respectively. The proof is the same as Case (A). □

Remark 3.2. In case $J(u_0) < 0$, the initial condition given by (3.7) (resp. (3.9); (3.11)) is obviously satisfied. But we obtain the upper bounds (3.1) and (3.8) (resp. (3.2) and (3.10); (3.3) and (3.12)) for the blow-up time T in Theorem 3.1 and Theorem 3.2 by using different methods. In fact, we can refine the lifespan T as follows

- Case (A)

$$T \leq \begin{cases} \dfrac{\|u_0\|^2_{H_0^1}}{4p_0(1-p_0)J(u_0)}, & \text{if } J(u_0) < -\dfrac{(p_0-1)^2\|u_0\|^2_{H_0^1}}{2p_0(3p_0+1)}, \\ \dfrac{2p_0\|u_0\|^2_{H_0^1}}{(p_0-1)^2[(p_0-1)\|u_0\|^2_{H_0^1} - 2p_0J(u_0)]}, & \text{if } -\dfrac{(p_0-1)^2\|u_0\|^2_{H_0^1}}{2p_0(3p_0+1)} \leq J(u_0) < 0. \end{cases}$$

- Case (B)

$$T \leq \begin{cases} \dfrac{\|u_0\|^2_{H_0^1}}{(1-p^2)J(u_0)}, & \text{if } J(u_0) < -\dfrac{(p-1)^2\|u_0\|^2_{H_0^1}}{2(p+1)(3p+5)}, \\ \dfrac{8(p+1)\|u_0\|^2_{H_0^1}}{(p-1)^2[(p-1)\|u_0\|^2_{H_0^1} - 2(p+1)J(u_0)]}, & \text{if } -\dfrac{(p-1)^2\|u_0\|^2_{H_0^1}}{2(p+1)(3p+5)} \leq J(u_0) < 0. \end{cases}$$

- Case (C)

$$T \leq \begin{cases} -\dfrac{\|u_0\|^2_{H_0^1}}{8J(u_0)}, & \text{if } J(u_0) < -\dfrac{1}{28}\|u_0\|^2_{H_0^1}, \\ \dfrac{4\|u_0\|^2_{H_0^1}}{\|u_0\|^2_{H_0^1} - 4J(u_0)]}, & \text{if } -\dfrac{1}{28}\|u_0\|^2_{H_0^1} \leq J(u_0) < 0. \end{cases}$$

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References

[1] T. D'Aprile, and D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear Stud., 4(3) (2004), 307–322.

[2] T. D'Aprile, and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A, 134(5) (2004), 893–906.

[3] I. Ianni, and D. Ruiz, Ground and bound states for a static Schrödinger-Poisson-Slater problem, Commun. Contemp. Math., 14(1) (2012), 1250003.

[4] I. Ianni, Local and global solutions for some parabolic nonlocal problems, Nonlinear Anal., 75(13) (2012), 4904–4913.

[5] M. O. Korpusov, and A. G. Sveshnikov, Three-dimensional nonlinear evolutionary pseudo-parabolic equations in mathematical physics, Zh. Vychisl. Mat. Fiz., 43(12) (2003), 1835–1869.

[6] T. W. Ting, Certain non-steady flows of second-order fluids, Arch. Ration. Mech. Anal., 14 (1963), 1–26.

[7] R. E. Showalter and T. W. Ting, Pseudoparabolic partial differential equations, SIAM J. Math. Anal., 1 (1970), 1–26.

[8] Y. Cao, J. Yin, and C. Wang, Cauchy problems of semilinear pseudo-parabolic equations, J. Differential Equations, 246(12) (2009), 4568–4590.

[9] G. Liu, and R. Zhao, Blow-up phenomena and lifespan for a quasi-linear pseudo-parabolic equation at arbitrary initial energy level, Bound. Value Probl., (2018), 159.

[10] P. Luo, Blow-up phenomena for a pseudo-parabolic equation, Math. Meth. Appl. Sci., 38(12) (2015), 2636–2641.

[11] R. Xu, and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, J. Funct. Anal., 264(12) (2013), 2732–2763.

[12] R. Xu, X. Wang, and Y. Yang, Blowup and blowup time for a class of semilinear pseudo-parabolic equations with high initial energy, Appl. Math. Lett., 83 (2018), 176–181.

[13] G. Xu, and J. Zhou, Lifespan for a semilinear pseudo-parabolic equation, Math. Meth. Appl. Sci., 41(2) (2018).

[14] X. Zhu, F, Li, and Y. Li, Some sharp result for the global existence and blow up of solutions to a class of pseudo-parabolic equation, Proc. Roy. Soc. Edinburgh Sect. A, 147 (2017), 1–21.

[15] X. Zhu, F, Li, and Y. Li, Global solutions and blow up solutions to a class of pseudo-parabolic equations with nonlocal term, Appl. Math. Comput., 329 (2018), 38–51.

[16] X. Zhu, F, Li, Z. Liang, and T. Rong, A sufficient condition for blowup of solutions to a class of pseudo-parabolic equations with a nonlocal term, Math. Meth. Appl. Sci., 39(13) (2016), 3591–3606.

[17] B. Liu, and L. Ma, Invariant sets and the blow up threshold for a nonlocal equation of parabolic type, Nonlinear Anal. Theor. Methods Appl., 110 (2014), 141–156.

[18] B. Liu, and L. Ma, Blow up threshold for a parabolic type equation involving space integral and variational structure, Commun. Pure. Appl. Anal., 14 (2015), 2169–2183.
[19] X. Li, and B. Liu, Vacuum isolating, blow up threshold, and asymptotic behavior of solutions for a nonlocal parabolic equation, J. Math. Phys., 58 (2017), 101–110.
[20] J. Zhou, Blow-up and lifespan of solutions to a nonlocal parabolic equation at arbitrary initial energy level, Appl. Math. Lett., 78 (2018), 118–125.