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Remarks on Gap Theorems for Complete Hypersurfaces with Constant Scalar Curvature

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Abstract. Assume that $M^n (n \geq 3)$ is a complete hypersurface in $\mathbb{R}^{n+1}$ with zero scalar curvature. Assume that B, H, g is the second fundamental form, the mean curvature and the induced metric of M, respectively. We prove that M is a hyperplane if

$$-P_1(\nabla H, \nabla |H|) \leq -\delta |H| |\nabla H|^2$$

for some positive constant δ, where $P_1 = nHg - B$ which denotes the first order Newton transformation, and

$$\left(\int_M |H|^n dv \right)^{\frac{1}{n}} < \alpha$$

for some small enough positive constant α which depends only on n and δ. We also derive similar result for complete hypersurfaces in $\mathbb{S}^{n+1}$ with constant scalar curvature $R = n(n-1)$.

AMS subject classifications: 53C24, 53C42

Key words: Hypersurfaces, constant scalar curvature, gap theorem.

1 Introduction

One of the most important research subject in differential geometry is the classification of hypersurfaces in $\mathbb{R}^{n+1}$ with (various) constant curvatures. The classical Bernstein theorem states that a minimal entire graph $M^n (n \leq 7)$ immersed in $\mathbb{R}^{n+1}$ must be a hyperplane [26]. Since then, this result has been generalized to the parametric minimal hypersurfaces which is stable or with finite total curvature [6,7,9,10,12–15,18,22,24,25,27,29,30]. A typical result which has intimate relation with the theme of this paper is the following theorem proved by Ni [22] and Yun [30] (see also [29]):

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Theorem 1.1 ([22,30]). *Assume that $M^n(n\geq 3)$ is a complete noncompact minimal hypersurface in $\mathbb{R}^{n+1}$. There exists a constant $C(n)$ depending only on n such that if*

$$\int_M |B|^n dv < C(n),$$

where B is the second fundamental form of M in $\mathbb{R}^{n+1}$, then M is a hyperplane.

Since the mean curvature is $\frac{1}{n}$ of the first elementary symmetric function of the second fundamental form, it is natural to consider hypersurfaces with zero r-th($r>1$) elementary symmetric functions of the second fundamental form [2,5,11,23]. The particular interesting case is complete hypersurfaces with zero scalar curvature [3,4,16,21]. Several years ago, Li, Xu and Zhou proved the following Bernstein type theorem:

Theorem 1.2 ([19]). *Let $M^n(n\geq 3)$ be a complete hypersurface immersed in $\mathbb{R}^{n+1}$ with zero scalar curvature. There exists a sufficiently small number κ which depends only on dimension n such that if*

$$\left(\int_M |H|^n dv\right)^{\frac{1}{n}} < \kappa, \tag{1.1}$$

then the following statement are equivalent:
(a) M is locally conformally flat;
(b) $|\nabla B|^2 = n^2|\nabla H|^2$;
(c) $nHtr(B^3) = n^4H^4$;
(d) M is flat.

In the above theorem B denotes the second fundamental form of M^n in $\mathbb{R}^{n+1}$ and $H := \frac{1}{n}trB$ denotes the mean curvature. As a direct corollary they indeed proved a Bernstein type theorem that a complete hypersurface in $\mathbb{R}^{n+1}$ with zero scalar curvature is a hyperplane if it satisfies (a) or (b) or (c) and has small enough total curvature $(\int_M |H|^n dv)^{\frac{1}{n}}$. It is very natural to ask whether one could find other Bernstein type theorems for complete hypersurfaces in $\mathbb{R}^{n+1}$ with zero scalar curvature. With this question we have

Theorem 1.3. *Let $(M^n,g)(n\geq 3)$ be a complete hypersurface immersed in $\mathbb{R}^{n+1}$ with zero scalar curvature. Assume that $P_1 = nHg - B$ and it satisfies that*

$$-P_1(\nabla H, \nabla |H|) \leq -\delta |H| |\nabla H|^2, \tag{1.2}$$

for some positive constant δ. Then there exists a sufficiently small number α which depends only on dimension n and δ such that if

$$\left(\int_M |H|^n dv\right)^{\frac{1}{n}} < \alpha, \tag{1.3}$$

then M is a hyperplane.