BANDED M-MATRIX SPLITTING PRECONDITIONER FOR RIESZ SPACE FRACTIONAL REACTION-DISPERSION EQUATION*

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Abstract

Based on the Crank-Nicolson and the weighted and shifted Gr"{u}wald operators, we present an implicit difference scheme for the Riesz space fractional reaction-dispersion equations and also analyze the stability and the convergence of this implicit difference scheme. However, after estimating the condition number of the coefficient matrix of the discretized scheme, we find that this coefficient matrix is ill-conditioned when the spatial mesh-size is sufficiently small. To overcome this deficiency, we further develop an effective banded M-matrix splitting preconditioner for the coefficient matrix. Some properties of this preconditioner together with its preconditioning effect are discussed. Finally, Numerical examples are employed to test the robustness and the effectiveness of the proposed preconditioner.


Key words: Riesz space fractional equations, Toeplitz matrix, conjugate gradient method, Incomplete Cholesky decomposition, Banded M-matrix splitting.

1. Introduction

We consider the following initial-boundary problem of Riesz space fractional reaction-dispersion equation (RSFRDE) [1]:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= -K u(x,t) + K_{\beta} \frac{\partial^{\beta} u(x,t)}{\partial |x|^\beta} + f(x,t), \quad x \in (a, b), \; t \in (0, T], \\
u(x,0) &= \phi(x), \quad x \in (a, b), \\
u(x,t) &= \psi(x,t), \quad x \in \mathbb{R}\backslash(a, b), \; t \in [0, T].
\end{align*}
\]

(1.1)

where $1 < \beta < 2$ and the coefficients $K$, $K_{\beta}$ are positive constants, $u(x,t)$ is an unknown function to be solved. In addition, $f(x,t)$ is the source term, and the Riesz space fractional

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operator $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ defined as

$$\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\Psi_\beta \left[ aD^\beta_x u(x,t) + xD^\beta_\delta u(x,t) \right], \quad (x,t) \in (a,b) \times (0,T)$$

(1.2)

was used for describing anomalous diffusion. Here, the coefficient $\Psi_\beta$ satisfies $\Psi_\beta = \frac{1}{2 \cos \frac{\pi \beta}{2}} < 0$ for $\beta \in (1,2)$. When $K = 0$, problem (1.1) reduces to the Riesz fractional diffusion equation [2].

The definition of the left- and the right-sided Riemann-Liouville fractional derivatives $aD^\beta_x$ and $xD^\beta_\delta$ are given as [3, 4],

$$\begin{align*}
aD^\beta_x u(x,t) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{2-\beta}} d\xi, \\
xD^\beta_\delta u(x,t) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\xi,t)}{((\xi-x)^{2-\beta})} d\xi,
\end{align*}$$

where $\Gamma(\cdot)$ is the Gamma function.

We should note that the boundary condition (BC) in (1.1) is defined on $\mathbb{R} \setminus (a,b)$ rather than $x = a, b$. This is because, take Lévy processes as an example, the paths of all proper Lévy processes, except Brownian motion with drift, are discontinuous, which means the boundary $x = a, b$ itself cannot be hit by the majority of discontinuous sample trajectories. Therefore, we consider the generalized Dirichlet type BC on the domain $\mathbb{R} \setminus (a,b)$, where $\psi(x,t)$ satisfy that there exist positive $M$ and $C$ such that when $|x| > M$,

$$\frac{\psi(x,t)}{|x|^{2-\beta}} < C \quad \text{for positive small } \varepsilon.$$  

(1.3)

In particular, we choose $\psi(x,t) \equiv 0$ in this work, which is the so-called absorbing boundary condition [5].

In recent decades, due to the development of natural science, it has been found that, compared with classical integer order model, the fractional order model has huge advantages in many fields. The space-fractional diffusion equation (SFDE) has obtained wide attention, and it has been successfully used for explaining the anomalous dispersion phenomena in the real world, such as groundwater contaminant transport [6, 7], turbulent flow [8, 9], nonlocal heat conduction [10], biological systems [11], finance [12,13], image processing [14] and so on. In addition, the SFDE was also used to model attenuation phenomena of acoustic waves in irregular porous random media, and acoustic wave propagation [15–17].

Since the fractional differential operator is nonlocal, it was shown that a native discretization of the SFDE, even though implicit, always leads to unstable numerical scheme. To overcome this difficulty, Meerschaet and Tadjeran in [18, 19] proposed a shifted Grünwald discretization (SGD) to approximate SFDEs. Their method has been proved to be unconditionally stable. However, most of the available numerical methods, including SGD, for SFDEs tend to generate full coefficient matrices, which require computational cost of $O(N^3)$ and storage of $O(N^2)$ for solving the corresponding numerical scheme, where $N$ is the number of grid points. Fortunately, Wang et al. [20] shown that the full coefficient matrix corresponding to the scheme possesses a special Toeplitz-like structure, which can be written as the sum of a scaled identity matrix and two diagonal-multiply-Toeplitz matrices. Hence, the storage requirement is significantly reduced from $O(N^2)$ to $O(N)$ and the complexity of the matrix-vector multiplication for the Toeplitz matrix is reduced from $O(N^3)$ to $O(N \log N)$ based on the fast Fourier transform [21–23].
addition, (preconditioned) iterative solvers used for solving the SGD-type schemes have been proposed and studied; see [24–29] for details.

On the other hand, the Riesz fractional derivative existed in (1.1) includes both the left- and the right-sided Riemann-Liouville derivatives. Different approximation methods for the Riesz fractional derivative can be found in Refs. [1, 30–32]. Recently, Tian et al. [33] proposed a second-order weighted and SGD (WSGD) formula to approximate the Riemann-Liouville derivative. Based on the Crank-Nicolson (CN) and the WSGD operators, we in this work present an implicit difference scheme (named as CN-WSGD scheme) for the RSFRDE (1.1). This difference scheme is proved to be unconditionally stable and convergent with the accuracy of $\mathcal{O}(h^2 + \Delta t^2)$. The coefficient matrix corresponding to this scheme is real symmetric Toeplitz, and also positive definite. Hence, we can use the conjugate gradient (CG) method to solve it. Unfortunately, after estimating the condition number of the coefficient matrix, we find that this coefficient matrix is ill-conditioned as the spatial mesh size $h$ approaches to zero. Thus, the iteration methods, including CG, used for solving this kind of linear systems convergence slowly.

Since preconditioning techniques play an important role in improving the convergence rate of the iteration method for solving FDEs [24,25,27–29,34]. Therefore, we try to find an efficient preconditioner for the symmetric positive definite Toeplitz matrix arising from the RSFRDE (1.1). Considering the coefficient matrix possesses off-diagonal decay property, we finally develop an effective banded $M$-matrix splitting preconditioner based on incomplete Cholesky decomposition and the preconditioner’s designing idea in Ref. [29]. Numerical results show that the proposed banded preconditioner is very efficient.

The remainder of this work is organized as follows. In Section 2 we first derive the CN-WSGD scheme for the RSFRDE (1.1). The matrix-vector form of this scheme is also presented. In Section 3 we further study the stability and the convergence of the numerical scheme. After knowing that the coefficient matrix is ill-conditioned by estimating its condition number, in Section 4, we propose a banded $M$-matrix splitting preconditioner to accelerate the convergence of the CG method for solving the symmetric positive definite Toeplitz linear systems. In Section 5, two numerical examples are employed to demonstrate the performance of the proposed preconditioner. Finally, in Section 6, we give a brief conclusion for this paper.

2. The Finite Difference Scheme of RSFRDE

To derive the numerical scheme for problem (1.1), we first introduce some notations. The spatial interval $[a, b]$ is uniformly divided into $N$ parts with space step $h = (b - a)/(N + 1)$ and the temporal interval is partitioned into $M$ parts using the grid-points $t_m = m\Delta t$, where the equidistant temporal step gives $\Delta t = T/M$. The set of the grid points are denoted by $\{x_j | x_j = a + jh, \ j = 0, 1, \ldots, N + 1\}$ and $\{t_m | t_m = m\Delta t, \ m = 0, 1, \ldots, M\}$.

Now, using the WSGD operator with shifting parameters $(p, q) = (1, 0)$ in [33] to approximate the left- and right-sided Riemann-Liouville fractional derivatives on the domain $[a, b]$ gives

$$aD^\beta_{x} u(x_j) = \frac{1}{h^\beta} \sum_{k=0}^{j+1} \omega_k^{(\beta)} u(x_{j-k+1}) + \mathcal{O}(h^2),$$

$$bD^\beta_{x} u(x_j) = \frac{1}{h^\beta} \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u(x_{j+k-1}) + \mathcal{O}(h^2),$$

(2.1)
where
\[ \omega_0^{(\beta)} = \frac{\beta}{2} g_0^{(\beta)}, \quad \omega_k^{(\beta)} = \frac{\beta}{2} g_k^{(\beta)} + \frac{2 - \beta}{2} g_{k-1}^{(\beta)} \] (2.2)
and
\[ g_0^{(\beta)} = 1, \quad g_k^{(\beta)} = (-1)^k \binom{\beta}{k} = \left(1 - \frac{\beta + 1}{k}\right) g_{k-1}^{(\beta)} \text{ for } k \geq 1. \]

The coefficients \( g_k^{(\beta)} \) and \( w_k^{(\beta)} \) have the following properties:

**Lemma 2.1 ([18, 19, 35, 36]).** For all \( 1 < \beta < 2 \), we have
\[
\begin{cases}
g_0^{(\beta)} = 1, & g_1^{(\beta)} = -\beta, \quad 1 > g_2^{(\beta)} > g_3^{(\beta)} > \cdots > 0, \\
\sum_{k=0}^{N} g_k^{(\beta)} = 0, & \sum_{k=0}^{N} g_k^{(\beta)} < 0, \quad 1 \leq N < \infty, \\
g_k^{(\beta)} = O(k^{-\beta-1}), & k \to \infty.
\end{cases}
\]

**Lemma 2.2 ([33]).** For \( 1 < \beta < 2 \), the sequence \( \{\omega_k^{(\beta)}\} \) satisfies
\[
\begin{cases}
\omega_0^{(\beta)} = \frac{\beta}{2} > 0, & \omega_1^{(\beta)} = \frac{2 - \beta - \beta^2}{2} < 0, & \omega_2^{(\beta)} = \frac{\beta^2 + \beta - 4}{4}, \\
1 \geq \omega_0^{(\beta)} \geq \omega_1^{(\beta)} \geq \omega_2^{(\beta)} \geq \cdots > 0, & \omega_0^{(\beta)} + \omega_2^{(\beta)} > 0, \\
\sum_{k=0}^{\infty} \omega_k^{(\beta)} = 0, & \sum_{k=0}^{N} \omega_k^{(\beta)} < 0, & N \geq 2.
\end{cases}
\]

Thus, we use WSGD formula (2.1) and CN technique to approximate the Riemann-Liouville fractional derivatives and the time derivative, respectively. The CN-WSGD finite difference scheme of the RSFRDE (1.1) at points \((x_j, t_{m+\frac{1}{2}})\) can be written as
\[
\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = \frac{1}{2} \left( -K u_j^{(m+1)} - \frac{K \Psi_{\beta} \Delta t}{h^\beta} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} u_{j-k+1}^{(m+1)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u_{j+k-1}^{(m+1)} \right) \right)
+ \frac{1}{2} \left( -K u_j^{(m)} - \frac{K \Psi_{\beta} \Delta t}{h^\beta} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} u_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u_{j+k-1}^{(m)} \right) \right) + f_j^{(m+\frac{1}{2})}, \tag{2.3}
\]
where \( f(x_j, t_{m+\frac{1}{2}}) = \frac{1}{2} (f_j^{(m+1)} + f_j^{(m)}) \).

Multiplying (2.3) by \( 2\Delta t \), we have
\[
(2 + \Delta t K) u_j^{(m+1)} + \frac{K \Psi_{\beta} \Delta t}{h^\beta} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} u_{j-k+1}^{(m+1)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u_{j+k-1}^{(m+1)} \right)
= (2 - \Delta t K) u_j^{(m)} - \frac{K \Psi_{\beta} \Delta t}{h^\beta} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} u_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u_{j+k-1}^{(m)} \right)
+ 2\Delta t f_j^{(m+\frac{1}{2})}. \tag{2.4}
\]

In addition, the boundary and initial conditions are discretized as follows:
\[
u_0^{(m)} = u_N^{(m)} = 0, \quad u_j^{(0)} = \psi(a + jh), \quad m = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N + 1.
\]
Remark 2.1. For Thus, the proof is completed.

Thus, the proof is completed.

Denote \( u^{(m)} = [u_1^{(m)}, u_2^{(m)}, \ldots, u_N^{(m)}]^T \), \( f^{(m+\frac{1}{2})} = [f_1^{(m+\frac{1}{2})}, f_2^{(m+\frac{1}{2})}, \ldots, f_N^{(m+\frac{1}{2})}]^T \). The equivalent matrix-vector form of numerical scheme (2.4) can be written as

\[
A_N u^{(m+1)} := ((2 + \Delta t K) I_N + K_\beta T_N) u^{(m+1)} = \eta^{(m+1)},
\]

where

\[
\eta_\beta = -\frac{\Psi_\beta \Delta t}{h^3} > 0, \quad T_N = \eta_\beta (T_\beta + T_{\beta}^T),
\]

and

\[
\eta^{(m+1)} = ((2 - \Delta t K) I_N - K_\beta T_N) u^{(m)} + 2 \Delta tf^{(m+\frac{1}{2})}.
\]

Here, \( I_N \in \mathbb{R}^{N \times N} \) is an identity matrix, \( T_\beta \) is the Toeplitz matrix of the form

\[
T_\beta = -
\begin{pmatrix}
\omega_1^{(b)} & \omega_0^{(b)} & 0 & \cdots & 0 \\
\omega_2^{(b)} & \ddots & \ddots & \ddots & \ddots \\
\omega_3^{(b)} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \omega_k^{(b)} \\
\omega_N^{(b)} & \cdots & \omega_1^{(b)} & \omega_2^{(b)} & \omega_3^{(b)} \\
\end{pmatrix}
\]

(2.7)

Obviously, the coefficient matrix \( A_N = (2 + \Delta t K) I_N + K_\beta T_N \) is a Hermitian Toeplitz matrix [21, 22, 37].

**Theorem 2.1.** For \( 1 < \beta < 2 \), the matrix \( T_N \) in (2.5) is a strictly diagonally dominant \( M \)-matrix.

**Proof.** Denote the entries of matrix \( T_N \) as follows

\[
[T_N]_{jk} = -
\begin{cases}
2\eta_\beta \omega_1^{(b)}, & k = j, \\
\eta_\beta (\omega_0^{(b)} + \omega_2^{(b)}), & k = j - 1, \\
\eta_\beta (\omega_2^{(b)} + \omega_0^{(b)}), & k = j + 1, \\
\eta_\beta \omega_{j-k+1}^{(b)}, & k < j - 1, \\
\eta_\beta \omega_{j-k+1}^{(b)}, & k > j + 1.
\end{cases}
\]

(2.8)

By Lemma 2.2 and note that \( \omega_0^{(b)} + \omega_2^{(b)} > 0 \) and \( \eta_\beta > 0 \), we can obtain \( |T_N|_{jk} \leq 0 \) for all \( k \neq j \) and \( |T_N|_{jj} = -2\eta_\beta \omega_1^{(b)} > 0 \) for all \( k = j \). This means that the coefficient matrix \( T_N \) is an \( M \)-matrix [38]. Furthermore, combining Lemma 2.2 with (2.8), we immediately get

\[
[T_N]_{jj} - \sum_{k=1, k \neq j}^N |[T_N]_{jk}|
\geq -2\eta_\beta \omega_1^{(b)} - 2\eta_\beta \sum_{k=0, k \neq 1}^N \omega_k^{(b)} \geq -2\eta_\beta \omega_1^{(b)} - 2\eta_\beta \sum_{k=0, k \neq 1}^\infty \omega_k^{(b)}
\]

(2.9)

Thus, the proof is completed. \( \square \)

According to Theorem 2.1, it is easy to conclude the following corollary.

**Corollary 2.1.** For \( 1 < \beta < 2 \), the matrix \( T_N \) in (2.5) is nonsingular.

**Remark 2.1.** From Lemma 2.2, the matrix \( T_\beta \) given in (2.7) is also an \( M \)-matrix when \( \sqrt{2}Δt < \beta < 2 \).
3. Stability and Convergence Analysis

In this section, we intend to discuss the stability and the convergence of the CN-WSGD scheme (2.3).

**Theorem 3.1.** The difference scheme (2.3) for the problem (1.1) is unconditionally stable.

**Proof.** Let $\lambda(T_N)$ be any eigenvalue of the matrix $T_N$. Then, by Lemma 2.2 and Geršgorin’s circle theorem [38], it holds that

$$|\lambda(T_N) - [T_N]_{jj}| \leq R_j = \sum_{k=1, k \neq j}^{N} |\omega^{(j)}_k|,$$

or equivalently,

$$[T_N]_{jj} - \sum_{k=1, k \neq j}^{N} |\omega^{(j)}_k| \leq \lambda(T_N) \leq [T_N]_{jj} + \sum_{k=1, k \neq j}^{N} |\omega^{(j)}_k|. \tag{3.1}$$

Thus, from (2.9), we know $[T_N]_{jj} > \sum_{k=1, k \neq j}^{N} |\omega^{(j)}_k|$. This means that all the eigenvalues of matrix $T_N$, i.e., $\lambda(T_N)_{k}$ ($k = 1, \ldots, N$), are positive.

In addition, notice that the difference scheme (2.3) (or (2.5)) can be rewritten as

$$u^{(m+1)} = B_N u^{(m)} + G_N f^{(m+\frac{1}{2})}, \tag{3.2}$$

where

$$G_N = 2\Delta t \left( (2 + \Delta tK)I_N + K_\beta T_N \right)^{-1}, \tag{3.3}$$

$$B_N = \left( (2 + \Delta tK)I_N + K_\beta T_N \right)^{-1} \left( (2 - \Delta tK)I_N - K_\beta T_N \right). \tag{3.4}$$

Suppose that $\lambda(B_N)$ is any eigenvalue of matrix $B_N$ and note that $\Delta t > 0$ and $K > 0$, we immediately obtain

$$|\lambda(B_N)| = \frac{|(2 - \Delta tK) - K_\beta \lambda(T_N)|}{|(2 + \Delta tK) + K_\beta \lambda(T_N)|} < 1. \tag{3.5}$$

Therefore, the spectral radius of the matrix $B_N$ is less than one. Then, the finite difference scheme (2.5) is unconditionally stable, that is to say that the difference scheme (2.3) is unconditionally stable.

Before giving the convergence property of the finite difference scheme (2.3), we firstly give a useful lemma.

**Lemma 3.1 ([39]).** Let $A$ be an $N$ order positive definite matrix. Then, for any parameter $\theta \geq 0$, we have

$$\|(I + \theta A)^{-1}\|_\infty \leq 1 \quad \text{and} \quad \|(I + \theta A)^{-1}(I - \theta A)\|_\infty \leq 1.$$

Note that the local truncation of the CN scheme of Eq. (1.1) at $(x_j, t_{m+\frac{1}{2}})$, with $j = 1, \ldots, N$ and $m = 0, 1, \ldots, M$, yields

$$\frac{\partial u(x_j, t_{m+\frac{1}{2}})}{\partial t} = \frac{1}{2} \left( -Ku(x_j, t_{m+1}) + K_\beta \frac{\partial^\beta u(x_j, t_{m+1})}{\partial |x|^\beta} - Ku(x_j, t_{m}) + K_\beta \frac{\partial^\beta u(x_j, t_{m})}{\partial |x|^\beta} \right) + f(x_j, t_{m+\frac{1}{2}}) + O(\Delta t^2). \tag{3.6}$$
Meanwhile, the second order central difference scheme of the derivative \( \frac{\partial u(x_j, t_{m+1})}{\partial t} \) has the form of

\[
\frac{\partial u(x_j, t_{m+1})}{\partial t} = u(x_j, t_{m+1}) - u(x_j, t_m) \quad \text{or equivalently,}
\]

\[
\frac{\partial u(x_j, t_{m+1})}{\partial t} = u(x_j, t_{m+1}) - u(x_j, t_m) + O(\Delta t^2).
\]

From (1.2) and (2.1), it follows that

\[
K_\beta \left[ \frac{\partial^2 u(x_j, t_m)}{\partial x^2} \right] = \eta_\beta \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} u_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} u_{j+k-1}^{(m)} \right) + O(h^2).
\]

Thus, substituting (3.7) and (3.8) into (3.6) gives

\[
\frac{u_{j}^{(m+1)} - u_{j}^{(m)}}{\Delta t} = -\frac{1}{2} \left( K_{\Psi \beta} u_{j}^{(m)} + \frac{K_\beta \Psi \beta}{h^2} \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m+1)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m+1)} \right)
\]

\[
- \frac{1}{2} \left( K_{\omega} e_{j}^{(m)} + \frac{K_\beta \Psi \beta}{h^2} \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m)} \right) + O(h^2 + \Delta t^2).
\]

**Theorem 3.2.** The difference schemes (2.3) is unconditionally convergent as \( h \) and \( \Delta t \) tend to zero. Moreover, the convergence order is \( O(h^2 + \Delta t^2) \).

**Proof.** Let \( \tilde{u} \) and \( u \) be the exact solution and the numerical solution of Eq. (1.1), respectively. The error at grid points \( (x_j, t_m) \) can be defined as \( e_j^{(m)} = u_j^{(m)} - u_j^{(m)} \). Then, in view of (2.3) and (3.9), we can get the following error equation

\[
\frac{e_{j}^{(m+1)} - e_{j}^{(m)}}{\Delta t} = -\frac{1}{2} \left( K_{\Psi \beta} e_{j}^{(m)} + \frac{K_\beta \Psi \beta}{h^2} \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m+1)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m+1)} \right)
\]

\[
- \frac{1}{2} \left( K_{\omega} e_{j}^{(m)} + \frac{K_\beta \Psi \beta}{h^2} \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m)} \right) + O(h^2 + \Delta t^2),
\]

or equivalently,

\[
(2 + \Delta t K_\beta) e_{j}^{(m+1)} + \frac{K_\beta \Psi \beta \Delta t}{h^2} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m+1)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m+1)} \right)
\]

\[
= (2 - \Delta t K_\beta) e_{j}^{(m)} - \frac{K_\beta \Psi \beta \Delta t}{h^2} \left( \sum_{k=0}^{j+1} \omega_k^{(\beta)} e_{j-k+1}^{(m)} + \sum_{k=0}^{N-j+2} \omega_k^{(\beta)} e_{j+k-1}^{(m)} \right)
\]

\[
+ O(\Delta t h^2 + \Delta t^3).
\]

Using initial-boundary condition, we know \( e_j^{(0)} = 0 \) for \( j = 1, \ldots, N \) and \( e_0^{(m)} = e_N^{(m)} = 0 \), for \( m = 0, 1, \ldots, M \). Thus, the matrix-vector form of system (3.11) can be written as

\[
((2 + \Delta t K_\beta) I_N + K_\beta T_N) E^{(m+1)} = ((2 - \Delta t K_\beta) I_N - K_\beta T_N) E^{(m)} + O(\Delta t h^2 + \Delta t^3) 1,
\]
where \( E^{(0)} = 0, E^{(m)} = (e_1^{(m)}, e_2^{(m)}, \ldots, e_N^{(m)})^T \), and \( \mathbf{1} \) denotes a column vector whose elements are all 1.

Note that the coefficient matrix \( A_N \) is nonsingular, the matrix-vector form of Eq. (3.10) or (3.11) can be written as

\[
E^{(m)} = B_N E^{(m-1)} + q_N,  \tag{3.12}
\]

where \( B_N \) is given in (3.4) and \( q_N = \mathcal{O}(h^2 + \Delta t^2)G_N \mathbf{1} \) with \( G_N \) being given in (3.3). Using (3.12) recursively and noticing that \( E_0 = \mathbf{0} \), we can derive that

\[
E^{(m)} = (B_N^{m-1} + B_N^{m-2} + \cdots + I_N)q_N, \quad \text{for} \; m = 1, \cdots, M.  \tag{3.13}
\]

Thus, the infinity norm of \( E^{(m)} \) yields

\[
\| E^{(m)} \|_{\infty} \leq \left( \| B_N^{m-1} \|_{\infty} + \| B_N^{m-2} \|_{\infty} + \cdots + \| I_N \|_{\infty} \right) \| q_N \|_{\infty}.  \tag{3.14}
\]

Now, by (3.5), Lemma 3.1 and the proof of Theorem 3.1, we can obtain that \( \| B_N \|_{\infty} < 1 \) and \( \| (2 + \Delta t K) I + T_N \|_{\infty} < 1 \). Hence, it further gives

\[
\| E^{(m)} \|_{\infty} \leq m \Delta t \mathcal{O}(h^2 + \Delta t^2),
\]

i.e., \( \| E^{(m)} \|_{\infty} \leq C(h^2 + \Delta t^2) \). Therefore, we can immediately obtain \( \| E^{(m)} \|_{\infty} \to 0 \), as \( h \to 0 \) and \( \Delta t \to 0 \). Moreover, the convergence order is \( \mathcal{O}(h^2 + \Delta t^2) \). \( \square \)

4. Preconditioning Technique

In this section, we first estimate the condition number of the coefficient matrix of the CN-WSGD scheme, and then introduce a banded \( M \)-matrix splitting preconditioner for the coefficient matrix. The theoretical analysis for the preconditioning effect is also given.

Now, before estimating the condition number, we firstly give two useful lemmas.

**Lemma 4.1 ([40]).** Let \( R_\beta = \frac{3^{\beta+1}}{\beta(3-\beta)^{\beta+1}} \), \( L_\beta = \frac{e^{\pi^2/2} \Gamma(1-\beta)}{\pi^2/2 \beta \Gamma(1-\beta)} \) and \( g_j^{(\beta)} = (-1)^j \left( \begin{array}{c} \beta \\ j \end{array} \right) \).

It holds that

\[
\sum_{j=N+1}^{\infty} g_j^{(\beta)} < R_\beta \left( \frac{1}{N^\beta} \right) \quad \text{for} \; N \geq 2, \quad \text{and} \quad \sum_{j=N+1}^{\infty} g_j^{(\beta)} > L_\beta \left( \frac{1}{N^\beta} \right) \quad \text{for} \; N \geq 3.
\]

**Lemma 4.2.** The fractional binomial coefficients \( \omega_j^{(\beta)} = (\beta/2) g_j^{(\beta)} + \left( 1 - \beta/2 \right) g_{j-1}^{(\beta)} \) \((j \geq 1)\) defined in (2.2) yield

\[
\sum_{j=N+1}^{\infty} \omega_j^{(\beta)} < \hat{\Theta}_\beta h^\beta \quad \text{for} \; N \geq 3, \quad \text{and} \quad \sum_{j=N+1}^{\infty} \omega_j^{(\beta)} > \Theta_\beta h^\beta \quad \text{for} \; N \geq 3,
\]

where

\[
\hat{\Theta}_\beta = \frac{2^{\beta} R_\beta}{(b-a)^\beta} \quad \text{and} \quad \Theta_\beta = \frac{L_\beta}{(b-a)^\beta}.
\]

**Proof.** By (2.2) and Lemma 4.1, we have, for \( N \geq 3 \), that

\[
\sum_{j=N+1}^{\infty} \omega_j^{(\beta)} = \frac{\beta}{2} \sum_{j=N+1}^{\infty} g_j^{(\beta)} + \left( 1 - \frac{\beta}{2} \right) \sum_{j=N+1}^{\infty} g_{j-1}^{(\beta)} < \sum_{j=N+1}^{\infty} g_{j-1}^{(\beta)} < \frac{R_\beta}{(N-1)^\beta}.
\]
Note that the spatial step size $h = \frac{b-a}{N+1}$, we further obtain that
\[
\sum_{j=N+1}^{\infty} \omega_j^{(b)} < \frac{R_\beta}{(N-1)^\beta} \left( \frac{b-a}{N-1} \right)^\beta = \frac{R_\beta}{(b-a)^\beta} = \Theta_\beta h^\beta.
\] (4.1)

Similarly, using Lemma 4.1, it follows for $N \geq 3$ that
\[
\sum_{j=N+1}^{\infty} \omega_j^{(b)} = \frac{\beta}{2} \sum_{j=N+1}^{\infty} g_j^{(b)} + \frac{2-\beta}{2} \sum_{j=N+1}^{\infty} g_j^{(b)} - \omega_j^{(b)} > L_\beta \frac{1}{N^\beta}.
\]
\[
= \frac{L_\beta}{(b-a)^\beta} \frac{1}{N^\beta} \geq \frac{L_\beta}{(b-a)^\beta} \frac{1}{(N+1)^\beta}.
\]
\[
= \Theta_\beta h^\beta.
\]
\[\square\]

**Theorem 4.1.** Let $\mu_\beta = -\Psi_\beta K_\beta$. Then, the coefficient matrix $A_N = (2 + \Delta t K) I + K_\beta T_N$ of linear system (2.5) yield
\[
\|A_N\|_\infty < (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( (\beta + 2)(\beta - 1) - \Theta_\beta h^\beta \right),
\]
and
\[
\| (A_N)^{-1} \|_\infty < \frac{1}{(2 + \Delta t K) + 2\mu_\beta \Delta t \Theta_\beta}.
\]
Hence, the condition number of the matrix $A_N$ in infinity norm (denoted as $\kappa_\infty(A_N)$) satisfies
\[
\kappa_\infty(A_N) = \|A_N\|_\infty \|(A_N)^{-1}\|_\infty \leq 1 + O(\Delta th^{-\beta}).
\]

**Proof.** Using (2.6), (2.7) and Lemma 2.2, we can get, for $N \geq 4$, that
\[
\|A_N\|_\infty \leq (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( \|w_1^{(b)}\| + |w_0^{(b)} + w_2^{(b)}| + \sum_{j=3}^{N} |w_j^{(b)}| \right)
\]
\[
= (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( -w_1^{(b)} + w_0^{(b)} + w_2^{(b)} + \sum_{j=3}^{N} w_j^{(b)} \right)
\]
\[
= (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( -2w_1^{(b)} + \sum_{j=0}^{N} w_j^{(b)} \right)
\]
\[
= (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( \beta + \beta - 2 - \sum_{j=N+1}^{\infty} w_j^{(b)} \right).
\]

We further use Lemma 4.2, the infinity norm of $A_N$ yields
\[
\|A_N\|_\infty \leq (2 + \Delta t K) + \frac{2\mu_\beta \Delta t}{h^\beta} \left( (\beta + 2)(\beta - 1) - \Theta_\beta h^\beta \right).
\]
Let $A_N = ([A_N]_{jk}) \in \mathbb{R}^{N \times N}$. Noticing that $A_N$ is reversible and using the similar method as in Ref. [41], it follows that

$$
\| (A_N)^{-1} \|_\infty \leq \max_{1 < j < N} \frac{1}{\| [A_N]_{jj} \| - \sum_{k=1, k \neq j}^N \| [A_N]_{jk} \|} \leq \frac{1}{(2 + \Delta t K) - 2 \mu_β \frac{\Delta t}{\tau} \sum_{k=0}^N \omega^{(β)}_k}
$$

$$
= \frac{1}{(2 + \Delta t K) - 2 \mu_β \frac{\Delta t}{\tau} \sum_{k=N+1}^\infty \omega^{(β)}_k}
$$

Therefore, the condition number of the coefficient matrix $A_N$ can be estimated as

$$
\kappa_\infty (A_N) = \| A_N \|_\infty \| (A_N)^{-1} \|_\infty \leq \frac{(2 + \Delta t K) - 2 \mu_β \Theta \beta \Delta t^{- β}}{2 + \Delta t K} \leq 1 + O(\Delta t^{- β}).
$$

The proof of this theorem is completed. □

Note that the coefficient matrix $A_N$ given in (2.5) is symmetric positive definite. Many iteration methods, especially CG, can be used to solve the corresponding system of linear equations. However, from Theorem 4.1, we find that the coefficient matrix $A_N$ is ill-conditioned when the spatial mesh-size is sufficiently small. Thus, the iteration methods, including CG, will convergence slowly. Since preconditioning techniques [22] play important role in improving the convergence rate of the iteration method, a good preconditioner is necessary to reduce the condition number of the coefficient matrix $A_N$, thereby improving the convergence speed of the iteration method. So, in the following, we will introduce a kind of preconditioners (denoted as $P_N$) for the coefficient matrix $A_N$, and use the CG method to solve the following preconditioned linear system

$$
(P_N)^{-1} A_N u^{(m+1)} = (P_N)^{-1} b^{(m+1)}.
$$

From Theorem 2.1, the matrix $T_N$ is a symmetric and strictly diagonally dominant $M$-matrix with positive real part. We consider constructing a preconditioner based on banded approximation of the coefficient matrix [42, 43]. Now, we can split matrix $T_N = T_{N,k} + (T_N - T_{N,k})$, where

$$
T_{N,k} = -\eta_β (B_{N,k} + C_{N,k}).
$$

Denote $ω^{(β)}_1 := 2ω^{(β)}_1$, $ω^{(β)}_2 := ω^{(β)}_0 + ω^{(β)}_2$ and $ω^{(β)}_k := ω^{(β)}_k$ for $k = 3, \ldots, N$. Then, $B_{N,k}$ and
Banded $M$-Matrix Splitting Preconditioner for RSFRDE

$C_{N,k}$ can be written as

$$B_{N,k} = \begin{pmatrix}
\omega_1^{(\beta)} & \cdots & \omega_k^{(\beta)} \\
\vdots & \ddots & \vdots \\
\omega_k^{(\beta)} & \cdots & \omega_k^{(\beta)} \\
\vdots & \ddots & \vdots \\
\omega_k^{(\beta)} & \cdots & \omega_1^{(\beta)}
\end{pmatrix}$$  \quad (4.4)

and

$$C_{N,k} = \text{diag} \left( 0, \ldots, 0, \omega_{k+1}^{(\beta)}, \ldots, \sum_{j=k+1}^{N} \omega_j^{(\beta)} \right) + \text{diag} \left( \sum_{j=k+1}^{N} \omega_j^{(\beta)}, \omega_{k+1}^{(\beta)}, 0, \ldots, 0 \right).$$  \quad (4.5)

From Theorem 2.1, $T_N$ is a strictly diagonally dominant $M$-matrix, which implies that $T_{N,k}$ is a symmetric strictly diagonally dominant $M$-matrix.

The preconditioner for the symmetric positive definite system of linear equations (2.5) is chosen as

$$P_k = (2 + \Delta t K) I_N + K \beta T_{N,k}. \quad (4.6)$$

Since preconditioner $P_k$ is also a banded $M$-matrix and obtained from the splitting of matrix $T_N$, we named it as banded $M$-matrix splitting preconditioner. Note that $T_{N,k}$ is a symmetric strictly diagonally dominant $M$-matrix, and its diagonal elements and $2 + \Delta t K$ are positive, we can also know that $P_k$ is symmetric and nonsingular for $k = 2, 3, \ldots, N$. Moreover, it yields

**Theorem 4.2.** For $2 \leq k \leq N$ and $1 < \beta < 2$, the matrix $P_k$ should be an efficient preconditioner for coefficient matrix $A_N$ as $k$ becomes large enough.

**Proof.** Using Lemma 2.1 and (2.2), it is easy to derive that

$$\omega_k^{(\beta)} = \frac{\beta}{2} O(k^{-1}) + \frac{2}{2} \beta O((k - 1)^{-(\beta + 1)}) = O((k - 1)^{-(\beta + 1)}), \quad k > 1. \quad (4.7)$$

Thus, from (2.8) and (4.3), there exists a positive number $C_1 \geq 4K \beta$, such that

$$\|T_N - T_{N,k}\|_{\infty} \leq 4K \beta \eta \beta \sum_{k}^{N} \omega_{k+1}^{(\beta)} \leq C_1 \eta \beta \sum_{k}^{N} \omega_{k+1}^{(\beta)} \leq C_1 \eta \beta \int_{k}^{\infty} d\ell \frac{1}{(\ell - 1)^{1+\beta}} = C_1 \eta \beta \frac{1}{(k - 1)^{-\beta}}. \quad (4.8)$$

In addition, by Lemma 2.2, the matrix $T_N$ yields

$$\|T_N\|_{\infty} = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |T_N|_{ij}$$

$$\geq K \beta \eta \beta \left( 2|\omega_1^{\beta}| + |\omega_0^{\beta} + \omega_2^{\beta}| + \sum_{j=3}^{N} |\omega_j^{\beta}| \right)$$

$$\geq -2K \beta \eta \beta \omega_1^{\beta} > 0. \quad (4.9)$$
Combining (4.8) with (4.9), it follows that
\[ \frac{\|T_N - T_{N,k}\|_\infty}{\|T_N\|_\infty} \leq \frac{C}{(k - 1)^\beta} \quad (k > 1). \]

Therefore, we finally obtain that
\[ \frac{\|A_N - P_k\|_\infty}{\|A_N\|_\infty} \leq \frac{\|T_N - T_{N,k}\|_\infty}{\|T_N\|_\infty} \leq \frac{C}{(k - 1)^\beta} \quad (k > 1), \]
which implies that the relative difference between \( A_N \) and \( P_k \) is very small if \( k \) is sufficiently large. Thus, \( P_k \) should be an efficient preconditioner for the linear systems (2.5).

\[ \square \]

Remark 4.1. Theorem 4.2 shows that the matrix \( P_k \) should be an efficient preconditioner for the coefficient matrix \( A_N \) if \( k \) is sufficiently large. However, when we use the preconditioner \( P_k \) to improve the convergence rate of the Krylov subspace methods, such as CG, the larger values of \( k \) increase the computational costs since \( P_k \) will be inversed in each step. Hence, choosing the optimal value of \( k \) is a key issue. Unfortunately, we could not find the theoretically optimal value of \( k \) at present.

5. Numerical Experiments

In this section, we employ an example to test the performance of the banded \( M \)-matrix splitting preconditioner proposed in this paper by using it to accelerate the convergence rate of the CG method.

Consider the initial-boundary value problem of RSFRDE (1.1) with \( a = 0, b = 1, T = 1, \phi(x) = x^6(1 - x)^6 \) and the source term
\[ f(x,t) = (1 + K)e^t x^6(1 - x)^6 + K_\beta \frac{e^t}{2 \cos \frac{\beta \pi}{2}} \left\{ \frac{\Gamma(7)}{\Gamma(7 - \beta)} \left[ x^{6-\beta} + (1 - x)^{6-\beta} \right] 
- 6 \frac{\Gamma(8)}{\Gamma(8 - \beta)} \left[ x^{7-\beta} + (1 - x)^{7-\beta} \right] + 15 \frac{\Gamma(9)}{\Gamma(9 - \beta)} \left[ x^{8-\beta} + (1 - x)^{8-\beta} \right] 
- 20 \frac{\Gamma(10)}{\Gamma(10 - \beta)} \left[ x^{9-\beta} + (1 - x)^{9-\beta} \right] + 15 \frac{\Gamma(11)}{\Gamma(11 - \beta)} \left[ x^{10-\beta} + (1 - x)^{10-\beta} \right] 
- 6 \frac{\Gamma(12)}{\Gamma(12 - \beta)} \left[ x^{11-\beta} + (1 - x)^{11-\beta} \right] + \frac{\Gamma(13)}{\Gamma(13 - \beta)} \left[ x^{12-\beta} + (1 - x)^{12-\beta} \right] \right\}. \]

Then, the exact solution is \( u(x,t) = e^t x^6(1 - x)^6 \). Here, the coefficients \( K \) and \( K_\beta \) are chosen as

- Case I. \( K = 5, K_\beta = 0.5 \);
- Case II. \( K = 60, K_\beta = 1 \);
- Case III. \( K(x) = e^{8x + 0.1\beta}, K_\beta = 10 \).

When we approximate the solution of the initial-boundary value problem of RSFRDE (1.1) by using the CN-WSGD scheme (2.3), we need to solve a series of systems of linear equations \( A_N u^{(m)} = b^{(m)} \) with \( m = 1, \ldots, M \), where \( A_N = (2 + \Delta t K)I_N + K_\beta T_N \) for Cases I and II, and \( A_N = 2I_N + \Delta t \tilde{K} + K_\beta T_N \) for Case III, where \( \tilde{K} \) is a diagonal matrix arising from the discretization of \( K(x) \). Obviously, for each of the three cases, \( A_N \) is a symmetrically positive definite and strictly diagonally dominant Toeplitz matrix. For the real symmetric Toeplitz
matrix $T_N$, we denote $s(T_N)$ and $c(T_N)$ the Strang’s circulant preconditioner [21] and the T. Chan’s circulant preconditioner [22], respectively. Then, replacing the matrix $T_N$ in $A_N$ with $s(T_N)$ and $c(T_N)$, respectively, gives the following two circulant preconditioners of the matrix $A_N$:

$$P_s = (2 + \Delta t \tilde{K})I_N + K \beta s(T_N) \quad \text{and} \quad P_c = (2 + \Delta t \tilde{K})I_N + K \beta c(T_N),$$

where $\tilde{K} = K$ for Cases I and II, and $\tilde{K}$ is the mean value of the diagonal elements of $\tilde{K}$ for Case III. For comparison, the numerical efficiencies of the above two preconditioners will also be tested.

In the implementation, the matrix-vector multiplication of a circulant matrix and a vector is done by using the fast Fourier transformation (FFT). In the following, we employ Gauss elimination (denoted as GE), CG and preconditioned CG (denoted as PCG) methods to solve the linear systems $A_N u^{(m)} = b^{(m)}$. The preconditioners used in the PCG methods are chosen to be the banded $M$-matrix splitting preconditioner $P_k$, the two circulant preconditioners $P_s$ and $P_c$, respectively. For convenience, the three PCG methods are denoted as $P_k$-CG, $P_s$-CG and $P_c$-CG, respectively. Since the half bandwidth $k$ in $P_k$ is chosen to be 6 in the following tests, we also write $P_k$-CG as $P_6$-CG. The initial estimation for each of the five iteration methods at each time level $m$, with $m = 1, \ldots, M$, is chosen as

$$u_0^{(m)} = \begin{cases} u_0^{(0)}, & m = 1 \\ 2u^{(m-1)} - u^{(m-2)}, & m \geq 2. \end{cases}$$

The stopping criterion for the iteration methods at time level $m$ is chosen as

$$\frac{\|r_k^{(m)}\|_2}{\|r_0^{(m)}\|_2} = \frac{\|b^{(m)} - A_Nu_k^{(m)}\|_2}{\|b^{(m)} - A_Nu_0^{(m)}\|_2} \leq 10^{-7}.$$ In addition, we choose $M = N + 1$. All the computations are implemented in MATLAB [version R2015b] in double precision on a personal computer with 2.50GHz central processing unit [Intel(R) Core(TM) i5-2450] and 4.00GB memory.

Denote the maximum norm of the error between the exact and the numerical solutions at the last time step as

$$\text{Error} := E(h, \Delta t) = \max_{1 \leq j \leq N} \left| u(x_j, t_M) - u_j^{(M)} \right|,$$

where $u(x_j, t_M)$ is the exact solution and $u_j^{(M)}$ is the numerical solution at the grid point $(x_j, t_M)$ with the mesh step sizes $h$ and $\Delta t$. The convergence order (denoted as Order) is defined as

$$\text{Order} := \log_2 \frac{E(2h, 2\Delta t)}{E(h, \Delta t)}.$$ In Tables 5.1, 5.2 and 5.3, for different values of $\beta$ and $N$, we list the Errors and the Orders of the numerical scheme (2.3) solved by the five tested methods. The numerical results show that all the five tested methods have a satisfactory maximum norm error. Moreover, the convergence rates for all the tested cases are always second-order in both space and time directions, which agree with the theoretical result derived in Theorem 3.2.

Denote by IT the average number of iterations required for solving the series of systems of linear equations $A_N u^{(m)} = b^{(m)}$ with $m = 1, \ldots, M$, i.e.,

$$\text{IT} = \frac{1}{M} \sum_{m=1}^{M} \text{IT}(m),$$
Table 5.1: The Errors and the Orders of the CN-WSGD scheme solved by the tested methods for Case I at $T = 1$.

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<th>Error</th>
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Table 5.2: The Errors and the Orders of the CN-WSGD scheme solved by the tested methods for Case II at $T = 1$.

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where $\text{IT}(m)$ denotes the number of iteration steps required by the iteration method used for solving $A_k y^{(m)} = b^{(m)}$. In addition, we denote by CPU the total elapsed CPU time (in seconds) for solving the RSFRDE (1.1) based on the CN-WSGD scheme (2.3).

In order to test the preconditioning performance of the banded $M$-matrix splitting preconditioner $P_k$, we list the ITs and CPUs of the $P_k$-CG ($k = 6$) method used for solving the CN-WSGD scheme (2.3). For comparison, the numerical results of the other four methods, i.e., GE, CG, $P_2$-CG and $P_1$-CG, are also presented in Tables 5.4, 5.5 and 5.6. From the numerical results, we can see that all the five tested methods are convergent for solving the CN-WSGD scheme. The $P_2$-CG and the $P_1$-CG iteration methods perform much better than the GE and CG methods since they cost less numbers of iteration steps and computing times than the GE and the CG method. Moreover, among the five tested methods, the $P_k$-CG is always the most efficient one especially when $\beta$ is close to 2 and $K$ is a variable coefficient.
Table 5.3: The Errors and the Orders of the CN-WSGD scheme solved by the tested methods for Case III at $T = 1$.

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<th>$\text{CG Order}$</th>
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<th>$P_s^+ \text{CG Order}$</th>
<th>$P_s^- \text{CG Error}$</th>
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Table 5.4: The numerical results of the five tested methods for Case I.

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Table 5.5: The numerical results of the five tested methods for Case II.

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Therefore, the banded $M$-matrix splitting preconditioner proposed in this work is robust and efficient for the tested problems.
Table 5.6: The numerical results of the five tested methods for Case III.

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6. Conclusions

For the initial-boundary problem of Riesz space fractional reaction-dispersion equation (RS-FRDE) (1.1), based on the Crank-Nicolson (CN) and the weighted and shifted Grünwald (WSGD) operators, we proposed a CN-WSGD numerical scheme to approximate its solution and proved that this difference scheme is unconditionally stable and convergent with the accuracy of $O(h^2 + \Delta t^2)$. In order to know whether the CN-WSGD scheme, or equivalently, the series of systems of linear equations with the same coefficient matrix, is well-conditioned, we estimated the condition number of this coefficient matrix. The estimation result shows that this coefficient matrix is ill-conditioned when the spatial mesh-size is sufficiently small. So, we further introduced a preconditioner, named as banded $M$-matrix splitting preconditioner, for the coefficient matrix to improve the convergence rate of the conjugate gradient (CG) method used for solving the ill-conditioned systems of linear equations. The preconditioning effect of this preconditioner is theoretically analyzed and numerically verified. Both the theoretical and numerical results show that the banded $M$-matrix splitting preconditioner is robust and efficient.

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References


