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An Integral Equation Method for the Scattering by Core-Shell Structures in a Layered Medium

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Abstract. The core-shell structure design is an important subject in science and engineering, which also plays a key role in wave scattering and target reconstructions. This work aims to develop a novel boundary integral equation method for solving the acoustic scattering from a 3D core-shell structure in a two-layered lossy medium. The boundary integral equation contains continuous and weakly singular kernels. The well-posedness of the scattering problem is established by combining the integral equation, variational, and operator theory techniques. The study lays the groundwork for future numerical methods for layered obstacles and rough surfaces composite scattering and inverse scattering problems.

AMS subject classifications: 35Q60, 31B10, 45L05

Key words: Composite scattering, Helmholtz equation, existence and uniqueness, integral equation method.

1 Introduction

The core-shell structure scattering arises in a wide range of scientific fields, including sea radar target detection [6], underwater radar surveillance [3], and the design of optics devices [4,14,25]. Given an incident field, the scattering problem is to determine the scattered field from the governing differential equation, along with the boundary conditions. Recent progress has been made in the development of acoustic scattering from underwater or aerial core-shell structure vehicles. In [9], a strategy was investigated to control plasmonic resonances by core-shell geometries from analyzing the scattering response of a column of partially magnetized plasma. A great deal of studies deal with various aspects of the transient response of submerged or fluid-filled elastic shell structures with

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simple geometrical configurations. The closed-form analytic solutions for spherical or cylindrical shells have been reported in the literature, see, e.g., [10,12,21] and the references therein. However, little is known in mathematics about the scattering problems of a general shape of three-dimensional layered obstacles in an unbounded structure, which is vital for underwater exploration and many other engineering applications. The main challenges in analysis and computation include the unboundedness and complexity of interfaces, multiple scales, and multiple scattering between targets and rough interfaces.

Our goal in this work is to develop a mathematical model and to study its well-posedness for solving composite scattering problems. More specifically, we study the acoustic scattering of core-shell structures (such as low frequency plasma-coated obstacles) in a two-layered lossy medium, where the core is with an impedance boundary and the shell is with a transparent boundary. In particular, the scattering of a point source incidence is considered for the layered obstacle, interface, and shell. The wave field is governed by the three-dimensional Helmholtz equation, which describes the propagation of acoustic waves. Throughout, we assume that the medium is lossy and inhomogeneous with smooth interfaces. The assumptions are reasonable and general. In fact, in the target recognition applications above the sea surface, it is reasonable to assume that the medium is filled with two different lossy parts with smooth interfaces (air and water). The interface can be very general, including plane structures, periodic structures (cf. [1]) and general unbounded rough surfaces (cf. [26,27]). Also, the wavenumbers in the medium may have different positive imaginary parts. Recently, related problems have been studied extensively. In [8], a fast numerical method was proposed for calculating the electromagnetic scattering from a perfectly electric conducting object above a two-layered dielectric rough surface. The mode-expansion method for calculating electromagnetic waves scattered by objects on rough ocean surfaces was considered in [28]. In [11], the method of moments was used to rigorously analyze the wide-band VHF scattering from a perfectly conducting trihedral placed above a lossy, dispersive half-space. A Kirchhoff-type formula for transient elastic waves was originally introduced by Love [19] for the fluid-solid interaction scattering problem. Helmholtz-type integral formulas were systematically derived for elastic waves in isotropic and anisotropic solids by Pao [23]. Various systems of boundary integral equations over the interface between the fluid and the solid have been derived and analyzed by Luke and Martin [20]. Hsiao *et al.* [13] presented weak formulations of the fluid-solid interaction problem by coupling the field and boundary integral equation methods. The relevant functional analysis, which is necessary for the theory's treatment and the numerical solution of linear integral equations, can be found in [16,22]. More recently, a boundary integral equation has been proposed for solving 2D homogeneous obstacle acoustic (electromagnetic) composite scattering problems. Based on the energy estimates, the uniqueness of the solution or the scattering problem is established [2,18]. To our best knowledge, no mathematical study is available for the scattering of the core-shell structures in a layered medium.

In this paper, we derive a rigorous mathematical model for a class of scattering problems from 3D core-shell structures in a two-layered lossy medium with an unbounded interface. The model problem is formulated as a three-dimensional Helmholtz equation boundary value problem in Section 2. Based on boundary integral equation methods and the asymptotic properties of the Green functions, we present a novel integral equation system for the boundary value problem in Section 3. In contrast to the classical obstacle scattering problem, one must deal with the integral over an unbounded surface for the composite scattering problem here. The well-posedness of the proposed scattering problem is established by combining the boundary integral equation method and a variational method in Section 4. The paper is concluded with some general remarks in Section 5.

2 A model problem

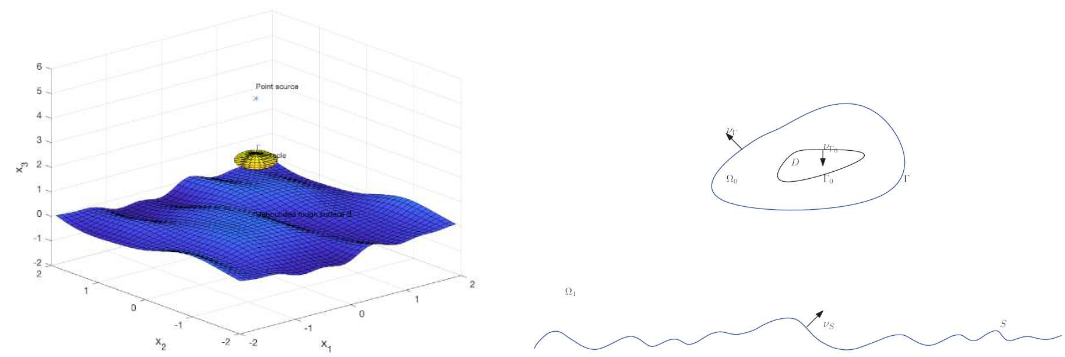
This section considers a mathematical model of the scattering problem from a layered obstacle in an unbounded structure (see Fig. 1). More specifically, we assume that the interface is the graph of a sufficiently smooth bounded continuous function

$$S := \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\},$$

where f and its first and second partial derivatives are all bounded continuously differentiable, i.e., $f \in \mathrm{BC}^2(\mathbb{R}^2)$, which separates $\mathbb{R}^3$ into two regions as

$$\Omega_1^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 > f(x_1, x_2)\}, \quad \Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 < f(x_1, x_2)\}.$$

Let D be an impenetrable bounded obstacle embedded in a bounded domain Ω_0^+, i.e. $D \subset\subset \Omega_0^+ \subset\subset \Omega_1^+$. Define $\Omega_0 = \Omega_0^+ \setminus \overline{D}$ and $\Omega_1 = \Omega_1^+ \setminus \overline{\Omega_0^+}$. The boundary $\Gamma \cup \Gamma_0$ of Ω_0 is of class C^2. $\boldsymbol{\nu}_\Gamma$ and $\boldsymbol{\nu}_{\Gamma_0}$ denote the unit normal vector on the boundary Γ and Γ_0 directed into the exterior of Ω_0, respectively. $\boldsymbol{\nu}_S$ denotes the unit normal vector on the boundary S



(a) A point source is incident on the obstacle and the interfaces from the top.

(b) Profile of a core-shell structure in a layered medium.

Figure 1: The geometry of a scattering problem with core-shell structures embedded in a layered medium.

pointing from region Ω_2 to region Ω_1. The relevant normal derivatives are defined as follows:

$$\begin{aligned}
&\frac{\partial u_1}{\partial \boldsymbol{\nu}_\Gamma}(\boldsymbol{x}) := \lim_{\sigma\to 0^+} \boldsymbol{\nu}_\Gamma(\boldsymbol{x})\cdot\nabla u_1\big(\boldsymbol{x}+\sigma\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\big), && \boldsymbol{x}\in\Gamma,\\
&\frac{\partial u_0}{\partial \boldsymbol{\nu}_\Gamma}(\boldsymbol{x}) := \lim_{\sigma\to 0^+} \boldsymbol{\nu}_\Gamma(\boldsymbol{x})\cdot\nabla u_0\big(\boldsymbol{x}-\sigma\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\big), && \boldsymbol{x}\in\Gamma,\\
&\frac{\partial u_1}{\partial \boldsymbol{\nu}_S}(\boldsymbol{x}) := \lim_{\sigma\to 0^+} \boldsymbol{\nu}_S(\boldsymbol{x})\cdot\nabla u_1\big(\boldsymbol{x}+\sigma\boldsymbol{\nu}_S(\boldsymbol{x})\big), && \boldsymbol{x}\in S,\\
&\frac{\partial u_2}{\partial \boldsymbol{\nu}_S}(\boldsymbol{x}) := \lim_{\sigma\to 0^+} \boldsymbol{\nu}_S(\boldsymbol{x})\cdot\nabla u_2\big(\boldsymbol{x}-\sigma\boldsymbol{\nu}_S(\boldsymbol{x})\big), && \boldsymbol{x}\in S,\\
&\frac{\partial u_0}{\partial \boldsymbol{\nu}_{\Gamma_0}}(\boldsymbol{x}) := \lim_{\sigma\to 0^+} \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\cdot\nabla u_0\big(\boldsymbol{x}-\sigma\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\big), && \boldsymbol{x}\in\Gamma_0.
\end{aligned}$$

Assume that the homogeneous isotropic medium with positive density ρ_j, speed of sound c_j, and damping coefficient γ_j in Ω_j, respectively. The wave motion can be described by a velocity potential $U_j(t,\boldsymbol{x})$ with the velocity field

$$\boldsymbol{v}_j = \frac{1}{\rho_j}\nabla U_j, \tag{2.1}$$

and the pressure

$$p_j = p_{j0} - \frac{\partial U_j}{\partial t} - \gamma_j U_j, \tag{2.2}$$

where $j=0,1,2$, and p_{j0} denotes the pressure of the undisturbed medium.

For the linearized model, the velocity potential U_j satisfies the dissipative wave equation

$$\frac{\partial^2 U_j}{\partial t^2} + \gamma_j \frac{\partial U_j}{\partial t} - c_j^2 \Delta U_j = 0.$$

In the time harmonic case, the wave field with the form

$$U_j(t,\boldsymbol{x}) = u_j(\boldsymbol{x}) e^{-\mathrm{i}\omega t}, \tag{2.3}$$

we have

$$\Delta u_j(\boldsymbol{x}) + \kappa_j^2 u_j(\boldsymbol{x}) = 0 \quad \text{in } \Omega_j, \quad j=0,1,2,$$

where $\omega>0$ is the angular frequency, the wavenumber κ_j satisfies $\kappa_j^2=\omega(\omega+\mathrm{i}\gamma_j)/c_j^2$ and $\Re(\kappa_j)>0,\Im(\kappa_j)>0$. For an impedance obstacle D, the excess pressure on the boundary is proportional to the normal velocity on the boundary

$$\boldsymbol{v}_0\cdot\boldsymbol{\nu}_{\Gamma_0} - \mu_0(p_0 - p_{00}) = 0.$$

Combining of (2.1)-(2.3), we have

$$\frac{\partial u_0}{\partial \boldsymbol{\nu}_{\Gamma_0}} - \mathrm{i}\lambda_0 u_0 = 0 \quad \text{on } \Gamma_0, \tag{2.4}$$

where μ_0 is the acoustic impedance coefficient of the obstacle D, and $\lambda_0 = \mu_0\rho_0(\omega + \mathrm{i}\gamma_0)$. For penetrable interfaces Γ and S, by the continuity of the normal velocity across the interface, we have

$$\boldsymbol{\nu}_\Gamma \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_0) = 0 \quad \text{on } \Gamma, \quad \boldsymbol{\nu}_S \cdot (\boldsymbol{v}_2 - \boldsymbol{v}_1) = 0 \quad \text{on } S.$$

From (2.1) and (2.3), it follows that:

$$\frac{1}{\rho_0}\frac{\partial u_0}{\partial \boldsymbol{\nu}_\Gamma} = \frac{1}{\rho_1}\frac{\partial u_1}{\partial \boldsymbol{\nu}_\Gamma} \quad \text{on } \Gamma, \quad \frac{1}{\rho_1}\frac{\partial u_1}{\partial \boldsymbol{\nu}_S} = \frac{1}{\rho_2}\frac{\partial u_2}{\partial \boldsymbol{\nu}_S} \quad \text{on } S. \tag{2.5}$$

Furthermore, based on the continuity of the pressure, we have

$$p_0 = p_1 \quad \text{on } \Gamma, \quad p_1 = p_2 \quad \text{on } S.$$

By (2.2), (2.3), $p_{00} = p_{10}$ on Γ, and $p_{10} = p_{20}$ on S, we further have

$$(\gamma_0 - \mathrm{i}\omega)u_0 = (\gamma_1 - \mathrm{i}\omega)u_1 \quad \text{on } \Gamma, \quad (\gamma_1 - \mathrm{i}\omega)u_1 = (\gamma_2 - \mathrm{i}\omega)u_2 \quad \text{on } S. \tag{2.6}$$

Throughout, assume that either $\gamma_0 = \gamma_1 = \gamma_2$ or $|\gamma_j| \ll \omega$, then

$$u_0 = u_1 \quad \text{on } \Gamma, \quad u_1 = u_2 \quad \text{on } S. \tag{2.7}$$

Let an incoming point source $u^i(\boldsymbol{x}) = G_1(\boldsymbol{x}, \boldsymbol{x_s})$ located at $\boldsymbol{x_s} \in \Omega_1$ be incident on the obstacle D and interface S from the above. The Green functions

$$G_j(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

for Helmholtz equation in Ω_j satisfies

$$\Delta G_j(\boldsymbol{x}, \boldsymbol{y}) + \kappa_j^2 G_j(\boldsymbol{x}, \boldsymbol{y}) = -\delta(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \Omega_j, \quad j = 0,1,2. \tag{2.8}$$

Therefore, the model can be described as the following boundary value problem:

$$\begin{cases} (\Delta + \kappa_1^2)u_1(\boldsymbol{x}) = -\delta(\boldsymbol{x}, \boldsymbol{x}_s) & \text{in } \Omega_1, \\ (\Delta + \kappa_2^2)u_2(\boldsymbol{x}) = 0 & \text{in } \Omega_2, \\ (\Delta + \kappa_0^2)u_0(\boldsymbol{x}) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial u_0}{\partial \boldsymbol{\nu}_{\Gamma_0}} - \mathrm{i}\lambda_0 u_0 = 0 & \text{on } \Gamma_0, \\ u_1 = u_0, \quad \dfrac{1}{\rho_1}\dfrac{\partial u_1}{\partial \boldsymbol{\nu}_\Gamma} = \dfrac{1}{\rho_0}\dfrac{\partial u_0}{\partial \boldsymbol{\nu}_\Gamma} & \text{on } \Gamma, \\ u_1 = u_2, \quad \dfrac{1}{\rho_1}\dfrac{\partial u_1}{\partial \boldsymbol{\nu}_S} = \dfrac{1}{\rho_2}\dfrac{\partial u_2}{\partial \boldsymbol{\nu}_S} & \text{on } S, \end{cases} \tag{2.9}$$

where the total field u_1 is the sum of the incident field u^i and the scattered field u^s, u_0 and u_2 are transmitted fields in Ω_0 and Ω_2, respectively.

The radiation conditions

$$\lim_{r\to+\infty}\int_{\partial B_r\cap\Omega_1}\left(|u^s|^2+\left|\frac{\partial u^s}{\partial \boldsymbol{\nu}}\right|^2\right)\mathrm{d}s=\lim_{r\to+\infty}\int_{\partial B_r\cap\Omega_2}\left(|u_2|^2+\left|\frac{\partial u_2}{\partial \boldsymbol{\nu}}\right|^2\right)\mathrm{d}s=0, \tag{2.10}$$

should be imposed in lossy mediums, where $\boldsymbol{\nu}$ denotes the unit outward normal vector on the boundary ∂B_r of the ball $B_r=\{\boldsymbol{x}\in\mathbb{R}^3:|\boldsymbol{x}|<r\}$.

We consider the following problem.

Scattering Problem: Given an incident field u^i which is located at $\boldsymbol{x_s}\in\Omega_1$, find (u_0,u_1,u_2) with $u^s\in C^2(\Omega_1)\cap C^{1,\alpha}(\overline{\Omega}_1)$ and $u_j\in C^2(\Omega_j)\cap C^{1,\alpha}(\overline{\Omega}_j),(0<\alpha<1),j=0,2$, which satisfy (2.9) and the radiation conditions (2.10), where $u_1=u^i+u^s$.

We first prove that the scattering problem has at most one solution, provided the above mention assumptions are the choice of κ_j and ρ_j, $j=0,1,2$.

Lemma 2.1. *The scattering problem has at most one solution.*

Proof. Denote by $\Omega_r^+=(B_r\cap\Omega_1)$ the region with boundary $\partial\Omega_r^+=\partial B_r^+\cup S_r\cup\Gamma$, where $\partial B_r^+=\partial B_r\cap\Omega_1$ and $S_r=S\cap B_r$. To prove the lemma, it suffices to show that $u_j=0$ in Ω_j, $j=0,1,2$ if $u^i=0$. Since u_1 is the solution of the Helmholtz equation in Ω_r^+, from Green's theorem for piecewise smooth curves [24], we have

$$0=\int_{\Omega_r^+}\left[-|\nabla u_1|^2+\kappa_1^2|u_1|^2\right]\mathrm{d}\boldsymbol{x}+\int_\Gamma \overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}}\mathrm{d}S_{\boldsymbol{x}}+\int_{S_r}\overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}}\mathrm{d}S_{\boldsymbol{x}}+\int_{\partial B_r^+}\overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}}\mathrm{d}S_{\boldsymbol{x}}, \tag{2.11}$$

where $\boldsymbol{\nu}(\boldsymbol{y})$ is the unit outward normal vector on $\partial\Omega_r^+$. Taking the limit $r\to+\infty$ in (2.11), by (2.10), we arrive at

$$\int_{\Omega_1}\left[|\nabla u_1|^2-\kappa_1^2|u_1|^2\right]\mathrm{d}\boldsymbol{x}+\int_\Gamma \overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}_\Gamma}\mathrm{d}S_{\boldsymbol{x}}+\int_S\overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}_S}\mathrm{d}S_{\boldsymbol{x}}=0. \tag{2.12}$$

Similarly, we also get the identities in Ω_2 and Ω_0

$$\begin{aligned}0&=\int_{\Omega_2}\left[|\nabla u_2|^2-\kappa_2^2|u_2|^2\right]\mathrm{d}\boldsymbol{x}-\int_S\overline{u}_2\frac{\partial u_2}{\partial\boldsymbol{\nu}_S}\mathrm{d}S_{\boldsymbol{x}}\\&=\int_{\Omega_2}\left[|\nabla u_2|^2-\kappa_2^2|u_2|^2\right]\mathrm{d}\boldsymbol{x}-\frac{\rho_2}{\rho_1}\int_S\overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}_S}\mathrm{d}S_{\boldsymbol{x}},\end{aligned} \tag{2.13}$$

$$\begin{aligned}0&=\int_{\Omega_0}\left[|\nabla u_0|^2-\kappa_0^2|u_0|^2\right]\mathrm{d}\boldsymbol{x}-\int_{\Gamma_0}\overline{u}_0\frac{\partial u_0}{\partial\boldsymbol{\nu}_{\Gamma_0}}\mathrm{d}S_{\boldsymbol{x}}-\int_\Gamma\overline{u}_0\frac{\partial u_0}{\partial\boldsymbol{\nu}_\Gamma}\mathrm{d}S_{\boldsymbol{x}}\\&=\int_{\Omega_0}\left[|\nabla u_0|^2-\kappa_0^2|u_0|^2\right]\mathrm{d}\boldsymbol{x}-\mathrm{i}\lambda_0\int_{\Gamma_0}|u_0|^2\mathrm{d}S_{\boldsymbol{x}}-\frac{\rho_0}{\rho_1}\int_\Gamma\overline{u}_1\frac{\partial u_1}{\partial\boldsymbol{\nu}_\Gamma}\mathrm{d}S_{\boldsymbol{x}}.\end{aligned} \tag{2.14}$$

Combining (2.12)-(2.14), we obtain

$$\begin{aligned}&\frac{\rho_1}{\rho_0}\int_{\Omega_0}\left(|\nabla u_0|^2-\kappa_0^2|u_0|^2\right)\mathrm{d}\boldsymbol{x}+\int_{\Omega_1}\left(|\nabla u_1|^2-\kappa_1^2|u_1|^2\right)\mathrm{d}\boldsymbol{x}\\&\quad+\frac{\rho_1}{\rho_2}\int_{\Omega_2}\left(|\nabla u_2|^2-\kappa_2^2|u_2|^2\right)\mathrm{d}\boldsymbol{x}-\mathrm{i}\lambda_0\frac{\rho_1}{\rho_0}\int_{\Gamma_0}|u_0|^2\mathrm{d}S_{\boldsymbol{x}}=0.\end{aligned}\tag{2.15}$$

Taking the imaginary part of (2.15), we have

$$\begin{aligned}&-\frac{\rho_1}{\rho_0}\Im\left(\kappa_0^2\right)\int_{\Omega_0}|u_0|^2\mathrm{d}\boldsymbol{x}-\Im\left(\kappa_1^2\right)\int_{\Omega_1}|u_1|^2\mathrm{d}\boldsymbol{x}-\frac{\rho_1}{\rho_2}\Im\left(\kappa_2^2\right)\int_{\Omega_2}|u_2|^2\mathrm{d}\boldsymbol{x}\\&-\frac{\rho_1}{\rho_0}\Im(\mathrm{i}\lambda_0)\int_{\Gamma_0}|u_0|^2\mathrm{d}S_{\boldsymbol{x}}=0.\end{aligned}\tag{2.16}$$

It follows from $\Im(\kappa_j^2)>0,\Im(\mathrm{i}\lambda_0)>0$ and (2.16) that

$$\int_{\Omega_j}|u_j|^2\mathrm{d}\boldsymbol{x}=0,\quad j=0,1,2,\tag{2.17}$$

which implies that u_j vanishes identically in Ω_j, $j=0,1,2$, respectively, if $u^i=0$. □

Remark 2.1. The assumption of parameters in studying the scattering problem is necessary. Even for the penetrable obstacle scattering problem, the uniqueness does not hold for all k_j and ρ_j, one can see a simple counterexample in [17].

The following asymptotic behaviors of the fundamental solution are helpful when analyzing the properties of the integral operator defined on the unbounded interfaces.

Lemma 2.2. *For any fixed* $\boldsymbol{y}\in\Omega_j, j=1,2$, *the Green functions* G_j *have the following asymptotic behavior:*

$$\begin{aligned}&|G_j(\boldsymbol{x},\boldsymbol{y})|,\left|\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})}\right|,\left|\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\right|,\left|\frac{\partial^2 G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})\partial\boldsymbol{\nu}(\boldsymbol{y})}\right|\\&\leq C\left(\frac{\exp\left(-(1/2)\Im(\kappa_j)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\right)\quad as\quad|\boldsymbol{x}|\ \to\ \infty.\end{aligned}\tag{2.18}$$

Proof. Take the normal derivative of Green's functions, we have

$$\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})}=\frac{\boldsymbol{\nu}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|-1\right)G_j(\boldsymbol{x},\boldsymbol{y}),\tag{2.19}$$

$$\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}=-\frac{\boldsymbol{\nu}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|-1\right)G_j(\boldsymbol{x},\boldsymbol{y}),\tag{2.20}$$

$$\begin{aligned}\frac{\partial^2 G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})\partial\boldsymbol{\nu}(\boldsymbol{y})}=&-\frac{[\boldsymbol{\nu}(\boldsymbol{x})\cdot\boldsymbol{\nu}(\boldsymbol{y})]}{|\boldsymbol{x}-\boldsymbol{y}|^2}\big(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|-1\big)G_j(\boldsymbol{x},\boldsymbol{y})\\&+\frac{[\boldsymbol{\nu}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})][\boldsymbol{\nu}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})]}{|\boldsymbol{x}-\boldsymbol{y}|^4}\\&\times\Big(\kappa_j^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|-3\Big)G_j(\boldsymbol{x},\boldsymbol{y}).\end{aligned}\tag{2.21}$$

Hence, as $|\boldsymbol{x}-\boldsymbol{y}|\to+\infty$, from (2.19)-(2.21) and $\Im(\kappa_j)>0$, $j=1,2$, we have

$$\begin{aligned}&\left|\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})}\right|,\left|\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{y})}\right|,\left|\frac{\partial^2 G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}(\boldsymbol{x})\partial\boldsymbol{\nu}(\boldsymbol{y})}\right|\\&\leq C|G_j(\boldsymbol{x},\boldsymbol{y})|\leq C\left[\frac{\exp(-\Im(\kappa_j)|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|}\right],\end{aligned}\tag{2.22}$$

where C is a constant independent of $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, for fixed $\boldsymbol{y}\in\Omega_j$, $j=1,2$, we note that

$$|\boldsymbol{x}-\boldsymbol{y}|\geq\frac{|\boldsymbol{x}|}{2}\quad\text{as}\quad|\boldsymbol{x}|\ \to\ \infty,\tag{2.23}$$

hence, the inequality (2.18) can be established by (2.22) and (2.23). □

Lemma 2.3. *For the twice continuously differentiable surface*

$$\partial\tilde{D}:=\big\{\big(x_1,x_2,f(x_1,x_2)\big)\,|\,(x_1,x_2)\in\mathbb{R}^2\big\},$$

$f\in BC^2(\mathbb{R}^2)$*, there exists a positive constant* $\mathbf{M}$ *such that*

$$|\boldsymbol{\nu}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})|\leq\mathbf{M}|\boldsymbol{x}-\boldsymbol{y}|^2,\quad|\boldsymbol{\nu}(\boldsymbol{x})-\boldsymbol{\nu}(\boldsymbol{y})|\leq\mathbf{M}|\boldsymbol{x}-\boldsymbol{y}|$$

for all $\boldsymbol{x},\boldsymbol{y}\in\partial\tilde{D}$.

For the bounded obstacle, the proof is given in [7, pp. 35]. For the unbounded case, we can refer to [5, Eq. (5.4)].

3 The boundary integral equations

The method of boundary integral equations has also played an essential role in studying boundary value problems. It has been widely used for solving scattering problems, especially those defined over unbounded domains, and are required to satisfy radiation conditions. For the convenience of description, we introduce the following integral operators:

$$\begin{aligned}(\mathbf{S}_{\Gamma_0}\phi)(\boldsymbol{x})&=2\int_{\Gamma_0}G_0(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}},\qquad \boldsymbol{x}\in\Gamma_0,\\(\mathbf{K}_{\Gamma_0}\phi)(\boldsymbol{x})&=2\int_{\Gamma_0}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}},\quad \boldsymbol{x}\in\Gamma_0,\end{aligned}$$

$$
\begin{aligned}
&(\mathbf{S}_{\Gamma_0,\Gamma}\phi)(\boldsymbol{x})=2\int_{\Gamma}G_0(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma_0,\\
&(\mathbf{K}_{\Gamma_0,\Gamma}\phi)(\boldsymbol{x})=2\int_{\Gamma}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma_0,\\
&(\mathbf{S}_{\Gamma}\phi)(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\rho_1}{\rho_0}G_1(\boldsymbol{x},\boldsymbol{y})-G_0(\boldsymbol{x},\boldsymbol{y})\right)\phi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{K}_{\Gamma}\phi)(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&\left(\mathbf{K}^{(*)}_{\Gamma}\phi\right)(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\rho_1}{\rho_0}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{T}_{\Gamma}\phi)(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{S}_{\Gamma,\Gamma_0}\phi)(\boldsymbol{x})=\int_{\Gamma_0}G_0(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{K}_{\Gamma,\Gamma_0}\phi)(\boldsymbol{x})=\int_{\Gamma_0}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&\left(\mathbf{K}^{(*)}_{\Gamma,\Gamma_0}\phi\right)(\boldsymbol{x})=\int_{\Gamma_0}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{T}_{\Gamma,\Gamma_0}\phi)(\boldsymbol{x})=\int_{\Gamma_0}\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{S}_{\Gamma,S}\phi)(\boldsymbol{x})=\int_{S}\frac{\rho_1}{\rho_2}G_1(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{K}_{\Gamma,S}\phi)(\boldsymbol{x})=\int_{S}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&\left(\mathbf{K}^{(*)}_{\Gamma,S}\phi\right)(\boldsymbol{x})=\int_{S}\frac{\rho_1}{\rho_2}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{T}_{\Gamma,S}\phi)(\boldsymbol{x})=\int_{S}\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in\Gamma,\\
&(\mathbf{S}_{S}\phi)(\boldsymbol{x})=\int_{S}\left(\frac{\rho_1}{\rho_2}G_1(\boldsymbol{x},\boldsymbol{y})-G_2(\boldsymbol{x},\boldsymbol{y})\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in S,\\
&(\mathbf{K}_{S}\phi)(\boldsymbol{x})=\int_{S}\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in S,\\
&\left(\mathbf{K}^{(*)}_{S}\phi\right)(\boldsymbol{x})=\int_{S}\left(\frac{\rho_1}{\rho_2}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{x})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{x})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in S,\\
&(\mathbf{T}_{S}\phi)(\boldsymbol{x})=\int_{S}\left(\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{x})\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\partial^2 G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{x})\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, && \boldsymbol{x}\in S,\\
&(\mathbf{S}_{S,\Gamma}\phi)(\boldsymbol{x})=\int_{\Gamma}\frac{\rho_1}{\rho_0}G_1(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}},
\end{aligned}
$$

$$(\mathbf{K}_{S,\Gamma}\phi)(\boldsymbol{x})=\int_\Gamma \frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in S,$$
$$\left(\mathbf{K}^{(*)}_{S,\Gamma}\phi\right)(\boldsymbol{x})=\int_\Gamma \frac{\rho_1}{\rho_0}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}},$$
$$(\mathbf{T}_{S,\Gamma}\phi)(\boldsymbol{x})=\int_\Gamma \frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in S.$$

Theorem 3.1. *If (u_0,u_1,u_2) is a solution of scattering problem, then it satisfies*

$$u_1(\boldsymbol{x})=\int_\Gamma\left[u_0(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}$$
$$+\int_S\left[u_2(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}+u^i(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega_1, \tag{3.1}$$
$$u_2(\boldsymbol{x})=-\int_S\left[u_2(\boldsymbol{y})\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_2(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in\Omega_2, \tag{3.2}$$
$$u_0(\boldsymbol{x})=-\int_{\Gamma_0}\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0 u_0(\boldsymbol{y})G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}$$
$$-\int_\Gamma\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in\Omega_0, \tag{3.3}$$

and the following boundary integral equations hold:

$$\mathbf{A}\mathbf{u}:=(\mathbf{I}-\mathbf{K}\mathbf{E})\mathbf{u}=\mathbf{b}, \tag{3.4}$$

where $\mathbf{I}=\mathbf{diag}(\mathbf{I}_{\Gamma_0},\mathbf{I}_\Gamma,\mathbf{I}^{(*)}_\Gamma,\mathbf{I}_S,\mathbf{I}^{(*)}_S)$ is a unit operator matrix

$$\mathbf{KE}=\begin{bmatrix} -\left(\mathbf{K}_{\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma_0}\right) & -\mathbf{K}_{\Gamma_0,\Gamma} & \mathbf{S}_{\Gamma_0,\Gamma} & \mathbf{0} & \mathbf{0}\\ -\left(\mathbf{K}_{\Gamma,\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma,\Gamma_0}\right) & \mathbf{K}_\Gamma & -\mathbf{S}_\Gamma & \mathbf{K}_{\Gamma,S} & -\mathbf{S}_{\Gamma,S}\\ \dfrac{-2\rho_0}{\rho_1+\rho_0}\left(\mathbf{T}_{\Gamma,\Gamma_0}-\mathrm{i}\lambda_0\mathbf{K}^{(*)}_{\Gamma,\Gamma_0}\right) & \dfrac{2\rho_0}{\rho_1+\rho_0}\mathbf{T}_\Gamma & \dfrac{-2\rho_0}{\rho_1+\rho_0}\mathbf{K}^{(*)}_\Gamma & \dfrac{2\rho_0}{\rho_1+\rho_0}\mathbf{T}_{\Gamma,S} & \dfrac{-2\rho_0}{\rho_1+\rho_0}\mathbf{K}^{(*)}_{\Gamma,S}\\ \mathbf{0} & \mathbf{K}_{S,\Gamma} & -\mathbf{S}_{S,\Gamma} & \mathbf{K}_S & -\mathbf{S}_S\\ \mathbf{0} & \dfrac{2\rho_2}{\rho_1+\rho_2}\mathbf{T}_{S,\Gamma} & \dfrac{-2\rho_2}{\rho_1+\rho_2}\mathbf{K}^{(*)}_{S,\Gamma} & \dfrac{2\rho_2}{\rho_1+\rho_2}\mathbf{T}_S & \dfrac{-2\rho_2}{\rho_1+\rho_2}\mathbf{K}^{(*)}_S \end{bmatrix},$$

and

$$\mathbf{u}=\begin{bmatrix} u_0|_{\Gamma_0}\\ u_0|_\Gamma\\ \dfrac{\partial u_0}{\partial \boldsymbol{\nu}_\Gamma}|_\Gamma\\ u_2|_S\\ \dfrac{\partial u_2}{\partial \boldsymbol{\nu}_S}|_S\end{bmatrix}, \quad \mathbf{b}=\begin{bmatrix} 0\\ u^i|_\Gamma\\ \dfrac{2\rho_0}{\rho_1+\rho_0}\dfrac{\partial u^i}{\partial \boldsymbol{\nu}_\Gamma}|_\Gamma\\ u^i|_S\\ \dfrac{2\rho_2}{\rho_1+\rho_2}\dfrac{\partial u^i}{\partial \boldsymbol{\nu}_S}|_S\end{bmatrix}.$$

Proof. For fixed $x\in\Omega_r^+$, applying Green's second theorem to u_1 and G_1 in the region Ω_r^+, we deduce that

$$\begin{aligned}
&\int_{\partial\Omega_r^+}\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu(y)}-\frac{\partial u_1(y)}{\partial\nu(y)}G_1(x,y)\right]\mathrm{d}S_y\\
=&\int_{\Omega_r^+}\left[u_1(y)\Delta G_1(x,y)-G_1(x,y)\Delta u_1(y)\right]\mathrm{d}y\\
=&\int_{\Omega_r^+}u_1(y)\left[\Delta G_1(x,y)+\kappa_1^2G_1(x,y)\right]\mathrm{d}y\\
&\quad-\int_{\Omega_r^+}\left[\Delta u_1(y)+\kappa_1^2u_1(y)\right]G_1(x,y)\mathrm{d}y\\
=&-u_1(x)+u^i(x),
\end{aligned}\tag{3.5}$$

where $\nu(y)$ is the unit outward normal vector on $\partial\Omega_r^+$. Take the limit in (3.5) as $r\to+\infty$. By Lemma 2.2, radiation conditions (2.10) and the continuity conditions on S,Γ, we arrive at

$$\begin{aligned}
u_1(x)=&\int_\Gamma\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}-\frac{\partial u_1(y)}{\partial\nu_\Gamma(y)}G_1(x,y)\right]\mathrm{d}S_y\\
&+\int_S\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial u_1(y)}{\partial\nu_S(y)}G_1(x,y)\right]\mathrm{d}S_y+u^i(x)\\
=&\int_\Gamma\left[u_0(y)\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}-\frac{\rho_1}{\rho_0}\frac{\partial u_0(y)}{\partial\nu_\Gamma(y)}G_1(x,y)\right]\mathrm{d}S_y\\
&+\int_S\left[u_2(y)\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(y)}{\partial\nu_S(y)}G_1(x,y)\right]\mathrm{d}S_y+u^i(x),\quad x\in\Omega_1.
\end{aligned}\tag{3.6}$$

Similarly, for the transmitted field u_0 and u_2, we have

$$u_2(x)=-\int_S\left[u_2(y)\frac{\partial G_2(x,y)}{\partial\nu_S(y)}-\frac{\partial u_2(y)}{\partial\nu_S(y)}G_2(x,y)\right]\mathrm{d}S_y,\qquad x\in\Omega_2,\tag{3.7}$$

$$\begin{aligned}
u_0(x)=&-\int_{\Gamma_0}\left[u_0(y)\frac{\partial G_0(x,y)}{\partial\nu_{\Gamma_0}(y)}-\mathrm{i}\lambda_0u_0(y)G_0(x,y)\right]\mathrm{d}S_y\\
&-\int_\Gamma\left[u_0(y)\frac{\partial G_0(x,y)}{\partial\nu_\Gamma(y)}-\frac{\partial u_0(y)}{\partial\nu_\Gamma(y)}G_0(x,y)\right]\mathrm{d}S_y,\qquad x\in\Omega_0.
\end{aligned}\tag{3.8}$$

From (3.8) and the jump relations, we obtain

$$\begin{aligned}
&u_0(x)+2\int_{\Gamma_0}\left[\left(\frac{\partial G_0(x,y)}{\partial\nu_{\Gamma_0}(y)}-\mathrm{i}\lambda_0G_0(x,y)\right)u_0(y)\right]\mathrm{d}S_y\\
&\quad+2\int_\Gamma\left[u_0(y)\frac{\partial G_0(x,y)}{\partial\nu_\Gamma(y)}-\frac{\partial u_0(y)}{\partial\nu_\Gamma(y)}G_0(x,y)\right]\mathrm{d}S_y=0,\quad x\in\Gamma_0.
\end{aligned}\tag{3.9}$$

Let $\boldsymbol{x}$ approach the boundary S in (3.6) and (3.7), respectively. By the jump relations and the continuity conditions on S, we have

$$\begin{aligned}\frac{1}{2}u_2(\boldsymbol{x}) =& \int_\Gamma \left[u_0(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}} \\ &+\int_S \left[u_2(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}} + u^i(\boldsymbol{x}), \quad \boldsymbol{x}\in S, \end{aligned} \tag{3.10}$$

$$\frac{1}{2}u_2(\boldsymbol{x}) = -\int_S \left[u_2(\boldsymbol{y})\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_2(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S. \tag{3.11}$$

Adding (3.10) and (3.11), we obtain the boundary integral equation

$$\begin{aligned}u^i(\boldsymbol{x}) = &u_2(\boldsymbol{x}) - \int_\Gamma \left[u_0(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}} \\ &- \int_S \left[u_2(\boldsymbol{y})\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\right)\right. \\ &\left.\qquad - \frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_2}G_1(\boldsymbol{x},\boldsymbol{y}) - G_2(\boldsymbol{x},\boldsymbol{y})\right)\right]\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S. \end{aligned} \tag{3.12}$$

Furthermore, taking the normal derivative of (3.6) and (3.7) on S, respectively, by using the jump relations and the continuity conditions on S, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} \\ =& \int_\Gamma \left[u_0(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}} \\ &+\int_S \left[u_2(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}} + \frac{\partial u^i(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}, \quad \boldsymbol{x}\in S, \end{aligned} \tag{3.13}$$

and

$$\frac{1}{2}\frac{\partial u_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} = -\int_S \left[u_2(\boldsymbol{y})\frac{\partial^2 G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S. \tag{3.14}$$

Adding (3.13) and (3.14), we arrive at the boundary integral equation

$$\begin{aligned}&\frac{\partial u_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} - \frac{2\rho_2}{\rho_1+\rho_2}\int_\Gamma \left[u_0(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}} \\ &\quad - \frac{2\rho_2}{\rho_1+\rho_2}\int_S \left[u_2(\boldsymbol{y})\left(\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\partial^2 G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\right)\right. \\ &\quad\qquad \left.- \frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_2}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} - \frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right)\right]\mathrm{d}S_{\boldsymbol{y}} \\ =& \frac{2\rho_2}{\rho_1+\rho_2}\frac{\partial u^i(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}, \quad \boldsymbol{x}\in S. \end{aligned} \tag{3.15}$$

Similarly, from (3.6) and (3.8), the corresponding boundary integral equations on Γ are obtained

$$
\begin{aligned}
&u_0(\boldsymbol{x})+\int_{\Gamma_0}\left[\left(\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0 G_0(\boldsymbol{x},\boldsymbol{y})\right)u_0(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\
&\quad-\int_{\Gamma}\left[u_0(\boldsymbol{y})\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\right)-\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_0}G_1(\boldsymbol{x},\boldsymbol{y})-G_0(\boldsymbol{x},\boldsymbol{y})\right)\right]\mathrm{d}S_{\boldsymbol{y}}\\
&\quad-\int_{S}\left[u_2(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}=u^i(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma, \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial u_0(\boldsymbol{x})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}+\frac{2\rho_0}{\rho_1+\rho_0}\int_{\Gamma_0}\left[\left(\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\right)u_0(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\
&\quad-\frac{2\rho_0}{\rho_1+\rho_0}\int_{\Gamma}\left[u_0(\boldsymbol{y})\left(\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\right)\right.\\
&\qquad\qquad\left.-\frac{\partial u_0(\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_0}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\right)\right]\mathrm{d}S_{\boldsymbol{y}}\\
&\quad-\frac{2\rho_0}{\rho_1+\rho_0}\int_{S}\left[u_2(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}\\
&=\frac{2\rho_0}{\rho_1+\rho_0}\frac{\partial u^i(\boldsymbol{x})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})},\quad \boldsymbol{x}\in\Gamma. \qquad (3.17)
\end{aligned}
$$

The proof is complete. □

We used the asymptotic analysis at the singular point to represent the singularity of the operators, then designed the kernels of operator $\mathbf{T}_\Gamma$ and $\mathbf{T}_S$ as

$$
\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})\partial \boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\quad\text{and}\quad\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{S}(\boldsymbol{x})\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\partial^2 G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{S}(\boldsymbol{x})\partial \boldsymbol{\nu}_{S}(\boldsymbol{y})}
$$

to ensure the weak singularity.

To simplify the presentation, let us introduce some notations: for $0<\alpha<1, C^{0,\alpha}(S)$ denotes a Banach space of real- or complex-valued bounded uniformly Hölder continuous functions decay sufficiently rapidly at infinity defined on S, with the norm

$$
\|\phi\|_{C^{0,\alpha}(S)}:=\sup_{x\in S}|\phi(\boldsymbol{x})|+\sup_{\substack{x,y\in S,\\ x\neq y}}\frac{|\phi(\boldsymbol{x})-\phi(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha}<\infty.
$$

$C^{1,\alpha}(S)$ denotes the set of bounded continuously differentiable real- or complex-valued functions with bounded and uniformly Hölder continuous first derivative and decay sufficiently rapidly at infinity on S, a Banach space with the norm

$$
\|\phi\|_{C^{1,\alpha}(S)}:=\sup_{x\in S}\big(|\phi(\boldsymbol{x})|+|\nabla\phi(\boldsymbol{x})|\big)+\sup_{\substack{x,y\in S,\\ x\neq y}}\frac{|\nabla\phi(\boldsymbol{x})-\nabla\phi(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha}<\infty.
$$

For surfaces Γ and Γ_0, the Hölder spaces are defined as usual, see, e.g., [16].

Theorem 3.2. *The densities* $u_{\Gamma_0}\in C^{1,\alpha}(\Gamma_0)$, $u_\Gamma\in C^{1,\alpha}(\Gamma)$, $\partial u_\Gamma/\partial\boldsymbol{\nu}_\Gamma\in C^{0,\alpha}(\Gamma)$, $u_S\in C^{1,\alpha}(S)$ *and* $\partial u_S/\partial\boldsymbol{\nu}_S\in C^{0,\alpha}(S)$ *are the solutions of boundary integral equations* (3.4)*, the layer potentials* u_j, $j=0,1,2$, (3.1)-(3.3)*, in terms of those densities, satisfy the scattering problem.*

Proof. For $u^s\in C^2(\Omega_1)\cap C^{1,\alpha}(\overline{\Omega}_1)$ and $u_j\in C^2(\Omega_j)\cap C^{1,\alpha}(\overline{\Omega}_j)$, where $0<\alpha<1$, by (2.8) and (3.1)-(3.4), we have $(\Delta+\kappa_1^2)u^s(\boldsymbol{x})=0$ in Ω_1 and $(\Delta+\kappa_j^2)u_j(\boldsymbol{x})=0$ in $\Omega_j, j=0,2$. Hence, applying Green's second theorem to u_0 and G_0 in Ω_0, we obtain

$$\begin{aligned}u_0(\boldsymbol{x})=&-\int_{\Gamma_0}\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{\Gamma}\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}},\quad \boldsymbol{x}\in\Omega_0.\end{aligned}\tag{3.18}$$

Thus, combining (3.3) and (3.18), we have $\partial u_0/\partial\boldsymbol{\nu}_{\Gamma_0}-\mathrm{i}\lambda_0u_0=0$ on Γ. Furthermore, with Lemma 2.2 and (3.1)-(3.2), we can derive the radiation conditions (2.10). From (3.1) and (3.3) we obtain the following equations:

$$\begin{aligned}u_1(\boldsymbol{x})=&\frac{1}{2}u_0(\boldsymbol{x})+\int_{\Gamma}\left[u_0(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&+\int_{S}\left[u_2(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}+u^i(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma,\end{aligned}\tag{3.19}$$

$$\begin{aligned}\frac{1}{2}u_0(\boldsymbol{x})=&-\int_{\Gamma_0}\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0u_0(\boldsymbol{y})G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{\Gamma}\left[u_0(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}},\qquad \boldsymbol{x}\in\Gamma.\end{aligned}\tag{3.20}$$

Add (3.19) and (3.20) together, we get

$$\begin{aligned}&u_1(\boldsymbol{x})+\int_{\Gamma_0}\left[\left(\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0G_0(\boldsymbol{x},\boldsymbol{y})\right)u_0(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{\Gamma}\left[u_0(\boldsymbol{y})\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\right)-\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_0}G_1(\boldsymbol{x},\boldsymbol{y})-G_0(\boldsymbol{x},\boldsymbol{y})\right)\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{S}\left[u_2(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial\boldsymbol{\nu}_{S}(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}=u^i(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma.\end{aligned}\tag{3.21}$$

Thus, combining (3.16) and (3.21), we have $u_0=u_1$ on Γ. Furthermore, from (3.1) and (3.3), we obtain

$$\frac{2\rho_0}{\rho_1+\rho_0}\left(\frac{\partial u_1(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}-\frac{1}{2}\frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}+\frac{1}{2}\frac{\partial u_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{x})}\right)$$

$$
\begin{aligned}
&+\frac{2\rho_0}{\rho_1+\rho_0}\int_{\Gamma_0}\left[\left(\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}\right)u_0(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\
&-\frac{2\rho_0}{\rho_1+\rho_0}\int_{\Gamma}\left[u_0(\boldsymbol{y})\left(\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\right)\right.\\
&\qquad\qquad\left.-\frac{\partial u_0(\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_0}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}-\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}\right)\right]\mathrm{d}S_{\boldsymbol{y}}\\
&-\frac{2\rho_0}{\rho_1+\rho_0}\int_{S}\left[u_2(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\partial\boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_2(\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}\\
=&\frac{2\rho_0}{\rho_1+\rho_0}\frac{\partial u^i(\boldsymbol{x})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})},\quad \boldsymbol{x}\in\Gamma. \qquad (3.22)
\end{aligned}
$$

By combining (3.17) and (3.22), we have

$$
\frac{2\rho_0}{\rho_1+\rho_0}\left(\frac{\partial u_1(\boldsymbol{x})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}-\frac{1}{2}\frac{\rho_1}{\rho_0}\frac{\partial u_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}+\frac{1}{2}\frac{\partial u_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}\right)=\frac{\partial u_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x})}, \qquad (3.23)
$$

which implies that

$$
\frac{1}{\rho_1}\frac{\partial u_1}{\partial\boldsymbol{\nu}_\Gamma}=\frac{1}{\rho_0}\frac{\partial u_0}{\partial\boldsymbol{\nu}_\Gamma}\quad\text{on }\ \Gamma.
$$

Similarly, we can obtain that $u_1=u_2$ and

$$
\frac{1}{\rho_1}\frac{\partial u_1}{\partial\boldsymbol{\nu}_S}=\frac{1}{\rho_2}\frac{\partial u_2}{\partial\boldsymbol{\nu}_S}\quad\text{on }\ S.
$$

The proof is complete. □

4 The well-posedness of boundary integral equations

In this section, we study the well-posedness of the scattering problem by the boundary integral equation method.

4.1 Compactness of integral operators

We assume that the boundary Γ_0 and Γ have the parametric form

$$
\begin{aligned}
&\Gamma_0:=\big\{\boldsymbol{x}(t_1,t_2)=\big(x_1(t_1,t_2),x_2(t_1,t_2),x_3(t_1,t_2)\big),(t_1,t_2)\in[0,\pi]\times[0,2\pi]\big\},\\
&\Gamma:=\big\{\hat{\boldsymbol{x}}(t_1,t_2)=\big(\hat{x}_1(t_1,t_2),\hat{x}_2(t_1,t_2),\hat{x}_3(t_1,t_2)\big),(t_1,t_2)\in[0,\pi]\times[0,2\pi]\big\}.
\end{aligned}
$$

For $\boldsymbol{x}\in\Gamma_0$ and Γ, set

$$
\begin{aligned}
&J_{\Gamma_0}(t_1,t_2):=\sqrt{\big|\boldsymbol{x}'_{t_1}\big|^2\cdot\big|\boldsymbol{x}'_{t_2}\big|^2-\big(\boldsymbol{x}'_{t_1}\cdot\boldsymbol{x}'_{t_2}\big)^2},\\
&J_{\Gamma}(t_1,t_2):=\sqrt{\big|\hat{\boldsymbol{x}}'_{t_1}\big|^2\cdot\big|\hat{\boldsymbol{x}}'_{t_2}\big|^2-\big(\hat{\boldsymbol{x}}'_{t_1}\cdot\hat{\boldsymbol{x}}'_{t_2}\big)^2},
\end{aligned}
$$

respectively, where

$$\boldsymbol{x}'_{t_j}=\boldsymbol{x}'_{t_j}(t_1,t_2)=\left(\frac{\partial x_1(t_1,t_2)}{\partial t_j},\frac{\partial x_2(t_1,t_2)}{\partial t_j},\frac{\partial x_3(t_1,t_2)}{\partial t_j}\right),\quad j=1,2.$$

Denote the interface by

$$S:=\big\{\boldsymbol{x}=\big(x_1,x_2,f(x_1,x_2)\big),(x_1,x_2)\in\mathbb{R}^2\big\},$$

and set

$$J_S(x_1,x_2):=\sqrt{1+|f'_{x_1}|^2+|f'_{x_2}|^2}.$$

Lemma 4.1. *The operators* $\mathbf{K}_{\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma_0},\mathbf{S}_\Gamma,\mathbf{K}_\Gamma^{(*)},\mathbf{T}_\Gamma,\mathbf{S}_S,\mathbf{K}_S^{(*)},\mathbf{T}_S$ *are the weakly singular integral operators, and the other integral operators in* $\mathbf{A}$ *of* (3.4) *have no singularity.*

Proof. For $\boldsymbol{x}\in\Gamma_0$, we have

$$\begin{aligned}&\big[(\mathbf{K}_{\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma_0})u_0\big]\big(\boldsymbol{x}(t_1,t_2)\big)\\&=\int_0^{2\pi}\int_0^{\pi}\mathrm{Ker}_{\Gamma_0}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)u_0\big(\boldsymbol{x}(\tau_1,\tau_2)\big)\mathrm{d}\tau_1\mathrm{d}\tau_2\\&=\int_0^{2\pi}\int_0^{\pi}\frac{1}{2\pi}\Bigg[\Bigg(-\frac{\boldsymbol{\nu}_{\Gamma_0}\big(\boldsymbol{x}(\tau_1,\tau_2)\big)\cdot\big(\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)\big)}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|^2}\big(\mathrm{i}\kappa_0|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|-1\big)-\mathrm{i}\lambda_0\Bigg)\\&\qquad\times\frac{e^{\mathrm{i}\kappa_0|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}J_{\Gamma_0}(\tau_1,\tau_2)\Bigg]u_0\big(\boldsymbol{x}(\tau_1,\tau_2)\big)\mathrm{d}\tau_1\mathrm{d}\tau_2.\end{aligned}\tag{4.1}$$

For all $\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\in\Gamma_0$ and $\boldsymbol{x}(t_1,t_2)\neq\boldsymbol{x}(\tau_1,\tau_2)$, by Lemma 2.3, the integral kernel $\mathrm{Ker}_{\Gamma_0}(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2))$ has the following estimate:

$$\big|\mathrm{Ker}_{\Gamma_0}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)\big|\le\frac{C}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}.\tag{4.2}$$

For all $\boldsymbol{x}\in\Gamma$, we consider the following operators:

$$\begin{aligned}\left[\mathbf{S}_\Gamma\frac{\partial u_0}{\partial\boldsymbol{\nu}_\Gamma}\right]\big(\boldsymbol{x}(t_1,t_2)\big)&=\int_0^{2\pi}\int_0^{\pi}\mathrm{Ker}_{\mathbf{S}_\Gamma}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)\frac{\partial u_0(\boldsymbol{x}(\tau_1,\tau_2))}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x}(\tau_1,\tau_2))}\mathrm{d}\tau_1\mathrm{d}\tau_2\\&=\int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{4\pi}\frac{\big((\rho_1/\rho_0)e^{\mathrm{i}\kappa_1|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}-e^{\mathrm{i}\kappa_0|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}\big)}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}J_\Gamma(\tau_1,\tau_2)\right]\\&\qquad\times\frac{\partial u_0(\boldsymbol{x}(\tau_1,\tau_2))}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{x}(\tau_1,\tau_2))}\mathrm{d}\tau_1\mathrm{d}\tau_2.\end{aligned}\tag{4.3}$$

For all $\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\in\Gamma$ and $\boldsymbol{x}(t_1,t_2)\neq\boldsymbol{x}(\tau_1,\tau_2)$, $\mathrm{Ker}_{\mathbf{S}_\Gamma}(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2))$ has the following estimate:

$$\big|\mathrm{Ker}_{\mathbf{S}_\Gamma}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)\big|\le\frac{C}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}.\tag{4.4}$$

Similarly, we can prove that $\mathrm{Ker}_{\mathbf{K}_\Gamma}(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2))$ has no singularity, and for all $\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\in\Gamma$, $\boldsymbol{x}(t_1,t_2)\neq\boldsymbol{x}(\tau_1,\tau_2)$, $\mathrm{Ker}_{\mathbf{K}_\Gamma^{(*)}}(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2))$,$\mathrm{Ker}_{\mathbf{T}_\Gamma}(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2))$ have the following estimate:

$$\big|\mathrm{Ker}_{\mathbf{K}_\Gamma^{(*)}}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)\big|,\ \big|\mathrm{Ker}_{\mathbf{T}_\Gamma}\big(\boldsymbol{x}(t_1,t_2),\boldsymbol{x}(\tau_1,\tau_2)\big)\big|\leq\frac{C}{|\boldsymbol{x}(t_1,t_2)-\boldsymbol{x}(\tau_1,\tau_2)|}. \tag{4.5}$$

For $\boldsymbol{x}\in S$, we consider the operator $\mathbf{T}_S$

$$[\mathbf{T}_S u_2](\boldsymbol{x})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{Ker}_{\mathbf{T}_S}(\boldsymbol{x},\boldsymbol{y})u_2(\boldsymbol{y})\mathrm{d}y_1\mathrm{d}y_2, \tag{4.6}$$

where

$$\begin{aligned}\mathrm{Ker}_{\mathbf{T}_S}(\boldsymbol{x},\boldsymbol{y})=\Big\{&-\frac{[\boldsymbol{\nu}_S(\boldsymbol{x})\cdot\boldsymbol{\nu}_S(\boldsymbol{y})]}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^2}\Big((\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}-(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big)\\&+\frac{[\boldsymbol{\nu}_S(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})][\boldsymbol{\nu}_S(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})]}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^4}\Big[\Big(\kappa_1^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-3\Big)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}\\&-\Big(\kappa_2^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-3\Big)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big]\Big\}J_S(y_1,y_2)\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}.\end{aligned} \tag{4.7}$$

From Lemma 2.3, we have

$$\begin{aligned}|\mathrm{Ker}_{\mathbf{T}_S}(\boldsymbol{x},\boldsymbol{y})|\leq C\Big\{&\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2}\Big|(\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}-(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big|\\&+\Big|(\kappa_1^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-3)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}\\&-(\kappa_2^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-3)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big|\Big\}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}.\end{aligned} \tag{4.8}$$

Taking the limit $y_j\to x_j$, $j=1,2$ for the right-hand side of (4.8) in the braces, we have

$$\begin{aligned}&\lim_{\substack{y_1\to x_1\\y_2\to x_2}}\Big\{\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2}\Big|(\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}-(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-1)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big|\\&\qquad+\Big|(\kappa_1^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-3)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}\\&\qquad\quad-(\kappa_2^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-3)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big|\Big\}\\=&\lim_{\substack{y_1\to x_1\\y_2\to x_2}}\Big\{\Big|\frac{(\mathrm{i}\kappa_1e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}-\mathrm{i}\kappa_2e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|})}{|\boldsymbol{x}-\boldsymbol{y}|}-\frac{(e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}-e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\Big|\\&\qquad+\Big|(\kappa_1^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|-3)e^{\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|}\end{aligned}$$

$$
\begin{aligned}
&\qquad -\big(\kappa_2^2|\boldsymbol{x}-\boldsymbol{y}|^2+3\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|-3\big)e^{\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|}\Big|\Big\} \\
&=\lim_{\substack{y_1\to x_1\\ y_2\to x_2}}\Big|\frac{\mathrm{i}(\kappa_1-\kappa_2)-(\kappa_1^2-\kappa_2^2)|\boldsymbol{x}-\boldsymbol{y}|-(\mathrm{i}/2)(\kappa_1^3-\kappa_2^3)|\boldsymbol{x}-\boldsymbol{y}|^2+\mathcal{O}(|\boldsymbol{x}-\boldsymbol{y}|^3)}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&\qquad -\frac{\mathrm{i}(\kappa_1-\kappa_2)|\boldsymbol{x}-\boldsymbol{y}|-(1/2)(\kappa_1^2-\kappa_2^2)|\boldsymbol{x}-\boldsymbol{y}|^2-(\mathrm{i}/6)(\kappa_1^3-\kappa_2^3)|\boldsymbol{x}-\boldsymbol{y}|^3+\mathcal{O}(|\boldsymbol{x}-\boldsymbol{y}|^4)}{|\boldsymbol{x}-\boldsymbol{y}|^2}\Big| \\
&=\frac{1}{2}\big|\kappa_1^2-\kappa_2^2\big|, \qquad (4.9)
\end{aligned}
$$

which implies that, for all $\boldsymbol{x},\boldsymbol{y}\in S$, $\boldsymbol{x}\neq\boldsymbol{y}$, $\mathrm{Ker}_{\mathbf{T}_S}(\boldsymbol{x},\boldsymbol{y})$ has the following estimate:

$$
|\mathrm{Ker}_{\mathbf{T}_S}(\boldsymbol{x},\boldsymbol{y})|\le\frac{C}{|\boldsymbol{x}-\boldsymbol{y}|}. \qquad (4.10)
$$

Similarly, for all $\boldsymbol{x},\boldsymbol{y}\in S$, $\boldsymbol{x}\neq\boldsymbol{y}$, we also can prove that

$$
|\mathrm{Ker}_{\mathbf{S}_S}(\boldsymbol{x},\boldsymbol{y})|,\,|\mathrm{Ker}_{\mathbf{K}_S^{(*)}}(\boldsymbol{x},\boldsymbol{y})|\le\frac{C}{|\boldsymbol{x}-\boldsymbol{y}|}, \qquad (4.11)
$$

and $\mathrm{Ker}_{\mathbf{K}_S}(\boldsymbol{x},\boldsymbol{y})$ has no singularity. □

The continuity properties for some integral operators in Hölder spaces have been considered with the analysis of singular convolutional integral operators in [15], [16, Section 7], [7, Section 2]. For the integral operators defined on unbounded domain S, we should consider the asymptotic properties of the kernels at the infinity. We note that the exponential decay properties of the Green functions at the infinity which proved in Lemma 2.2 play a key role in proving the following results.

Theorem 4.1. *For $0<\alpha<1$ and $S\in C^2$, the operators $\mathbf{S}_S,\mathbf{K}_S$ map $C^{0,\alpha}(S)$ continuously into $C^{1,\alpha}(S)$, and are compact from $C^{0,\alpha}(S)$ into itself. The operators $\mathbf{K}_S^{(*)}$ and $\mathbf{T}_S$ are compact from $C^{0,\alpha}(S)$ into itself.*

Proof. Since the proofs are similar for these operators, we show the details for the operator $\mathbf{K}_S$. For fixed $\boldsymbol{x}\in S$ and $\forall\phi\in C^{0,\alpha}(S)$, we have

$$
\begin{aligned}
\big[(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\big](\boldsymbol{x}) &=\int_{S\backslash S_{2L}}\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}=\sum_{j=1}^{8}I_j \\
&:=\bigg(\iint_{(-\infty,-L)\times(L,+\infty)}+\iint_{(-L,L)\times(L,+\infty)}+\iint_{(L,+\infty)\times(L,+\infty)} \\
&\qquad+\iint_{(-\infty,-L)\times(-L,L)}+\iint_{(L,+\infty)\times(-L,L)}+\iint_{(-\infty,-L)\times(-\infty,-L)} \\
&\qquad+\iint_{(-L,L)\times(-\infty,-L)}+\iint_{(L,+\infty)\times(-\infty,-L)}\bigg) \\
&\quad\times\Psi\big(\boldsymbol{x},y_1,y_2,f(y_1,y_2)\big)\mathrm{d}y_1\mathrm{d}y_2, \qquad (4.12)
\end{aligned}
$$

where $S_{2L}=\{\boldsymbol{x}\in S|-L<x_j<L, j=1,2\}$,

$$\Psi(\boldsymbol{x},y_1,y_2,f(y_1,y_2))=\left[\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\Big|_{y_3=f(y_1,y_2)}\right]J_S(y_1,y_2).$$

By Lemma 2.2, for fixed $\boldsymbol{x}\in S$, let $L\to+\infty$, we have $|x_j-L|\to+\infty, j=1,2$, and

$$\begin{aligned}
|I_1|\le&\iint_{(-\infty,-L)\times(L,+\infty)}\left|\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\Big|_{y_3=f(y_1,y_2)}\right| \\
&\qquad\qquad\times J_S(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 \\
\le& C\|\phi\|_\infty\int_L^{+\infty}\int_{-\infty}^{-L}\left(\frac{\exp\left(-(1/2)\Im(\kappa_1)|\boldsymbol{y}|\right)}{|\boldsymbol{y}|}\Big|_{y_3=f(y_1,y_2)}\right. \\
&\qquad\qquad\left.+\frac{\exp\left(-(1/2)\Im(\kappa_2)|\boldsymbol{y}|\right)}{|\boldsymbol{y}|}\Big|_{y_3=f(y_1,y_2)}\right)\mathrm{d}y_1\mathrm{d}y_2 \\
\le& C\|\phi\|_\infty\int_L^{+\infty}\int_{-\infty}^{-L}\left(\frac{\exp\left(-(1/2)\hat{\kappa}\sqrt{y_1^2+y_2^2}\right)}{\sqrt{y_1^2+y_2^2}}\right)\mathrm{d}y_1\mathrm{d}y_2 \\
\le& C\|\phi\|_\infty\left(\frac{1}{L}\right)\int_L^{+\infty}\int_{-\infty}^{-L}\exp\left(-\frac{1}{4}\hat{\kappa}|y_1|\right)\exp\left(-\frac{1}{4}\hat{\kappa}|y_2|\right)\mathrm{d}y_1\mathrm{d}y_2 \\
=& C\|\phi\|_\infty\left(\frac{1}{L}\right)\left[\int_L^{+\infty}\exp\left(-\frac{1}{4}\hat{\kappa}|y_2|\right)\mathrm{d}y_2\right]\left[\int_{-\infty}^{-L}\exp\left(-\frac{1}{4}\hat{\kappa}|y_1|\right)\mathrm{d}y_1\right] \\
=& C\|\phi\|_\infty\left(\frac{4}{\hat{\kappa}}\right)^2\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{2}L\right)\right) \\
\le& C\|\phi\|_\infty\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty,
\end{aligned}\tag{4.13}$$

where $C>0$ is a constant independent of L, $\hat{\kappa}=\min\{\Im(\kappa_1),\Im(\kappa_2)\}>0$. Similarly, for $j=2,\ldots,8$, we may also show that

$$|I_j|\le C\|\phi\|_\infty\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty.\tag{4.14}$$

Combining (4.12)-(4.14), we obtain

$$\left|[(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi](\boldsymbol{x})\right|\le\sum_{j=1}^{8}|I_j|\le C\|\phi\|_\infty\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty,\tag{4.15}$$

which implies that

$$\|(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\|_\infty\le C\|\phi\|_\infty\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\to\ +\infty.\tag{4.16}$$

Note that

$$\begin{aligned}&\big[\nabla_{\boldsymbol{x}}(\mathbf{K}_S\phi)(\boldsymbol{x})\big]-\big[\nabla_{\boldsymbol{x}}(\mathbf{K}_{S_{2L}}\phi)(\boldsymbol{x})\big]\\&=\int_{S\backslash S_{2L}}\nabla_{\boldsymbol{x}}\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)\phi(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}.\end{aligned}\tag{4.17}$$

For fixed $\boldsymbol{x}\in S$, by Lemma 2.2, similar to (4.13) and (4.14), we obtain

$$\|(\nabla_{\boldsymbol{x}}\mathbf{K}_S-\nabla_{\boldsymbol{x}}\mathbf{K}_{S_{2L}})\phi\|_\infty\le C\|\phi\|_\infty\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty.\tag{4.18}$$

Furthermore, for fixed $\boldsymbol{x},\tilde{\boldsymbol{x}}\in S$ and $\boldsymbol{x}\neq\tilde{\boldsymbol{x}}$, similar to (4.12), we define

$$\nabla_{\boldsymbol{x}}\big((\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\big)(\boldsymbol{x})-\nabla_{\tilde{\boldsymbol{x}}}\big((\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\big)(\tilde{\boldsymbol{x}}):=\sum_{j=1}^{8}\hat{I}_j.\tag{4.19}$$

From the Mean Value Theorem and Lemma 2.2, when $L\to+\infty$, we have $|x_j-L|\to+\infty$, $|\tilde{x}_j-L|\to+\infty,j=1,2$, and

$$\begin{aligned}&\left|\nabla_{\boldsymbol{x}}\left(\frac{\partial G_j(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)-\nabla_{\tilde{\boldsymbol{x}}}\left(\frac{\partial G_j(\tilde{\boldsymbol{x}},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)\right|\\&\le C\frac{\exp\big(-(1/2)\Im(\kappa_j)|\boldsymbol{y}|\big)}{|\boldsymbol{y}|}|\boldsymbol{x}-\tilde{\boldsymbol{x}}|\quad\text{as}\quad L\ \to\ +\infty.\end{aligned}\tag{4.20}$$

Hence, from (4.20) we obtain

$$\begin{aligned}|\hat{I}_1|\le&\iint_{(-\infty,-L)\times(L,+\infty)}\Bigg|\left[\nabla_{\boldsymbol{x}}\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)\right.\\&\qquad\left.-\nabla_{\tilde{\boldsymbol{x}}}\left(\frac{\partial G_1(\tilde{\boldsymbol{x}},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\tilde{\boldsymbol{x}},\boldsymbol{y})}{\partial\boldsymbol{\nu}_S(\boldsymbol{y})}\right)\right]\phi(\boldsymbol{y})\Big|_{y_3=f(y_1,y_2)}\Bigg|\\&\qquad\times J_S(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2\\\le&\,C\|\phi\|_\infty|\boldsymbol{x}-\tilde{\boldsymbol{x}}|\int_L^{+\infty}\int_{-\infty}^{-L}\left(\frac{\exp\left(-(1/2)\hat{\kappa}\sqrt{y_1^2+y_2^2}\right)}{\sqrt{y_1^2+y_2^2}}\right)\mathrm{d}y_1\mathrm{d}y_2\\\le&\,C\|\phi\|_\infty|\boldsymbol{x}-\tilde{\boldsymbol{x}}|\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty.\end{aligned}\tag{4.21}$$

Similarly, for $j=2,\ldots,8$, we have

$$|\hat{I}_j|\le C\|\phi\|_\infty|\boldsymbol{x}-\tilde{\boldsymbol{x}}|\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\ \to\ 0\quad\text{as}\quad L\ \to\ +\infty.\tag{4.22}$$

For $0<\alpha<1$, combining (4.19)-(4.22), we obtain

$$
\begin{aligned}
&\frac{\left|\nabla_{\boldsymbol{x}}\left((\mathbf{K}_S-\mathbf{K}_{2L})\phi\right)(\boldsymbol{x})-\nabla_{\tilde{\boldsymbol{x}}}\left((\mathbf{K}_S-\mathbf{K}_{2L})\phi\right)(\tilde{\boldsymbol{x}})\right|}{|\boldsymbol{x}-\tilde{\boldsymbol{x}}|^{\alpha}}\\
\leq & C\left(|\boldsymbol{x}-\tilde{\boldsymbol{x}}|^{1-\alpha}\right)\|\phi\|_{\infty}\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\\
\leq & CL^{1-\alpha}\|\phi\|_{\infty}\left(\frac{1}{L}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right)\\
= & C\|\phi\|_{\infty}\left(L^{-\alpha}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right) \;\rightarrow\; 0 \quad \text{as} \quad L \;\rightarrow\; +\infty.
\end{aligned}
\tag{4.23}
$$

For $0<\alpha<1$ and $\forall\phi(\boldsymbol{x})\in C^{0,\alpha}(S)$, by (4.16), (4.18) and (4.23), it can be deduced that

$$
\begin{aligned}
&\|(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\|_{C^{1,\alpha}(S)}\\
=&\|(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\|_{\infty}+\|\nabla_{\boldsymbol{x}}(\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\|_{\infty}\\
&+\sup_{\substack{\boldsymbol{x},\tilde{\boldsymbol{x}}\in S\\ \boldsymbol{x}\neq\tilde{\boldsymbol{x}}}}\frac{\left|\nabla_{\boldsymbol{x}}\left((\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\right)(\boldsymbol{x})-\nabla_{\tilde{\boldsymbol{x}}}\left((\mathbf{K}_S-\mathbf{K}_{S_{2L}})\phi\right)(\tilde{\boldsymbol{x}})\right|}{|\boldsymbol{x}-\tilde{\boldsymbol{x}}|^{\alpha}}\\
\leq & C\left(L^{-\alpha}\exp\left(-\frac{\hat{\kappa}}{4}L\right)\right) \;\rightarrow\; 0 \quad \text{as} \quad L \;\rightarrow\; +\infty,
\end{aligned}
\tag{4.24}
$$

which implies that $\mathbf{K}_{S_{2L}}$ is convergent to $\mathbf{K}_S$ as $L\to+\infty$. We can find the compactness of the related finite truncation operator in [15, 16]. Therefore, we obtain that the operator $\mathbf{K}_S$ is compact. Similarly, the operators $\mathbf{S}_S, \mathbf{K}_S^{(*)}$ and $\mathbf{T}_S$ are compact. □

4.2 Existence and uniqueness

Let X be a product space, i.e.,

$$X:=C^{1,\alpha}(\Gamma_0)\times C^{1,\alpha}(\Gamma)\times C^{0,\alpha}(\Gamma)\times C^{1,\alpha}(S)\times C^{0,\alpha}(S).$$

From Lemma 4.1 and Theorem 4.1, we find that the system (3.4) is of the Fredholm type. Consequently, a unique solution to (3.4) exists if the corresponding homogeneous system

$$\mathbf{Ap}=\mathbf{0} \tag{4.25}$$

has only the trivial solution.

Theorem 4.2. *There exists a unique solution of the boundary integral equations* (3.4) *in the product space X.*

Proof. Proof by contradiction. Let $(u_{\Gamma_0}, u_\Gamma, \partial u_\Gamma/\partial \boldsymbol{\nu}_\Gamma, u_S, \partial u_S/\partial \boldsymbol{\nu}_S)^\top$ be a nontrivial solution of the system (4.25), where $u_{\Gamma_0}\in C^{1,\alpha}(\Gamma_0)$, $u_\Gamma\in C^{1,\alpha}(\Gamma)$, $\partial u_\Gamma/\partial\boldsymbol{\nu}_\Gamma\in C^{0,\alpha}(\Gamma)$, $u_S\in C^{1,\alpha}(S)$ and $\partial u_S/\partial\boldsymbol{\nu}_S\in C^{0,\alpha}(S)$. Define the following functions by, respectively:

$$\begin{aligned} w_1(\boldsymbol{x}) = & \int_\Gamma \left[u_\Gamma(\boldsymbol{y}) \frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} G_1(\boldsymbol{x},\boldsymbol{y})\right] \mathrm{d}S_{\boldsymbol{y}} \\ & + \int_S \left[u_S(\boldsymbol{y}) \frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} G_1(\boldsymbol{x},\boldsymbol{y})\right] \mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in\Omega_1, \end{aligned} \tag{4.26}$$

$$w_2(\boldsymbol{x}) = -\int_S \left[u_S(\boldsymbol{y}) \frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} - \frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} G_2(\boldsymbol{x},\boldsymbol{y})\right] \mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in\Omega_2, \tag{4.27}$$

and

$$\begin{aligned} & -\int_{\Gamma_0} \left[\left(\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})} - \mathrm{i}\lambda_0 G_0(\boldsymbol{x},\boldsymbol{y}) \right) u_{\Gamma_0}(\boldsymbol{y}) \right] \mathrm{d}S_{\boldsymbol{y}} \\ & -\int_\Gamma \left[u_\Gamma(\boldsymbol{y}) \frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} G_0(\boldsymbol{x},\boldsymbol{y})\right] \mathrm{d}S_{\boldsymbol{y}} = \begin{cases} w_0(\boldsymbol{x}), & \text{if} \quad \boldsymbol{x}\in\Omega_0, \\ \hat{w}_0(\boldsymbol{x}), & \text{if} \quad \boldsymbol{x}\in D, \end{cases} \end{aligned} \tag{4.28}$$

which satisfy

$$w_1(\boldsymbol{x}) = w_0(\boldsymbol{x}), \quad \frac{1}{\rho_1}\frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{x})} = \frac{1}{\rho_0}\frac{\partial w_0(\boldsymbol{x})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{x})} \quad \text{on } \Gamma, \tag{4.29}$$

$$w_1(\boldsymbol{x}) = w_2(\boldsymbol{x}), \quad \frac{1}{\rho_1}\frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} = \frac{1}{\rho_2}\frac{\partial w_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})} \quad \text{on } S. \tag{4.30}$$

By the properties of the single-layer and double-layer potentials and $\Gamma_0,\Gamma,S\in C^2$, we know that $w_j\in C^2(\Omega_j)\cap C^{1,\alpha}(\overline{\Omega}_j)$, $j=0,1,2$, and $\hat{w}_0\in C^2(D)\cap C^{1,\alpha}(\overline{D})$. Clearly, it follows from (2.8) and (4.26) that:

$$\begin{aligned} & \Delta w_1(\boldsymbol{x}) + \kappa_1^2 w_1(\boldsymbol{x}) \\ = & \int_\Gamma \left[u_\Gamma(\boldsymbol{y}) \frac{\partial\big(\Delta_{\boldsymbol{x}} G_1(\boldsymbol{x},\boldsymbol{y}) + \kappa_1^2 G_1(\boldsymbol{x},\boldsymbol{y})\big)}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} - \frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})} \big(\Delta_{\boldsymbol{x}} G_1(\boldsymbol{x},\boldsymbol{y}) + \kappa_1^2 G_1(\boldsymbol{x},\boldsymbol{y})\big)\right] \mathrm{d}S_{\boldsymbol{y}} \\ & + \int_S \left[u_S(\boldsymbol{y}) \frac{\partial\big(\Delta_{\boldsymbol{x}} G_1(\boldsymbol{x},\boldsymbol{y}) + \kappa_1^2 G_1(\boldsymbol{x},\boldsymbol{y})\big)}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} \right. \\ & \qquad \left. - \frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})} \big(\Delta_{\boldsymbol{x}} G_1(\boldsymbol{x},\boldsymbol{y}) + \kappa_1^2 G_1(\boldsymbol{x},\boldsymbol{y})\big)\right] \mathrm{d}S_{\boldsymbol{y}} = 0 \quad \text{in } \Omega_1. \end{aligned} \tag{4.31}$$

Similarly, $w_2, w_0, \hat{w}_0$ are also the solutions of the homogeneous Helmholtz equation in their respective domains. Furthermore, with Lemma 2.2 and (4.26)-(4.27), we can derive

$$\lim_{r\to+\infty} \int_{\partial B_r\cap\Omega_j} \left(|w_j|^2 + \left| \frac{\partial w_j}{\partial \boldsymbol{\nu}} \right|^2 \right) \mathrm{d}s = 0, \quad j=1,2. \tag{4.32}$$

Letting $\boldsymbol{x}$ tend to the boundary Γ_0 from D and Ω_0 in (4.28), respectively, then, it follows from the jump relations and (4.25) that:

$$\begin{aligned}\hat{w}_0(\boldsymbol{x})=&-\frac{1}{2}u_{\Gamma_0}(\boldsymbol{x})-\int_{\Gamma_0}\left[\left(\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0 G_0(\boldsymbol{x},\boldsymbol{y})\right)u_{\Gamma_0}(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{\Gamma}\left[u_\Gamma(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\=&-\frac{1}{2}\left[(\mathbf{I}_{\Gamma_0}+\mathbf{K}_{\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma_0})u_{\Gamma_0}+\mathbf{K}_{\Gamma_0,\Gamma}u_\Gamma-\mathbf{S}_{\Gamma_0,\Gamma}\frac{\partial u_\Gamma}{\partial \boldsymbol{\nu}_\Gamma}\right](\boldsymbol{x})=0,\quad \boldsymbol{x}\in\Gamma_0,\end{aligned}\tag{4.33}$$

$$\begin{aligned}w_0(\boldsymbol{x})=&\frac{1}{2}u_{\Gamma_0}(\boldsymbol{x})-\int_{\Gamma_0}\left[\left(\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0 G_0(\boldsymbol{x},\boldsymbol{y})\right)u_{\Gamma_0}(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_{\Gamma}\left[u_\Gamma(\boldsymbol{y})\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_0(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\=&u_{\Gamma_0}(\boldsymbol{x})-\frac{1}{2}\left[(\mathbf{I}_{\Gamma_0}+\mathbf{K}_{\Gamma_0}-\mathrm{i}\lambda_0\mathbf{S}_{\Gamma})u_{\Gamma_0}+\mathbf{K}_{\Gamma_0,\Gamma}u_\Gamma-\mathbf{S}_{\Gamma_0,\Gamma}\frac{\partial u_\Gamma}{\partial \boldsymbol{\nu}_\Gamma}\right](\boldsymbol{x})\\=&u_{\Gamma_0}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_0.\end{aligned}\tag{4.34}$$

In addition, since $\Delta\hat{w}_0+\kappa_0^2\hat{w}_0=0$ in D. Combining (4.33) and $\Im(\kappa_0^2)>0$, by using Green's first theorem, we have $\hat{w}_0=0$ in D. In particular, $\partial\hat{w}_0/\partial\boldsymbol{\nu}_{\Gamma_0}=0$ on Γ_0. On the other hand, using (4.28), we deduce that the normal derivatives of $\hat{w}_0,w_0$ on Γ_0, respectively, as

$$\begin{aligned}0=\frac{\partial\hat{w}_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}=&\frac{-\mathrm{i}\lambda_0 u_{\Gamma_0}(\boldsymbol{x})}{2}-\int_{\Gamma_0}\left(\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}\right)u_{\Gamma_0}(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}\\&-\int_\Gamma\left[u_\Gamma(\boldsymbol{y})\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}},\quad \boldsymbol{x}\in\Gamma_0,\end{aligned}\tag{4.35}$$

and

$$\begin{aligned}\frac{\partial w_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}=&\frac{\mathrm{i}\lambda_0}{2}u_{\Gamma_0}(\boldsymbol{x})-\int_{\Gamma_0}\left[\left(\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{y})}-\mathrm{i}\lambda_0\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}\right)u_{\Gamma_0}(\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\&-\int_\Gamma\left[u_\Gamma(\boldsymbol{y})\frac{\partial^2 G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})\partial\boldsymbol{\nu}_{\Gamma}(\boldsymbol{y})}-\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\frac{\partial G_0(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}\\=&\mathrm{i}\lambda_0 u_{\Gamma_0}(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma_0.\end{aligned}\tag{4.36}$$

Then, combining (4.34) and (4.36), we find

$$\frac{\partial w_0(\boldsymbol{x})}{\partial\boldsymbol{\nu}_{\Gamma_0}(\boldsymbol{x})}-\mathrm{i}\lambda_0 w_0(\boldsymbol{x})=0\quad\text{on }\ \Gamma_0.\tag{4.37}$$

In (4.26) and (4.27), letting $\boldsymbol{x}$ tend to the boundary S, respectively, then by the jump relations we find that

$$w_1(\boldsymbol{x})=\frac{1}{2}u_S(\boldsymbol{x})+\int_\Gamma\left[u_\Gamma(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial\boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}$$

$$+\int_S \left[u_S(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in S, \tag{4.38}$$

$$w_2(\boldsymbol{x})=\frac{1}{2}u_S(\boldsymbol{x})-\int_S \left[u_S(\boldsymbol{y})\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}G_2(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S. \tag{4.39}$$

Hence, it follows from (4.25), (4.38) and (4.39) that:

$$\begin{aligned}
&w_1(\boldsymbol{x})+w_2(\boldsymbol{x})\\
=&u_S(\boldsymbol{x})+\int_\Gamma \left[u_\Gamma(\boldsymbol{y})\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}G_1(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}S_{\boldsymbol{y}}\\
&+\int_S \left[u_S(\boldsymbol{y})\left(\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\right)-\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\left(\frac{\rho_1}{\rho_2}G_1(\boldsymbol{x},\boldsymbol{y})-G_2(\boldsymbol{x},\boldsymbol{y})\right)\right]\mathrm{d}S_{\boldsymbol{y}}\\
=&2u_S(\boldsymbol{x})-\left[-\mathbf{K}_{S,\Gamma}u_\Gamma+\mathbf{S}_{S,\Gamma}\frac{\partial u_\Gamma}{\partial \boldsymbol{\nu}_\Gamma}+(\mathbf{I}_S-\mathbf{K}_S)u_S+\mathbf{S}_S\frac{\partial u_S}{\partial \boldsymbol{\nu}_S}\right](\boldsymbol{x})\\
=&2u_S(\boldsymbol{x}), \quad \boldsymbol{x}\in S.
\end{aligned} \tag{4.40}$$

Furthermore, using (4.26) and (4.27), we deduce that the normal derivatives of w_1,w_2 on S, respectively, as

$$\begin{aligned}
\frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}=&\frac{1}{2}\frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}+\int_\Gamma \left[u_\Gamma(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}-\frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{y})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}\\
&+\int_S \left[u_S(\boldsymbol{y})\frac{\partial^2 G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\frac{\partial G_1(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in S,
\end{aligned} \tag{4.41}$$

$$\frac{\partial w_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}=\frac{1}{2}\frac{\partial u_S(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}-\int_S \left[u_S(\boldsymbol{y})\frac{\partial^2 G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})\partial \boldsymbol{\nu}_S(\boldsymbol{y})}-\frac{\partial u_S(\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{y})}\frac{\partial G_2(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\right]\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S. \tag{4.42}$$

Then, it follows from (4.25), (4.41) and (4.42) that:

$$\begin{aligned}
&\frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}+\frac{\partial w_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}\\
=&\frac{\rho_1+\rho_2}{\rho_2}\frac{\partial u_S(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}-\frac{\rho_1+\rho_2}{2\rho_2}\left[-\frac{2\rho_2}{\rho_1+\rho_2}\mathbf{T}_{S,\Gamma}u_\Gamma+\frac{2\rho_2}{\rho_1+\rho_2}\mathbf{K}^{(*)}_{S,\Gamma}\frac{\partial u_\Gamma}{\partial \boldsymbol{\nu}_\Gamma}\right.\\
&\left.-\frac{2\rho_2}{\rho_1+\rho_2}\mathbf{T}_S u_S+\left(\mathbf{I}^{(*)}_S+\frac{2\rho_2}{\rho_1+\rho_2}\mathbf{K}^{(*)}_S\right)\frac{\partial u_S}{\partial \boldsymbol{\nu}_S}\right](\boldsymbol{x})\\
=&\frac{\rho_1+\rho_2}{\rho_2}\frac{\partial u_S(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}, \quad \boldsymbol{x}\in S,
\end{aligned} \tag{4.43}$$

Considering

$$w_1|_S=w_2|_S, \quad \frac{1}{\rho_1}\frac{\partial w_1}{\partial \boldsymbol{\nu}_S}|_S=\frac{1}{\rho_2}\frac{\partial w_2}{\partial \boldsymbol{\nu}_S}|_S$$

in (4.30), then from (4.40) and (4.43), we have

$$w_1(\boldsymbol{x})=w_2(\boldsymbol{x})=u_S(\boldsymbol{x}),\quad \frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}=\frac{\rho_1}{\rho_2}\frac{\partial w_2(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})}=\frac{\rho_1}{\rho_2}\frac{\partial u_S(\boldsymbol{x})}{\partial \boldsymbol{\nu}_S(\boldsymbol{x})},\quad \boldsymbol{x}\in S. \tag{4.44}$$

Similarly, we obtain

$$w_1(\boldsymbol{x})=w_0(\boldsymbol{x})=u_\Gamma(\boldsymbol{x}),\quad \frac{\partial w_1(\boldsymbol{x})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{x})}=\frac{\rho_1}{\rho_0}\frac{\partial w_0(\boldsymbol{x})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{x})}=\frac{\rho_1}{\rho_0}\frac{\partial u_\Gamma(\boldsymbol{x})}{\partial \boldsymbol{\nu}_\Gamma(\boldsymbol{x})},\quad \boldsymbol{x}\in \Gamma. \tag{4.45}$$

The above analysis shows the (w_0,w_1,w_2) is a solution of the following problem:

$$\begin{cases} \left(\Delta+\kappa_1^2\right)w_1(\boldsymbol{x})=0 & \text{in } \Omega_1,\\ \left(\Delta+\kappa_2^2\right)w_2(\boldsymbol{x})=0 & \text{in } \Omega_2,\\ \left(\Delta+\kappa_0^2\right)w_0(\boldsymbol{x})=0 & \text{in } \Omega_0,\\ \dfrac{\partial w_0}{\partial \boldsymbol{\nu}_{\Gamma_0}}-\mathrm{i}\lambda_0 w_0=0 & \text{on } \Gamma_0,\\ w_1=w_0,\quad \dfrac{1}{\rho_1}\dfrac{\partial w_1}{\partial \boldsymbol{\nu}_\Gamma}=\dfrac{1}{\rho_0}\dfrac{\partial w_0}{\partial \boldsymbol{\nu}_\Gamma} & \text{on } \Gamma,\\ w_1=w_2,\quad \dfrac{1}{\rho_1}\dfrac{\partial w_1}{\partial \boldsymbol{\nu}_S}=\dfrac{1}{\rho_2}\dfrac{\partial w_2}{\partial \boldsymbol{\nu}_S} & \text{on } S \end{cases} \tag{4.46}$$

with the radiation conditions (4.32). Therefore, by Lemma 2.1, we obtain $w_j=0,j=0,1,2$. Thus, from (4.34), (4.44) and (4.45), we have shown that the homogeneous system (4.25) has only a trivial solution, this is

$$u_{\Gamma_0}=0,\quad u_\Gamma=0,\quad \frac{\partial u_\Gamma}{\partial \boldsymbol{\nu}_\Gamma}=0,\quad u_S=0,\quad \frac{\partial u_S}{\partial \boldsymbol{\nu}_S}=0. \tag{4.47}$$

Consequently, due to (3.4) is Fredholm integral equations, a unique solution to (3.4) exists. The proof is complete. □

Then, the well-posedness of Scattering Problem follows immediately from combining Theorems 3.1, 3.2 and 4.2.

5 Concluding remarks

The goal of this work is to conduct mathematical modeling and analysis of the acoustic scattering by three-dimensional core-shell structures in a two-layered medium. It is motivated by the acoustic scattering from underwater or aerial layered obstacles. We have derived a scattering model under the weak absorption (damping) medium assumption. An integral equation method is developed for solving the scattering problem. Compared with the usual potential operators in the classical obstacle scattering problem, our integral operators must be defined on an unbounded interface. The properties of these operators are established from the asymptotic properties of Green's functions at the infinity. Furthermore, to reduce the singularity of the integral operators, a novel boundary integral equation formulation is introduced for the scattering problem that leads to an equivalent well-posed integral operator system. The well-posedness of the scattering problem is proved by combining the integral equation operator theory and a variational technique.

One future direction is to develop numerical methods for solving the scattering problem. The integral equation method developed here may be used to serve the purpose. The numerical analysis may also be studied. Another direction is to solve the scattering problem by more general core-shell structures in a layered medium, particularly more complex class of structures with general damping coefficients or viscoelastic core-shell structures. In fact, the anisotropic medium model involved in describing core-shell scattering in a heterogeneous atmosphere is more complex. More general plasma shell models need to introduce the nonlinear Boltzmann transfer equations.

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