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Conservative Three-Level Linearized Finite Difference Schemes for the Fisher Equation and Its Maximum Error Estimates

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Abstract. A three-level linearized difference scheme for solving the Fisher equation is firstly proposed in this work. It has the good property of discrete conservative energy. By the discrete energy analysis and mathematical induction method, it is proved to be uniquely solvable and unconditionally convergent with the second-order accuracy in both time and space. Then another three-level linearized compact difference scheme is derived along with its discrete energy conservation law, unique solvability and unconditional convergence of order two in time and four in space. The resultant schemes preserve the maximum bound principle. The analysis techniques for convergence used in this paper also work for the Euler scheme, the Crank-Nicolson scheme and others. Numerical experiments are carried out to verify the computational efficiency, conservative law and the maximum bound principle of the proposed difference schemes.

AMS subject classifications: 65M06, 65M12, 65M15
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1. Introduction

The Fisher equation belongs to the class of reaction-diffusion equation: in fact, it is one of the simplest semilinear reaction-diffusion equations, the one which has the inhomogeneous term

$$f(u,x,t)=\lambda u(1-u),$$

which can exhibit traveling wave solutions that switch between equilibrium states given by $f(u)=0$. Such equation describes a balance between linear diffusion and nonlinear

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reaction, and it occurs, e.g., in ecology, biology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems. Fisher proposed this equation in 1937 to describe the spatial spread of an advantageous allele and explored its travelling wave solutions [6]. In the same year as Fisher, Kolmogorov *et al.* introduced a more general reaction-diffusion equation [9].

The wider use of this equation in many applications of engineering has been found by researchers. There have been many numerical and approximate methods in the literature to solve this equation, such as the finite difference method, the collocation method, the finite element technique, the wavelet Galerkin method, the pseudospectral method, the various differentiation quadrature method and so on. Here we mainly recall some relevant discretizations based on finite differences. In 1985, Aggarwal [1] compared various difference numerical methods for solving the Fisher equation, including the standard implicit, the quasi linear implicit, the time-linearization implicit, the Crank-Nicolson implicit, the predictor-corrector explicit and two forms of operator-splitting schemes, by the technique of plotting an optimized error-norm versus CPU time. The conclusion that the two-step operator splitting procedure is the most effective method has been drawn. A highly accurate finite difference approach for the second-order spatial derivative in conjunction with a TVD-RK3 method in time was presented to solve the Fisher equation in [2], but there was no any theoretical analysis on the derived scheme. Hasnain *et al.* [8] discussed three difference schemes for solving the Fisher equation: the forward Euler central space scheme, the Lax Wendroff central space scheme and the nonlinear Crank-Nicolson scheme, then the Richardson extrapolation technique was used to improve the numerical accuracy. The Neumann stability analysis was made for the linear form of the resultant difference equation. Chandraker *et al.* [3] proposed two implicit difference schemes to solve the Fisher equation: one is the modified Crank-Nicolson scheme and the other one is the modified Keller box scheme, where the nonlinearity is handled by the method of lagging. The accuracy and stability of the proposed schemes are both discussed based on the numerical experiments.

The considered equation (1.1) is a semilinear parabolic equation and satisfies the maximum principle, or say the maximum bound principle (MBP), i.e., the solution has the range in the set $[0,1]$ at any time if the initial and boundary values have the same property. Such a problem has been discussed under a systematical framework in [5] along with some provable MBP-preserving numerical schemes. It is always expected that discrete numerical formats have this property. The authors in [10] pointed out that there are few works to study the capability of the numerical methods for solving the Fisher equation to preserve the structure of solutions although abundant numerical schemes can be found in the literature. They proposed a finite difference scheme in a logarithmic form based on the logarithmic form of the continuous model and showed that the scheme can preserve the positivity, the boundedness and the monotonicity of the numerical approximations. The accuracy is of order 1 in time and order 2 in space. Sun *et al.* [14] constructed several difference schemes to solve the Fisher equation and analyzed some conditions to preserve the boundedness and monotone property of these schemes. In [4], the authors mentioned that although mathematical properties of Fisher's equation and plenty of discussion are available in the literature, majority of them do not address the important properties such as stability analysis, order of convergence and consistence of the underlying numerical method. They derived a compact difference scheme to solve the Fisher equation, which has the fourth-order accuracy in space. The stability was shown by the von-Neumann's method and the Richardson extrapolation to the sixth-order accuracy in space has been made. Gao and Yang [7] established two finite difference schemes, where one is explicit-implicit (E-I) and the other is implicit-explicit (I-E), and provided the convergence analysis, whereas, a coarse assumption on the nonlinear term has been made for the analysis.

In [19], the authors constructed high-order energy dissipative and conservative local discontinuous Galerkin methods for the Fornberg-Whitham type equations. Then they gave the proof for the dissipation or conservation of related conservative quantities. The capability of their schemes for different types of solutions was shown via several numerical experiments. Ranocha *et al.* [11] developed a general framework for designing conservative numerical methods based on summation by parts operators and split forms in space, combined with relaxation Runge-Kutta methods in time. They applied this framework to create new classes of fully-discrete conservative methods for several nonlinear dispersive wave equations. Wu *et al.* [17] considered the Crank-Nicolson Fourier collocation method for the nonlinear fractional Schrödinger equation, which has the second-order accuracy in time and the spectral accuracy in space. They proved that at each discrete time the method preserves the discrete mass and energy conservation laws. Zhang and Sun [18] studied a linearized CCD method with weighted approximation in time for nonlinear time fractional Klein-Gordon equations. The method can achieve at least sixth-order spatial accuracy and second-order temporal accuracy. As we know, linearization is a common technique for solving nonlinear problems numerically, which can simplify the calculation of nonlinear problems, so that, it is meaningful to consider a linearized conservative numerical method for nonlinear problems.

In this paper, we consider the following initial-boundary value problem of one-dimensional Fisher equation:

$$u_t - u_{xx} = \lambda u(1-u), \qquad 0 < x < L, \quad 0 < t \le T, \tag{1.1}$$

$$u(x,0) = \varphi(x), \qquad 0 \le x \le L, \tag{1.2}$$

$$u(0,t) = \alpha(t), \quad u(L,t) = \beta(t), \quad 0 < t \le T, \tag{1.3}$$

where λ is a positive constant, functions $\varphi(x), \alpha(t), \beta(t)$ are all given and $\varphi(0) = \alpha(0)$, $\varphi(L) = \beta(0)$. Suppose that the problem (1.1)-(1.3) has a smooth solution. To the authors' knowledge, no conservative finite difference scheme has been discussed in the literature for the numerical solution of (1.1)-(1.3). This work will make some efforts in this respect and two conservative three-level linearized finite difference schemes will be derived, analyzed and numerically verified. The major contribution of this work lies in the construction of two conservative and linearized finite difference schemes together with the rigorous analysis on the conservative property, the unique solvability and unconditional convergence in the maximum norm.

The rest of this work is organized as follows. Section 2 gives a priori estimate on the continuous problem and prepares some notations and useful lemmas. A three-level linearized difference scheme is derived in Section 3 along with its conservative property, unique solvability and unconditional convergence. Section 4 devotes to another three-level linearized compact difference scheme. Three numerical experiments are implemented to test the numerical accuracy, discrete conservative law and the MBP of the proposed two difference schemes in Section 5. A brief conclusion ends this work finally.

2. Some preparations

Before introducing the difference scheme, a priori estimate on the solution of the problem (1.1)-(1.3) is given.

Theorem 2.1. *Let $u(x,t)$ be the solution of the problem* (1.1)-(1.3) *with $\alpha(t)\equiv 0, \beta(t)\equiv 0$. Denote*

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + 2\int_0^t \left[\int_0^L u_x^2(x,s)\mathrm{d}x + \lambda\int_0^L \big[u^3(x,s) - u^2(x,s)\big]\mathrm{d}x\right]\mathrm{d}s,$$

$$F(t) = \int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\int_0^L \left[\frac{2}{3}u^3(x,t) - u^2(x,t)\right]\mathrm{d}x + 2\int_0^t \left[\int_0^L u_s^2(x,s)\mathrm{d}x\right]\mathrm{d}s.$$

Then

$$E(t) = E(0), \quad F(t) = F(0), \quad 0 < t \le T. \tag{2.1}$$

Proof. (I) Multiplying (1.1) by u gives

$$u(x,t)u_t(x,t) - u(x,t)u_{xx}(x,t) + \lambda\big[u^3(x,t) - u^2(x,t)\big] = 0,$$

that is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[u^2(x,t)\big] - \big(u(x,t)u_x(x,t)\big)_x + u_x^2(x,t) + \lambda\big[u^3(x,t) - u^2(x,t)\big] = 0.$$

Integrating this equation with respect to x on the interval $[0,L]$ and noticing (1.3) with $\alpha(t) = \beta(t) = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2(x,t)\mathrm{d}x + \int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\int_0^L \big[u^3(x,t) - u^2(x,t)\big]\mathrm{d}x = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^L u^2(x,t)\mathrm{d}x + 2\int_0^t \left[\int_0^L u_x^2(x,s)\mathrm{d}x + \lambda\int_0^L \big(u^3(x,s) - u^2(x,s)\big)\mathrm{d}x\right]\mathrm{d}s\right\} = 0.$$

Then $E(t) = E(0)$ is obtained.

(II) Multiplying (1.1) by u_t gives

$$u_t^2(x,t) - u_t(x,t)u_{xx}(x,t) - \lambda\big[u(x,t) - u^2(x,t)\big]u_t(x,t) = 0,$$

that is

$$u_t^2(x,t) - \big(u_t(x,t)u_x(x,t)\big)_x + \left(\frac{1}{2}u_x^2(x,t)\right)_t + \lambda\left[\frac{1}{3}u^3(x,t) - \frac{1}{2}u^2(x,t)\right]_t = 0.$$

Integrating this equation with respect to x on the interval $[0,L]$ and noticing (1.3) with $\alpha(t) = \beta(t) = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left[\frac{1}{3}u^3(x,t) - \frac{1}{2}u^2(x,t)\right]\mathrm{d}x + \int_0^L u_t^2(x,t)\mathrm{d}x = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L u_x^2(x,t)\mathrm{d}x + \lambda\int_0^L\left(\frac{2}{3}u^3(x,t) - u^2(x,t)\right)\mathrm{d}x + 2\int_0^t\left(\int_0^L u_s^2(x,s)\mathrm{d}x\right)\mathrm{d}s\right] = 0,$$

that is

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = 0, \quad 0 < t \le T.$$

Thus, $F(t) = F(0)$ is followed. □

It is noted that these two invariants have their physical meanings: when u represents concentration, $E(t)$ reflects total mass or total number of particles, and when u represents potential, $F(t)$ can be understood as the potential energy.

Remark 2.1. Equipped with the time-independent Dirichlet, the homogeneous Neumann, or the periodic boundary condition, (1.1) can also be viewed as the L^2 gradient flow with respect to the energy functional

$$E(u) = \int_0^L\left[\frac{1}{2}(u_x)^2 + \lambda\left(\frac{1}{3}u^3 - \frac{1}{2}u^2\right)\right]\mathrm{d}x, \tag{2.2}$$

and thus satisfies the energy dissipation law in the sense that

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t} = \left(\frac{\delta E}{\delta u}, u_t\right) = -\left\|\frac{\delta E}{\delta u}\right\|^2 \le 0,$$

where $\delta E/\delta u$ is the variational derivative of the energy functional $E(u)$. Indeed, the energy dissipation law can also be obtained from the proof for $F(t) = F(0)$.

In order to derive the difference scheme, we firstly divide the domain $[0,L]\times[0,T]$. Take two positive integers m,n. Divide $[0,L]$ into m equal subintervals, and $[0,T]$ into n subintervals. Denote $h = L/m, \tau = T/n; x_i = ih, 0 \le i \le m; t_k = k\tau, 0 \le k \le n;$

$\Omega_h = \{x_i \,|\, 0 \le i \le m\}, \Omega_\tau = \{t_k \,|\, 0 \le k \le n\}; \Omega_{h\tau} = \Omega_h \times \Omega_\tau$. In addition, denote $r = \tau/h^2$.

Define the mesh function spaces

$$\mathcal{U}_h = \{u \,|\, u = (u_0, u_1, \cdots, u_m) \text{ is the grid function defined on } \Omega_h\},$$
$$\overset{\circ}{\mathcal{U}}_h = \{u \,|\, u \in \mathcal{U}_h, u_0 = u_m = 0\}.$$

For any grid function $u \in \mathcal{U}_h$, introduce the following notation:

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i),$$
$$\delta_x^2 u_i = \frac{1}{h}\big(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}\big),$$
$$(\mathcal{A}u)_i = \begin{cases} \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \le i \le m-1, \\ u_i, & i = 0, m. \end{cases}$$

For any $u, v \in \overset{\circ}{\mathcal{U}}_h$, introduce the inner products and norms (or semi-norms) as

$$(u, v) = h\sum_{i=1}^{m-1} u_i v_i, \qquad \|u\| = \Big(h\sum_{i=1}^{m-1} u_i^2\Big)^{\frac{1}{2}},$$
$$\|u\|_\infty = \max_{0\le i\le m} |u_i|, \qquad |u|_1 = \Big(h\sum_{i=1}^{m} \big(\delta_x u_{i-\frac{1}{2}}\big)^2\Big)^{\frac{1}{2}},$$
$$(u, v)_{1,\mathcal{A}} = -(\mathcal{A}^{-1}\delta_x^2 u, v), \quad |u|_{1,\mathcal{A}} = \big((u, u)_{1,\mathcal{A}}\big)^{\frac{1}{2}}.$$

Denote

$$\mathcal{S}_\tau = \{w \,|\, w = (w^0, w^1, \cdots, w^n) \text{ is the grid function defined on } \Omega_\tau\}.$$

For any $w \in \mathcal{S}_\tau$, introduce the following notation:

$$w^{k+\frac{1}{2}} = \frac{1}{2}(w^k + w^{k+1}), \qquad w^{\bar{k}} = \frac{1}{2}(w^{k+1} + w^{k-1}),$$
$$\delta_t w^{k+\frac{1}{2}} = \frac{1}{\tau}(w^{k+1} - w^k), \quad \Delta_t w^k = \frac{1}{2\tau}(w^{k+1} - w^{k-1}).$$

It is easy to see that

$$\Delta_t w^k = \frac{1}{2}\big(\delta_t w^{k-\frac{1}{2}} + \delta_t w^{k+\frac{1}{2}}\big).$$

Lemma 2.1 ([12, 15, 16]). *(a) Suppose $u, v \in \mathcal{U}_h$, then*

$$-h\sum_{i=1}^{m-1} \big(\delta_x^2 u_i\big) v_i = h\sum_{i=1}^{m} \big(\delta_x u_{i-\frac{1}{2}}\big)\big(\delta_x v_{i-\frac{1}{2}}\big) + \big(\delta_x u_{\frac{1}{2}}\big) v_0 - \big(\delta_x u_{m-\frac{1}{2}}\big) v_m.$$

(b) Suppose $u \in \overset{\circ}{\mathcal{U}}_h$*, then*

$$-h\sum_{i=1}^{m-1}(\delta_x^2 u_i)u_i = |u|_1^2, \quad \|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1, \quad \|u\| \le \frac{L}{\sqrt{6}}|u|_1,$$

$$\|\mathcal{A}u\| \le \|u\|, \quad |u|_1 \le |u|_{1,\mathcal{A}} \le \frac{\sqrt{6}}{2}|u|_1.$$

(c) Suppose $u \in \overset{\circ}{\mathcal{U}}_h$*, then*

$$|u|_1^2 \le \frac{4}{h^2}\|u\|^2.$$

Next we will give several commonly used numerical differential formulas.

Lemma 2.2 ([15]). *Let* c, h *be given constants and* $h > 0$*.*

(a) If $g \in C^2[c-h, c+h]$*, then*

$$g(c) = \frac{1}{2}[g(c-h) + g(c+h)] - \frac{h^2}{2}g''(\xi_1), \quad c-h < \xi_1 < c+h.$$

(b) If $g \in C^3[c-h, c+h]$*, then*

$$g'(c) = \frac{1}{2h}[g(c+h) - g(c-h)] - \frac{h^2}{6}g'''(\xi_2), \quad c-h < \xi_2 < c+h.$$

(c) If $g \in C^4[c-h, c+h]$*, then*

$$g''(c) = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] - \frac{h^2}{12}g^{(4)}(\xi_3), \quad c-h < \xi_3 < c+h.$$

(d) If $g \in C^6[c-h, c+h]$*, then*

$$\begin{aligned}&\frac{1}{12}[g''(c-h) + 10g''(c) + g''(c+h)]\\ =&\frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] + \frac{h^4}{240}g^{(6)}(\xi_4), \quad c-h < \xi_4 < c+h.\end{aligned}$$

Now we introduce an important Gronwall inequality.

Lemma 2.3 ([15]). *Suppose* $\{F^k\}_{k=0}^\infty$ *is a nonnegative sequence,* c *and* g *are two nonnegative constants satisfying*

$$F^{k+1} \le (1 + c\tau)F^k + \tau g, \quad k = 0, 1, 2, \dots,$$

then

$$F^k \le \mathrm{e}^{ck\tau}\left(F^0 + \frac{g}{c}\right), \quad k = 0, 1, 2, \dots .$$

3. A three-level linearized difference scheme

This part will focus on an unconditionally convergent and conservative difference scheme for solving (1.1)-(1.3) with the convergence order $\mathcal{O}(\tau^2+h^2)$.

3.1. Derivation of the difference scheme

Define the grid function $U=\{U_i^k \,|\, 0\le i\le m, 0\le k\le n\}$ on $\Omega_{h\tau}$, where

$$U_i^k=u(x_i,t_k),\quad 0\le i\le m,\quad 0\le k\le n.$$

Denote

$$c_0=\max_{\substack{0\le x\le L\\0\le t\le T}}|u(x,t)|.$$

Considering Eq. (1.1) at point $(x_i,t_{1/2})$, we have

$$u_t\big(x_i,t_{\frac{1}{2}}\big)-u_{xx}\big(x_i,t_{\frac{1}{2}}\big)=\lambda\Big[u\big(x_i,t_{\frac{1}{2}}\big)-u^2\big(x_i,t_{\frac{1}{2}}\big)\Big],\quad 1\le i\le m-1.$$

By the Taylor expansion, we have

$$\begin{aligned}u(x_i,t_0)&=u\big(x_i,t_{\frac{1}{2}}\big)-\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2),\\ u(x_i,t_1)&=u\big(x_i,t_{\frac{1}{2}}\big)+\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2),\end{aligned}$$

or

$$\begin{aligned}u\big(x_i,t_{\frac{1}{2}}\big)&=u(x_i,t_0)+\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2),\\ u\big(x_i,t_{\frac{1}{2}}\big)&=u(x_i,t_1)-\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2).\end{aligned}$$

Consequently,

$$\begin{aligned}u^2\big(x_i,t_{\frac{1}{2}}\big)&=\Big[U_i^0+\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2)\Big]\Big[U_i^1-\frac{\tau}{2}u_t\big(x_i,t_{\frac{1}{2}}\big)+\mathcal{O}(\tau^2)\Big]\\&=U_i^0U_i^1+\mathcal{O}(\tau^2).\end{aligned}$$

By Lemma 2.2 and the above equality, we get

$$\delta_tU_i^{\frac{1}{2}}-\delta_x^2U_i^{\frac{1}{2}}=\lambda\Big(U_i^{\frac{1}{2}}-U_i^0U_i^1\Big)+(R_1)_i^0,\quad 1\le i\le m-1,\tag{3.1}$$

where there is a constant c_1 such that

$$\big|(R_1)_i^0\big|\le c_1(\tau^2+h^2),\quad 1\le i\le m-1.\tag{3.2}$$

Considering Eq. (1.1) at point (x_i,t_k), we have

$$u_t(x_i,t_k)-u_{xx}(x_i,t_k)=\lambda\big[u(x_i,t_k)-u^2(x_i,t_k)\big],\quad 1\le i\le m-1,\quad 1\le k\le n-1.$$

By Lemma 2.2, we have

$$\Delta_t U_i^k - \delta_x^2 U_i^{\bar{k}} = \lambda \left[U_i^{\bar{k}} - \frac{1}{3}\big(U_i^{k-1} + U_i^k + U_i^{k+1}\big) U_i^k \right] + (R_1)_i^k, \quad 1 \le i \le m-1, \quad 1 \le k \le n-1, \tag{3.3}$$

where there is a constant c_2 such that

$$\big|(R_1)_i^k\big| \le c_2(\tau^2 + h^2), \quad 1 \le i \le m-1, \quad 1 \le k \le n-1. \tag{3.4}$$

Noticing the initial-boundary value conditions (1.2)-(1.3), we have

$$U_i^0 = \varphi(x_i), \qquad 0 \le i \le m, \tag{3.5}$$

$$U_0^k = \alpha(t_k), \quad U_m^k = \beta(t_k), \quad 1 \le k \le n. \tag{3.6}$$

Neglecting the small term $(R_1)_i^k$ in (3.1) and (3.3), and replacing the exact solution U_i^k by its numerical one u_i^k, the following difference scheme can be derived:

$$\delta_t u_i^{\frac{1}{2}} - \delta_x^2 u_i^{\frac{1}{2}} = \lambda \Big(u_i^{\frac{1}{2}} - u_i^0 u_i^1 \Big), \qquad 1 \le i \le m-1, \tag{3.7}$$

$$\Delta_t u_i^k - \delta_x^2 u_i^{\bar{k}} = \lambda \left[u_i^{\bar{k}} - \frac{1}{3} u_i^k \big(u_i^{k-1} + u_i^k + u_i^{k+1}\big) \right], \quad 1 \le i \le m-1, \quad 1 \le k \le n-1, \tag{3.8}$$

$$u_i^0 = \varphi(x_i), \qquad 0 \le i \le m, \tag{3.9}$$

$$u_0^k = \alpha(t_k), \quad u_m^k = \beta(t_k), \qquad 1 \le k \le n. \tag{3.10}$$

3.2. Conservative law of the difference scheme

The next result illustrates the conservative property of this difference scheme.

Theorem 3.1. *Suppose* $\{u_i^k \,|\, 0 \le i \le m, 0 \le k \le n\}$ *is the solution of the difference scheme* (3.7)-(3.10) *and* $\alpha(t) \equiv 0, \beta(t) \equiv 0$. *Denote*

$$\begin{aligned} E^k = {} & \frac{1}{2}\big(\|u^{k+1}\|^2 + \|u^k\|^2\big) + 2\tau \left(\frac{1}{2} |u^{\frac{1}{2}}|_1^2 + \sum_{l=1}^{k} |u^{\bar{l}}|_1^2 \right) \\ & + 2\lambda\tau \Bigg\{ \frac{1}{2}\big[(u^0 u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2\big] \\ & \qquad + \sum_{l=1}^{k} \left[\left(\frac{1}{3}(u^{l-1} + u^l + u^{l+1}) u^l, u^{\bar{l}} \right) - \|u^{\bar{l}}\|^2 \right] \Bigg\}, \quad 0 \le k \le n-1, \\ F^k = {} & \frac{1}{2}\big(|u^{k+1}|_1^2 + |u^k|_1^2\big) + \lambda \left\{ \frac{1}{3}\big[\big(u^k, (u^{k+1})^2\big) + \big((u^k)^2, u^{k+1}\big)\big] - \frac{1}{2}\big(\|u^{k+1}\|^2 + \|u^k\|^2\big) \right\} \\ & + 2\tau \left(\frac{1}{2} \|\delta_t u^{\frac{1}{2}}\|^2 + \sum_{l=1}^{k} \|\Delta_t u^l\|^2 \right), \qquad 0 \le k \le n-1. \end{aligned}$$

Then, we have

$$E^k = \|u^0\|^2, \quad 0 \le k \le n-1, \tag{3.11}$$
$$F^k = \hat{F}^0, \qquad 0 \le k \le n-1, \tag{3.12}$$

where

$$\hat{F}^0 = |u^0|_1^2 + \lambda\left[\frac{4}{3}\big((u^0)^2, u^1\big) - \frac{2}{3}\big(u^0, (u^1)^2\big) - \|u^0\|^2\right].$$

Proof. (I) Taking the inner product of (3.7) with $u^{1/2}$ gives

$$\big(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}\big) - \big(\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}\big) = \lambda\big[\|u^{\frac{1}{2}}\|^2 - (u^0 u^1, u^{\frac{1}{2}})\big].$$

Noticing

$$\big(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}\big) = \frac{1}{2\tau}\big(\|u^1\|^2 - \|u^0\|^2\big), \quad -\big(\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}\big) = |u^{\frac{1}{2}}|_1^2,$$

we have

$$\frac{1}{2\tau}\big(\|u^1\|^2 - \|u^0\|^2\big) + |u^{\frac{1}{2}}|_1^2 + \lambda\big[(u^0 u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2\big] = 0,$$

which can be rewritten as

$$\frac{1}{2}\big(\|u^1\|^2 + \|u^0\|^2\big) + \tau|u^{\frac{1}{2}}|_1^2 + \lambda\tau\big[(u^0 u^1, u^{\frac{1}{2}}) - \|u^{\frac{1}{2}}\|^2\big] = \|u^0\|^2,$$

that is

$$E^0 = \|u^0\|^2. \tag{3.13}$$

Taking the inner product of (3.8) with $u^{\bar{k}}$ yields

$$\begin{aligned}&\big(\Delta_t u^k, u^{\bar{k}}\big) - \big(\delta_x^2 u^{\bar{k}}, u^{\bar{k}}\big)\\ =&\ \lambda\left[\|u^{\bar{k}}\|^2 - \left(\frac{1}{3}(u^{k-1} + u^k + u^{k+1})u^k, u^{\bar{k}}\right)\right], \quad 1 \le k \le n-1.\end{aligned}$$

Noticing

$$\big(\Delta_t u^k, u^{\bar{k}}\big) = \frac{1}{4\tau}\big(\|u^{k+1}\|^2 - \|u^{k-1}\|^2\big), \quad -\big(\delta_x^2 u^{\bar{k}}, u^{\bar{k}}\big) = |u^{\bar{k}}|_1^2,$$

we have

$$\begin{aligned}&\frac{1}{2\tau}\left(\frac{1}{2}\big(\|u^{k+1}\|^2 + \|u^k\|^2\big) - \frac{1}{2}\big(\|u^k\|^2 + \|u^{k-1}\|^2\big)\right) + |u^{\bar{k}}|_1^2\\ &\quad + \lambda\left[\left(\frac{1}{3}(u^{k-1} + u^k + u^{k+1})u^k, u^{\bar{k}}\right) - \|u^{\bar{k}}\|^2\right] = 0, \quad 1 \le k \le n-1.\end{aligned}$$

Replacing k by l in the equality above and summing over l from 1 to k will arrive at

$$\begin{aligned}&\frac{1}{2}\big(\|u^{k+1}\|^2+\|u^k\|^2\big)+2\tau\sum_{l=1}^{k}|u^{\bar{l}}|_1^2\\&\quad+2\lambda\tau\sum_{l=1}^{k}\left[\left(\frac{1}{3}(u^{l-1}+u^l+u^{l+1})u^l,u^{\bar{l}}\right)-\|u^{\bar{l}}\|^2\right]\\&=\frac{1}{2}\big(\|u^1\|^2+\|u^0\|^2\big),\quad 1\le k\le n-1.\end{aligned}$$

Adding $\tau|u^{1/2}|_1^2+\lambda\tau[(u^0u^1,u^{1/2})-\|u^{1/2}\|^2]$ to both hand sides of the equality above yields

$$E^k=E^0,\quad 1\le k\le n-1. \tag{3.14}$$

Then (3.11) is followed from (3.13) and (3.14).

(II) Taking the inner product of (3.7) with $\delta_t u^{1/2}$ gives

$$\big\|\delta_t u^{\frac{1}{2}}\big\|^2-\big(\delta_x^2u^{\frac{1}{2}},\delta_tu^{\frac{1}{2}}\big)=\lambda\big[\big(u^{\frac{1}{2}},\delta_tu^{\frac{1}{2}}\big)-\big(u^0u^1,\delta_tu^{\frac{1}{2}}\big)\big].$$

Noticing

$$\begin{aligned}-\big(\delta_x^2u^{\frac{1}{2}},\delta_tu^{\frac{1}{2}}\big)&=\frac{1}{2\tau}\big(|u^1|_1^2-|u^0|_1^2\big),\\\big(u^{\frac{1}{2}},\delta_tu^{\frac{1}{2}}\big)&=\frac{1}{2\tau}\big(\|u^1\|^2-\|u^0\|^2\big),\\\big(u^0u^1,\delta_tu^{\frac{1}{2}}\big)&=\frac{1}{\tau}\big[\big(u^0,(u^1)^2\big)-\big((u^0)^2,u^1\big)\big],\end{aligned}$$

we have

$$\begin{aligned}&\big\|\delta_tu^{\frac{1}{2}}\big\|^2+\frac{1}{2\tau}\big(|u^1|_1^2-|u^0|_1^2\big)\\&\quad+\lambda\left\{\frac{1}{\tau}\big[\big(u^0,(u^1)^2\big)-\big((u^0)^2,u^1\big)\big]-\frac{1}{2\tau}\big(\|u^1\|^2-\|u^0\|^2\big)\right\}=0,\end{aligned}$$

which can be rewritten as

$$F^0=|u^0|_1^2+\lambda\left[\frac{4}{3}\big((u^0)^2,u^1\big)-\frac{2}{3}\big(u^0,(u^1)^2\big)-\|u^0\|^2\right]\equiv\hat{F}^0. \tag{3.15}$$

Taking the inner product of (3.8) with Δ_tu^k yields

$$\begin{aligned}&\big\|\Delta_tu^k\big\|^2-\big(\delta_x^2u^{\bar{k}},\Delta_tu^k\big)\\&=\lambda\left[\big(u^{\bar{k}},\Delta_tu^k\big)-\frac{1}{3}\big((u^{k-1}+u^k+u^{k+1})u^k,\Delta_tu^k\big)\right],\quad 1\le k\le n-1.\end{aligned}$$

Noticing

$$-\big(\delta_x^2u^{\bar{k}},\Delta_tu^k\big)=\frac{1}{4\tau}\big(|u^{k+1}|_1^2-|u^{k-1}|_1^2\big),$$

$$\big(u^{\bar{k}},\Delta_t u^k\big)=\frac{1}{4\tau}\big(\|u^{k+1}\|^2-\|u^{k-1}\|^2\big),$$

and

$$\begin{aligned}&\frac{1}{3}\big((u^{k-1}+u^k+u^{k+1})u^k,\Delta_t u^k\big)\\=&\frac{1}{6\tau}\big[\big((u^{k+1}+u^{k-1})u^k,u^{k+1}-u^{k-1}\big)+\big((u^k)^2,u^{k+1}-u^{k-1}\big)\big]\\=&\frac{1}{6\tau}\big[\big(u^k,(u^{k+1})^2-(u^{k-1})^2\big)+\big((u^k)^2,u^{k+1}-u^{k-1}\big)\big]\\=&\frac{1}{6\tau}\big[\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)-\big(u^{k-1},(u^k)^2\big)-\big((u^{k-1})^2,u^k\big)\big],\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2\tau}\bigg\{\bigg[\frac{|u^{k+1}|_1^2+|u^k|_1^2}{2}+\lambda\bigg(\frac{\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)}{3}-\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}\bigg)\bigg]\\&\qquad-\bigg[\frac{|u^k|_1^2+|u^{k-1}|_1^2}{2}+\lambda\bigg(\frac{\big(u^{k-1},(u^k)^2\big)+\big((u^{k-1})^2,u^k\big)}{3}-\frac{\|u^k\|^2+\|u^{k-1}\|^2}{2}\bigg)\bigg]\bigg\}\\&\quad+\|\Delta_t u^k\|^2=0,\quad 1\le k\le n-1.\end{aligned}$$

Replacing k by l in the equality above and summing over l from 1 to k, we arrive at

$$\begin{aligned}&\frac{1}{2}\big(|u^{k+1}|_1^2+|u^k|_1^2\big)+\lambda\bigg\{\frac{1}{3}\big[\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)\big]\\&\qquad\qquad-\frac{1}{2}(\|u^{k+1}\|^2+\|u^k\|^2)\bigg\}+2\tau\sum_{l=1}^{k}\|\Delta_t u^l\|^2\\=&\frac{1}{2}\big(|u^1|_1^2+|u^0|_1^2\big)+\lambda\bigg\{\frac{1}{3}\big[\big(u^0,(u^1)^2\big)+\big((u^0)^2,u^1\big)\big]\\&\qquad\qquad-\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)\bigg\},\quad 1\le k\le n-1.\end{aligned}$$

Adding $\tau\|\delta_t u^{1/2}\|^2$ to both hand sides of the equality above gives

$$F^k=F^0,\quad 1\le k\le n-1. \tag{3.16}$$

Combining (3.15) and (3.16), we arrive at (3.12). □

Remark 3.1. Let

$$\begin{aligned}\tilde{E}(u^{k+1},u^k)=\frac{1}{2}\bigg[&\frac{|u^{k+1}|_1^2+|u^k|_1^2}{2}+\lambda\bigg(\frac{\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)}{3}\\&-\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}\bigg)\bigg],\quad 0\le k\le n-1.\end{aligned}$$

Then it is a discrete counterpart of the free energy (2.2). Then (3.12) implies

$$\tilde{E}(u^{k+1},u^k) \le \tilde{E}(u^k,u^{k-1}) \le \frac{1}{2}\hat{F}^0, \quad 1 \le k \le n-1,$$

i.e. the scheme (3.7)-(3.10) preserves the energy dissipation law with respect to the discrete energy above.

3.3. Solvability and convergence of the difference solution

Theorem 3.2. *Let* $\{U_i^k \,|\, 0 \le i \le m, 0 \le k \le n\}$ *and* $\{u_i^k \,|\, 0 \le i \le m, 0 \le k \le n\}$ *be solutions of the problem* (1.1)-(1.3) *and the difference scheme* (3.7)-(3.10)*, respectively. Denote*

$$\begin{aligned} &e_i^k = U_i^k - u_i^k, \quad 0 \le i \le m, \quad 0 \le k \le n, \\ &c_3 = \left(2TLc_1^2 + \frac{3}{(c_0+1)^2\lambda^2 L}c_2^2\right)^{\frac{1}{2}} \mathrm{e}^{(c_0+1)^2\lambda^2 L^2 T}. \end{aligned}$$

Then when

$$\left(\frac{1}{2}+c_0\right)\lambda\tau \le 1, \quad \left[1+\frac{2}{3}(c_0+1)^2\right]\lambda^2 L^2 \tau \le 1, \quad c_3\frac{\sqrt{L}}{2}(\tau^2+h^2) \le 1$$

it holds that:

(I) The difference scheme (3.7)-(3.10) *is uniquely solvable.*

(II)

$$|e^k|_1 \le c_3(\tau^2+h^2), \quad 0 \le k \le n. \tag{3.17}$$

Proof. Subtracting (3.7)-(3.10) from (3.1), (3.3), (3.5)-(3.6), respectively, the error system reads

$$\delta_t e_i^{\frac{1}{2}} - \delta_x^2 e_i^{\frac{1}{2}} = \lambda\Big[e_i^{\frac{1}{2}} - \big(U_i^0U_i^1 - u_i^0u_i^1\big)\Big] + (R_1)_i^0, \quad 1 \le i \le m-1, \tag{3.18}$$

$$\begin{aligned} \Delta_t e_i^k - \delta_x^2 e_i^{\bar{k}} = {}& \lambda\left[e_i^{\bar{k}} - \frac{1}{3}\big(U_i^{k-1} + U_i^k + U_i^{k+1}\big)U_i^k + \frac{1}{3}\big(u_i^{k-1} + u_i^k + u_i^{k+1}\big)u_i^k\right] \\ &+ (R_1)_i^k, \quad 1 \le i \le m-1, \quad 1 \le k \le n-1, \end{aligned} \tag{3.19}$$

$$e_i^0 = 0, \qquad 0 \le i \le m, \tag{3.20}$$

$$e_0^k = 0, \quad e_m^k = 0, \qquad 1 \le k \le n. \tag{3.21}$$

The value of u^0 is uniquely determined by (3.9) and the truth of (3.17) for $k=0$ is obvious in view of (3.20).

(A1) Proof for the unique solvability of u^1.

From (3.7) and (3.10), the system in u^1 can be obtained. Consider its homogeneous one

$$\frac{1}{\tau}u_i^1 - \frac{1}{2}\delta_x^2 u^1 = \lambda\left(\frac{1}{2}u_i^1 - u_i^0 u_i^1\right), \quad 1 \le i \le m-1, \tag{3.22}$$

$$u_0^1 = 0, \quad u_m^1 = 0. \tag{3.23}$$

Taking the inner product of (3.22) with u^1 gives

$$\frac{1}{\tau}\|u^1\|^2 - \frac{1}{2}(\delta_x^2 u^1, u^1) = \frac{1}{2}\lambda\|u^1\|^2 - \lambda(u^0 u^1, u^1).$$

Noticing $-(\delta_x^2 u^1, u^1) = |u^1|_1^2$ and $\|u^0\|_\infty \le c_0$, we have

$$\frac{1}{\tau}\|u^1\|^2 + \frac{1}{2}|u^1|_1^2 \le \left(\frac{1}{2} + \|u^0\|_\infty\right)\lambda\|u^1\|^2 \le \left(\frac{1}{2} + c_0\right)\lambda\|u^1\|^2.$$

When $(1/2 + c_0)\lambda\tau \le 1$, it follows $|u^1|_1 = 0$. Thus, (3.7) and (3.10) uniquely determine the value of u^1.

(B1) Proof for (3.17) with $k = 1$.

Taking the inner product of (3.18) with $\delta_t e^{1/2}$ gives

$$\begin{aligned}
&\left\|\delta_t e^{\frac{1}{2}}\right\|^2 - \left(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) \\
&= \lambda\big(e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\big) - \lambda\big(u^0 e^1 + e^0 U^1, \delta_t e^{\frac{1}{2}}\big) + \big((R_1)^0, \delta_t e^{\frac{1}{2}}\big) \\
&\le \lambda\|e^{\frac{1}{2}}\| \cdot \|\delta_t e^{\frac{1}{2}}\| + \lambda\|u^0\|_\infty \cdot \|e^1\| \cdot \|\delta_t e^{\frac{1}{2}}\| + \|(R_1)^0\| \cdot \|\delta_t e^{\frac{1}{2}}\|.
\end{aligned}$$

Noticing

$$\left\|\delta_t e^{\frac{1}{2}}\right\|^2 - \left(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) = \left\|\delta_t e^{\frac{1}{2}}\right\|^2 + \frac{1}{2\tau}|e^1|_1^2 \ge 2\frac{1}{\sqrt{2\tau}}\left\|\delta_t e^{\frac{1}{2}}\right\| \cdot |e^1|_1,$$

we have

$$\begin{aligned}
\frac{2}{\sqrt{2\tau}}|e^1|_1 &\le \lambda\|e^{\frac{1}{2}}\| + \lambda\|u^0\|_\infty \cdot \|e^1\| + \|(R_1)^0\| \\
&\le \left(\frac{1}{2} + c_0\right)\lambda\|e^1\| + \|(R_1)^0\| \\
&\le \left(\frac{1}{2} + c_0\right)\frac{L}{\sqrt{6}}\lambda|e^1|_1 + c_1\sqrt{L}(\tau^2 + h^2).
\end{aligned}$$

When $(1/2 + c_0)(L/\sqrt{6})\sqrt{2\tau}\lambda \le 1$, i.e., $(1/2 + c_0)^2\lambda^2 L^2\tau/3 \le 1$, it follows:

$$|e^1|_1 \le \sqrt{2\tau}c_1\sqrt{L}(\tau^2 + h^2) \le \sqrt{2TL}c_1(\tau^2 + h^2), \tag{3.24}$$

which implies that (3.17) holds for $k = 1$.

Now assume that the values of $u^0, u^1, \cdots, u^l$ $(l \geq 1)$ have been determined and (3.17) is true for $0 \leq k \leq l$, that is

$$|e^k|_1 \leq c_3(\tau^2 + h^2), \quad 0 \leq k \leq l.$$

Then by Lemma 2.1, we have

$$\|e^k\|_\infty \leq \frac{\sqrt{L}}{2}|e^k|_1 \leq \frac{\sqrt{L}}{2}c_3(\tau^2 + h^2) \leq 1, \quad 0 \leq k \leq l, \tag{3.25}$$

$$\|u^k\|_\infty \leq \|U^k\|_\infty + \|e^k\|_\infty \leq c_0 + 1, \qquad 0 \leq k \leq l. \tag{3.26}$$

(A2) Proof for the unique solvability of u^{l+1}.

From (3.8) ($k = l$) and (3.10) ($k = l+1$), the linear system in u^{l+1} can be obtained. Consider its homogeneous one

$$\frac{1}{2\tau}u_i^{l+1} - \frac{1}{2}\delta_x^2 u_i^{l+1} = \lambda\left(\frac{1}{2}u_i^{l+1} - \frac{1}{3}u_i^l u_i^{l+1}\right), \quad 1 \leq i \leq m-1, \tag{3.27}$$

$$u_0^{l+1} = 0, \quad u_m^{l+1} = 0. \tag{3.28}$$

Taking the inner product of (3.27) with u^{l+1} gives

$$\begin{aligned}\frac{1}{2\tau}\|u^{l+1}\|^2 + \frac{1}{2}|u^{l+1}|_1^2 &\leq \frac{1}{2}\lambda\|u^{l+1}\|^2 + \frac{\lambda}{3}\|u^l\|_\infty\|u^{l+1}\|^2 \\ &\leq \lambda\left[\frac{1}{2} + \frac{1}{3}(c_0+1)\right]\|u^{l+1}\|^2.\end{aligned}$$

Noticing

$$\frac{1}{2\tau}\|u^{l+1}\|^2 + \frac{1}{2}|u^{l+1}|_1^2 \geq \frac{1}{\sqrt{\tau}}\|u^{l+1}\| \cdot |u^{l+1}|_1,$$

we have

$$\begin{aligned}|u^{l+1}|_1 &\leq \lambda\left[\frac{1}{2} + \frac{1}{3}(c_0+1)\right]\sqrt{\tau}\|u^{l+1}\| \\ &\leq \lambda\left[\frac{1}{2} + \frac{1}{3}(c_0+1)\right]\sqrt{\tau}\frac{L}{\sqrt{6}}|u^{l+1}|_1.\end{aligned}$$

When

$$\frac{1}{6}\left[\frac{1}{2} + \frac{1}{3}(c_0+1)\right]^2 \lambda^2 L^2 \tau < 1,$$

it follows $|u^{l+1}|_1 = 0$. Hence, (3.8) and (3.10) determine u^{l+1} uniquely.

(B2) Proof for (3.17) with $k=l+1$.
Taking the inner product of (3.19) with $\Delta_t e^k$ yields

$$\begin{aligned}
&\|\Delta_t e^k\|^2+\frac{1}{4\tau}\big(|e^{k+1}|_1^2-|e^{k-1}|_1^2\big)\\
=&\lambda(e^{\bar{k}},\Delta_t e^k)-\frac{1}{3}\lambda h\sum_{i=1}^{m-1}\Big[u_i^k(e_i^{k-1}+e_i^k+e_i^{k+1})+e_i^k\big(U_i^{k-1}+U_i^k+U_i^{k+1}\big)\Big]\Delta_t e_i^k\\
&+\big((R_1)^k,\Delta_t e^k\big)\\
\le&\lambda\|e^{\bar{k}}\|\cdot\|\Delta_t e^k\|+\frac{1}{3}\lambda\|u^k\|_\infty\cdot\|e^{k-1}+e^k+e^{k+1}\|\cdot\|\Delta_t e^k\|\\
&+\frac{1}{3}\lambda\|e^k\|\cdot\|U^{k-1}+U^k+U^{k+1}\|_\infty\cdot\|\Delta_t e^k\|+\|(R_1)^k\|\cdot\|\Delta_t e^k\|\\
\le&\lambda\|e^{\bar{k}}\|\cdot\|\Delta_t e^k\|+\frac{1}{3}\lambda(1+c_0)\|e^{k-1}+e^k+e^{k+1}\|\cdot\|\Delta_t e^k\|\\
&+\frac{1}{3}\lambda(3c_0)\|e^k\|\cdot\|\Delta_t e^k\|+\|(R_1)^k\|\cdot\|\Delta_t e^k\|\\
\le&\left(\frac{1}{4}\|\Delta_t e^k\|^2+\lambda^2\|e^{\bar{k}}\|^2\right)+\left(\frac{1}{4}\|\Delta_t e^k\|^2+\frac{1}{9}\lambda^2(c_0+1)^2\|e^{k-1}+e^k+e^{k+1}\|^2\right)\\
&+\left(\frac{1}{4}\|\Delta_t e^k\|^2+\lambda^2c_0^2\|e^k\|^2\right)+\frac{1}{4}\|\Delta_t e^k\|^2+\|(R_1)^k\|^2,\quad 1\le k\le l.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{4\tau}\big(|e^{k+1}|_1^2-|e^{k-1}|_1^2\big)\\
\le&\frac{\lambda^2}{2}\big(\|e^{k+1}\|^2+\|e^{k-1}\|^2\big)+\frac{\lambda^2}{3}(c_0+1)^2\big(\|e^{k-1}\|^2+\|e^k\|^2+\|e^{k+1}\|^2\big)\\
&+\lambda^2c_0^2\|e^k\|^2+c_2^2L(\tau^2+h^2)^2\\
=&\left[\frac{\lambda^2}{2}+\frac{\lambda^2}{3}(c_0+1)^2\right]\|e^{k+1}\|^2+\left[\frac{\lambda^2}{2}+\frac{\lambda^2}{3}(c_0+1)^2\right]\|e^{k-1}\|^2\\
&+\left[\frac{\lambda^2}{3}(c_0+1)^2+\lambda^2c_0^2\right]\|e^k\|^2+c_2^2L(\tau^2+h^2)^2\\
\le&\left[\frac{1}{2}+\frac{1}{3}(c_0+1)^2\right]\lambda^2\frac{L^2}{6}|e^{k+1}|_1^2+\left[\frac{1}{2}+\frac{1}{3}(c_0+1)^2\right]\lambda^2\frac{L^2}{6}|e^{k-1}|_1^2\\
&+\left[\frac{1}{3}(c_0+1)^2+c_0^2\right]\lambda^2\frac{L^2}{6}|e^k|_1^2+c_2^2L(\tau^2+h^2)^2,\quad 1\le k\le l,
\end{aligned}$$

that is

$$\begin{aligned}
&\left\{1-\left[1+\frac{2}{3}(c_0+1)^2\right]\lambda^2\frac{L^2}{3}\tau\right\}|e^{k+1}|_1^2\\
\le&\left\{1+\left[1+\frac{2}{3}(c_0+1)^2\right]\lambda^2\frac{L^2}{3}\tau\right\}|e^{k-1}|_1^2+\left[\frac{2}{3}(c_0+1)^2+2c_0^2\right]\lambda^2\frac{L^2}{3}\tau|e^k|_1^2\\
&+4c_2^2L\tau(\tau^2+h^2)^2,\quad 1\le k\le l.
\end{aligned}$$

When

$$\left[1+\frac{2}{3}(c_0+1)^2\right]\lambda^2\frac{L^2}{3}\tau\le\frac{1}{3},$$

it follows:

$$\begin{aligned}|e^{k+1}|_1^2\le&\left\{1+\left[1+\frac{2}{3}(c_0+1)^2\right]\lambda^2L^2\tau\right\}|e^{k-1}|_1^2\\&+\left[\frac{1}{3}(c_0+1)^2+c_0^2\right]\lambda^2L^2\tau|e^k|_1^2+6c_2^2L\tau(\tau^2+h^2)^2\\\le&\left[1+2(c_0+1)^2\lambda^2L^2\tau\right]\max\left\{|e^k|_1^2,|e^{k-1}|_1^2\right\}\\&+6c_2^2L\tau(\tau^2+h^2)^2,\quad 1\le k\le l,\end{aligned}$$

so that

$$\begin{aligned}\max\left\{|e^{k+1}|_1^2,|e^k|_1^2\right\}\le&\left[1+2(c_0+1)^2\lambda^2L^2\tau\right]\max\left\{|e^k|_1^2,|e^{k-1}|_1^2\right\}\\&+6c_2^2L\tau(\tau^2+h^2)^2,\quad 1\le k\le l.\end{aligned}$$

Applying the Gronwall inequality in Lemma 2.3 and noticing (3.24), we get

$$\begin{aligned}\max\left\{|e^{l+1}|_1^2,|e^l|_1^2\right\}\le&\,\mathrm{e}^{2(c_0+1)^2\lambda^2L^2l\tau}\left[\max\left\{|e^1|_1^2,|e^0|_1^2\right\}+\frac{3c_2^2(\tau^2+h^2)^2}{(c_0+1)^2\lambda^2L}\right]\\\le&\,\mathrm{e}^{2(c_0+1)^2\lambda^2L^2T}\left[2TLc_1^2+\frac{3c_2^2}{(c_0+1)^2\lambda^2L}\right](\tau^2+h^2)^2.\end{aligned}$$

Then

$$|e^{l+1}|_1\le\mathrm{e}^{(c_0+1)^2\lambda^2L^2T}\left(2TLc_1^2+\frac{3}{(c_0+1)^2\lambda^2L}c_2^2\right)^{\frac{1}{2}}(\tau^2+h^2)=c_3(\tau^2+h^2),$$

which implies that (3.17) is also true for $k=l+1$. By induction, the theorem is proved. □

In view of Lemma 2.1(b), the difference scheme (3.7)-(3.10) is unconditionally convergent in the maximum norm and the convergence order is also $\mathcal{O}(\tau^2+h^2)$.

Remark 3.2. The Fisher equation (1.1) satisfies the maximum bound principle (MBP) [5], i.e., the solution has the range in the set $[0,1]$ at any time if the initial and boundary values have the same property, so the numerical methods are always expected to preserve the MBP. For the proposed difference scheme (3.7)-(3.10), the unconditional convergence in the maximum norm has been proved rigorously, which means that if the mesh step sizes h and τ are sufficiently small, so is the difference between the numerical solution u_i^k and the exact one U_i^k. For any small positive ε, it follows:

$$\max_{\substack{0\le i\le m\\0\le k\le n}}\left|U_i^k(h,\tau)-u_i^k(h,\tau)\right|\le c(\tau^2+h^2)=\varepsilon,$$

then $u_i^k(h,\tau) \in [-\varepsilon, 1+\varepsilon]$ if $U_i^k(h,\tau) \in [0,1]$, so that

$$\lim_{\substack{h\to 0\\ \tau\to 0}} u_i^k(h,\tau) \in [0,1].$$

That is, the difference scheme (3.7)-(3.10) satisfies the MBP when the mesh step sizes are sufficiently small.

4. A three-level linearized compact difference scheme

This part will concern on an unconditionally convergent and conservative compact difference scheme for solving (1.1)-(1.3) with the convergence order $\mathcal{O}(\tau^2+h^4)$.

4.1. Derivation of the compact difference scheme

Considering Eq. (1.1) at point $(x_i, t_{1/2})$, we have

$$u_t(x_i,t_{\frac12}) - u_{xx}(x_i,t_{\frac12}) = \lambda\Big[u(x_i,t_{\frac12}) - u^2(x_i,t_{\frac12})\Big], \quad 0\le i\le m.$$

By Lemma 2.2, we have

$$\delta_t U_i^{\frac12} - \frac12[u_{xx}(x_i,t_1)+u_{xx}(x_i,t_0)] = \lambda\Big(U_i^{\frac12} - U_i^0U_i^1\Big) + \mathcal{O}(\tau^2), \quad 0\le i\le m.$$

Performing the operator $\mathcal{A}$ on both hand sides and noticing Lemma 2.2(d), we obtain

$$\mathcal{A}\delta_t U_i^{\frac12} - \delta_x^2 U_i^{\frac12} = \lambda\mathcal{A}\Big(U_i^{\frac12} - U_i^0U_i^1\Big) + (R_2)_i^0, \quad 1\le i\le m-1, \tag{4.1}$$

where there is a constant c_4 such that

$$\big|(R_2)_i^0\big| \le c_4(\tau^2+h^4), \quad 1\le i\le m-1. \tag{4.2}$$

Considering Eq. (1.1) at point (x_i,t_k), we have

$$u_t(x_i,t_k) - u_{xx}(x_i,t_k) = \lambda\big[u(x_i,t_k) - u^2(x_i,t_k)\big], \quad 0\le i\le m, \quad 1\le k\le n-1.$$

By Lemma 2.2, we have

$$\begin{aligned}&\Delta_t U_i^k - \frac12[u_{xx}(x_i,t_{k+1}) + u_{xx}(x_i,t_{k-1})]\\ &= \lambda\Big[U_i^{\bar k} - \frac13\big(U_i^{k-1}+U_i^k+U_i^{k+1}\big)U_i^k\Big] + \mathcal{O}(\tau^2), \quad 0\le i\le m, \quad 1\le k\le n-1.\end{aligned}$$

Performing the operator $\mathcal{A}$ on both hand sides and noticing Lemma 2.2(d), we obtain

$$\begin{aligned}\mathcal{A}\Delta_t U_i^k - \delta_x^2 U_i^{\bar k} = {}&\lambda\mathcal{A}\Big[U_i^{\bar k} - \frac13\big(U_i^{k-1}+U_i^k+U_i^{k+1}\big)U_i^k\Big]\\ &+ (R_2)_i^k, \quad 1\le i\le m-1, \quad 1\le k\le n-1,\end{aligned} \tag{4.3}$$

where there is a constant c_5 such that

$$\big|(R_2)_i^k\big| \le c_5(\tau^2+h^4), \quad 1\le i\le m-1, \quad 1\le k\le n-1. \tag{4.4}$$

Noticing the initial-boundary value conditions (1.2)-(1.3), we have

$$U_i^0=\varphi(x_i), \qquad 0\le i\le m, \tag{4.5}$$

$$U_0^k=\alpha(t_k), \quad U_m^k=\beta(t_k), \quad 1\le k\le n. \tag{4.6}$$

Neglecting the small term $(R_2)_i^k$ in (4.1) and (4.3), and replacing the exact solution U_i^k by its numerical one u_i^k, the following compact difference scheme can be produced:

$$\mathcal{A}\delta_t u_i^{\frac12}-\delta_x^2 u_i^{\frac12}=\lambda\mathcal{A}\big(u_i^{\frac12}-u_i^0u_i^1\big), \quad 1\le i\le m-1, \tag{4.7}$$

$$\mathcal{A}\Delta_t u_i^k-\delta_x^2 u_i^{\bar k}=\lambda\mathcal{A}\Big[u_i^{\bar k}-\frac13 u_i^k\big(u_i^{k-1}+u_i^k+u_i^{k+1}\big)\Big], \tag{4.8}$$
$$1\le i\le m-1, \quad 1\le k\le n-1,$$

$$u_i^0=\varphi(x_i), \qquad 0\le i\le m, \tag{4.9}$$

$$u_0^k=\alpha(t_k), \quad u_m^k=\beta(t_k), \qquad 1\le k\le n. \tag{4.10}$$

4.2. Conservative law of the compact difference scheme

Theorem 4.1. *Suppose* $\{u_i^k\,|\,0\le i\le m, 0\le k\le n\}$ *is the solution of the difference scheme* (4.7)-(4.10) *with* $\alpha(t)\equiv 0, \beta(t)\equiv 0$. *Denote*

$$\begin{aligned}
P^k=&\frac12\big(\|u^{k+1}\|^2+\|u^k\|^2\big)+2\tau\Big(\frac12|u^{\frac12}|_{1,\mathcal{A}}^2+\sum_{l=1}^{k}|u^{\bar l}|_{1,\mathcal{A}}^2\Big)\\
&+2\lambda\tau\Bigg\{\frac12\big[(u^0u^1,u^{\frac12})-\|u^{\frac12}\|^2\big]\\
&\qquad+\sum_{l=1}^{k}\bigg[\Big(\frac13(u^{l-1}+u^l+u^{l+1})u^l,u^{\bar l}\Big)-\|u^{\bar l}\|^2\bigg]\Bigg\}, \quad 0\le k\le n-1,\\
Q^k=&\frac12\big(|u^{k+1}|_{1,\mathcal{A}}^2+|u^k|_{1,\mathcal{A}}^2\big)\\
&+\lambda\Big\{\frac13\big[\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)\big]-\frac12\big(\|u^{k+1}\|^2+\|u^k\|^2\big)\Big\}\\
&+2\tau\Big(\frac12\big\|\delta_t u^{\frac12}\big\|^2+\sum_{l=1}^{k}\big\|\Delta_t u^l\big\|^2\Big), \qquad 0\le k\le n-1.
\end{aligned}$$

Then, we have

$$P^k=\|u^0\|^2, \quad 0\le k\le n-1, \tag{4.11}$$

$$Q^k=\hat{Q}^0, \qquad 0\le k\le n-1, \tag{4.12}$$

where

$$\hat{Q}^0=|u^0|_{1,\mathcal{A}}^2+\lambda\Big[\frac{4}{3}\big((u^0)^2,u^1\big)-\frac{2}{3}\big(u^0,(u^1)^2\big)-\|u^0\|^2\Big].$$

Proof. Applying the operator $\mathcal{A}^{-1}$ to both hand sides of (4.7) and (4.8), one can obtain

$$\begin{aligned}&\delta_t u_i^{\frac{1}{2}}-\mathcal{A}^{-1}\delta_x^2 u_i^{\frac{1}{2}}=\lambda\big(u_i^{\frac{1}{2}}-u_i^0u_i^1\big), && 1\le i\le m-1,\\ &\Delta_t u_i^k-\mathcal{A}^{-1}\delta_x^2 u_i^{\bar{k}}=\lambda\Big[u_i^{\bar{k}}-\frac{1}{3}u_i^k\big(u_i^{k-1}+u_i^k+u_i^{k+1}\big)\Big], && 1\le i\le m-1,\quad 1\le k\le n-1.\end{aligned}$$

Noticing

$$\begin{aligned}&-\big(\mathcal{A}^{-1}\delta_x^2u^{\frac{1}{2}},u^{\frac{1}{2}}\big)=|u^{\frac{1}{2}}|_{1,\mathcal{A}}^2,\\ &-\big(\mathcal{A}^{-1}\delta_x^2u^{\frac{1}{2}},\delta_tu^{\frac{1}{2}}\big)=\frac{1}{2\tau}\big(|u^1|_{1,\mathcal{A}}^2-|u^0|_{1,\mathcal{A}}^2\big),\\ &-\big(\mathcal{A}^{-1}\delta_x^2u^{\bar{k}},u^{\bar{k}}\big)=|u^{\bar{k}}|_{1,\mathcal{A}}^2,\\ &-\big(\mathcal{A}^{-1}\delta_x^2u^{\bar{k}},\Delta_tu^k\big)=\frac{1}{4\tau}\big(|u^{k+1}|_{1,\mathcal{A}}^2-|u^{k-1}|_{1,\mathcal{A}}^2\big),\end{aligned}$$

similar to the proof of Theorem 3.1, one can easily get this theorem. The details are omitted for brevity. ☐

It is worth noting that the norm $|\cdot|_{1,\mathcal{A}}$ is equivalent to the usual H^1 norm $|\cdot|_1$ in view of Lemma 2.1(b).

Remark 4.1. Let

$$\begin{aligned}\hat{E}(u^{k+1},u^k)=\frac{1}{2}\Bigg[&\frac{|u^{k+1}|_{1,\mathcal{A}}^2+|u^k|_{1,\mathcal{A}}^2}{2}+\lambda\Bigg(\frac{\big(u^k,(u^{k+1})^2\big)+\big((u^k)^2,u^{k+1}\big)}{3}\\ &-\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}\Bigg)\Bigg],\quad 0\le k\le n-1\end{aligned}$$

be another discrete counterpart of the free energy (2.2). Then (4.12) implies

$$\hat{E}(u^{k+1},u^k)\le\hat{E}(u^k,u^{k-1})\le\frac{1}{2}\hat{Q}^0,\quad 1\le k\le n-1,$$

i.e. the scheme (4.7)-(4.10) also preserves the energy dissipation law with respect to this discrete energy.

4.3. Solvability and convergence of the compact difference solution

Theorem 4.2. *Let* $\{U_i^k\,|\,0\le i\le m, 0\le k\le n\}$ *and* $\{u_i^k\,|\,0\le i\le m, 0\le k\le n\}$ *be solutions of the problem* (1.1)-(1.3) *and the compact difference scheme* (4.7)-(4.10), *respectively. Denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \le i \le m, \quad 0 \le k \le n,$$
$$c_6 = \left(3TLc_4^2 + \frac{3c_5^2}{(c_0+1)^2\lambda^2 L}\right)^{\frac{1}{2}} \mathrm{e}^{\frac{3}{2}(c_0+1)^2\lambda^2 L^2 T}.$$

Then when

$$\frac{3}{4}(1+2c_0)\lambda\tau \le 1, \quad \big[3+2(c_0+1)^2\big]\lambda^2 L^2\tau \le 2, \quad \frac{\sqrt{L}}{2}c_6(\tau^2+h^4) \le 1$$

it holds that:

(I) The difference scheme (4.7)-(4.10) *is uniquely solvable.*

(II)

$$|e^k|_1 \le c_6(\tau^2+h^4), \quad 0 \le k \le n. \tag{4.13}$$

Proof. Subtracting (4.7)-(4.10) from (4.1), (4.3), (4.5)-(4.6), respectively, the error system reads

$$\mathcal{A}\delta_t e_i^{\frac{1}{2}} - \delta_x^2 e_i^{\frac{1}{2}} = \lambda\mathcal{A}\Big[e_i^{\frac{1}{2}} - \big(U_i^0 U_i^1 - u_i^0 u_i^1\big)\Big] + (R_2)_i^0, \quad 1 \le i \le m-1, \tag{4.14}$$

$$\begin{aligned}\mathcal{A}\Delta_t e_i^k - \delta_x^2 e_i^{\bar{k}} = {}& \lambda\mathcal{A}\left[e_i^{\bar{k}} - \frac{1}{3}\big(U_i^{k-1} + U_i^k + U_i^{k+1}\big)U_i^k + \frac{1}{3}(u_i^{k-1} + u_i^k + u_i^{k+1})u_i^k\right] \\ & + (R_2)_i^k, \quad 1 \le i \le m-1, \quad 1 \le k \le n-1,\end{aligned} \tag{4.15}$$

$$e_i^0 = 0, \qquad 0 \le i \le m, \tag{4.16}$$

$$e_0^k = 0, \quad e_m^k = 0, \qquad 1 \le k \le n. \tag{4.17}$$

The value of u^0 is uniquely determined by (4.9). And (4.13) holds obviously for $k=0$ in view of (4.16).

(A1) Proof for the unique solvability of u^1.

From (4.7) and (4.10), the system in u^1 is obtained. Consider its homogeneous one

$$\frac{1}{\tau}\mathcal{A}u_i^1 - \frac{1}{2}\delta_x^2 u^1 = \lambda\mathcal{A}\left(\frac{1}{2}u_i^1 - u_i^0 u_i^1\right), \quad 1 \le i \le m-1, \tag{4.18}$$

$$u_0^1 = 0, \quad u_m^1 = 0. \tag{4.19}$$

Taking the inner product of (4.18) with u^1 gives

$$\frac{1}{\tau}(\mathcal{A}u^1, u^1) - \frac{1}{2}\big(\delta_x^2 u^1, u^1\big) = \frac{1}{2}\lambda(\mathcal{A}u^1, u^1) - \lambda\big(\mathcal{A}(u^0 u^1), u^1\big).$$

By Lemma 2.1(b) and the Cauchy-Schwarz inequality, it follows:

$$\begin{aligned}&(\mathcal{A}u^1, u^1) = \left(\left(I + \frac{h^2}{12}\delta_x^2\right)u^1, u^1\right) = \|u^1\|^2 - \frac{h^2}{12}|u^1|_1^2 \ge \frac{2}{3}\|u^1\|^2, \\ &(\mathcal{A}u^1, u^1) \le \|\mathcal{A}u^1\| \cdot \|u^1\| \le \|u^1\|^2, \\ &\big(\mathcal{A}(u^0 u^1), u^1\big) \le \|\mathcal{A}(u^0 u^1)\| \cdot \|u^1\| \le \|u^0 u^1\| \cdot \|u^1\| \le c_0\|u^1\|^2,\end{aligned}$$

then

$$\frac{2}{3\tau}\|u^1\|^2 + \frac{1}{2}|u^1|_1^2 \le \left(\frac{1}{2} + c_0\right)\lambda\|u^1\|^2.$$

When $3(1 + 2c_0)\lambda\tau/4 \le 1$, it follows $|u^1|_1 = 0$. Thus, (4.7) and (4.10) uniquely determine u^1.

(B1) Proof for (4.13) with $k = 1$.

Taking the inner product of (4.14) with $\delta_t e^{1/2}$ gives

$$\begin{aligned}&\left(\mathcal{A}\delta_t e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) - \left(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right)\\ =&\lambda\Big[\left(\mathcal{A}e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) - \left(\mathcal{A}(u^0 e^1 + e^0 U^1), \delta_t e^{\frac{1}{2}}\right)\Big] + \left((R_2)^0, \delta_t e^{\frac{1}{2}}\right).\end{aligned}$$

Noticing

$$\begin{aligned}&\left(\mathcal{A}\delta_t e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) - \left(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) \ge \frac{2}{3}\|\delta_t e^{\frac{1}{2}}\|^2 + \frac{1}{2\tau}|e^1|_1^2 \ge 2\frac{1}{\sqrt{3\tau}}\|\delta_t e^{\frac{1}{2}}\| \cdot |e^1|_1,\\ &\left(\mathcal{A}e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) \le \frac{1}{2}\|\mathcal{A}e^1\| \cdot \|\delta_t e^{\frac{1}{2}}\| \le \frac{1}{2}\|e^1\| \cdot \|\delta_t e^{\frac{1}{2}}\|,\\ &\left(\mathcal{A}(u^0 e^1 + e^0 U^1), \delta_t e^{\frac{1}{2}}\right) \le \|\mathcal{A}(u^0 e^1)\| \cdot \|\delta_t e^{\frac{1}{2}}\| \le \|u^0 e^1\| \cdot \|\delta_t e^{\frac{1}{2}}\| \le c_0\|e^1\| \cdot \|\delta_t e^{\frac{1}{2}}\|,\\ &\left((R_2)^0, \delta_t e^{\frac{1}{2}}\right) \le \|(R_2)^0\| \cdot \|\delta_t e^{\frac{1}{2}}\|,\end{aligned}$$

we have

$$\begin{aligned}\frac{2}{\sqrt{3\tau}}|e^1|_1 &\le \left(\frac{1}{2} + c_0\right)\lambda\|e^1\| + \|(R_2)^0\|\\ &\le \left(\frac{1}{2} + c_0\right)\frac{L}{\sqrt{6}}\lambda|e^1|_1 + c_4\sqrt{L}(\tau^2 + h^4).\end{aligned}$$

When

$$\left(\frac{1}{2} + c_0\right)\frac{L}{\sqrt{6}}\sqrt{3\tau}\lambda \le 1, \quad \text{i.e.,} \quad \frac{1}{2}\left(\frac{1}{2} + c_0\right)^2\lambda^2 L^2\tau \le 1,$$

it follows:

$$|e^1|_1 \le \sqrt{3\tau}c_4\sqrt{L}(\tau^2 + h^4) \le \sqrt{3TL}c_4(\tau^2 + h^4), \tag{4.20}$$

which implies that (4.13) holds for $k = 1$.

Now assume that the values of $u^0, u^1, \cdots, u^l$ $(l \ge 1)$ have been determined and (4.13) is true for $0 \le k \le l$, that is

$$|e^k|_1 \le c_6(\tau^2 + h^4), \quad 0 \le k \le l.$$

Then by Lemma 2.1(b), we have

$$\|e^k\|_\infty \le \frac{\sqrt{L}}{2}|e^k|_1 \le \frac{\sqrt{L}}{2}c_6(\tau^2 + h^4) \le 1, \quad 0 \le k \le l, \tag{4.21}$$

$$\|u^k\|_\infty \le \|U^k\|_\infty + \|e^k\|_\infty \le c_0 + 1, \qquad 0 \le k \le l. \tag{4.22}$$

(A2) Proof for the unique solvability of u^{l+1}.

From (4.8) ($k=l$) and (4.10) ($k=l+1$), the linear system in u^{l+1} is determined. Consider its homogeneous one

$$\frac{1}{2\tau}\mathcal{A}u_i^{l+1}-\frac{1}{2}\delta_x^2u_i^{l+1}=\lambda\mathcal{A}\left(\frac{1}{2}u_i^{l+1}-\frac{1}{3}u_i^lu_i^{l+1}\right),\quad 1\le i\le m-1, \tag{4.23}$$

$$u_0^{l+1}=0,\quad u_m^{l+1}=0. \tag{4.24}$$

Taking the inner product of (4.23) with u^{l+1} produces

$$\frac{1}{2\tau}(\mathcal{A}u^{l+1},u^{l+1})+\frac{1}{2}|u^{l+1}|_1^2=\frac{\lambda}{2}(\mathcal{A}u^{l+1},u^{l+1})-\frac{\lambda}{3}\big(\mathcal{A}(u^lu^{l+1}),u^{l+1}\big).$$

Then

$$\frac{1}{3\tau}\|u^{l+1}\|^2+\frac{1}{2}|u^{l+1}|_1^2\le\lambda\left[\frac{1}{2}+\frac{1}{3}(c_0+1)\right]\|u^{l+1}\|^2.$$

Noticing

$$\frac{1}{3\tau}\|u^{l+1}\|^2+\frac{1}{2}|u^{l+1}|_1^2\ge\frac{2}{\sqrt{6\tau}}\|u^{l+1}\|\cdot|u^{l+1}|_1,$$

we have

$$\begin{aligned}|u^{l+1}|_1&\le\frac{\sqrt{6\tau}}{2}\left[\frac{1}{2}+\frac{1}{3}(c_0+1)\right]\lambda\|u^{l+1}\|\\&\le\frac{\sqrt{\tau}}{2}\left[\frac{1}{2}+\frac{1}{3}(c_0+1)\right]\lambda L|u^{l+1}|_1.\end{aligned}$$

When

$$\frac{1}{4}\left[\frac{1}{2}+\frac{1}{3}(c_0+1)\right]^2\lambda^2L^2\tau<1,$$

it follows $|u^{l+1}|_1=0$. Hence, (4.8) and (4.10) determine u^{l+1} uniquely.

(B2) Proof for (4.13) with $k=l+1$.

Taking the inner product of (4.15) with Δ_te^k yields

$$\begin{aligned}&\big(\mathcal{A}\Delta_te^k,\Delta_te^k\big)+\frac{1}{4\tau}\big(|e^{k+1}|_1^2-|e^{k-1}|_1^2\big)\\&=\lambda(\mathcal{A}e^{\bar{k}},\Delta_te^k)-\frac{1}{3}\lambda\Big(\mathcal{A}\big(u^k(e^{k-1}+e^k+e^{k+1})+e^k(U^{k-1}+U^k+U^{k+1})\big),\Delta_te^k\Big)\\&\quad+\big((R_2)^k,\Delta_te^k\big)\\&\le\lambda\|\mathcal{A}e^{\bar{k}}\|\cdot\|\Delta_te^k\|+\frac{1}{3}\lambda\Big[\big\|\mathcal{A}\big(u^k(e^{k-1}+e^k+e^{k+1})\big)\big\|\\&\qquad+\big\|\mathcal{A}\big(e^k(U^{k-1}+U^k+U^{k+1})\big)\big\|\Big]\|\Delta_te^k\|+\|(R_2)^k\|\cdot\|\Delta_te^k\|\\&\le\lambda\|e^{\bar{k}}\|\cdot\|\Delta_te^k\|+\frac{1}{3}\lambda\Big[\|u^k(e^{k-1}+e^k+e^{k+1})\|+\|e^k(U^{k-1}+U^k+U^{k+1})\|\Big]\|\Delta_te^k\|\\&\quad+\|(R_2)^k\|\cdot\|\Delta_te^k\|\end{aligned}$$

$$
\begin{aligned}
&\le \lambda\|e^{\bar{k}}\|\cdot\|\Delta_t e^k\| + \frac{1}{3}\lambda(c_0+1)\|e^{k-1}+e^k+e^{k+1}\|\cdot\|\Delta_t e^k\| \\
&\quad + \frac{1}{3}\lambda(3c_0)\|e^k\|\cdot\|\Delta_t e^k\| + \|(R_2)^k\|\cdot\|\Delta_t e^k\| \\
&\le \left(\frac{1}{6}\|\Delta_t e^k\|^2 + \frac{3}{2}\lambda^2\|e^{\bar{k}}\|^2\right) + \left(\frac{1}{6}\|\Delta_t e^k\|^2 + \frac{1}{6}\lambda^2(c_0+1)^2\|e^{k-1}+e^k+e^{k+1}\|^2\right) \\
&\quad + \left(\frac{1}{6}\|\Delta_t e^k\|^2 + \frac{3}{2}\lambda^2 c_0^2\|e^k\|^2\right) + \frac{1}{6}\|\Delta_t e^k\|^2 + \frac{3}{2}\|(R_2)^k\|^2, \quad 1\le k\le l.
\end{aligned}
$$

Noticing

$$
\big(\mathcal{A}\Delta_t e^k, \Delta_t e^k\big) \ge \frac{2}{3}\|\Delta_t e^k\|^2
$$

and (4.4), it follows:

$$
\begin{aligned}
&\frac{1}{4\tau}\big(|e^{k+1}|_1^2 - |e^{k-1}|_1^2\big) \\
\le\ & \frac{3}{4}\lambda^2\big(\|e^{k+1}\|^2 + \|e^{k-1}\|^2\big) + \frac{\lambda^2}{2}(c_0+1)^2\big(\|e^{k-1}\|^2 + \|e^k\|^2 + \|e^{k+1}\|^2\big) \\
&\quad + \frac{3}{2}\lambda^2 c_0^2\|e^k\|^2 + \frac{3}{2}c_5^2 L(\tau^2+h^4)^2 \\
=\ & \left[\frac{3}{4}\lambda^2 + \frac{\lambda^2}{2}(c_0+1)^2\right]\|e^{k+1}\|^2 + \left[\frac{3}{4}\lambda^2 + \frac{\lambda^2}{2}(c_0+1)^2\right]\|e^{k-1}\|^2 \\
&\quad + \left[\frac{\lambda^2}{2}(c_0+1)^2 + \frac{3}{2}\lambda^2 c_0^2\right]\|e^k\|^2 + \frac{3}{2}c_5^2 L(\tau^2+h^4)^2 \\
\le\ & \left[\frac{3}{4}\lambda^2 + \frac{\lambda^2}{2}(c_0+1)^2\right]\frac{L^2}{6}|e^{k+1}|_1^2 + \left[\frac{3}{4}\lambda^2 + \frac{\lambda^2}{2}(c_0+1)^2\right]\frac{L^2}{6}|e^{k-1}|_1^2 \\
&\quad + \left[\frac{\lambda^2}{2}(c_0+1)^2 + \frac{3}{2}\lambda^2 c_0^2\right]\frac{L^2}{6}|e^k|_1^2 + \frac{3}{2}c_5^2 L(\tau^2+h^4)^2, \quad 1\le k\le l,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\left\{1 - \frac{1}{6}\big[3+2(c_0+1)^2\big]\lambda^2L^2\tau\right\}|e^{k+1}|_1^2 \\
\le\ & \left\{1 + \frac{1}{6}\big[3+2(c_0+1)^2\big]\lambda^2L^2\tau\right\}|e^{k-1}|_1^2 + \frac{1}{3}\big[(c_0+1)^2+3c_0^2\big]\lambda^2L^2\tau|e^k|_1^2 \\
&\quad + 6c_5^2L\tau(\tau^2+h^4)^2, \quad 1\le k\le l.
\end{aligned}
$$

When $[3+2(c_0+1)^2]\lambda^2L^2\tau \le 2$, it follows:

$$
\begin{aligned}
|e^{k+1}|_1^2 &\le \left\{1 + \frac{1}{2}\big[3+2(c_0+1)^2\big]\lambda^2L^2\tau\right\}|e^{k-1}|_1^2 \\
&\quad + \frac{1}{2}\big[(c_0+1)^2+3c_0^2\big]\lambda^2L^2\tau|e^k|_1^2 + 9c_5^2L\tau(\tau^2+h^4)^2 \\
&\le \Big[1+3(c_0+1)^2\lambda^2L^2\tau\Big]\max\big\{|e^k|_1^2, |e^{k-1}|_1^2\big\}
\end{aligned}
$$

$$+ 9c_5^2 L\tau(\tau^2+h^4)^2, \quad 1 \le k \le l,$$

so that

$$\begin{aligned}\max\left\{|e^{k+1}|_1^2, |e^k|_1^2\right\} \le &\left[1+3(c_0+1)^2\lambda^2L^2\tau\right]\max\left\{|e^k|_1^2, |e^{k-1}|_1^2\right\}\\ &+ 9c_5^2L\tau(\tau^2+h^4)^2, \quad 1\le k\le l.\end{aligned}$$

Applying the Gronwall inequality in Lemma 2.3 and noticing (4.20), we get

$$\begin{aligned}\max\left\{|e^{l+1}|_1^2, |e^l|_1^2\right\} &\le \mathrm{e}^{3(c_0+1)^2\lambda^2L^2l\tau}\left[\max\left\{|e^1|_1^2, |e^0|_1^2\right\} + \frac{3c_5^2(\tau^2+h^4)^2}{(c_0+1)^2\lambda^2L}\right]\\ &\le \mathrm{e}^{3(c_0+1)^2\lambda^2L^2T}\left[3TLc_4^2 + \frac{3c_5^2}{(c_0+1)^2\lambda^2L}\right](\tau^2+h^4)^2.\end{aligned}$$

Then

$$\begin{aligned}|e^{l+1}|_1 &\le \mathrm{e}^{\frac{3}{2}(c_0+1)^2\lambda^2L^2T}\left(3TLc_4^2 + \frac{3c_5^2}{(c_0+1)^2\lambda^2L}\right)^{\frac{1}{2}}(\tau^2+h^4)\\ &= c_6(\tau^2+h^4),\end{aligned}$$

which implies that (4.13) is true for $k = l+1$. By induction, the theorem is proved. □

In view of Lemma 2.1(b), the difference scheme (4.7)-(4.10) is unconditionally convergent in the maximum norm with the convergence order $\mathcal{O}(\tau^2+h^4)$.

Remark 4.2. Similar to the discussion in Remark 3.2, the difference scheme (4.7)-(4.10) also satisfies the MBP when the mesh step sizes are sufficiently small based on the convergence Theorem 4.2.

5. Numerical experiments

In this part, we are concerned with the numerical test for the above two difference schemes. Three numerical examples are used to test the numerical accuracy, conservative property and the MBP of the difference scheme (3.7)-(3.10) and the compact difference scheme (4.7)-(4.10). Denote

$$E_\infty(h,\tau) = \max_{\substack{0\le i\le m\\0\le k\le n}}\left|U_i^k(h,\tau) - u_i^k(h,\tau)\right|, \quad \mathcal{O}_h = \log_2\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}, \quad \mathcal{O}_\tau = \log_2\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)},$$

$$F_\infty(\tau) = \max_{\substack{0\le i\le m\\0\le k\le n}}\left|u_i^k(h,\tau) - u_i^{2k}\left(h,\frac{\tau}{2}\right)\right|, \quad Ord_\tau = \log_2\frac{F_\infty(2\tau)}{F_\infty(\tau)},$$

$$G_\infty(h) = \max_{\substack{0\le i\le m\\0\le k\le n}}\left|u_i^k(h,\tau) - u_{2i}^k\left(\frac{h}{2},\tau\right)\right|, \quad Ord_h = \log_2\frac{G_\infty(2h)}{G_\infty(h)}.$$

Example 5.1 ([13]). In the problem (1.1)-(1.3), take $L = 1, T = 1, \lambda = 6$,

$$\varphi(x) = \frac{1}{(1+\mathrm{e}^x)^2}, \quad \alpha(t) = \frac{1}{(1+\mathrm{e}^{-5t})^2}, \quad \beta(t) = \frac{1}{(1+\mathrm{e}^{1-5t})^2}.$$

The exact solution is given by

$$u(x,t) = \frac{1}{(1+\mathrm{e}^{x-5t})^2}.$$

We fix a sufficiently small h and vary τ to observe the temporal convergence, and fix a sufficiently small τ and vary h to observe the spatial convergence. The maximum absolute error $E_\infty(h,\tau)$ and related convergence orders are presented in Tables 1 and 2 respectively. From these two tables, it is clear that we obtain approximate second-order accuracy in both the temporal and the spatial directions for the difference scheme (3.7)-(3.10), while for the compact difference scheme (4.7)-(4.10), the second-order and fourth-order accuracy in the temporal and the spatial directions, respectively, can be read off, which is consistent with our theoretical results in Sections 3.3 and 4.3.

Table 1: Example 5.1. Maximum errors and convergence orders in time.

	Scheme (3.7)-(3.10) ($m = 400$)		Scheme (4.7)-(4.10) ($m = 400$)	
τ	$E_\infty(h,\tau)$	$\mathcal{O}_\tau$	$E_\infty(h,\tau)$	$\mathcal{O}_\tau$
1/20	2.388066e-03	–	2.388080e-03	–
1/40	5.624444e-04	2.09	5.624561e-04	2.09
1/80	1.380640e-04	2.03	1.380756e-04	2.03
1/160	3.423045e-05	2.01	3.424205e-05	2.01
1/320	8.517390e-06	2.01	8.528982e-06	2.01

Table 2: Example 5.1. Maximum errors and convergence orders in space.

	Scheme (3.7)-(3.10) ($n = 10000$)		Scheme (4.7)-(4.10) ($n = 500000$)	
h	$E_\infty(h,\tau)$	$\mathcal{O}_h$	$E_\infty(h,\tau)$	$\mathcal{O}_h$
1/10	2.158202e-05	–	4.087641e-08	–
1/20	5.405155e-06	2.00	2.548302e-09	4.00
1/40	1.348020e-06	2.00	1.564399e-10	4.03
1/80	3.309353e-07	2.03	6.959544e-12	4.49

Example 5.2. In the problem (1.1)-(1.3), take $L = 1, T = 1, \lambda = \pi^2, \varphi(x) = \sin(\pi x)$, $\alpha(t) = \beta(t) = 0$.

The exact solution is unknown. Take a sufficiently small h and different temporal step sizes $\tau = 1/80, 1/160, 1/320$, $1/640, 1/1280, 1/2560$, respectively. Table 3 presents the numerical errors $F_\infty(\tau)$ and temporal convergence orders Ord_τ in the maximum

Table 3: Example 5.2. Maximum errors and convergence orders in time.

τ	Scheme (3.7)-(3.10) ($m=1000$)		Scheme (4.7)-(4.10) ($m=1000$)	
	$F_\infty(\tau)$	Ord_τ	$F_\infty(\tau)$	Ord_τ
$1/80$	3.002033e-03	–	3.002049e-03	–
$1/160$	8.445769e-04	1.83	8.445860e-04	1.83
$1/320$	1.987258e-04	2.09	1.987297e-04	2.09
$1/640$	4.493577e-05	2.14	4.493665e-05	2.14
$1/1280$	1.039646e-05	2.11	1.039656e-05	2.11
$1/2560$	–	–	–	–

Table 4: Example 5.2. Maximum errors and convergence orders in space.

h	Scheme (3.7)-(3.10) ($n=1000$)		Scheme (4.7)-(4.10) ($n=10000$)	
	$G_\infty(h)$	Ord_h	$G_\infty(h)$	Ord_h
$1/20$	8.985675e-04	–	1.628731e-06	–
$1/40$	2.241644e-04	2.00	1.012496e-07	4.01
$1/80$	5.601100e-05	2.00	6.319529e-09	4.00
$1/160$	1.400086e-05	2.00	4.113894e-10	3.94
$1/320$	3.500102e-06	2.00	2.706591e-11	3.93
$1/640$	–	–	–	–

norm, demonstrating that both the difference scheme (3.7)-(3.10) and the compact difference scheme (4.7)-(4.10) generate the temporal convergence of order two.

Fix a sufficiently small τ and take different spatial step sizes $h = 1/20, 1/40, 1/80, 1/160, 1/320, 1/640$, respectively. Table 4 records the numerical errors $G_\infty(h)$ and spatial convergence orders Ord_h in the maximum norm, verifying that the difference scheme (3.7)-(3.10) generates the spatial convergence of order two, while the difference scheme (4.7)-(4.10) generates the fourth-order spatial convergence.

In order to verify the conservation of the difference scheme (3.7)-(3.10) and the compact difference scheme (4.7)-(4.10), we compute the discrete energy E^k and F^k in Theorem 3.1, and P^k and Q^k in Theorem 4.1 ($0 \le k \le n-1$). Fig. 1 collects the energy error curves of E^k and P^k under different step sizes, where the exact energy

$$E(t) = E(0) = \|u(\cdot,0)\|^2 = \int_0^1 \sin^2(\pi x)\mathrm{d}x = \frac{1}{2}.$$

Note that the discrete energy E^k and P^k is almost equal to the exact energy $E(0)$ in this example. Indeed, the conservative law equalities (3.11) and (4.11) imply that

$$E^k = P^k = \|u^0\|^2 = h\sum_{i=1}^{m-1}(u_i^0)^2 = h\sum_{i=1}^{m-1}\varphi^2(x_i),$$

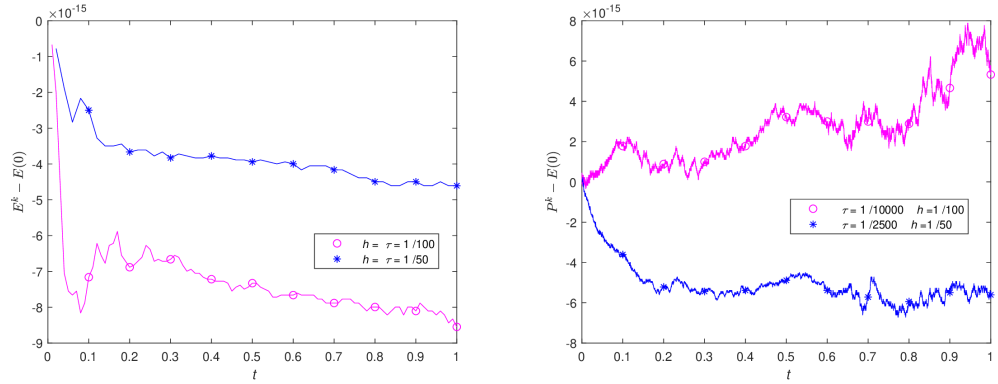


Figure 1: The energy errors of E^k and P^k.

which is precisely the result of composite trapzoidal formula applied to the integral

$$\int_0^1 \varphi^2(x)\,\mathrm{d}x = E(0).$$

Due to the special choice of $\varphi(x)$ in this example, it is easy to find that

$$h\sum_{i=1}^{m-1}\varphi^2(x_i) = \int_0^1 \varphi^2(x)\,\mathrm{d}x.$$

The error curves of F^k and Q^k are plugged into Fig. 2, where the exact energy

$$F(t) = F(0) = \int_0^1 \pi^2\cos^2(\pi x)\mathrm{d}x + \pi^2\int_0^1 \left[\frac{2}{3}\sin^3(\pi x) - \sin^2(\pi x)\right]\mathrm{d}x = \frac{8}{9}\pi.$$

The conservation property of the discrete energy of these two difference schemes is numerically verified.

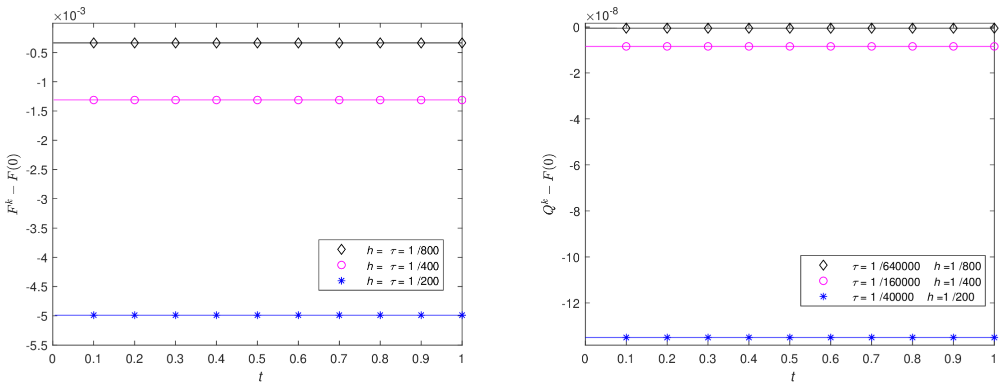
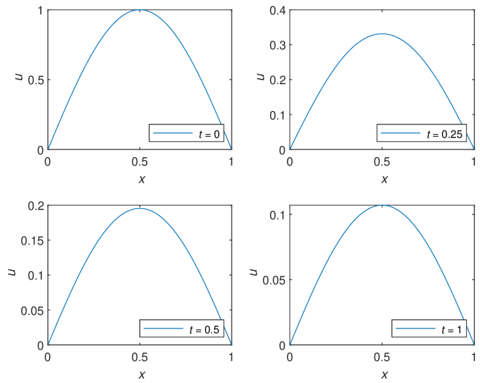
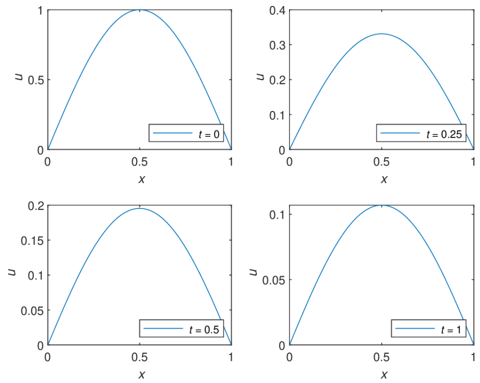


Figure 2: The energy errors of F^k and Q^k.

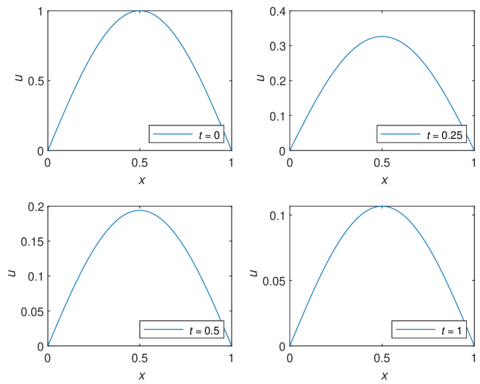
Next, the performance of the numerical solution at some moments, including the evolution of its maximum and minimum value, is displayed in Figs. 3 and 4 with $h = 0.01$ and varying τ. Numerically we verified the proposed two numerical schemes both satisfy the MBP unconditionally.



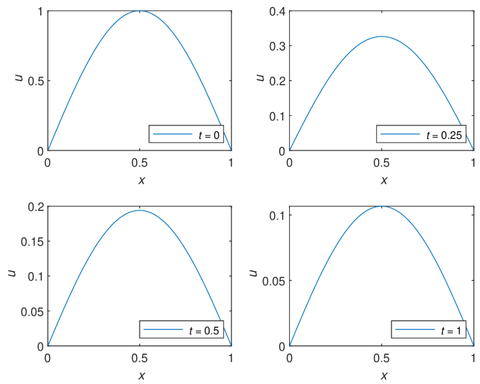
(a) $\tau = 0.01$



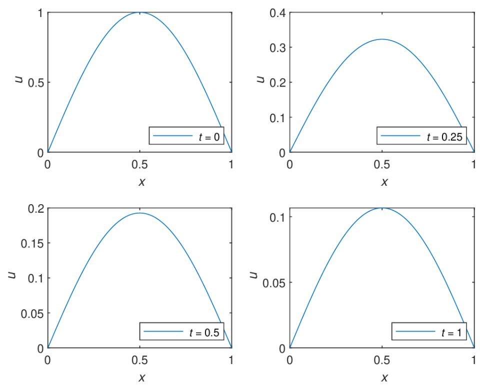
(b) $\tau = 0.01$



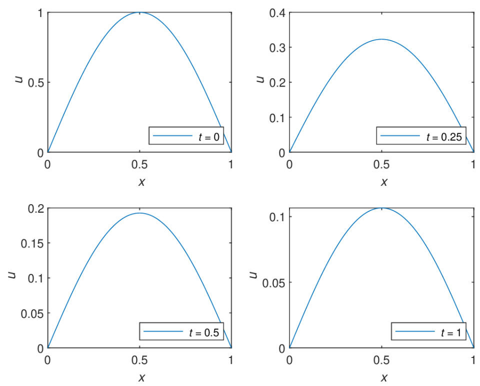
(c) $\tau = 0.005$



(d) $\tau = 0.005$

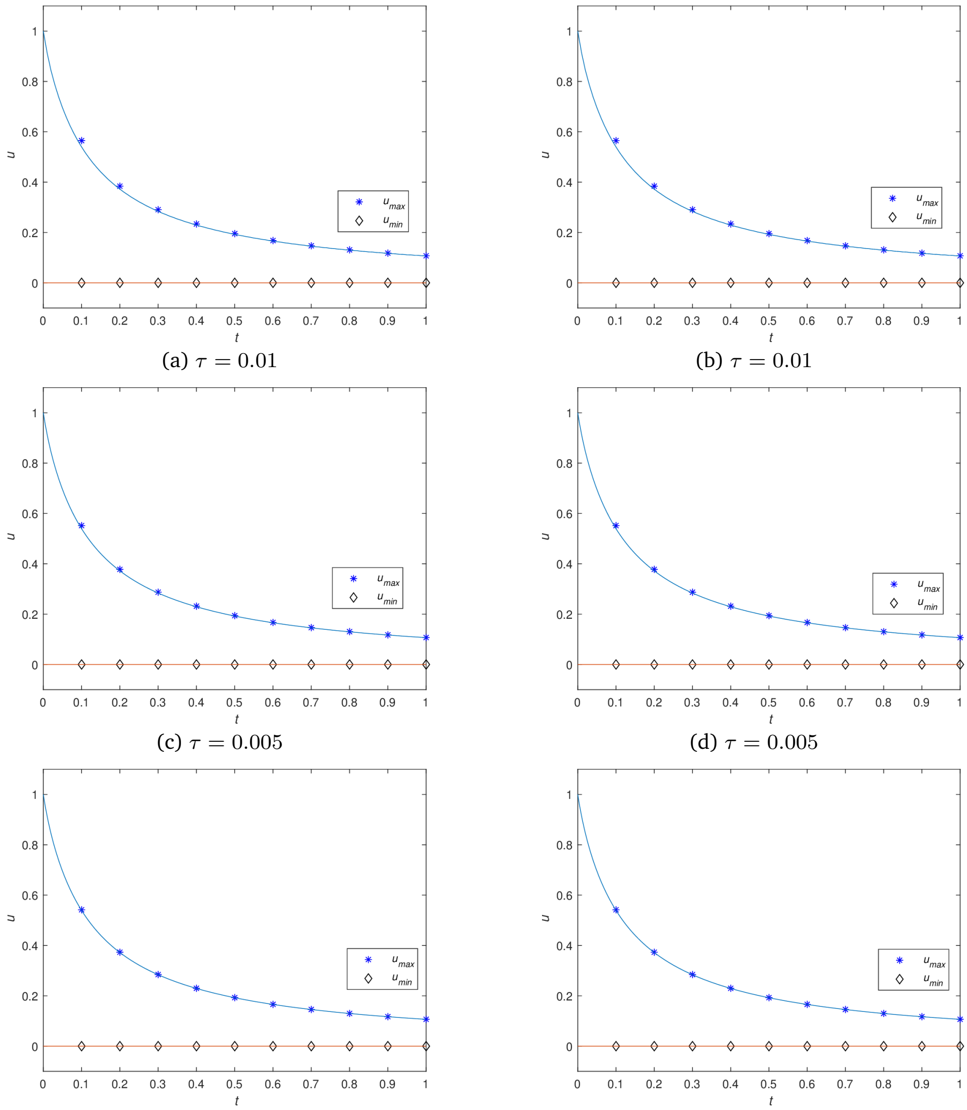


(e) $\tau = 0.001$



(f) $\tau = 0.001$

Figure 3: The numerical solution u at different moments. (Left: scheme (3.7)-(3.10); Right: scheme (4.7)-(4.10)).



(a) $\tau = 0.01$ (b) $\tau = 0.01$

(c) $\tau = 0.005$ (d) $\tau = 0.005$

(e) $\tau = 0.001$ (f) $\tau = 0.001$

Figure 4: Max and min of u at different moments. (Left: scheme (3.7)-(3.10); Right: scheme (4.7)-(4.10)).

Example 5.3. In the problem (1.1)-(1.3), take $L = 1, T = 1, \lambda = 1, \varphi(x) = 108x^2(1 - 2x)^2(1 - x)^2$, $\alpha(t) = \beta(t) = 0$.

The exact solution is also unknown. We numerically calculate this example using the difference scheme (3.7)-(3.10) and the compact difference scheme (4.7)-(4.10), respectively. The discrete invariants are shown in Figs. 5 and 6. In addition, taking $h = 0.001$ and varying τ, the calculated numerical solution and its maximum/minimum values at different moments are plotted in Figs. 7 and 8, respectively, which shows that the proposed schemes satisfy the MBP unconditionally in this example.

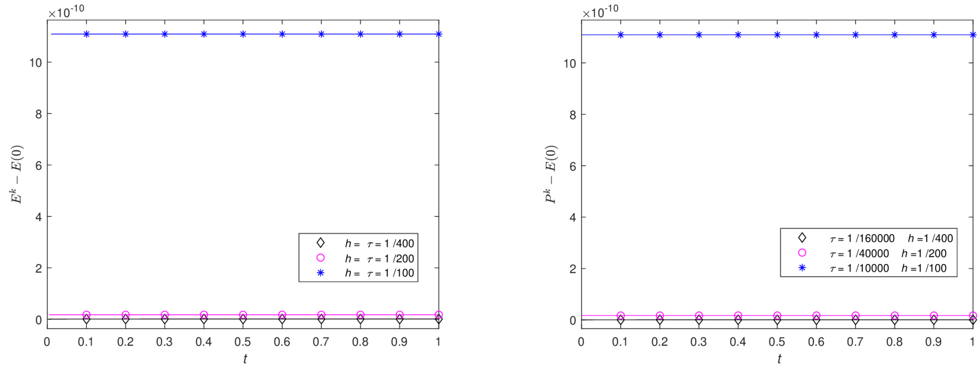


Figure 5: The energy errors of E^k and P^k.

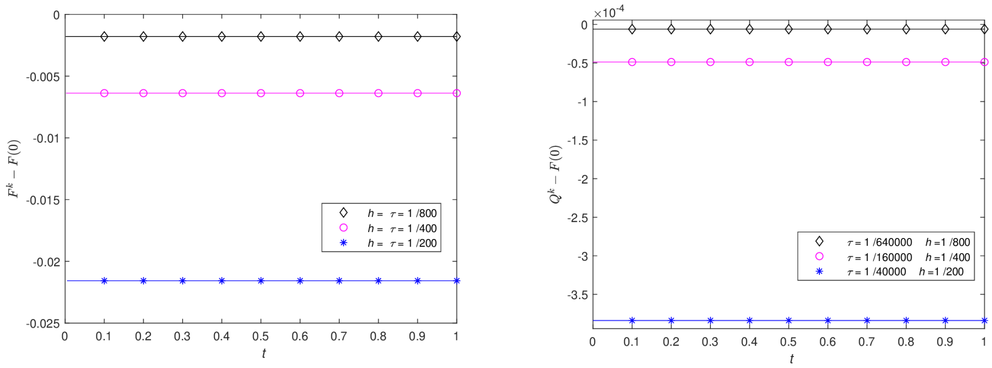


Figure 6: The energy errors of F^k and Q^k.

6. Conclusion

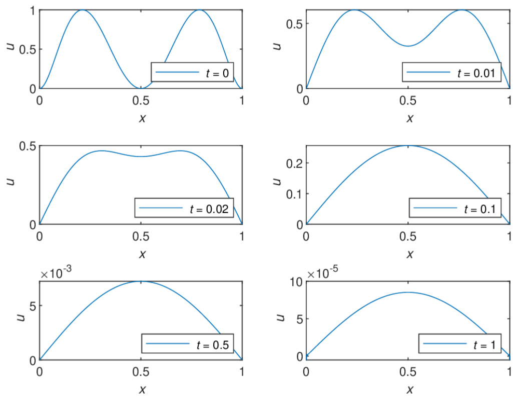
Two conservative difference schemes are proposed to solve the Fisher equation, which are both three-level linearized and implicit. The coefficient matrices of these two implicit difference schemes are both tri-diagonal, which can be solved using the Thomas algorithm. The unique solvability and unconditional convergence of the two difference schemes are rigorously proved by the energy analysis and mathematical induction method. Numerical experiments are used to support the theoretical results.

The current work has its highlights:

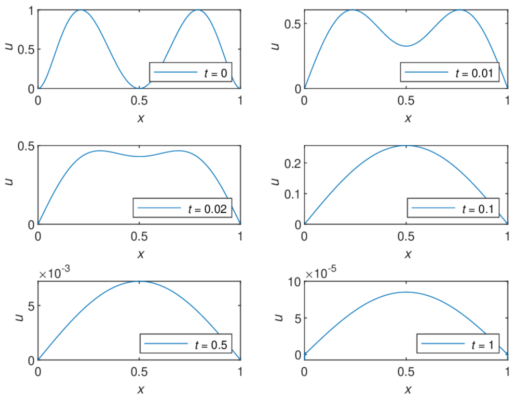
(I) The numerical accuracy of the proposed algorithms can reach second-order in time and second- or fourth-order in space, which is quite ideal and superior to the existing results.

(II) The constructed numerical schemes are able to maintain two conservation invariants, energy stable for the free energy of the Fisher equation as a gradient flow and satisfy the MBP unconditionally.

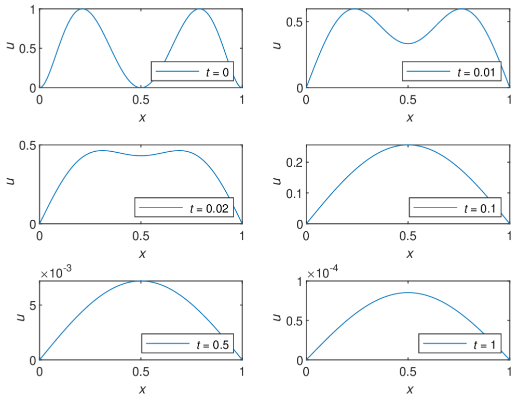
(III) The schemes we construct are linearized and computationally simple.



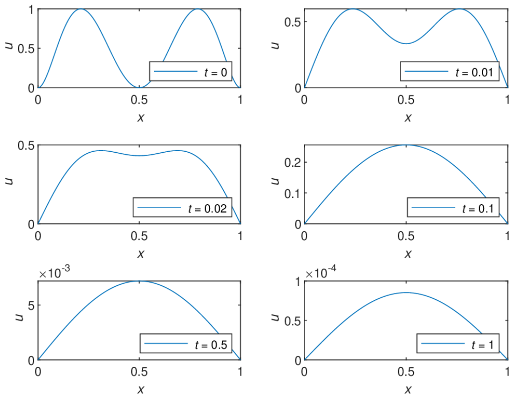
(a) $\tau = 0.0005$



(b) $\tau = 0.0005$



(c) $\tau = 0.0001$

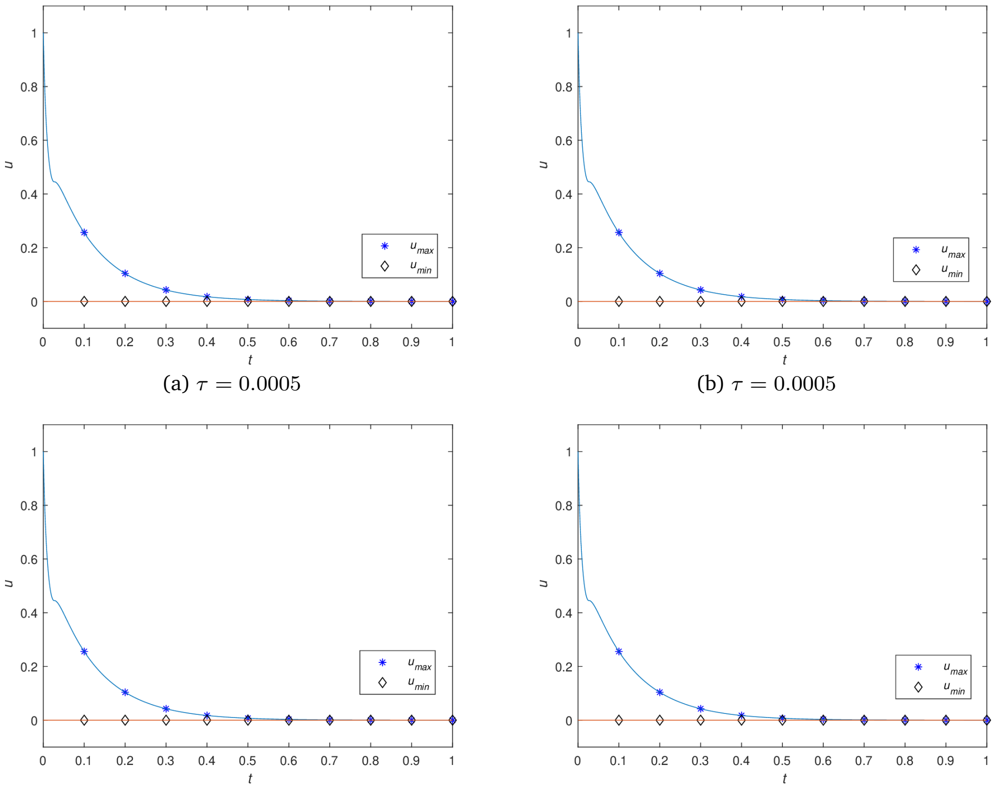


(d) $\tau = 0.0001$

Figure 7: The numerical solution u at different moments. (Left: scheme (3.7)-(3.10); Right: scheme (4.7)-(4.10)).

(IV) We make clever use of the energy analysis and mathematical induction method to rigorously prove the unique solvability and convergence of the schemes. There is little convergence analysis in the existing literature. In particular, for the analysis on the conservation invariants of compact difference scheme, a new equivalent norm has been defined to get the corresponding results. There is one fly in the ointment that we can not prove that the current numerical algorithms satisfy the MBP directly rather than relying on the convergence result, which will be one of our future research.

It is worth mentioning that the analysis techniques developed in this work also work for the forward Euler scheme, the backward Euler scheme, and the Crank-Nicolson scheme to solve the Fisher equation.



(a) $\tau = 0.0005$ (b) $\tau = 0.0005$

(c) $\tau = 0.0001$ (d) $\tau = 0.0001$

Figure 8: Max and min of u at different moments (Left: scheme (3.7)-(3.10); Right: scheme (4.7)-(4.10)).

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