Analysis of Weakly Nonlinear Evolution Characteristics of Flow in the Constant Curvature Bend

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Abstract. The meandering river is an unstable system with the characteristic of non-linearity, which results from the instability of the flow and boundary. Focusing on the hydrodynamic nonlinearity of the bend, we use the weakly nonlinear theory and perturbation method to construct the nonlinear evolution equations of the disturbance amplitude and disturbance phase of two-dimensional flow in meandering bend. The influence of the curvature, $Re$ and the disturbance wave number on the evolution of disturbance amplitude and disturbance phase are analyzed. Then, the spatial and temporal evolution of the disturbance vorticity is expounded. The research results show: that the curvature makes the flow more stable; that in the evolution of the disturbance amplitude effected by curvature, $Re$ and the disturbance wave number, exist nonlinear attenuation with damping disturbances, and nonlinear explosive growth with positive disturbances; that the asymmetry distribution of the disturbance velocities increases with the curvature; that the location of the disturbance vorticity’s core area changes periodically with disturbance phase, and the disturbance vorticity gradually attenuates/increases with the decrease of the disturbance phase in the evolution process of damping/positive disturbances. These results shed light on the construction of the interaction model of hydrodynamic nonlinearity and geometric nonlinearity of bed.

AMS subject classifications: 76E30, 34C60

Key words: Curvature bend, hydrodynamics, weakly nonlinearity, disturbance vorticity.

1 Introduction

Since the 1980s, scholars have kept investigating the characteristics of the stability and nonlinearity of meandering rivers. Callander [1] considered that the instability is the
cause of the river bending or braiding. In the instability analysis by Ikeda, Parker and Sawai [2], the most unstable wavelength was considered as the finite amplitude wavelength of the meandering river, which assumed the curvature ratio (the ratio of the half width to the curvature radius) is far less than 1. Parker, Sawai and Ikeda [3] analyzed the geometrical nonlinear stability of meandering river, ignoring the nonlinear dynamic terms. In the weakly nonlinear analysis of the meandering river by Seminara and Tubino [4], the geometric weakly nonlinear analysis was carried out near the resonance state, and the suppression of the nonlinear effect was revealed. Imran, Parker and Pirmez [5] pointed out that the study of Sun [6] has a fundamental defect in simulating the evolution of the meandering river because the nonlinear effect of flow dynamic is ignored. The meandering river has the instability mechanism, which refers to the instability of the flow [7]. Pittaluga, Nobile and Seminara [8] thought that the linear theory of the meandering river explains the resonance mechanism. However, the complete nonlinear theory of the meandering river has not been established yet, and the nonlinear effect has a certain influence on the flow field. Pittaluga and Seminara [9] argued that nonlinearity and instability are the important characteristics of the meandering river, while the effects of nonlinearity have been seriously ignored. Bai et al. [10–12] suggested that the nonlinear hydrodynamics theory is important to investigate the evolution of rivers under disturbances. Nelson, Pittaluga and Seminara [13] presented a nonlinear asymptotic theory of fully developed flow and bed topography subjected to unerodible bedrock layer, but they ignored the nonlinearity of the flow itself.

The previous studies mainly focused on the geometric nonlinearity of the meandering river and the bed disturbance. However, the study on the characteristics of the nonlinearity and evolution of the flow dynamics in the meandering river with different curvatures is insufficient. Different from the weakly nonlinear instability in the plane Poiseuille flow and shear layers [14, 15], in this paper, the instability and nonlinear evolution of the flow dynamics affected by the curved boundary with damping and positive disturbances are analyzed by constructing the control equation under the small flow disturbance. Under the conditions of the time mode, the Orr-Sommerfeld equation is used to analyze the stability characteristics of the hydrodynamic in the bend. And the Landau-Stuart equations are used with the weakly nonlinear theory of flow stability [16, 17]. Then the nonlinear evolution equations of the hydrodynamic of the constant curvature bend under the nonlinear effect of the flow are derived. The influences of the material composition of the river bank and the form of the bed surface are ignored. Generally, the river’s curvature is constant along the bend, and the width is limited by the walls, such as in the canyon channel of Jing River in Shaanxi Province, China (Fig. 1). And in order to reduce the complexity of equations, we take the constant curvature form to investigate the stability and nonlinearity of hydrodynamic in the meandering river. The hydrodynamic study in the constant curvature bend provides the nonlinear hydrodynamic basis for further exploring the complete nonlinear relationship between river hydrodynamic and bed morphology. And in natural rivers, a high-order method is an accurate tool to study three-dimensional hydrodynamics [18].
2 Theoretical model

2.1 Coordinate transformations

In order to simplify the governing equations of the meandering river possibly, Parker [2, 3] and Smith [19] proposed an orthogonal curvilinear coordinate system (Fig. 2). The bend’s width is proposed to be constant along the river ($2B_r$). The values of the $x_0$ and the $y_0$ are the values on the X-axis and the Y-axis of the river centerline in the Cartesian coordinate system, respectively, shown in the following figure. Then the transformation relationship between orthogonal curvilinear coordinate system and Cartesian coordinate system are as follows:

$$x = x_0 + \Delta x = x_0 - n \frac{dy_0}{ds}, \quad (2.1a)$$
$$y = y_0 + \Delta y = y_0 + n \frac{dx_0}{ds}. \quad (2.1b)$$

The radius of curvature $R(s)$:

$$R(s) = \left( \frac{d x_0}{d s} \frac{d^2 y_0}{d s^2} - \frac{d y_0}{d s} \frac{d^2 x_0}{d s^2} \right)^{-1}. \quad (2.2)$$

The relevant metric coefficients are:

$$h_s = 1 - \frac{n}{R(s)} = 1 - N, \quad h_n = 1, \quad h_z = 1. \quad (2.3)$$

2.2 Control equations of the meandering bend

The object investigated in this paper is the constant curvature bend in the two-dimensional case. Under the conditions of the orthogonal curvilinear coordinates, we
Figure 2: Orthogonal curvilinear coordinate system in the meandering river.

take \( B^* \), \( R^* \), and \( U^*_m \) as the half-width of the bend, the minimum curve radius and the maximum value of the base flow velocity respectively, "*" means the dimensional quantity. The scaling of the length, velocity and time are expressed by the \( B^* \), \( U^*_m \), \( B^* / U^*_m \), respectively. Then we have:

\[
(s^*, n^*) = B^* (s, n), \quad R^* = R^*_m R, \quad (u^*_s, u^*_n) = U^*_m (u_s, u_n),
\]

\[
p^* = \rho U^*_m^2 p, \quad t^* = B^* U^*_m t, \quad N = B^* R^*_m, \quad \psi = \frac{B^*}{R^*}.
\]

In the above equations, \( p^* \), \( \rho \), and \( t^* \) represent the pressure, the density of the flow and the time respectively. \( \psi \), called the curvature ratio, which refers to the ratio between the half-width and the minimum curve radius, is the critical characteristic parameter of the meandering river. In the case of the constant curvature, \( R = 1 \), so \( h_s = 1 - \psi n \). Then we can derive the dimensionless governing equations of the meandering river with the constant curvature:

\[
\frac{1}{h_s} \frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} - \psi u_n = 0, \quad (2.4a)
\]

\[
\frac{\partial u_s}{\partial t} + \frac{u_s \partial u_s}{h_s} + \frac{u_n \partial u_s}{h_n} - \frac{\psi u_s u_n}{h_s} = - \frac{1}{h_s} \frac{\partial p}{\partial s} + \frac{1}{\text{Re}} \frac{1}{h_s^2} \frac{\partial^2 u_s}{\partial s^2} + \frac{1}{\text{Re}} \frac{1}{h_n} \frac{\partial^2 u_s}{\partial n^2} + \psi u_s, \quad (2.4b)
\]

\[
\frac{\partial u_n}{\partial t} + \frac{u_s \partial u_n}{h_s} + \frac{u_n \partial u_n}{h_n} - \frac{\psi u_n^2}{h_s} = - \frac{\partial p}{\partial n} + \frac{1}{\text{Re}} \frac{1}{h_s} \frac{\partial^2 u_n}{\partial s^2} + \psi \frac{2}{\text{Re}} \frac{1}{h_n} \frac{\partial^2 u_n}{\partial n^2} + \psi u_n, \quad (2.4c)
\]

In the above equations: \( \text{Re} (= \frac{B^* U^*_m}{\nu}) \) is the Reynolds number, \( \nu \) is the kinematic viscosity, \( \frac{\Delta}{\psi} = \frac{1}{h_s^2} \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} \).

2.3 Perturbation analysis

The study of Zachmann and Lagasse [20] shows that the value of \( \psi \) in natural rivers is mainly distributed at 0.05 \~ 0.20. Therefore, in our study, \( \psi \) is regarded as a small parameter; Eqs. (2.4a)-(2.4c) are solved with the perturbation method; and the velocity
and pressure are divided into three parts:

\[
\begin{bmatrix}
u_s \\
u_n \\
p 
\end{bmatrix} = \begin{bmatrix}
u_{s0} \\
u_{n0} \\
p_{\psi0} 
\end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix}
u_{s1} \\
u_{n1} \\
p_{\psi1} 
\end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix}
u_{s2} \\
u_{n2} \\
p_{\psi2} 
\end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix}
u_{s3} \\
u_{n3} \\
p_{\psi3} 
\end{bmatrix}.
\]

At the right of the above equation, the first part is the base flow of the open channel; the second part is the curvature correction terms, which relate to \(\psi\); the third part is the flow disturbance terms, which consider the nonlinear effect arising from the flow disturbance in the meandering river, and \(\varepsilon\) is an any small quantity. In the analysis of the base flow terms and the curvature correction terms, because \(\psi\) is far less than 1, thus the influence caused by weak curvature can only consider the effect of the first-order of \(\psi\).

Considering the base flow terms of the open channel in Eq. (2.5), we set \(\psi = 0\), \(u_{n0} = 0\), \(\partial / \partial s = \partial / \partial t = 0\), and the corresponding boundary conditions as \(u_{n0} = u_{s0} = 0\) \((n = \pm 1)\). And then the solutions of the velocity and pressure are as follow:

\[
u_{n0} = 0, \quad u_{s0} = 1 - n^2, \quad p_{\psi0} = 0.
\]

The curvature correction terms, \(u_{s1}, u_{n1}, p_{\psi1}\) in the constant curvature bend are:

\[
u_{n1} = 0, \quad u_{s1} = -\frac{1}{3}(n - n^3), \quad p_{\psi1} = -n - \frac{1}{5}n^5 + \frac{2}{3}n^3.
\]

The base flow terms and the first-order curvature correction terms are combined as the base flow of the constant curvature bend. And then, the base flow terms and the flow disturbance terms are brought into the control equations of the constant curvature bend thus we have:

\[
\begin{bmatrix}
u_s \\
u_n \\
p 
\end{bmatrix} = \begin{bmatrix}
u_{s1} \\
u_{n1} \\
p_{\psi1} 
\end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix}
u_{s2} \\
u_{n2} \\
p_{\psi2} 
\end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix}
u_{s3} \\
u_{n3} \\
p_{\psi3} 
\end{bmatrix}.
\]
In Eq. (2.8), \( \overline{u}_s \), \( \overline{u}_n \), and \( \overline{p} \) are the streamwise velocity, cross-section velocity, and pressure of the base flow in the constant curvature bend, respectively. We take Eq. (2.8) to Eqs. (2.4a)-(2.4c), and then the perturbation control equations of any order can be obtained

\[
\sum_{i=1}^{\infty} \left\{ \frac{1}{h_s} \frac{\partial u_{\delta i}}{\partial s} + \frac{\partial u_{\delta i}}{\partial n} - \frac{\psi u_{\delta i}}{h_s} \right\} = 0, \quad (2.9a) \\
\sum_{i=1}^{\infty} \left\{ \frac{\partial u_{\delta i}}{\partial t} + L u_{\delta i} + u_{\nu i} \frac{\partial \overline{u}_s}{\partial n} + \frac{1}{h_s} \frac{\partial p_i}{\partial s} - \frac{\psi \overline{u}_s u_{\delta i}}{h_s} + \frac{\psi}{Re} \left[ \frac{2}{h_s} \frac{\partial u_{\delta i}}{\partial s} + \frac{1}{h_s} \frac{u_{\delta i}}{\partial n} + \frac{\psi u_{\delta i}}{h_s^2} \right] \right\} = \epsilon^i, \quad (2.9b) \\
\sum_{i=1}^{\infty} \left\{ \frac{\partial u_{\nu i}}{\partial t} + L u_{\nu i} + \frac{\partial p_i}{\partial n} + \frac{2 \psi \overline{u}_s u_{\delta i}}{h_s} + \frac{\psi}{Re} \left[ -\frac{2}{h_s} \frac{\partial u_{\delta i}}{\partial s} + \frac{1}{h_s} \frac{u_{\delta i}}{\partial n} + \frac{\psi u_{\delta i}}{h_s^2} \right] \right\} = \epsilon^i \quad (2.9c)
\]

In the above equations,

\[
L = \frac{\overline{u}_s}{h_s} \frac{\partial}{\partial s} - \frac{1}{Re} \left( \frac{\partial^2}{h_s^2} + \frac{\partial^2}{\partial n^2} \right)
\]

and \( i \) means the order \( (i = 1, 2, \ldots) \).

### 2.4 Linear stability analysis

In the linear instability analysis of shear flow, Wu [21] introduced a small amplitude perturbation based on the base flow and obtained the solution form by separating variables based on the approximate local parallel flow. In this paper, as \( \overline{u}_s \), \( \overline{u}_n \), and \( \overline{p} \) in Eq. (2.8) are merely the function of \( n \), and the flow disturbance is considered a small amplitude quantity. Therefore, the first-order disturbance quantity of \( u_{\delta i}, u_{\nu i}, p_1 \) can be written as:

\[
\begin{bmatrix}
  u_{\delta i} \\
  u_{\nu i} \\
  p_1
\end{bmatrix}
= \begin{bmatrix}
  \hat{u}_{\delta i(n)} \\
  \hat{u}_{\nu i(n)} \\
  \hat{p}_{1(n)}
\end{bmatrix} \exp[i(\alpha s - \omega t)] + c.c. \quad (2.10)
\]

In the above equations, \( \hat{u}_{\delta i(n)}, \hat{u}_{\nu i(n)}, \hat{p}_{1(n)} \) are the shape functions about \( n \); \( \alpha \) is the disturbance wave number; \( \omega \) is the disturbance frequency; and \( c.c. \) is the conjugate complex number. We take Eq. (2.10) into Eqs. (2.9a)-(2.9c) and then get the governing equations of the first-order disturbance. Our goal is to investigate the flow’s instability and nonlinear evolution process in the constant curvature bend under the time mode. Therefore, the disturbance frequency can be written as \( \omega = \omega_r + i \omega_i \). The imaginary part of the disturbance frequency \( (\omega_i) \) is related to the increase or decrease of the disturbance. Then the
disturbance amplitude and disturbance phase can be written as $a = \exp(\omega_i t)$ and $\theta = -\omega_r t$, respectively, and both of them satisfy the relation as follows:

$$F(a, \omega_r, Re, \phi) = 0. \quad (2.11)$$

The Eq. (2.11) is the Orr-Sommerfeld equation for hydrodynamic characteristics of the meandering channel with the constant curvature. Then under the influence of curvature and the corresponding variation of the disturbance wave number and Reynolds number, the neutral curve distribution can be obtained. The central difference method and Muller method are used to solve Eq. (2.11). It can be calculated with the change of the $\omega_r$, $\omega_i$ under specific $Re$ and $\alpha$. When $\omega_i = 0$, it is called the neutral state.

### 2.5 Nonlinear evolution equations under time mode

Wu [21] pointed out that the weakly nonlinear theory is closely related to the development of shear flow at high Reynolds number. Due to the influence of nonlinearity, the evolution process of the flow hydrodynamics can be expressed by the disturbance amplitude function. Moreover, higher harmonic terms are generated due to the nonlinear interaction. In the nonlinear evolution of the hydrodynamics process in the constant curvature bend, we use the classical weakly nonlinear theory of the Landau-Stuart equation [16] under the time mode. Because of the effect of nonlinearity, the evolution of the disturbance amplitude and disturbance phase can be expressed as follows:

$$\frac{da}{dt} = \omega_i a + \sum_{m=1}^{\infty} A_m a^m = A_r, \quad \frac{d\theta}{dt} = -\omega_r + \sum_{m=1}^{\infty} B_m a^m = B. \quad (2.12)$$

And according to the chain rule, we have:

$$\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial a} \left( \omega_i a + \sum_{m=1}^{\infty} A_m a^m \right) + \frac{\partial u_i}{\partial \theta} \left( -\omega_r + \sum_{m=1}^{\infty} B_m a^m \right). \quad (2.13)$$

In Eq. (2.12), $A_0 = \omega_i a$, $B_0 = -\omega_r$, $a = \exp(\omega_i t)$, $\theta = -\omega_r t$. $A_m$ and $B_m$, $(m = 1, 2, \cdots)$ are the Landau coefficients. In Eq. (2.13), $u_i$, $(i = 1, 2, \cdots)$ is the disturbance velocity component at any order. If we take Eq. (2.13) back to Eqs. (2.9a)-(2.9c), then the weakly nonlinear evolution equations of the constant curvature bend can be obtained at any order. Semnara [9] pointed out that the analysis of the meandering river needs complete nonlinear governing equations for simulation, and the effect of the sidewall boundary layer cannot be ignored. The corresponding boundary conditions of the flow disturbance on the sidewall in our study are:

$$u_{n_i} = u_s_i = 0, \quad (n = \pm 1, \quad i = 1, 2, \cdots). \quad (2.14)$$

Then the perturbed control equations of the first order to the fifth order under the weakly nonlinear evolution of the constant curvature bend under the time mode are obtained. The perturbed control equations are solved by the central difference method.
3 Results

3.1 Model validation and neutral curve of stability theory

The critical Reynolds number ($Re_{cr}$) and neutral curve calculated when $\psi = 0$ are compared with the experiment results of the plane Poiseuille flow made by Nishioka [22]. And we take the results of $Re$ from Orszag [23] as the critical Reynolds number ($Re_{cr} = 5772.22$ and $\alpha = 1.02059$). And the results in our model, $Re_{cr} = 5772.2222$ and $\alpha = 1.02059$, are consistent with the results of Nishioka and Orzag (Fig. 4(a)). In terms of the nonlinear evolution of the disturbance velocity, we compare our results with that of Zhou and You [14] in the evolution of $u_1$, $u_{20}$ and $u_{22}$ under the conditions of $\psi = 0$, $Re = 10000$, $\alpha = 1.096$, $\omega_r = 0.270284$, $\omega_i = 0.000088$ (Fig. 4(b)), which are consistent well.

The black dots in Fig. 5 and Fig. 6 show the distribution of $Re_{cr}$ under different $\psi$. $c_r (= \omega_r / \alpha)$ is the real part of the disturbance wave velocity (Fig. 6). The results show that with the increase of $\psi$, the neutral curve moves toward the direction of the increasing $Re$. The region outside the neutral curve is stable while the region inside the neutral curve is unstable, which means with the increase of $\psi$, the area of the unstable region reduces, the flow in the bend tends to be more stable and loses the stability at the higher Reynolds number, which is consistent to the result of Bai et al. [24].

Except at $Re_{cr}$, there are two different neutral points at the neutral curve for the same $Re$ at the upper and lower branches, respectively. And with the increase of $\psi$, $\alpha$, and $c_r$ get closer to the lower branch. With the limit of the weak curvature ratio, $\psi$ increases from 0 to 0.20, $Re_{cr}$ increases exponentially, while $\alpha$ and $c_r$ show a downward trend monotonously (Fig. 7). It is because the curved boundary impacts on the flow, which relates to the interaction among $\psi$ and $\alpha$, $c_r$, as well as $Re$ in Eq. (2.11). Besides, the results also reveal that the river boundary is often curved in nature.

![Figure 4: Verification of neutral curve $Re-a$ (a) and the disturbance velocity (b).](image-url)
4 Discussion

4.1 Evolution characteristics of the disturbance amplitude and disturbance phase

After obtaining Landau coefficients of $A$ and $B$ by using the solvable condition, the initial disturbance amplitude ($a_0$) is set to be 0.01 for analysis. The values of $\alpha$ and $c_r$ near the
neutral curve (see Supplement A) are taken to analyze the influence of different $\psi$ on the evolution of the disturbance amplitude ($a$) and disturbance phase ($\theta$) near the neutral curve with damping and positive disturbances, respectively. In the following analysis, the imaginary part of the disturbance wave velocity is taken as $c_i = \frac{\omega_i}{\alpha} = 0.005$ and $-0.005$, respectively. In a high-Reynolds number shear layer, the nonlinear evolution of a pair of initially linear oblique waves can induce the explosive growth by nonlinear effects [15]. So we aim to reveal the evolution characteristics of the positive and damping disturbances under the weak nonlinear effect.

When $Re$ keeps the same, with the increase of $\psi$, the attenuation rate of $a$ increases when $c_i = -0.005$ (Fig. 8). However, when time increases to a certain value, the value of the disturbance amplitude decreases to 0, the initial disturbance will disappear after a certain period under negative $c_i$. This effect can be called nonlinear attenuation. While $c_i = 0.005$, the disturbance amplitude increases with time, and explosively grows when time reaches a critical value. And we found that the larger $\psi$ is, the larger $a$ is. However, when time reaches another critical value, the magnitude of $a$ under $\psi = 0$ is larger than that of $\psi=0.05\sim0.10$, which means that the existence of $\psi$ will reduce the increase rate of $a$ under positive $c_i$. $\theta$ presents a monotonic decreasing trend with time when $c_i$ is negative. When $c_i$ is positive and the time value is larger than critical value, with the nonlinear effect, $\theta$ is explosively increased when $\psi \leq 0.05$ and explosively decreases when $\psi \geq 0.07$. When $\psi \leq 0.05$, the lower $\psi$ is, the larger $\theta$ is, while when $\psi \geq 0.07$, the result is reversed, which is related to the Landau coefficient.

We take $\psi = 0.01$, $Re = 6000$ to $12000$ to analyze the evolution characteristics of $a$ and $\theta$ with $Re$. When $c_i = -0.005$, both $a$ and $\theta$ decrease with $Re$, and the larger $Re$ is, the larger the decreasing rate of $a$ and $\theta$ is. While $c_i$ is positive, the larger $Re$ is, the lower value of $a$ is. However, $\theta$ decreases with the increase of time. When time reaches critical values, $\theta$
is in explosive growth and the critical time value increases with $Re$.

There are two different values of the disturbance wave number or disturbance wave velocity at the neutral curve, except for $Re_{cr}$. In order to investigate the influence of the disturbance wave number on the disturbance amplitude and disturbance phase, we take $\psi = 0.01$ and 0.07, $Re = 12000$, $\alpha = 1.0859$ and 1.0614 at the upper branch, and $\alpha = 0.7579$ and 0.7715 at the lower branch and then analyze respectively, as shown in Fig. 10. With the same $Re$, time, and $c_i$, the larger $\alpha$ is, the larger the absolute value of $a$ is. And when $c_i = 0.005$, $a$ increases faster with high values of $\psi$, but when time reaches a certain value, $a$ increases slower under the effect of curvature. In terms of $\theta$, the larger $\alpha$ is, the larger the magnitude of $\theta$ is. When $c_i$ is positive, the critical time value of the explosive growth/decrease point is smaller when $\alpha$ is larger.

### 4.2 Distribution of the disturbance velocity

The real part and the imaginary part of the components of the disturbance velocity can be written as:

\[
\begin{align*}
    u_{sr} &= \text{Real} \left( \sum_{i=1}^{5} \epsilon^i u_{si} \right), & u_{si} &= \text{Img} \left( \sum_{i=1}^{5} \epsilon^i u_{si} \right), \\
    u_{nr} &= \text{Real} \left( \sum_{i=1}^{5} \epsilon^i u_{ni} \right), & u_{ni} &= \text{Img} \left( \sum_{i=1}^{5} \epsilon^i u_{ni} \right).
\end{align*}
\]

In Eq. (4.1), $\text{Real}$ and $\text{Img}$ mean the real part and imaginary part, respectively, and $\epsilon$ is any small quantity. We take $a$ (disturbance amplitude) to replace $\epsilon$, neglect the effect of the correction of the base flow and then analyze the distribution of the disturbance velocity.
4.2.1 Influence of the disturbance phase on the disturbance velocity

We aim to investigate the evolution characteristics of the disturbance velocity under different $\theta$ ranging from $-4\pi$ to $-6\pi$. And we take $\psi = 0.10$, $s = 0$, $k = \theta$, $a_0 = 0.01$.

The distribution of the streamwise disturbance velocity ($u_{sr}$) in Figs. 11(a) and (c) presents the sigmoid shape, which means that the maximum value is close to both sides. Nevertheless, there is a turning point near the region of $n = -1$. When $\theta$ ranges from $-4\pi$ to $-6\pi$, $u_{sr}$ near $n = 1$ decreases to $\theta = -5\pi$ and then increases in other periods. As $u_{sr}$ approaches to $n = -1$, the turning point increase to the maximum till $\theta = -4\pi - 2\pi/3$. The shape function of $u_{sr}$ is the same when $c_i$ is positive and negative.

The distribution of the cross-section disturbance velocity ($u_{nr}$) presents parabolic shape. The location of the maximum absolute value of the disturbance velocity ranges from $n = 0.2$ to $0.4$, which is influenced by $\psi$. And this position changes with $\theta$, which moves to $n = 1$ when $\theta$ ranges from $-4\pi - \pi/3$ to $-5\pi$. In some certain values of $\theta$ ($= -4\pi$), there exists the sigmoid shape of $u_{nr}$, and the magnitude of $u_{nr}$ near $n = 1$ is larger than that near $n = -1$.

4.2.2 Influence of the curvature ratio on the shape function of the disturbance velocity

For investigating the effect of $\psi$ on the shape of the disturbance velocity, we take $\psi = 0.03 \sim 0.15$, $Re = 10000$, $s = 0$, $c_i = -0.005$ and $0.005$. $\alpha$ and $c_r$ from the neutral curve (Supplement A) and $\theta = -4\pi$ to $-5\pi$.

The streamwise disturbance velocities ($u_{sr}$ and $u_{si}$) at the cross-section of $s = 0$ present the sigmoid shape under different $\psi$. When $\psi$ is large, such as $\psi = 0.15$, the distribution of $u_{sr}$ near $n = -1$ shows a velocity turning point. Below the point, the direction of $u_{sr}$ keeps the same with $u_{sr}$ near $n = 1$. Apart from this turning point, an extremum value near $n = -1$ occurs corresponds to the extremum value of velocity near $n = 1$. With the
increase of $\psi$, the symmetry of $u_{sr}$ and $u_{si}$ decreases gradually. The extremum value of $u_{sr}$ near $n = -1$ decreases with the increase of $\psi$, and when $\psi = 0.15$ and $\theta = -4\pi$, the magnitude of maximum value of $u_{sr}$ near $n = -1$ is only half of that near $n = 1$.

The transverse disturbance velocities ($u_{nr}$ and $u_{ni}$) at the cross-section of $s = 0$ present the parabolic shape. But when $\psi$ is larger ($\psi = 0.10, 0.15$), $\theta = -4\pi$ and $-5\pi$, the distribution of $u_{nr}$ presents the sigmoid shape, and the magnitude of $u_{nr}$ near $n < 0$ is lower than that on the other side. This phenomenon also exists in the distribution of $u_{ni}$. These results show that the distribution of $u_{nr}$ and $u_{ni}$ will move toward the direction of $n = 1$ with the increase of $\psi$, and the symmetry of the distribution of $u_{nr}/u_{ni}$ also transfers to asymmetry gradually. The shape function of the disturbance velocity keeps the same when $c_i = -0.005$ and 0.005.
4.2.3 Influence of the Reynolds number on the disturbance velocity

In order to investigate the effect of $Re$ on the disturbance velocity under different $\theta$, we take $\psi = 0.10$, $Re = 9000, 10000, 12000$, respectively, $c_i = -0.005$ and $0.005$, $s = 0$, $\alpha$ and $c_r$ from the neutral curve (Supplement A), and $\theta$ ranging from $-4\pi$ to $-5\pi$.

In Fig. 13, the characteristics of both the disturbance velocity in streamwise and transverse are as follow. Firstly, at the same $\theta$, with the increase of $Re$, the extreme values of $u_{sr}$
and $u_{ij}$ decrease when $c_i = -0.005$ and $\theta = -4\pi, -5\pi$, while when $c_i = 0.005$, the condition is reversed. The changes of $Re$ do not influence the shape function of the disturbance velocity. Secondly, with the increase of $Re$, the extreme values of $u_{nr}$ and $u_{ni}$ decrease when $c_i = -0.005$ and $\theta = -4\pi - \pi/2$, while when $c_i = 0.005$, the condition is reversed. From the analysis above, we can conclude that with the increase of $Re$, the extreme value of the disturbance velocity decreases when $c_i$ is negative and increases when $c_i$ is positive. However, the changes of $Re$ have not influenced the shape function of the disturbance velocity.
4.3 Distribution characteristics of the disturbance vorticity

For the investigation of the distribution of the velocity and vorticity in the whole constant curvature bend, we take $\psi = 0.10$, $Re = 10000$, $c_t = -0.005$ and 0.005, $\alpha$ and $c_r$ from the neutral curve (Supplement A) and $\theta$ ranging from $-4\pi$ to $-5\pi$.

The distribution of the disturbance vorticity ($\Omega = \frac{1}{h_s} \frac{\partial u_n}{\partial s} - \frac{\partial u_s}{\partial n} + u_s h_s R$) in the whole constant curvature bend is characterized that $\Omega$ is negative (the clockwise is positive) at the entrance. However, it is positive at the end of the bend when $\theta = -4\pi$ (Fig. 14). With the decrease of $\theta$ (increasing in the negative direction), the negative $\Omega$ at the entrance begins to migrate downstream. When $\theta$ decreases to $-5\pi$, $\Omega$ is positive at the entrance of the

Figure 14: The distribution of the disturbance vorticity under different $\theta$ in the whole constant curvature bend.
bend, and negative at the end of the bend. Therefore, in the whole period of $\theta$, the negative $\Omega$ begins to occupy the center of the bend and gradually moves downstream. The positive disturbance vorticity begins to develop, moving downstream and occupying the whole bend in the next half period. And the core area of $\Omega$ changes with $\theta$. When $c_i$ is negative, the core area's intensity of $\Omega$ decreases gradually with $\theta$, while it increases with $\theta$ when $c_i$ is positive.

In order to investigate the influence of $Re$ on the distribution of the disturbance vorticity in the whole constant curvature bend, we take $\psi = 0.10$, $Re = 9000, 10000, 12000$, $c_i = -0.005$ and $0.005$, $\theta = -4\pi - \pi/2$, $\alpha$ and $c_r$ from the neutral curve (Supplement A). The results are shown in Fig. 15. Through the distribution of $\Omega$ at different $Re$, it can be seen that near the center area of bend apex, with the increase of $Re$, the magnitude value of $\Omega$ decreases as $c_i = -0.005$ and increases as $c_i = 0.005$. The higher magnitude value of $\Omega$ is located near both sides. However, the variation of $Re$ does not affect the spatial and temporal evolution trend of the disturbance vorticity distribution but changes the magnitude of the disturbance vorticity.

Figure 15: The distribution of the disturbance vorticity under different $Re$ in the whole constant curvature bend.
Then, we investigate the influence of $\psi$ on the disturbance vorticity’s distribution in the whole constant curvature bend, we take $\psi = 0.05, 0.07, 0.10$, respectively, $Re = 10000$, $c_i = -0.005$ and $0.005$, $\theta = -4\pi - \pi/2$, $\alpha$ and $c_r$ from the neutral curve (Supplement A) (Fig. 16). With the increase of $\psi$ from 0.05 to 0.10, the strength of $\Omega$ around the bend center area decreases gradually, and the negative value of $\Omega$ approaches the concave band ($n = 1$) gradually. From the whole distribution of $\Omega$, the existence of $\psi$ does not cause any phase lag. The locations of the maximum value of $\Omega$ are near both side walls. $\Omega$ rapidly decays when it gets away from the side wall. Combined with Fig. 11, it can be seen that with the increase of $\psi$, $\Omega$ near the side of $n = 1$ gradually strengthens, while it weakens on the side of $n = -1$.

5 Conclusions

The characteristics of flow stability and nonlinear evolution under the influence of $\psi$ (curvature ratio), $Re$ (Reynolds number) and $\theta$ (disturbance phase) in the constant curvature
bend under damping/positive disturbances are studied in our paper. The conclusions are as follows:

1. In the constant curvature bend, with the increase of $\psi$, the neutral curve moves in the direction of the increasing $Re$, and the corresponding $Re_{cr}$ (the critical Reynolds number) also increases, making the bend tend to be stable. Under the small values of $\psi$, $\alpha$ (the disturbance wave number) and $c_r$ (the real part of the disturbance wave velocity) corresponding to $Re_{cr}$ monotonically decrease with the increase of $\psi$.

2. The nonlinear evolution characteristics of $a$ (disturbance amplitude) and $\theta$ in time mode are as follows: when $c_i$ (the imaginary part of the disturbance wave velocity) is negative, the attenuation rates of $a$ and $\theta$ are larger with $\psi$. The attenuation rates of $a$ and $\theta$ (increases in negative direction) are lower with the increase of $Re$. When $\alpha$ is larger, the attenuation rate of $a$ and $\theta$ is faster. When $c_i$ is positive and time reaches the critical value under the nonlinear effect, $a$ grows explosively, and the increasing rate is larger when $\alpha$, $\psi$, and $Re$ is larger. Moreover, the explosive growth exists in the evolution of $\theta$, and the critical time value increases with the increase of $Re$ and decreases of $\alpha$ and $\psi$, while $\psi \geq 0.07$, the explosive growth changed to explosive decrease.

3. The distribution of the streamwise disturbance velocity shows the sigmoid shape, while the parabola style in the transverse disturbance velocity, both of them changing periodically with $\theta$. The symmetry of the disturbance velocities in both the streamwise and transverse directions decreases gradually with $\psi$, and shifts toward the convex bank ($n = 1$). With the increase of $Re$, the magnitudes of the disturbance velocity decrease/increase both in the streamwise and transverse direction when $c_i$ is negative/positive. The values of $c_i$ and $Re$ rarely influence on the shape function of the disturbance velocity.

4. The characteristics of the disturbance vorticity ($\Omega$)'s evolution in the constant curvature bend are as follows: (1) In the half period of $\theta(-4\pi \sim -5\pi)$, $\Omega$ moves from the entrance to the exit gradually; in the other half period, the direction of disturbance vortex turns, and the location of the disturbance vortex core changes with $\theta$. (2) Around the center bend, $\Omega$ decays gradually and shifts to both sides with $Re$, but it would not have much influence on the spatial-temporal evolution of $\Omega$. (3) With the increase of $\psi$, $\Omega$ around the center will gradually decays, and moves to the convex bank. And $\Omega$ increases near the side of the convex bank ($n = 1$) and decays near the side of the concave bank ($n = -1$) when $\psi$ is larger.

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References