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The Neumann Problem for a Class of Fully Nonlinear Elliptic Partial Differential Equations

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Abstract. In this paper, we establish global C^2 estimates to the Neumann problem for a class of fully nonlinear elliptic equations. As an application, we prove the existence and uniqueness of k-admissible solutions to the Neumann problems.

Key Words: Neumann problem, fully nonlinear, elliptic equation.

AMS Subject Classifications: 35J60, 35J09, 35J40

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain and $\nu(x)$ be the outer unit normal at $x \in \partial\Omega$. Suppose $f \in C^2(\Omega)$ is a positive function and $a, b \in C^3(\partial\Omega)$ with $a > 0$. In this paper, we consider the Neumann problem of the fully nonlinear equation

$$\begin{cases} S_k(W) = f(x), & \Omega, \\ u_\nu = -a(x)u + b(x), & \partial\Omega, \end{cases} \tag{1.1}$$

where the matrix $W = (w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m})_{C_n^m \times C_n^m}$ with the elements as follows, for $1 \leq m \leq n-1$,

$$w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m} = \sum_{\gamma=1}^{n}\sum_{i=1}^{m} u_{\gamma\alpha_i}\delta^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}, \tag{1.2}$$

a linear combination of u_{ij}, where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\delta^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}$ is the generalized Kronecker symbol. All indexes $\alpha_1, \beta_1, \cdots$ come from 1 to n. For each $1 \leq k \leq C_n^m$, we define

$$S_k(W) = S_k\big(\lambda(W)\big) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq C_n^m} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

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where $\lambda(W)=(\lambda_1,\lambda_2,\cdots,\lambda_{C_n^m})$ is the set of eigenvalues of W. We also set $S_0(W)=1$.

In fact, the matrix W comes from the following operator $U^{[m]}$ as in [2] and [10]. First, we note that $(u_{ij})_{n\times n}$ induces an operator U on $\mathbb{R}^n$ by

$$U(e_i)=\sum_{j=1}^n u_{ij}e_j,\quad \forall 1\le i\le n,$$

where $\{e_1,e_2,\cdots,e_n\}$ is the standard basis of $\mathbb{R}^n$. We further extend U to act on the real vector space $\wedge^m\mathbb{R}^n$ by

$$U^{[m]}(e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_m})=\sum_{i=1}^m e_{\alpha_1}\wedge\cdots\wedge U(e_{\alpha_i})\wedge\cdots\wedge e_{\alpha_m},$$

where $\{e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_m}|1\le\alpha_1<\cdots<\alpha_m\le n\}$ is the standard basis for $\wedge^m\mathbb{R}^n$. Then W is the matrix of $U^{[m]}$ under this standard basis. It is convenient to denote the multi-index by $\bar{\alpha}=(\alpha_1\cdots\alpha_m)$. We only consider the admissible multi-index, that is, $1\le\alpha_1<\alpha_2\cdots<\alpha_m\le n$. By the dictionary arrangement, we can arrange all admissible multi-indexes from 1 to C_n^m and use $N_{\bar{\alpha}}$ denote the order number of the multi-index $\bar{\alpha}=(\alpha_1\cdots\alpha_m)$, i.e., $N_{\bar{\alpha}}=1$ for $\bar{\alpha}=(12\cdots m)$, $\cdots$. We also use $\bar{\alpha}$ denote the index set $\{\alpha_1,\cdots,\alpha_m\}$. It is not hard to see that

$$W_{N_{\bar{\alpha}}N_{\bar{\alpha}}}=w_{\bar{\alpha}\bar{\alpha}}=\sum_{i=1}^m u_{\alpha_i\alpha_i}, \tag{1.3a}$$

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}}=w_{\bar{\alpha}\bar{\beta}}=(-1)^{|i-j|}u_{\alpha_i\beta_j}, \tag{1.3b}$$

when the index set $\{\alpha_1,\cdots,\alpha_m\}\setminus\{\alpha_i\}$ equals to the index set $\{\beta_1,\cdots,\beta_m\}\setminus\{\beta_j\}$ but $\alpha_i\ne\beta_j$; and also

$$W_{N_{\bar{\alpha}}N_{\bar{\beta}}}=w_{\bar{\alpha}\bar{\beta}}=0, \tag{1.4}$$

when the index sets $\{\alpha_1,\cdots,\alpha_m\}$ and $\{\beta_1,\cdots,\beta_m\}$ have more than one different element. It follows that W is symmetrical and diagonal with $(u_{ij})_{n\times n}$ diagonal. The eigenvalues of W are the m-sums of eigenvalues of $(u_{ij})_{n\times n}$.

Define the Gårding's cone in $\mathbb{R}^n$ by

$$\Gamma_k=\{\mu\in\mathbb{R}^n|S_i(\mu)>0,\ \forall 1\le i\le k\},\quad 1\le k\le n.$$

For $\mu\in\mathbb{R}^n$, we let

$$\Psi=\Psi(\mu)=\{\mu_{i_1}+\cdots+\mu_{i_m}|1\le i_1<\cdots<i_m\le n\}\in\mathbb{R}^{C_n^m}.$$

Then we define the generalized Gårding's cone as follows,

$$\Gamma_k^{(m)}=\{\mu\in\mathbb{R}^n|S_i(\Psi)>0,\ \forall 1\le i\le k\},\quad 1\le m\le n,\quad 1\le k\le C_n^m. \tag{1.5}$$

Obviously,

$$\Gamma_k = \Gamma_k^{(1)} \quad \text{and} \quad \Gamma_n \subset \Gamma_k^{(m)} \subset \Gamma_1.$$

If the set of eigenvalues of D^2u, denoted by $\mu(D^2u)$, is contained in $\Gamma_k^{(m)}$ for any $x \in \Omega$, then equivalently $\lambda(W) \in \Gamma_k$, such that the operator $S_k(W)$ is elliptic (see [2] or [19]). It is naturally to define k-admissible function.

Definition 1.1. *We say u is k-admissible if* $\mu(D^2u) \in \Gamma_k^{(m)}$.

If $m = 1$, (1.1) is known as the k-Hessian equation. In particular, (1.1) is the Poisson equation if $k = 1$ and the Monge-Ampère equation if $k = n$, $m = 1$.

For the Dirichlet problem in $\mathbb{R}^n$, many results are known. For example, the Dirichlet problem of Laplace equation is studied in [8], Caffarelli-Nirenberg-Spruck [1] and Ivochkina [16] solved the Dirichlet problem of Monge-Ampère equation and Caffarelli-Nirenberg-Spruck [2] solved the Dirichlet problem of general Hessian equations even including the case considered here. For the general Hessian quotient equation, the Dirichlet problem is solved by Trudinger in [33]. Finally, Guan [9] treated the Dirichlet problem for general fully nonlinear elliptic equation on the Riemannian manifolds.

Also, the Neumann or oblique derivative problem of partial differential equations was widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book [8]. Also, we can see the book written by Lieberman [20] for the Neumann or oblique derivative problem of linear and quasilinear elliptic equations. In 1987, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equation in the celebrated paper [23]. For the the Neumann problem of k-Hessian equations, Trudinger [32] established the existence theorem when the domain is a ball. Recently, Ma and Qiu [24] solved the the Neumann problem of k-Hessian equations in uniformly convex domains. After their work, the research on the Neumann problem of other equations has made many progresses (see [4, 5, 25, 35]). Furthermore, Jiang and Trudinger [17, 18] studied the general oblique boundary value problems for augmented Hessian equations with some regular conditions and concavity conditions.

For general m, the W-matrix is quite related to the "m-convexity" or "m-positivity" in differential geometry and partial differential equations. We say a C^2 function u is m-convex if the sum of any m eigenvalues of its Hessian is nonnegative, equivalently, $\mu(D^2u) \in \overline{\Gamma_{C_n^m}^{(m)}}$ or $\lambda(W) \in \overline{\Gamma_{C_n^m}}$. Similarly, we can formulate the notion of m-convexity for curvature operator and second fundamental forms of hypersurfaces. There are large amount literature in differential geometry on this subject. For example, Sha [26] and Wu [36] introduced the m-convexity of the sectional curvature of Riemannian manifolds and studied the topology for these manifolds. In a series interesting papers, Harvey and Lawson [11–13] introduced some generalized convexities on the solutions of the nonlinear elliptic Dirichlet problem, m-convexity is a special case. Han-Ma-Wu [10] obtained an existence theorem of m-convex starshaped hypersurface with prescribed mean curvature.

In the complex space $\mathbb{C}^n$, we consider W with the Hessian replaced by the Hermitian, and the Monge-Ampère equation of W is related to the Gauduchon conjecture which is solved by [29–31].

From the above geometry and analysis reasons, it is naturally to study the Neumann problem for general equation (1.1).

The methods of Ma and Qiu [24] for the problem with $m = 1$ can be generalized to our case. The key ingredient in the present paper is to understand the structure of W, precisely, to replace the eigenvalues of D^2u by the sums of them. For $k \leq C_{n-1}^{m-1}$, we obtain an existence theorem of the k-admissible solution.

It seems that as the degree of nonlinearity of the Eq. (1.1) increases, i.e., k becomes larger, the problem becomes more difficult to solve. Particularly, for $m = n-1$, we get the existence of the k-admissible solution for $k \leq n-1$ only except that of the strictly $(n-1)$-convex solution for $k = n$. The latter is settled in [7] when the domain is almost a ball. It is interesting to solve the latter case for general convex domain.

Theorem 1.1. *Suppose $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with C^4 boundary, $2 \leq m \leq n-1$ and $2 \leq k \leq C_{n-1}^{m-1}$. Denote $\kappa_{\min}(x)$ the minimum principal curvature with respect to $\nu(x)$ at $x \in \partial\Omega$. Let $f \in C^2(\Omega)$ is a positive function and $a, b \in C^3(\partial\Omega)$ with $a > 0$, $a + 2\kappa_{\min} > 0$. Then there exists a unique k-admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ of the Neumann problem* (1.1).

The rest of this paper is arranged as follows. In Section 2, we give some basic properties of the elementary symmetric functions. In Sections 3 and 4, we establish C^0 estimates and the gradient estimates, interior and global. In Section 5, we show the proof of the global estimates of second order derivatives. Finally, we can prove the existence theorem by the method of continuity in Section 6.

2 Preliminary

In this section, we give some basic properties of elementary symmetric functions.

First, we denote by $S_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$ and $S_k(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Proposition 2.1. *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \cdots, n$, then*

$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n, \tag{2.1a}$$

$$\sum_{i=1}^n \lambda_i S_{k-1}(\lambda|i) = kS_k(\lambda), \tag{2.1b}$$

$$\sum_{i=1}^n S_k(\lambda|i) = (n-k)S_k(\lambda). \tag{2.1c}$$

We denote by $S_k(W|i)$ the symmetric function with W deleting the i-row and i-column and $S_k(W|ij)$ the symmetric function with W deleting the i,j-rows and i,j-columns. Then we have the following identities.

Proposition 2.2. *Suppose $A=(a_{ij})_{n\times n}$ is diagonal and k is a positive integer, then*

$$\frac{\partial S_k(A)}{\partial a_{ij}}=\begin{cases} S_{k-1}(A|i), & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases} \tag{2.2}$$

Furthermore, suppose $W=(w_{\bar{\alpha}\bar{\beta}})_{C_n^m\times C_n^m}$ defined as in (1.2) *is diagonal, then*

$$\frac{\partial S_k(W)}{\partial u_{ij}}=\begin{cases} \sum_{i\in\bar{\alpha}} S_{k-1}(W|N_{\bar{\alpha}}), & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases} \tag{2.3}$$

Proof. For (2.2), see a proof in [19]. Note that

$$\frac{\partial S_k(W)}{\partial u_{ij}}=\sum_{\bar{\alpha},\bar{\beta}}\frac{\partial S_k(W)}{\partial w_{\bar{\alpha}\bar{\beta}}}\frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{ij}}.$$

Using (1.3a), (1.3b) and (1.4), the identigy (2.3) is immediately a consequence of (2.2). □

Proposition 2.3. *Let $\lambda\in\Gamma_k$ and $k\in\{1,\cdots,n\}$. Suppose that*

$$\lambda_1\geq\cdots\geq\lambda_k\geq\cdots\geq\lambda_n,$$

then we have

$$S_{k-1}(\lambda|n)\geq\cdots\geq S_{k-1}(\lambda|k)\geq\cdots\geq S_{k-1}(\lambda|1)>0, \tag{2.4a}$$

$$\lambda_1\geq\cdots\geq\lambda_k>0,\quad S_{k-1}(\lambda|k)\geq C(n,k)S_k(\lambda), \tag{2.4b}$$

$$\lambda_1 S_{k-1}(\lambda|1)\geq\frac{k}{n}S_k(\lambda), \tag{2.4c}$$

$$S_k^{\frac{1}{k}}(\lambda)\ \text{ is concave in }\ \Gamma_k. \tag{2.4d}$$

where $C(n,k)$ is a positive constant depends only on n and k.

Proof. All the properties are well known. For example, see [19] or [15] for a proof of (2.4a), [22] for (2.4b), [6] or [14] for (2.4c) and [2] for (2.4d). □

The Newton-Maclaurin inequality is as follows.

Proposition 2.4. *For $\lambda\in\Gamma_k$ and $k>l>0$, we have*

$$\Big(\frac{S_k(\lambda)}{C_n^k}\Big)^{\frac{1}{k}}\leq\Big(\frac{S_l(\lambda)}{C_n^l}\Big)^{\frac{1}{l}}, \tag{2.5}$$

where $C_n^k=\frac{n!}{k!(n-k)!}$. Furthermore we have

$$\sum_{i=1}^n\frac{\partial S_k^{\frac{1}{k}}}{\partial\lambda_i}\geq[C_n^k]^{\frac{1}{k}}. \tag{2.6}$$

Proof. See [28] for a proof of (2.5). For (2.6), we use (2.5) and Proposition 2.1 to get

$$\sum_{i=1}^{n}\frac{\partial S_k^{\frac{1}{k}}(\lambda)}{\partial\lambda_i}=\frac{1}{k}S_k^{\frac{1}{k}-1}\sum_{i=1}^{n}S_{k-1}(\lambda|i)=\frac{n-k+1}{k}S_k^{\frac{1}{k}-1}S_{k-1}(\lambda)\geq[C_n^k]^{\frac{1}{k}}.$$

Thus, we complete the proof. □

Proposition 2.5. *Suppose $\lambda=(\lambda_1,\cdots,\lambda_n)\in\Gamma_k$, $k\geq 1$, satisfing $\lambda_1<0$. Then we have*

$$\frac{\partial S_k(\lambda)}{\partial\lambda_1}\geq\frac{1}{n-k+1}\sum_{i=1}^{n}\frac{\partial S_k}{\partial\lambda_i}, \tag{2.7a}$$

$$\sum_{i=1}^{n}\frac{\partial S_k(\lambda)}{\partial\lambda_i}\geq(-\lambda_1)^{k-1},\quad\forall 1\leq k\leq n. \tag{2.7b}$$

Proof. See Lemma 3.9 in [3] for the proof of (2.7a) and [4] or [6] for (2.7b). □

Throughout the rest of this paper, we always admit the Einstein's summation convention. All repeated indices come from 1 to n. We always denote C a positive constant depends only on some known data and C may change line by line. We will denote $F(D^2u)=S_k(W)$ and

$$F^{ij}=\frac{\partial F(D^2u)}{\partial u_{ij}}=\frac{\partial S_k(W)}{\partial w_{\bar\alpha\bar\beta}}\frac{\partial w_{\bar\alpha\bar\beta}}{\partial u_{ij}}. \tag{2.8}$$

From (1.3a) and (2.3) we have, for any $1\leq i\leq n$,

$$F^{ii}=\sum_{i\in\bar\alpha}\frac{\partial S_k(W)}{\partial w_{\bar\alpha\bar\alpha}}. \tag{2.9}$$

We always denote

$$\mathcal{F}=\sum_{i=1}^{n}F^{ii}=m\sum_{N_{\bar\alpha}=1}^{C_n^m}S_{k-1}(W|N_{\bar\alpha})$$

for simplicity.

3 Zero order estimate

Following the idea of Lions-Trudinger-Urbas [23], we prove the following theorem.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ be a bounded domain with C^1 boundary. Suppose that $u\in C^2(\overline{\Omega})\cap C^3(\Omega)$ is a k-admissible solution of the following Neumann boundary problem,*

$$\begin{cases}S_k(W)=f(x), & \Omega,\\ u_\nu=-a(x)u+b(x), & \partial\Omega,\end{cases} \tag{3.1}$$

where $f > 0$ and $a, b \in C^3(\partial\Omega)$ with

$$\inf_{\partial\Omega} a(x) > \sigma.$$

Then

$$\sup_{\overline{\Omega}} |u| \leq \frac{C}{\sigma}, \tag{3.2}$$

where C depends only on k, n, a, b, f and $diam(\Omega)$.

Proof. Because $f > 0$, the comparison principle tells us that u attains its maximum on the boundary. At the maximum point $x_0 \in \partial\Omega$ we have

$$0 \leq u_\nu(x_0) = (-au + b)(x_0).$$

It implies that

$$u(x) \leq u(x_0) \leq \frac{\sup\limits_{\partial\Omega} b}{\inf\limits_{\partial\Omega} a}.$$

Assume $0 \in \Omega$ and let $w = u - A|x|^2$. We obtain, denoting

$$\begin{aligned} &F(D^2u) = S_k(W), \\ &F(2A\delta_{ij}) \geq f = F(D^2u), \end{aligned}$$

by choosing A large enough depending on k, n and $\sup f$. Similarly w attains its minimum on the boundary by comparison principle. At the minimum point $x_1 \in \partial\Omega$ we have

$$0 \geq w_\nu(x_1) = (-au + b)(x_1) - 2Ax_0 \cdot \nu.$$

We use $w(x) \geq w(x_1)$ to get

$$u(x) \geq -\frac{\left|\inf\limits_{\partial\Omega} b - 2AL(L+1)\right|}{\sup\limits_{\partial\Omega} a} \geq -\frac{\left|\inf\limits_{\partial\Omega} b - 2AL(L+1)\right|}{\inf\limits_{\partial\Omega} a},$$

where $L = diam(\Omega)$. Then we complete the proof of Theorem 3.1. □

4 Gradient estimate

4.1 Interior gradient estimate

Chou-Wang [6] gave the interior gradient estimates for k-Hessian equations. In a similar way, we will prove the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $2 \leq k \leq C_n^m$. Suppose that $u \in C^3(\Omega)$ is a k-admissible solution of the following equation,*

$$S_k(W) = f(x,u) \quad in \;\; \Omega, \tag{4.1}$$

where $f(x,z) \in C^1(\overline{\Omega} \times [-M_0, M_0])$ is a positive function,

$$M_0 = \sup_{\overline{\Omega}} |u|.$$

We also assume that

$$|f|_{C^1(\overline{\Omega} \times [-M_0, M_0])} \leq L_1 \tag{4.2}$$

for some constant L_1. Then, for any $B_r(y) \subset \Omega$, we have

$$\sup_{B_{\frac{r}{2}}(y)} |Du| \leq C_1 + C_2 \frac{M_0}{r}, \tag{4.3}$$

where C_1 depends only on M_0, L_1, $\min f$, n, m and k and C_2 depends only on L_1, $\min f$, n, m and k. Moreover, if $f \equiv$ constant, then $C_1 = 0$.

Proof. Assume $y = 0 \in \Omega$ and $B_r(0) \subset \Omega$. Choose the auxiliary function as

$$G(x) = \rho(x)\varphi(u)|Du|^2,$$

where $\rho(x) = (1 - \frac{x^2}{r^2})^2$ such that

$$|D\rho| \leq b_0 \rho^{\frac{1}{2}} \quad \text{and} \quad |\nabla^2 \rho| \leq b_0^2$$

with $b_0 = \frac{4}{r}$ and $\varphi(u) = (M-u)^{-\frac{1}{2}}$ with $M = 4M_0$. It is easy to see that

$$\varphi'' - \frac{2(\varphi')^2}{\varphi} \geq \frac{1}{16} M^{-\frac{5}{2}}. \tag{4.4}$$

Suppose G attains its maximum at the point $x_0 \in \Omega = B_r(0)$. In the following, all the calculations are at x_0. First, we have

$$0 = G_i(x_0) = \rho_i \varphi |Du|^2 + \rho u_i \varphi' |Du|^2 + 2\rho\varphi u_k u_{ki}, \quad i = 1, \ldots, n.$$

After a rotation of the coordinates, we may assume that the matrix $(u_{ij})_{n\times n}$ is diagonal at x_0, so are W and $(F^{ij})_{n\times n}$. The above identity can be rewrote as

$$u_i u_{ii} = -\frac{1}{2\rho\varphi}(\varphi\rho_i + \rho\varphi' u_i)|Du|^2, \quad i = 1,\dots,n. \tag{4.5}$$

We also have

$$\begin{aligned} G_{ij}(x_0) =& 2\rho\varphi u_k u_{kij} + 2\rho\varphi u_{ki} u_{kj} + 2\rho\varphi'(u_i u_k u_{kj} + u_j u_k u_{ki}) \\ &+ 2\varphi(\rho_i u_k u_{kj} + \rho_j u_k u_{ki}) + \rho u_{ij}\varphi'|Du|^2 + \rho\varphi''|Du|^2 u_i u_j \\ &+ \varphi''|Du|^2(\rho_i u_j + \rho_j u_i) + \rho_{ij}\varphi|Du|^2. \end{aligned} \tag{4.6}$$

Using the maximum principle to get

$$\begin{aligned} 0 \geq& F^{ij}G_{ij} = F^{ii}G_{ii} \\ =& 2\rho\varphi u_k F^{ii} u_{iik} + 2\rho\varphi F^{ii} u_{ii}^2 + 4\rho\varphi' F^{ii} u_i^2 u_{ii} + 4\varphi F^{ii}\rho_i u_i u_{ii} \\ &+ \rho\varphi'|Du|^2 F^{ii} u_{ii} + \rho\varphi''|Du|^2 F^{ii} u_i^2 + 2\varphi'|Du|^2 F^{ii}\rho_i u_i + F^{ii}\rho_{ii}\varphi|Du|^2. \end{aligned} \tag{4.7}$$

From the facts that

$$F^{ii}u_{ii} = kf, \quad F^{ii}u_{iil} = f_{x_l} + f_z u_l, \tag{4.8}$$

we have

$$\begin{aligned} 0 \geq& 2\rho\varphi u_l(f_l + f_z u_l) + 2\rho\varphi F^{ii}u_{ii}^2 + 4\rho\varphi' F^{ii} u_i^2 u_{ii} + 4\varphi F^{ii}\rho_i u_i u_{ii} \\ &+ mf\rho\varphi'|Du|^2 + \rho\varphi''|Du|^2 F^{ii}u_i^2 + 2\varphi'|Du|^2 F^{ii}\rho_i u_i + F^{ii}\rho_{ii}\varphi|Du|^2. \end{aligned} \tag{4.9}$$

Assume $|Du|(x_0) \geq b_0 + 1$, otherwise we have (4.3). By (4.2) and (4.5), we obtain

$$\begin{aligned} 0 \geq& -4L_1\varphi|Du|^2 + 2\rho\varphi F^{ii}u_{ii}^2 - 2\varphi'|Du|^2 F^{ii}u_i\rho_i - \frac{2\varphi|Du|^2}{\rho}F^{ii}\rho_i^2 \\ &+ \Big(\varphi'' - \frac{2\varphi'^2}{\varphi}\Big)\rho|Du|^2 F^{ii}u_i^2 + \varphi|Du|^2 F^{ii}\rho_{ii}. \end{aligned} \tag{4.10}$$

By (4.4) and properties of ρ we have

$$0 \geq 2\rho\varphi F^{ii}u_{ii}^2 - 2b_0\varphi'\rho^{\frac{1}{2}}|Du|^3\mathcal{F} - 3b_0^2\varphi|Du|^2\mathcal{F} - 4L_1\varphi|Du|^2. \tag{4.11}$$

Assume

$$G(x_0) \geq 20nb_0^2 M^{\frac{3}{2}},$$

otherwise we have (4.3), which implies that

$$|Du| \geq \frac{2\sqrt{5n}b_0 M^{\frac{3}{4}}}{\rho^{\frac{1}{2}}\varphi^{\frac{1}{2}}}$$

at x_0. There exists at least one index i_0 such that $|u_{i_0}| \geq \frac{|Du|}{\sqrt{n}}$. By (4.5), it is not hard to get

$$\begin{aligned} u_{i_0i_0} =& -\Big(\frac{\varphi'}{2\varphi}+\frac{\rho_{i_0}}{2\rho u_{i_0}}\Big)|Du|^2 \\ \leq& -(\frac{\varphi'}{2\varphi}-\frac{1}{20M})|Du|^2 \\ \leq& -\frac{\varphi'}{4\varphi}|Du|^2. \end{aligned} \tag{4.12}$$

Let $u_{11} \geq \cdots \geq u_{nn}$, from (2.4a) and (4.12) we have

$$u_{nn} \leq -\frac{\varphi'|Du|^2}{4\varphi}, \quad F^{11} \leq \cdots \leq F^{nn}. \tag{4.13}$$

The second part implies that

$$\begin{aligned} F^{nn} \geq& \frac{1}{n}\mathcal{F} = \frac{m}{n}\sum_{i=1}^{C_n^m} S_{k-1}(\lambda|i) \\ =& \frac{m(C_n^m-k+1)}{n} S_{k-1}(\lambda). \end{aligned} \tag{4.14}$$

By the Newton-Maclaurin inequality, we have

$$F^{nn} \geq \frac{1}{n}\mathcal{F} \geq cS_k^{\frac{1}{k}}(\lambda) \geq c(\min f)^{\frac{1}{k}}, \tag{4.15}$$

where $c=c(n,m,k)$ is a universal constant. Returning to (4.11) we have

$$\begin{aligned} 0 \geq& 2\rho\varphi F^{nn}u_{nn}^2 - 2b_0\varphi'\rho^{\frac{1}{2}}|Du|^3\mathcal{F} - 3b_0^2\varphi|Du|^2\mathcal{F} - 4L_1\varphi|Du|^2 \\ \geq& \frac{\rho\varphi'^2}{8n\varphi}|Du|^4\mathcal{F} - 2b_0\varphi'\rho^{\frac{1}{2}}|Du|^3\mathcal{F} - 3b_0^2\varphi|Du|^2\mathcal{F} - 4L_1\varphi|Du|^2. \end{aligned} \tag{4.16}$$

Both sides of (4.16) multiplied by $\rho\varphi^3$, then we have

$$0 \geq \Big(\frac{2G^2}{125nM^3} - \frac{8b_0G^{\frac{3}{2}}}{3M^{\frac{9}{4}}} - \frac{6b_0^2G}{M^{\frac{3}{2}}}\Big)\mathcal{F} - \frac{4L_1}{M^{\frac{5}{4}}}G. \tag{4.17}$$

Plugging (4.15) into (4.17), then

$$0 \geq \frac{2G}{125nM^3} - \frac{8b_0G^{\frac{1}{2}}}{3M^{\frac{9}{4}}} - \frac{6b_0^2}{M^{\frac{3}{2}}} - \frac{4c^{-1}(\min f)^{-\frac{1}{k}}L_1}{M^{\frac{5}{4}}}.$$

Thus we have

$$\sup_{B_{\frac{r}{2}}}|Du| \leq C_1 + C_2\frac{M}{r},$$

where C_1 depends only on M_0, L_1, $\min f$, n, m and k and C_2 depends only on L_1, $\min f$, n, m and k. It is not hard to see that $C_1 = 0$ when $f \equiv$ constant. □

4.2 Gradient estimate near boundary

In this subsection, we will establish a gradient estimate in the small neighborhood near boundary. We use a similar method as in Ma-Qiu [24] with minor changes. We define

$$d(x) = dist(x, \partial\Omega), \quad \Omega_\mu = \{x \in \Omega | \, d(x) < \mu\}. \tag{4.18}$$

It is well known that there exists a small positive universal constant μ_0 such that $d(x) \in C^4(\Omega_\mu)$, $\forall 0 < \mu \le \mu_0$, provided $\partial\Omega \in C^4$. As in Simon-Spruck [27] or Lieberman [20] (in page 331), we can extend ν by $\nu = -Dd$ in Ω_μ and note that ν is a $C^4(\overline{\Omega_\mu})$ vector field. As mentioned in the book [20], we also have the following formulas, in Ω_μ,

$$\begin{cases} |D\nu| + |D^2\nu| \le C(n, \Omega), \\ \sum_{i=1}^n \nu^i D_j \nu^i = \sum_{i=1}^n \nu^i D_i \nu^j = \sum_{i=1}^n d_i d_{ij} = 0, \quad |\nu| = |Dd| = 1. \end{cases} \tag{4.19}$$

Theorem 4.2. *Suppose $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ is a bounded domain with C^3 boundary and $2 \le k \le C_n^m$. Let $f(x,z) \in C^1(\overline{\Omega} \times [-M_0, M_0])$ is a positive function and*

$$\phi \in C^3(\overline{\Omega} \times [-M_0, M_0]), \quad M_0 = \sup_{\overline{\Omega}} |u|.$$

We also assume that there exists constants L_1 and L_2 such that

$$|f|_{C^1(\overline{\Omega} \times [-M_0, M_0])} \le L_1,$$
$$|\phi|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \le L_2.$$

If $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ is a k-admissible solution of the equation

$$\begin{cases} S_k(W) = f(x,u), & \Omega, \\ u_\nu = \phi(x,u), & \partial\Omega. \end{cases} \tag{4.20}$$

Then we have

$$\sup_{\Omega_\mu} |Du| \le C, \tag{4.21}$$

with C depending only on n, k, m, μ, M_0, L_1, L_2 and Ω.

Proof. Let $0 < \mu \le \mu_0$ and

$$G(x) := \log |Dw|^2 + h(u) + \alpha_0 d(x), \quad \Omega_\mu,$$

where

$$w(x) = u(x) + \phi(x,u) d(x), \tag{4.22a}$$

$$h(x) = -\frac{1}{2} \log(1 + 4M_0 - u), \quad h'' - 2h'^2 = 0, \tag{4.22b}$$

and α_0 is a constant to be determined.

Case 1: G attains its maximum on the boundary $\partial\Omega$. If we assume that $|Du| > 8nL_2$ and $\mu \leq \frac{1}{2L_2}$, it follows from (4.26) that

$$\frac{1}{4}|Du| \leq |Dw| \leq 2|Du|.$$

Assume x_0 is the maximum point of G, then we have

$$\begin{aligned} 0 \leq G_\nu(x_0) =& \frac{D(|Dw|^2)\cdot\nu}{|Dw|^2} + h'u_\nu + \alpha_0 Dd\cdot\nu \\ =& \frac{D(|Dw|^2)\cdot\nu}{|Dw|^2} + h'\phi - \alpha_0, \end{aligned} \tag{4.23}$$

since $\nu = -Dd$.

On the boundary $\partial\Omega$, by the Neumann condition,

$$\begin{aligned} D(|Dw|^2)\cdot\nu =& - w_i w_{ij} d_j \\ =& - (u_i + \phi d_i)(u_{ij} + D_{ij}\phi d + D_i\phi d_j + D_j\phi d_i + \phi d_{ij})d_j \\ =& - (u_i + \phi d_i)(D_i(u_j d_j) - u_j d_{ij} + D_i\phi + D_j\phi d_i d_j) \\ =& (u_i + \phi d_i)(u_j d_{ij} - \phi_z u_j d_i d_j - \phi_{x_j} d_i d_j) \\ \leq& C(|Dw|^2 + |Dw|), \end{aligned} \tag{4.24}$$

where $C = C(|d|_{C^2}, |\phi|_{C^1})$. Plugging (4.24) into (4.23) to get

$$\begin{aligned} 0 \leq& G_\nu \leq C + \frac{C}{|Dw|} + h'|\phi| - \alpha_0 \\ \leq& - C + \frac{C}{|Dw|}, \end{aligned} \tag{4.25}$$

provided

$$\alpha_0 = 2C + \frac{2L_2}{1+M} + 1.$$

Thus, we have

$$|Dw|(x_0) \leq 1 \quad \text{and} \quad G(x_0) \leq \alpha_0.$$

Case 2: G attains its maximum on the interior boundary $\partial\Omega_\mu \cap \Omega$. It follows from the interior gradient estimate (4.3) that

$$\sup_{\partial\Omega_\mu\cap\Omega} |Dw|(x_0) \leq C,$$

where C depends only on M, L_1, μ, n, m and k. Thus we also have an upper bound for $G(x_0)$.

Case 3: G attains its maximum at some point $x_0 \in \Omega_\mu$. We have

$$w_i = (1+\phi_z d)u_i + R_i, \tag{4.26a}$$
$$R_i = \phi_i d + \phi d_i, \tag{4.26b}$$

and the second derivatives

$$w_{ij} = (1+\phi_z d)u_{ij} + R_{ij} \tag{4.27}$$

with

$$\begin{aligned} R_{ij} =& d\phi_{zz}u_i u_j + (d\phi_{iz}u_j + d\phi_{zj}u_i + d_i\phi_z u_j + d_i\phi_z u_i) \\ &+ (d\phi_{ij} + d_i\phi_j + d_j\phi_i + d_{ij}\phi). \end{aligned} \tag{4.28}$$

It is easy to see that

$$|R_i| \le 2L_2, \quad |R_{ij}| \le C(\mu|Du|^2 + |Du| + 1), \tag{4.29}$$

where $C = C(L_2, n, |d|_{C^3})$. The third derivatives are more complicated,

$$\begin{aligned} w_{ijl} =& (1+\phi_z d)u_{ijl} + d\phi_{zzz}u_i u_j u_l + R_{ijl} \\ &+ (d\phi_{zz}u_j u_{il} + d\phi_{zz}u_i u_{jl} + d\phi_{zz}u_l u_{ij}) \\ &+ (d\phi_{iz}u_{jl} + d\phi_{jz}u_{il} + d_i\phi_z u_{jl} + d_j\phi_z u_{il} + d\phi_{zl}u_{ij} + d_l\phi_z u_{ij}), \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} R_{ijl} =& (d\phi_{izz}u_l u_j + d\phi_{jzz}u_l u_i + d\phi_{zzl}u_i u_j + d_i\phi_{zz}u_l u_j + d_j\phi_{zz}u_l u_i + d_l\phi_{zz}u_i u_j) \\ &+ (d\phi_{ijz}u_l + d\phi_{izl}u_j + d\phi_{jzl}u_i + d_l\phi_{zj}u_i + d_l\phi_{iz}u_j + d_i\phi_{jz}u_l + d_i\phi_{zl}u_j \\ &+ d_j\phi_{iz}u_l + d_j\phi_{zl}u_i + d_{ij}\phi_z u_l + d_{il}\phi_z u_j + d_{jl}\phi_z u_i) \\ &+ (d\phi_{ijp} + d_l\phi_{ij} + d_j\phi_{il} + d_i\phi_{jl} + d_{ij}\phi_l + d_{jl}\phi_i + d_{il}\phi_j + d_{ijl}\phi). \end{aligned} \tag{4.31}$$

So we have

$$|R_{ijl}| \le C(|Du|^2 + |Du| + 1) \quad \text{with} \quad C = C(|d|_{C^3}, L_2).$$

We compute at the maximum point $x_0 \in \Omega_\mu$,

$$0 = G_i(x_0) = \frac{2w_l w_{li}}{|Dw|^2} + \alpha_0 d_i + h' u_i, \quad i = 1, \dots, n, \tag{4.32}$$

and

$$G_{ij}(x_0) = \frac{2w_{li}w_{lj}}{|Dw|^2} + \frac{2w_l w_{lij}}{|Dw|^2} - \frac{4w_l w_{li} w_q w_{qj}}{|Dw|^4} + \alpha_0 d_{ij} + h'' u_i u_j + h' u_{ij}. \tag{4.33}$$

By the maximum principle we have

$$\begin{aligned}0 \geq& F^{ij}G_{ij} = F^{ii}G_{ii}\\ =&\frac{2F^{ii}w_{li}^2}{|Dw|^2} + \frac{2w_lF^{ii}w_{iil}}{|Dw|^2} - \frac{4F^{ii}(w_lw_{li})^2}{|Dw|^4} + \alpha_0F^{ii}d_{ii} + h''F^{ii}u_i^2 + h'F^{ii}u_{ii}.\end{aligned} \tag{4.34}$$

The (4.32) implies that

$$2w_lw_{li} = -(\alpha_0d_i + h'u_i)|Dw|^2.$$

By the Cauchy-Schwartz inequality, then

$$\begin{aligned}\frac{4F^{ii}(w_lw_{li})^2}{|Dw|^4} =&\alpha_0F^{ii}d_i^2 + 2\alpha_0h'F^{ii}u_id_i + h'^2F^{ii}u_i^2\\ \leq&2h'^2F^{ii}u_i^2 + C\mathcal{F},\end{aligned} \tag{4.35}$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3})$. Combining (4.8), (4.22b), (4.35) with (4.34), we get

$$0 \geq \frac{2F^{ii}w_{li}^2}{|Dw|^2} + \frac{2w_lF^{ii}w_{iil}}{|Dw|^2} - C\mathcal{F}. \tag{4.36}$$

We may assume that

$$\mu \leq \frac{1}{2L_2} \quad \text{and} \quad |Du|(x_0) \geq 16nL_2 + 1,$$

so that

$$\frac{1}{2} \leq 1 + \phi_zd \leq 1 \quad \text{and} \quad \frac{1}{8}|Du|^2 \leq |Dw|^2 \leq \frac{3}{2}|Du|^2.$$

By (4.30), we have

$$\begin{aligned}\frac{2w_lF^{ii}w_{iil}}{|Dw|^2} =&\frac{1}{|Dw|^2}\Big(2(1+\phi_zd)w_iD_if + 2d\phi_{zzz}w_lu_lF^{ii}u_i^2\\ &+ 4d\phi_{zz}F^{ii}u_{ii}u_iw_i + (2d\phi_{zz}w_lu_l + 2d\phi_{zl}w_l + \phi_zd_lw_l)F^{ii}u_{ii}\\ &+ 4d\phi_{iz}F^{ii}u_{ii}w_i + 4\phi_zd_iF^{ii}u_{ii}w_i + 2F^{ii}R_{iil}w_l\Big)\\ \geq& -\frac{C}{|Dw|^2}\Big(\mu|Dw|^4 + |Dw|^3 + \Big(\mu + \frac{1}{\mu}\Big)|Dw|^2 + |Dw|\Big)\mathcal{F}\\ &- \frac{C\mu F^{ii}u_{ii}^2}{|Dw|^2} + \frac{2(1+\phi_zd)}{|Dw|^2}w_iD_if,\end{aligned} \tag{4.37}$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2)$. Here we used the Cauchy inequality and the fact that $|R_{ijl}| \leq C(|Du|^2 + |Du| + 1)$. Now we deal with the last term. By (4.27) and (4.32), we have

$$\begin{aligned}\frac{2(1+\phi_zd)}{|Dw|^2}|w_iD_if| =&\Big|\frac{2(1+\phi_zd)}{|Dw|^2}w_i(f_{x_i} + f_zu_i)\\ \leq&4L_1(1 + |Du|^{-1}).\end{aligned} \tag{4.38}$$

Plugging (4.37) and (4.38) into (4.36), we have

$$0 \geq \frac{2F^{ii}w_{li}^2}{|Dw|^2} - \frac{C\mu F^{ii}u_{ii}^2}{|Dw|^2} - C\mu|Dw|^2\mathcal{F} - C(|Dw|\mathcal{F}+1), \tag{4.39}$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2, \mu)$. By (4.27), (4.29) and the inequality (see [14])

$$(a+b)^2 \geq \epsilon a^2 - \frac{\epsilon}{1-\epsilon}b^2,$$

by choosing $\epsilon = \frac{1}{2}$, we obtain

$$\begin{aligned} w_{ii}^2 \geq& \frac{1}{4}u_{ii}^2 - R_{ii}^2 \\ \geq& \frac{1}{4}u_{ii}^2 - C(\mu^2|Dw|^4 + |Dw|^2 + 1). \end{aligned} \tag{4.40}$$

It follows that

$$0 \geq \left(\frac{1}{8} - C\mu\right)\frac{F^{ii}u_{ii}^2}{|Dw|^2} - C\mu|Dw|^2\mathcal{F} - C(|Dw|+1)\mathcal{F}. \tag{4.41}$$

There exists at least a index l_0 such that $u_{l_0} \geq \frac{|Du|}{\sqrt{n}}$. We rewrite the (4.32) as

$$2w_{l_0}w_{l_0l_0} + 2\sum_{q\neq l_0} w_q w_{ql_0} = -(\alpha_0 d_{l_0} + h' u_{l_0})|Dw|^2.$$

From (4.27) we have

$$2(1+\phi_z d)w_{l_0}u_{l_0l_0} = -(\alpha_0 d_{l_0} + h' u_{l_0})|Dw|^2 - 2w_q R_{ql}.$$

Since $|R_l| \leq 2L_2 \leq \frac{u_l}{4}$, from (4.26), we have $w_l \geq \frac{u_l}{4}$. If we assume that

$$|Du| \geq \frac{2\sqrt{n}\alpha_0|Dd|}{h'}$$

and use the facts that

$$1+\phi_z d \geq \frac{1}{2} \quad \text{and} \quad |R_{ij}| \leq C(\mu|Du|^2 + |Du| + 1),$$

then

$$u_{l_0l_0} \leq -2h'|Dw|^2 + 12\sqrt{n}C(\mu|Dw|^2 + |Dw|).$$

If we assume that

$$|Dw| \geq \frac{2}{h'} \geq 10M + 2 \quad \text{and} \quad \mu \leq \frac{h'}{12\sqrt{n}C},$$

then

$$u_{l_0 l_0} \leq -\frac{h'}{2}|Dw|^2. \tag{4.42}$$

Denote $u_{11} \geq \cdots \geq u_{nn}$. Then

$$u_{nn} \leq -\frac{h'}{2}|Dw|^2, \quad F^{nn} \geq \frac{1}{n}\mathcal{F}. \tag{4.43}$$

By the Newton-Maclaurin inequality, we have

$$F^{nn} \geq \frac{1}{n}\mathcal{F} \geq cS_k^{\frac{1}{k}}(\lambda) \geq c(\min f)^{\frac{1}{k}}, \tag{4.44}$$

where $c = c(n,m,k)$ is a universal constant. We assume that $\mu \leq \min\{\frac{1}{16C}, \frac{h'^2}{128nC}\}$. By (4.41) we obtain

$$0 \geq \frac{h'^2}{128n}|Dw|^2\mathcal{F} - C(|Dw|+1)\mathcal{F}.$$

By (4.44), we have

$$0 \geq \frac{h'^2}{128n}|Dw|^2 - C|Dw| - C.$$

It is easy to get a bound for $|Dw|(x_0)$, then a bound for $G(x_0)$. Anyway we have the bound

$$G(x_0) = \sup_{\overline{\Omega}_\mu} G(x) \leq C,$$

where $C = C(\alpha_0, M, n, m, |d|_{C^3}, L_1, L_2, \mu)$. Thus we obtain

$$\sup_{\overline{\Omega}_\mu} |Du| \leq C + \log(1+2M) + \alpha_0\mu.$$

This completes the proof. □

5 Global second order estimate

5.1 Reduce to double normal derivative estimate on boundary

Using the method of Lions-Trudinger-Urbas [23], we can reduce the second order estimate of the solution into the boundary double normal estimate.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary. Assume $f(x,z) \in C^2(\overline{\Omega} \times \mathbb{R})$ is positive and $\phi(x,z) \in C^3(\overline{\Omega} \times \mathbb{R})$ with $\phi_z - 2\kappa_{\min} < 0$. If $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is a k-admissible solution of the Neumann problem*

$$\begin{cases} S_k(W) = f(x,u), & \Omega, \\ u_\nu = \phi(x,u), & \partial\Omega. \end{cases} \tag{5.1}$$

Denote $N = \sup_{\partial\Omega} |u_{\nu\nu}|$, then

$$\sup_{\overline{\Omega}} |D^2 u| \leq C_0(1+N), \tag{5.2}$$

where C_0 depends only on n, m, k, $|u|_{C^1(\overline{\Omega})}$, $|f|_{C^2(\overline{\Omega}\times[-M_0,M_0])}$, $\min f$, $|\phi|_{C^3(\overline{\Omega}\times[-M_0,M_0])}$ and Ω. Here $M_0 = \sup_{\overline{\Omega}} |u|$.

Proof. Write equation (5.1) in the form of

$$\begin{cases} S_k^{\frac{1}{k}}(W) = \widetilde{f}(x,u), & \Omega, \\ u_\nu = \phi(x,u), & \partial\Omega, \end{cases} \tag{5.3}$$

where $\widetilde{f} = f^{\frac{1}{k}}$. Since $\lambda(W) \in \Gamma_k \subset \Gamma_2$ in $\mathbb{R}^{C_n^m}$, we have

$$\sum_{i\neq j} |u_{ij}| \leq c(n,m) S_1(W) = mc(n,m) S_1(D^2 u), \tag{5.4}$$

where $c(n,m)$ is a universal number independent of u. Thus, it is sufficiently to prove (5.2) for any direction $\xi \in \mathbb{S}^{n-1}$, that is

$$u_{\xi\xi} \leq C_0(1+N). \tag{5.5}$$

We consider the following auxiliary function in $\Omega \times \mathbb{S}^{n-1}$,

$$v(x,\xi) := u_{\xi\xi} - v'(x,\xi) + K_1|x|^2 + K_2|Du|^2, \tag{5.6}$$

where

$$v'(x,\xi) = a^l u_l + b := 2(\xi\cdot\nu)\xi' \cdot (D_x\phi + \phi_z Du - u_l D\nu^l)$$

with

$$\xi' = \xi - (\xi\cdot\nu)\nu \quad \text{and} \quad a^l = 2(\xi\cdot\nu)(\xi'^l \phi_z - \xi'^i D_i \nu^l), \quad b = 2(\xi\cdot\nu)\xi'^l \phi_{x_l}.$$

K_1, K_2 are positive constants to be determined. By a direct computation, we have

$$v_i = u_{\xi\xi i} - D_i a^l u_l - a^l u_{li} - D_i b + 2K_1 x_i + 2K_2 u_l u_{li}, \tag{5.7a}$$

$$\begin{aligned} v_{ij} = {} & u_{\xi\xi ij} - D_{ij} a^l u_l - D_i a^l u_{lj} - D_j a^l u_{li} - a^l u_{lij} - D_{ij} b \\ & + 2K_1 \delta_{ij} + 2K_2 u_{li} u_{lj} + 2K_2 u_l u_{lij}. \end{aligned} \tag{5.7b}$$

Denote $\widetilde{F}(D^2u) = S_k^{\frac{1}{k}}(W)$ and

$$\widetilde{F}^{ij} = \frac{\partial \widetilde{F}}{\partial u_{ij}} = \frac{\partial S_k^{\frac{1}{k}}(W)}{\partial w_{\bar{\alpha}\bar{\beta}}}\frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{ij}},$$
$$\widetilde{F}^{pq,rs} = \frac{\partial^2 \widetilde{F}}{\partial u_{pq}\partial u_{rs}} = \frac{\partial^2 S_k^{\frac{1}{k}}(W)}{\partial w_{\bar{\alpha}\bar{\beta}}\partial w_{\overline{\eta}\overline{\xi}}}\frac{\partial w_{\bar{\alpha}\bar{\beta}}}{\partial u_{pq}}\frac{\partial w_{\overline{\eta}\overline{\xi}}}{\partial u_{rs}},$$

since $w_{\bar{\alpha}\bar{\beta}}$ is a linear combination of u_{ij}, $1 \leq i,j \leq n$. Differentiating Eq. (5.3) twice, we have

$$\widetilde{F}^{ij}u_{ijl} = D_l\widetilde{f}, \tag{5.8a}$$
$$\widetilde{F}^{pq,rs}u_{pq\xi}u_{rs\xi} + \widetilde{F}^{ij}u_{ij\xi\xi} = D_{\xi\xi}\widetilde{f}. \tag{5.8b}$$

By the concavity of $S_k^{\frac{1}{k}}(W)$ operator with respect to W, we have

$$D_{\xi\xi}\widetilde{f} = \widetilde{F}^{pq,rs}u_{pq\xi}u_{rs\xi} + \widetilde{F}^{ij}u_{ij\xi\xi} \leq \widetilde{F}^{ij}u_{ij\xi\xi}. \tag{5.9}$$

Now we contract (5.7b) with $\widetilde{F}^{ij}$ to get, using (5.8a)-(5.9),

$$\begin{aligned}\widetilde{F}^{ij}v_{ij} =&\widetilde{F}^{ij}u_{ij\xi\xi} - \widetilde{F}^{ij}D_{ij}a^lu_l - 2\widetilde{F}^{ij}D_ia^lu_{lj} - \widetilde{F}^{ij}u_{ijl}a^l - \widetilde{F}^{ij}D_{ij}b\\ &+ 2K_1\widetilde{\mathcal{F}} + 2K_2\widetilde{F}^{ij}u_{il}u_{jl} + 2K_2\widetilde{F}^{ij}u_{ijl}u_l\\ \geq& D_{\xi\xi}\widetilde{f} - \widetilde{F}^{ij}D_{ij}a^lu_l - 2\widetilde{F}^{ij}D_ia^lu_{ij} - a^lD_l\widetilde{f} - \widetilde{F}^{ij}D_{ij}b\\ &+ 2K_1\widetilde{\mathcal{F}} + 2K_2\widetilde{F}^{ij}u_{il}u_{jl} + 2K_2u_lD_l\widetilde{f},\end{aligned}$$

where $\widetilde{\mathcal{F}} = \sum_{i=1}^n \widetilde{F}^{ii}$. Note that

$$\begin{aligned}D_{\xi\xi}\widetilde{f} &= \widetilde{f}_{\xi\xi} + 2\widetilde{f}_{\xi z}u_\xi + \widetilde{f}_zu_{\xi\xi},\\ D_{ij}a^l &= 2(\xi\cdot\nu)\xi'^l\phi_{zz}u_{ij} + r_{ij}^l,\\ D_{ij}b &= 2(\xi\cdot\nu)\xi'^l\phi_{x_lz}u_{ij} + r_{ij},\end{aligned}$$

with

$$|r_{ij}^l|, |r_{ij}| \leq C(|u|_{C^1}, |\phi|_{C^3}, |\partial\Omega|_{C^4}).$$

At the maximum point $x_0 \in \Omega$ of v, we can assume $(u_{ij})_{n\times n}$ is diagonal. It follows that, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\widetilde{F}^{ij}v_{ij} \geq& -C(\widetilde{\mathcal{F}} + K_2 + 1) - C\widetilde{F}^{ii}|u_{ii}| + \widetilde{f}_zu_{\xi\xi} + 2K_1\widetilde{\mathcal{F}} + 2K_2\widetilde{F}^{ii}u_{ii}^2\\ \geq& -C(\widetilde{\mathcal{F}} + K_2 + 1) + \widetilde{f}_zu_{\xi\xi} + 2K_1\widetilde{\mathcal{F}} + (2K_2 - 1)\widetilde{F}^{ii}u_{ii}^2,\end{aligned} \tag{5.10}$$

where $C = C(|u|_{C^1}, |\phi|_{C^3}, |\partial\Omega|_{C^4}, |f|_{C^2})$.

Assume $u_{11} \geq u_{22} \cdots \geq u_{nn}$ and denote $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{C_n^m}$ the eigenvalues of the matrix $(w_{\bar{\alpha}\bar{\beta}})_{C_n^m \times C_n^m}$. It is easy to see

$$\lambda_1 = u_{11} + \sum_{i=2}^{m} u_{ii} \leq m u_{11}.$$

Then we have, by (2.3) in Proposition 2.2 and (2.4d) in Proposition 2.3,

$$\begin{aligned}
\widetilde{F}^{11} u_{11}^2 =& \sum_{1 \in \bar{\alpha}} \frac{1}{k} S_k^{\frac{1}{k}-1} S_{k-1}(\lambda | N_{\bar{\alpha}}) u_{11}^2 \\
\geq& \frac{1}{mk} S_k^{\frac{1}{k}-1} S_{k-1}(\lambda|1) \lambda_1 u_{11} \\
\geq& \frac{1}{m C_n^m} S_k^{\frac{1}{k}} u_{11} = \frac{\widetilde{f}}{m C_n^m} u_{11}.
\end{aligned} \tag{5.11}$$

We can assume $u_{\xi\xi} \geq 0$, otherwise we have (5.5). Plug (5.11) into (5.10) and use the Cauchy-Schwartz inequality, then

$$\widetilde{F}^{ii} v_{ii} \geq (K_2 - 1) \sum_{i=1}^{n} \widetilde{F}^{ii} u_{ii}^2 + \Big(\frac{K_2 \widetilde{f}}{m C_n^m} + \widetilde{f}_z \Big) u_{\xi\xi} + (2K_1 - C)\widetilde{\mathcal{F}} - C(K_2 + 1).$$

Choose

$$K_2 = \frac{m C_n^m |\max f_z|}{k \min f} + 1 \quad \text{and} \quad K_1 = C(K_2 + 2) + 1.$$

It follows that

$$\widetilde{F}^{ii} v_{ii} \geq (2K_1 - C)\widetilde{\mathcal{F}} - C(K_2 + 1) > 0,$$

since we have $\widetilde{\mathcal{F}} \geq 1$ from (2.6). This implies that $v(x, \xi)$ attains its maximum on the boundary by the maximum principle. Now we assume $(x_0, \xi_0) \in \partial\Omega \times \mathbb{S}^{n-1}$ is the maximum pint of $v(x, \xi)$ in $\overline{\Omega} \times \mathbb{S}^{n-1}$. Then we consider two cases as follows.

Case 1. ξ_0 is a tangential vector at $x_0 \in \partial\Omega$. We directly have $\xi_0 \cdot \nu = 0$, $\nu = -Dd$, $v'(x_0, \xi_0) = 0$ and $u_{\xi_0, \xi_0}(x_0) > 0$. As in [20], we define

$$c^{ij} = \delta_{ij} - \nu^i \nu^j \quad \text{in } \Omega_\mu,$$

and it is easy to see that $c^{ij} D_j$ is a tangential direction on $\partial\Omega$. We compute at (x_0, ξ_0). From the boundary condition, we have

$$\begin{aligned}
u_{li} \nu^l =& (c^{ij} + \nu^i \nu^j) \nu^l u_{lj} \\
=& c^{ij} u_j \phi_z + c^{ij} \phi_{x_j} - c^{ij} u_l D_j \nu^l + \nu^i \nu^j \nu^l u_{lj}.
\end{aligned} \tag{5.12}$$

It follows that

$$\begin{aligned}u_{lip}\nu^l =&[c^{pq}+\nu^p\nu^q]u_{liq}\nu^l\\ =&c^{pq}D_q(c^{ij}u_j\phi_z+c^{ij}\phi_{x_j}-c^{ij}u_lD_j\nu^l+\nu^i\nu^j\nu^lu_{lj})\\ &-c^{pq}u_{li}D_q\nu^l+\nu^p\nu^q\nu^lu_{liq},\end{aligned}$$

then we obtain

$$\begin{aligned}u_{\xi_0\xi_0\nu}=&\sum_{ilp=1}^n\xi_0^i\xi_0^pu_{lip}\nu^l\\ =&\sum_{i=1}^n\xi_0^i\xi_0^q[D_q(c^{ij}u_j\phi_z+c^{ij}\phi_{x_j}-c^{ij}u_lD_j\nu^l+\nu^i\nu^j\nu^lu_{lj})-u_{li}D_q\nu^l]\\ \le&\phi_zu_{\xi_0\xi_0}-2\xi_0^i\xi_0^qu_{li}D_q\nu^l+C(1+|u_{\nu\nu}|).\end{aligned}\tag{5.13}$$

We assume $\xi_0=e_1$, it is easy to get the bound for $u_{1i}(x_0)$ for $i>1$ from the maximum of $v(x,\xi)$ in the ξ_0 direction. In fact, we can assume $\xi(t)=\frac{(1,t,0,\cdots,0)}{\sqrt{1+t^2}}$. Then

$$\begin{aligned}0=&\frac{dv(x_0,\xi(t))}{dt}\Big|_{t=0}\\ =&2u_{12}(x_0)-2\nu^2(\phi_zu_1-u_lD_l\nu^l),\end{aligned}\tag{5.14}$$

so

$$|u_{12}|(x_0)\le C+C|Du|.$$

Similarly, we have for $\forall i>1$,

$$|u_{1i}|(x_0)\le C+C|Du|.\tag{5.15}$$

Thus we have, by $D_1\nu^1\ge\kappa_{\min}$,

$$\begin{aligned}u_{\xi_0\xi_0\nu}\le&\phi_zu_{\xi_0\xi_0}-2D_1\nu^1u_{11}+C(1+|u_{\nu\nu}|)\\ \le&(\phi_z-2\kappa_{\min})u_{\xi_0\xi_0}+C(1+|u_{\nu\nu}|).\end{aligned}\tag{5.16}$$

On the other hand, we have from the Hopf lemma, (5.7a) and (5.15),

$$\begin{aligned}0\le&v_\nu(x_0,\xi_0)\\ =&u_{\xi_0\xi_0\nu}-D_\nu a^lu_l-a^lu_{\nu\nu}-D_\nu b+2K_1x_i\nu^i+2K_2u_lu_{l\nu}\\ \le&(\phi_z-2\kappa_{\min})u_{\xi_0\xi_0}+C(1+|u_{\nu\nu}|).\end{aligned}\tag{5.17}$$

Then we get, since $2\kappa_{\min}-\phi_z\ge c>0$,

$$u_{\xi_0\xi_0}(x_0)\le C(1+|u_{\nu\nu}|).$$

Case 2. ξ_0 is non-tangential. We can find a tangential vector τ, such that $\xi_0 = \alpha\tau + \beta\nu$, with $\alpha^2 + \beta^2 = 1$. Then we have

$$\begin{aligned} u_{\xi_0\xi_0}(x_0) =&\alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2\alpha\beta u_{\tau\nu}(x_0) \\ =&\alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2(\xi_0 \cdot \nu)\xi_0' \cdot (\phi_z Du - u_l D\nu^l). \end{aligned} \tag{5.18}$$

By the definition of $v(x_0, \xi_0)$,

$$\begin{aligned} v(x_0, \xi_0) =&\alpha^2 v(x_0, \tau) + \beta^2 v(x_0, \nu) \\ \leq&\alpha^2 v(x_0, \xi_0) + \beta^2 v(x_0, \nu). \end{aligned} \tag{5.19}$$

Thus,

$$\begin{aligned} &v(x_0, \xi_0) = v(x_0, \nu), \\ &u_{\xi_0\xi_0}(x_0) \leq |u_{\nu\nu}| + C. \end{aligned}$$

In conclusion, we have (5.5) in both cases. □

5.2 Global second order estimate by double normal estimate on boundary

As in [23] and [24], we construct sub and super barrier function to give lower and upper bounds for $u_{\nu\nu}$ on the boundary. Then we give the global second order estimate.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary, $2 \leq m \leq n-1$ and $2 \leq k \leq C_{n-1}^{m-1}$. Assume $f(x,z) \in C^2(\overline{\Omega} \times \mathbb{R})$ is positive and $\phi(x,z) \in C^3(\overline{\Omega} \times \mathbb{R})$ with $\phi_z - 2\kappa_{\min} < 0$. If $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ is a k-admissible solution of the Neumann problem* (5.1). *Then we have*

$$\sup_{\overline{\Omega}} |D^2 u| \leq C, \tag{5.20}$$

where C depends only on n, m, k, $|u|_{C^1(\overline{\Omega})}$, $|f|_{C^2(\overline{\Omega}\times[-M_0,M_0])}$, $\min f$, $|\phi|_{C^3(\overline{\Omega}\times[-M_0,M_0])}$ and Ω, where $M_0 = \sup_\Omega |u|$.

First, we denote $d(x) = dist(x, \partial\Omega)$ and define

$$h(x) := -d(x) + K_3 d^2(x), \tag{5.21}$$

where K_3 is large constant to be determined later. Then we give the following key Lemma.

Lemma 5.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary, $2 \leq m \leq n-1$ and $2 \leq k \leq C_{n-1}^{m-1}$. Let $u \in C^2(\overline{\Omega})$ is a k-admissible solution of Eq.* (5.1) *and h is defined as in* (5.21). *Then, there exists K^*, a sufficiently large number depending only on n, m, k, $\min f$ and Ω, such that,*

$$F^{ij} h_{ij} \geq K_3^{\frac{1}{2}} (1 + \mathcal{F}) \quad in\ \Omega_\mu, \quad (0 < \mu \leq \widetilde{\mu}), \tag{5.22}$$

for any $K_3 \geq K^*$, where $\widetilde{\mu} = \min\{\frac{1}{4K_3}, \mu_0\}$, μ_0 is mentioned in (4.19).

Proof. For $x_0 \in \Omega_\mu$, there exists $y_0 \in \partial\Omega$ such that $|x_0 - y_0| = d(x_0)$. Then, in terms of a principal coordinate system at y_0, we have (see [8, Lemmas 14.17]),

$$\begin{aligned}
&[D^2 d(x_0)] = -diag\Big[\frac{\kappa_1}{1-\kappa_1 d}, \cdots, \frac{\kappa_{n-1}}{1-\kappa_{n-1}d}, 0\Big],\\
&Dd(x_0) = -\nu(x_0) = (0, \cdots, 0, -1).
\end{aligned}$$

Observe that

$$[D^2 h(x_0)] = diag\Big[\frac{((1-2K_3 d)\kappa_1}{1-\kappa_1 d}, \cdots, \frac{(1-2K_3 d)\kappa_{n-1}}{1-\kappa_{n-1}d}, 2K_3\Big]. \tag{5.23}$$

Denote

$$\mu_i = \frac{(1-K_3 d)\kappa_i}{1-\kappa_i d}, \quad \forall 1 \le i \le n-1 \quad \text{and} \quad \mu_n = 2K_3,$$

for simplicity. Then we define $\lambda(D^2 h) = \{\mu_{i_1} + \cdots + \mu_{i_m} | \, 1 \le i_1 < \cdots < i_m \le n\}$ and assume $\lambda_1 \ge \cdots \ge \lambda_{C_n^m}$, it is easy to see that

$$\lambda_k \ge \lambda_{C_{n-1}^{m-1}} \ge 2K_3 + \sum_{l=1}^{m-1} \mu_{i_l} \ge K_3,$$

if we choose K_3 sufficiently large and $\mu \le \frac{1}{4K_3}$. It follows that, for $\forall 1 \le l \le k$,

$$S_l(\lambda) \ge K_3^l - C(n,m,\kappa) K_3^{l-1} \ge \frac{K_3^l}{2}, \tag{5.24}$$

such that h is k-admissible. Similarly, $w = h - \frac{K_3}{2n}|x|^2$ is also k-admissible if we choose K_3 sufficiently large. By the concavity of $\widetilde{F}$, we have

$$\begin{aligned}
\widetilde{F}^{ij} w_{ij} \ge& \widetilde{F}[D^2 u + D^2 w] - \widetilde{F}[D^2 u]\\
\ge& \widetilde{F}[D^2 w] \ge \frac{K_3}{4}.
\end{aligned} \tag{5.25}$$

Then we have

$$\widetilde{F}^{ij} h_{ij} = \widetilde{F}^{ij}\Big(h - \frac{K_3}{2n}|x|^2 + \frac{K_3}{2n}|x|^2\Big)_{ij} \ge \frac{K_3}{4n}(1 + \widetilde{\mathcal{F}}).$$

If we choose $K_3 \ge (\frac{4n \max f^{\frac{1}{k}}}{k \min f})^2$, then we have

$$F^{ij} h_{ij} \ge K_3^{\frac{1}{2}}(1 + \mathcal{F}).$$

Thus, we complete the proof. □

Remark 5.1. Compare to the Hessian case ($m = 1$, see Ma-Qiu [24, Lemma 14]), the constant K_3 in (5.22) can be large as we want while one can only get $F^{ij}h_{ij} \geq \delta(1+\mathcal{F})$ for a small δ in [24]. The Lemma 5.2 leads to more simple proofs of lower and upper bounds for $u_{\nu\nu}$ on the boundary than those in [24].

Now we can use Lemma 5.2 to prove Theorem 5.1.

Proof of Theorem 5.1. We define

$$P(x) = Du\cdot\nu - \phi(x,u),$$

with $\nu = -Dd$. Differentiating P twice to obtain

$$P_{ij} = -u_{rij}d_r - u_{ri}d_{rj} - u_{rj}d_{ri} - u_r d_{rij} - D_{ij}\phi. \tag{5.26}$$

Then we obtain

$$\begin{aligned} F^{ij}P_{ij} =& -F^{ij}(u_{rij}d_r + 2u_{ri}d_{rj} + u_r d_{rij} - D_{ij}\phi) \\ \leq& -F^{ii}u_{ii}d_{ii} + C_1(1+\mathcal{F}), \end{aligned} \tag{5.27}$$

where $C_1 = C_1(|u|_{C^1}, |\partial\Omega|_{C^3}, |\phi|_{C^2}, |f|_{C^1}, n)$. From (5.2) in Lemma 5.1, we have

$$|u_{ii}| \leq C_0(1+N).$$

It follows that

$$F^{ij}P_{ij} \leq C_2(1+N)(1+\mathcal{F}),$$

where $C_2 = C_1 + C_0|d|_{C^2}$. On the other hand, using Lemma 5.2, we have

$$\begin{aligned} \Big(A+\frac{1}{2}N\Big)F^{ij}h_{ij} \geq& \Big(A+\frac{1}{2}N\Big)K_3^{\frac{1}{2}}(1+\mathcal{F}) \\ \geq& C_2(1+N)(1+\mathcal{F}) \\ \geq& F^{ij}P_{ij}, \end{aligned} \tag{5.28}$$

by choosing $K_3 = K^* + (2C_2)^2 + 1$ and $A \geq C_2 + 1$. On $\partial\Omega$, it is easy to see

$$P = 0.$$

On $\partial\Omega_\mu \cap \Omega$, we have

$$|P| \leq C_3(|u|_{C^1}, |\phi|_{C^0}) \leq \Big(A+\frac{1}{2}N\Big)\frac{\mu}{2},$$

provided $A=\max\{\frac{2C_3}{\mu}, C_2+1\}$. Finally the maximum principle tells us that

$$-\Big(A+\frac{1}{2}N\Big)h(x)\le P(x)\le\Big(A+\frac{1}{2}N\Big)h(x)\quad\text{in }\ \Omega_\mu.$$

Suppose

$$u_{\nu\nu}(y_0)=\sup_{\partial\Omega}u_{\nu\nu}>0,$$

we have

$$\begin{aligned}0\ge&P_\nu(y_0)-\Big(A+\frac{1}{2}N\Big)h_\nu\\=&u_{\nu\nu}-D_\nu\phi-\Big(A+\frac{1}{2}N\Big)\\\ge&u_{\nu\nu}(y_0)-C(|u|_{C^1},|\partial\Omega|_{C^2},|\phi|_{C^2})-\Big(A+\frac{1}{2}N\Big).\end{aligned}\tag{5.29}$$

Then we get

$$\sup_{\partial\Omega}u_{\nu\nu}\le C+\frac{1}{2}N.\tag{5.30}$$

Similarly, doing this at the minimum point of $u_{\nu\nu}$, we have

$$\inf_{\partial\Omega}u_{\nu\nu}\ge -C-\frac{1}{2}N.\tag{5.31}$$

It follows that

$$\sup_{\partial\Omega}|u_{\nu\nu}|\le C.\tag{5.32}$$

Combining (5.32) with (5.2) in Lemma 5.1, we obtain

$$\sup_{\overline{\Omega}}|D^2u|\le C.$$

This completes the proof. □

6 Existence of the Neumann boundary problem

Now we prove the existence and uniqueness theorem for Neumann problem (1.1).

Proof of Theorem 1.1. Consider a family of equations with parameter t,

$$\begin{cases}S_k(W)=tf+(1-t)\dfrac{(C_n^m)!m^k}{(C_n^m-k)!k!}, & \Omega,\\ u_\nu=-au+tb+(1-t)(x\cdot\nu+\dfrac{a}{2}|x|^2), & \partial\Omega.\end{cases}\tag{6.1}$$

From Theorems 3.1, 4.1, 4.2 and 5.1, we get a global C^2 estimate independent of t for Eq. (6.1) in both cases of Theorem 1.1. It follows that Eq. (6.1) is uniformly elliptic. Due to the concavity of $S_k^{\frac{1}{k}}(W)$ with respect to D^2u (see [2]), we can get the global Hölder estimates of second derivatives following the arguments in [21], that is, we can get

$$|u|_{C^{2,\alpha}} \leq C, \tag{6.2}$$

where C depends only on $n, m, k, |u|_{C^1}, |f|_{C^2}$,$\min f$, $|\phi|_{C^3}$ and Ω. It is easy to see that $\frac{1}{2}|x|^2$ is a k-admissible solution to (6.1) for $t=0$. Applying the method of continuity (see [8, Theorem 17.28]), the existence of the classical solution holds for $t=1$. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity. The uniqueness is a simple consequence of the comparison principle. Thus, we complete the proof. □

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