ONE-PARAMETER FINITE DIFFERENCE METHODS AND THEIR ACCELERATED SCHEMES FOR SPACE-FRACTIONAL SINE-GORDON EQUATIONS WITH DISTRIBUTED DELAY*

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Abstract

This paper deals with numerical methods for solving one-dimensional (1D) and two-dimensional (2D) initial-boundary value problems (IBVPs) of space-fractional sine-Gordon equations (SGEs) with distributed delay. For 1D problems, we construct a kind of one-parameter finite difference (OPFD) method. It is shown that, under a suitable condition, the proposed method is convergent with second order accuracy both in time and space. In implementation, the preconditioned conjugate gradient (PCG) method with the Strang circulant preconditioner is carried out to improve the computational efficiency of the OPFD method. For 2D problems, we develop another kind of OPFD method. For such a method, two classes of accelerated schemes are suggested, one is alternative direction implicit (ADI) scheme and the other is ADI-PCG scheme. In particular, we prove that ADI scheme can arrive at second-order accuracy in time and space. With some numerical experiments, the computational effectiveness and accuracy of the methods are further verified. Moreover, for the suggested methods, a numerical comparison in computational efficiency is presented.


Key words: Fractional sine-Gordon equation with distributed delay, One-parameter finite difference methods, Convergence analysis, ADI scheme, PCG method.

1. Introduction

The sine-Gordon equations (SGEs) are a kind of important partial differential equations used to model some practical problems in electrodynamics [1–3], nonlinear optics [4], particle physics [5] and the other related scientific fields. In order to give a detailed description to the physical background of SGEs, as an example, we recall the following SGE used to characterize the long Josephson junction in an electrodynamic system ([2]):

$$\frac{\partial^2}{\partial t^2} u(x, t) = \kappa \frac{\partial^2}{\partial x^2} u(x, t) + \beta \sin(u(x, t)), \quad (1.1)$$

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where $u(x, t)$ denotes the superconducting phase difference across the Josephson junction and $\kappa > 0$ and $\beta$ are the two real parameters of the system. Due to the existence of quasi-particle tunnel current and external bias current, dissipative term $\sigma \frac{\partial}{\partial t} u(x, t)$ ($\sigma \geq 0$) and source term $f(x, t)$ were introduced and thus an extended SGE was derived as follows ([11]):

$$\frac{\partial^2}{\partial t^2} u(x, t) + \sigma \frac{\partial}{\partial t} u(x, t) = \kappa \frac{\partial^2}{\partial x^2} u(x, t) + \beta \sin(u(x, t)) + f(x, t).$$  \hspace{1cm} (1.2)

In view of the fact that time-dependent problems generally have the aftereffect phenomenon, some researchers introduced distributed delay $\int_{t-s}^{t} e^{-\alpha(t-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta$ into the above equation to produce the extended SGE ([6]):

$$\frac{\partial^2}{\partial t^2} u(x, t) + \sigma \frac{\partial}{\partial t} u(x, t) = \kappa \frac{\partial^2}{\partial x^2} u(x, t) - \int_{t-s}^{t} e^{-\alpha(t-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta + \beta \sin(u(x, t)) + f(x, t).$$  \hspace{1cm} (1.3)

On the other hand, in order to model the nonlocal Josephson junction, by replacing the term $\frac{\partial^2}{\partial x^2} u(x, t)$ in (1.2) with a nonlocal operator

$$\mathcal{H}(u_x(x, t)) := \frac{1}{\pi} \mathrm{v.p.} \int_{-\infty}^{+\infty} \frac{1}{x-x'} \frac{\partial u(x', t)}{\partial x'} d\hat{x},$$

the following nonlocal SGE was presented ([7]):

$$\frac{\partial^2}{\partial t^2} u(x, t) + \sigma \frac{\partial}{\partial t} u(x, t) = \kappa \mathcal{H}(u_x(x, t)) + \beta \sin(u(x, t)) + f(x, t).$$  \hspace{1cm} (1.4)

In Ray [9], the author pointed out that $\frac{\partial^2}{\partial x^2} u(x, t)$ and $\mathcal{H}(u_x(x, t))$ are just the special cases when the order $\gamma$ ($1 < \gamma \leq 2$) of the Riesz space-fractional derivative tends to 2 and 1, respectively, where the $\gamma$-order Riesz fractional derivative $\frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t)$ is defined by

$$\frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) = -\frac{1}{2 \cos(\frac{\pi \gamma}{2}) \Gamma(2-\gamma)} \frac{\partial^2}{\partial x^2} \left( \int_{-b}^{b} \frac{u(\zeta, t)}{(x-\zeta)^{1-\gamma}} d\zeta + \int_{-a}^{a} \frac{u(\zeta, t)}{(x-\zeta)^{1-\gamma}} d\zeta \right), \hspace{1cm} a \leq x \leq b,$$  \hspace{1cm} (1.5)

in which $\Gamma(\cdot)$ is the Gamma function. Based on this finding, Eqs. (1.2) and (1.4) were further generalized into the following space-fractional SGE ([8–12]):

$$\frac{\partial^2}{\partial t^2} u(x, t) + \sigma \frac{\partial}{\partial t} u(x, t) = \kappa \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) + \beta \sin(u(x, t)) + f(x, t), \hspace{1cm} 1 < \gamma \leq 2.$$  \hspace{1cm} (1.6)

Similar to Eq. (1.3), we consider the aftereffect phenomenon and introduce distributed delay $\int_{t-s}^{t} e^{-\alpha(t-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta$ into Eq. (1.6). This generates the following space-fractional SGE with distributed delay:

$$\frac{\partial^2}{\partial t^2} u(x, t) + \sigma \frac{\partial}{\partial t} u(x, t) = \kappa \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) - \int_{t-s}^{t} e^{-\alpha(t-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta + \beta \sin(u(x, t)) + f(x, t).$$  \hspace{1cm} (1.7)
In recent years, numerically solving the space-fractional SGEs has aroused great interest of many researchers. For example, Alfimov, Pierantozzi & Vázquez [8] studied the rotating wave approximation of the solution, Ray [9,10] proposed the modified homotopy method and decomposition method, Xing & Wen [11] constructed a conservative and stable difference method, Xing, Wen & Wang [12] gave an explicit fourth-order energy-preserving method. Although these numerical research on space-fractional SGEs have been actively carried out, the numerical methods directly for the time-delay equation (1.7) have not been presented in the existing references. In fact, time-delay is an inevitable natural phenomenon for a time-dependent problem. A differential equation with time-delay means the variation of the characterized problem depends not only on the current state but also on the past one. Therefore, the time-delay models are more relevant to the actual situation of the real world and their numerical treatments are more difficult than the standard differential equations. For these reasons, the present paper will focus on the numerical studies on the space-fractional SGE (1.7) with distributed delay.

The rest of this paper is organized as follows. In Section 2, for 1D problems, we construct a kind of OPFD methods and prove that the methods are convergent of order two in time and space under the suitable condition. In order to reduce the computational cost of OPFD methods, by combining Strang circulant preconditioner (see, e.g., [13]) and conjugate gradient (CG) method (see e.g. [14]), we also propose the so-called PCG scheme. In Section 3, for 2D problems, we develop another kind of OPFD methods. For improving the computational efficiency of such methods, two classes of accelerated schemes are suggested, one is ADI scheme and the other is ADI-PCG scheme. In particular, it is shown that ADI scheme can arrive at second-order accuracy in time and space. In the end, in Section 4, by performing some numerical experiments, we further illustrate the computational effectiveness and accuracy of the methods and given a numerical comparison of the presented methods in computational efficiency.

2. OPFD Method and Its PCG Scheme for 1D Problems

In this section, we will present a class of OPFD methods for solving the following 1D IBVPs of space-fractional SGEs with distributed delay:

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u(x,t) + \sigma \frac{\partial}{\partial t} u(x,t) &= \kappa \frac{\partial^\gamma}{\partial |x|^\gamma} u(x,t) - \int_{t-s}^{t} e^{-\alpha(t-\eta)} \frac{\partial}{\partial \eta} u(x,\eta) d\eta + \beta \sin(u(x,t)) + f(x,t), \\
&\quad (x,t) \in (a,b) \times (0,T], \\
u(x,t) &= \psi(x,t), \\
u(a,t) = \psi(b,t) = 0,
\end{aligned}
\]

(2.1)

(2.2)

(2.3)

where \( \sigma, \kappa, \alpha, s, T > 0, a,b \) and \( \beta \) are some real constants, \( f(x,t) \) and \( \psi(x,t) \) are two given functions.

2.1. Discretization for distributed delay term and Riesz fractional derivative

Let temporal stepsize \( \tau = s/m \) \((m \in \mathbb{N})\), \([\cdot]\) denotes the floor function and temporal mesh points \( t_n = n\tau \) \((-m \leq n \leq M := \lfloor T/\tau \rfloor\)). For the discretization of distributed delay term \( \int_{t_n-s}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x,\eta) d\eta \), on each small interval \([t_{i-1},t_i]\) we apply the linear interpolation:

\[
L_i^u(x,\eta) := \frac{t_i - \eta}{\tau} u(x,t_{i-1}) + \frac{\eta - t_{i-1}}{\tau} u(x,t_i)
\]
Lemma 2.1. Assume that

\[
\int_{t_n-m}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta = \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta \approx \sum_{i=n-m}^{n} b_{n-i} u(x, t_i),
\]

where

\[
b_i = \begin{cases} 
a_0, & i = 0, 
a_i - a_{i-1}, & 1 \leq i \leq m - 1, 
-a_{m-1}, & i = m,
\end{cases}
\]

Moreover, by some simple calculations, we have that

\[
\sum_{i=0}^{m} |b_i| = a_0 + \sum_{i=1}^{m-1} (a_{i-1} - a_i) + a_{m-1} = 2a_0 = 2 - \frac{e^{-\alpha T}}{\alpha T} > a_{i+1} > 0.
\]

An error estimate to the above approximation can be stated as follows.

**Lemma 2.1.** Assume that \(u(\cdot, t) \in C^2[-s, T]\). Then

\[
\left| \int_{t_n-m}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta - \sum_{i=n-m}^{n} b_{n-i} u(x, t_i) \right| \leq \frac{\alpha s}{12} (x, t) \max_{(x, t) \in [a, b] \times [t_{n-m}, t_n]} \left| \frac{\partial^2}{\partial t^2} u(x, t) \right| \tau^2.
\]

**Proof.** A common property of polynomial interpolation (see, e.g., [15, chap.8]) gives that

\[
u(x, \eta) - L_i^n(x, \eta) = \frac{1}{2} \frac{\partial^2}{\partial t^2} u(x, \xi_i(\eta)) (\eta - t_{i-1}(\eta - t_i), \quad \xi_i(\eta) \in (t_{i-1}, t_i), \quad \eta \in [t_{i-1}, t_i].
\]

It follows from the definition of \(b_i\) that

\[
\int_{t_n-m}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta - \sum_{i=n-m}^{n} b_{n-i} u(x, t_i)
\]

\[
= \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta - \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} e^{-\alpha(t_n-\eta)} u(x, t_i) - u(x, t_{i-1}) \frac{\partial}{\partial \eta} d\eta
\]

\[
= \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} (u(x, \eta) - L_i^n(x, \eta)) d\eta.
\]

Integrating (2.5) by parts yields

\[
\int_{t_n-m}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta - \sum_{i=n-m}^{n} b_{n-i} u(x, t_i)
\]

\[
= \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} \alpha e^{-\alpha(t_n-\eta)} (u(x, \eta) - L_i^n(x, \eta)) d\eta
\]

\[
= \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} \alpha e^{-\alpha(t_n-\eta)} \frac{1}{2} \frac{\partial^2}{\partial t^2} u(x, \xi_i(\eta)) (\eta - t_{i-1}(\eta - t_i) d\eta.
\]
Taking absolute value of above equation gives that
\[
\left| \int_{t_{n-m}}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta - \sum_{i=n-m}^{n} b_{n-i} u(x, t_i) \right|
\]
\[
\leq \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} \frac{\alpha}{2} \frac{\partial^2}{\partial t^2} u(x, \xi_i(\eta)) (\eta - t_{i-1})(t_i - \eta) d\eta
\]
\[
\leq \frac{\alpha}{2} \max_{(x,t)\in[a,b] \times [t_{n-m}, t_n]} \left| \frac{\partial^2}{\partial t^2} u(x, t) \right| \sum_{i=n-m+1}^{n} \int_{t_{i-1}}^{t_i} (\eta - t_{i-1})(t_i - \eta) d\eta
\]
\[
= \frac{\alpha \gamma}{12} \max_{(x,t)\in[a,b] \times [t_{n-m}, t_n]} \left| \frac{\partial^2}{\partial t^2} u(x, t) \right| \gamma^2.
\]
This completes the proof.

This lemma indicates the following relation:
\[
\int_{t_{n-m}}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, \eta) d\eta = \sum_{i=n-m}^{n} b_{n-i} u(x, t_i) + O(\gamma^2). \tag{2.6}
\]
Besides the above lemma, the following preliminary results will also be useful for the subsequent analysis.

**Lemma 2.2** ([16]). Let
\[
g_k^{(\gamma)} := \frac{(-1)^k \Gamma(\gamma + 1)}{\Gamma(\gamma/2 - k + 1) \Gamma(\gamma/2 + k + 1)} \quad \text{with} \quad \gamma \in (1, 2].
\]
Then
\[
g_0^{(\gamma)} = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma/2 + 1)} > 0, \quad g_k^{(\gamma)} = g_{-k}^{(\gamma)} \leq 0 \quad (k \neq 0), \quad \sum_{k=-\infty}^{+\infty} g_k^{(\gamma)} = 0
\]
and
\[
g_k^{(\gamma)} = \left(1 - \frac{\gamma + 1}{2 + k}\right) g_{k-1}^{(\gamma)} \quad (k = 1, 2, ...).
\]

**Lemma 2.3.** For all \( l \geq 2, \)
\[
\sum_{j=l}^{+\infty} |g_j^{(\gamma)}| \leq \frac{|g_1^{(\gamma)}|(\gamma/2 + 2)^{\gamma+1}}{\gamma(\gamma/2 + l)^\gamma}.
\]

**Proof.** According to Lemma 2.2, we have
\[
|g_j^{(\gamma)}| = \left(1 - \frac{\gamma + 1}{\gamma/2 + j}\right) |g_{j-1}^{(\gamma)}| \quad \text{for} \quad j \geq 2.
\]
Using inequality \( 1 - x \leq e^{-x}, \) the following result holds for \( j \geq 2: \)
\[
|g_j^{(\gamma)}| \leq e^{-\frac{\gamma + 1}{2 + j}} |g_{j-1}^{(\gamma)}| \leq e^{-\sum_{k=1}^{j} \frac{\gamma + 1}{2 + k}} |g_1^{(\gamma)}| \leq e^{-\int_{0}^{l} \sum_{k=1}^{\gamma+1} \frac{\gamma + 1}{2 + k} dx} |g_1^{(\gamma)}| = \left(\frac{\gamma + 2}{2 + j + 1}\right)^{\gamma+1} |g_1^{(\gamma)}|.
\]
Summing above inequality with respect to \( j \) from \( l \) to \( +\infty \) yields
\[
\sum_{j=l}^{+\infty} \left| g_j^{(\gamma)} \right| \leq \left| g_1^{(\gamma)} \right| \sum_{j=l}^{+\infty} \left( \frac{2 + 2}{\gamma j + 1} \right)^{\gamma + 1}
\leq \left| g_1^{(\gamma)} \right| \sum_{j=l}^{+\infty} \int_{j-1}^{j} \left( \frac{2 + 2}{\gamma x + 1} \right)^{\gamma + 1} \, dx = \frac{\left| g_1^{(\gamma)} \right| (\frac{2}{\gamma} + 2)^{\gamma + 1}}{\gamma (\frac{2}{\gamma} + l)^{\gamma}}.
\]
This completes the proof. \( \square \)

**Lemma 2.4** ([17]). Let \( \hat{\Theta} \) be the Fourier transform of function \( \Theta(x) \) and
\[
\ell^{2+\gamma}(\mathbb{R}) = \left\{ \Theta \in L^1(\mathbb{R}) \mid \int_{-\infty}^{+\infty} (1 + |\omega|)^{2+\gamma} |\hat{\Theta}(\omega)| \, d\omega < +\infty \right\}.
\]
Assume that functions \( z(x) \in C^5[a, b] \) and
\[
z^*(x) := \begin{cases} 
z(x), & x \in [a, b], \\
0, & x \not\in [a, b], \end{cases} \in \ell^{2+\gamma}(\mathbb{R}).
\]
Then
\[
\frac{\partial^\gamma}{\partial|x|^\gamma} z(x) = -\frac{1}{h^\gamma} \sum_{k=-\left\lfloor \frac{x-n}{h} \right\rfloor}^{\left\lceil \frac{x-n}{h} \right\rceil} g_k^{(\gamma)} z(x - kh) + \mathcal{O}(h^2),
\]
where \( h = \frac{b-a}{N+1} \) (\( N \in \mathbb{N} \)).

**2.2. Construction of OPFD method**

Let \( \Omega_{ht} = \Omega_h \times \Omega_T \) with \( \Omega_h = \{ x_j = a + jh \mid 0 \leq j \leq N + 1 \} \) and \( \Omega_T = \{ t_n \mid -m \leq n \leq M \} \). For all grid functions \( v \in \mathcal{W} := \{ v \mid v = \{ v^n_j \mid 0 \leq j \leq N + 1, -m \leq n \leq M \} \} \) on \( \Omega_{ht} \), we define following difference operators:
\[
\delta_t v_j^{n+\frac{1}{2}} = v_j^{n+1} - v_j^n, \quad \delta_t^2 v_j^n = \frac{\delta_t v_j^{n+\frac{1}{2}} - \delta_t v_j^{n-\frac{1}{2}}}{\tau}, \quad D_t v_j^n = \frac{v_j^{n+1} - v_j^{n-1}}{2\tau}.
\]

Throughout this paper, we assume that problem (2.1)–(2.3) has a unique solution \( u(x, t) \in C^{5,4}([a, b] \times [-s, T]) \) and
\[
u^*(x, \cdot) := \begin{cases} u(x, \cdot), & x \in [a, b], \\
0, & x \not\in \mathbb{R} \setminus [a, b], \end{cases} \in \ell^{2+\gamma}(\mathbb{R}).
\]
It follows from Lemma 2.4 that
\[
\frac{\partial^\gamma}{\partial|x|^\gamma} U_j^n = \delta_t^2 U_j^n + \mathcal{O}(h^2),
\]
where \( U_j^n = u(x_j, t_n) \) and
\[
\delta_t U_j^n = -h^{-\gamma} \sum_{k=j-N}^{j-1} g_k^{(\gamma)} U_{j-k}.
\]
Moreover, using Taylor formula, we have that
\begin{equation}
\frac{\partial^2}{\partial t^2} U_j^n = \delta_2^2 U_j^n + O(\tau^2), \quad \frac{\partial}{\partial t} U_j^n = D_t U_j^n + O(\tau^2),
\end{equation}

\begin{equation}
\frac{\partial^\gamma}{\partial |x|^\gamma} U_j^n = \theta \frac{\partial^\gamma}{\partial |x|^\gamma} U_j^{n+1} + \frac{1-2\theta}{\partial |x|^\gamma} U_j^n + \theta \frac{\partial^\gamma}{\partial |x|^\gamma} U_j^{n-1} + O(\tau^2), \quad \theta \in [0, 1].
\end{equation}

Inserting (2.9) to (2.11) leads to
\begin{equation}
\frac{\partial^\gamma}{\partial |x|^\gamma} U_j^n = \theta \delta_2^\gamma U_j^{n+1} + (1-2\theta) \theta \delta_2^\gamma U_j^n + \theta \delta_2^\gamma U_j^{n-1} + O(\tau^2 + h^2), \quad \theta \in [0, 1].
\end{equation}

Substituting (2.6), (2.10) and (2.12) into (2.1) yields for \(1 \leq j \leq N, 0 \leq n \leq M-1\) and \(0 \leq \theta \leq 1\) that
\begin{equation}
\delta_1^2 U_j^n + \sigma D_t U_j^n = \kappa [\theta \delta_2^\gamma U_j^{n+1} + (1-2\theta) \theta \delta_2^\gamma U_j^n + \theta \delta_2^\gamma U_j^{n-1}] - \sum_{i=-m}^n b_{n-i} u_j^i + \beta \sin(u_j^n) + f_j^n + R_j^n,
\end{equation}

where \(f_j^n = f(x_j, t_n)\) and \(R_j^n = O(\tau^2 + h^2)\), which implies that there exists a constant \(c_0 > 0\) independent of \(\tau\) and \(h\) such that \(|R_j^n| \leq c_0(\tau^2 + h^2)\).

Omitting the remainder \(R_j^n\) in (2.13) and using the initial and boundary conditions (2.2)–(2.3), we obtain the following OPFD methods for 1D IBVPs (2.1)–(2.3):

\begin{equation}
\begin{cases}
\delta_1^2 u_j^n + \sigma D_t u_j^n = \kappa [\theta \delta_2^\gamma u_j^{n+1} + (1-2\theta) \theta \delta_2^\gamma u_j^n + \theta \delta_2^\gamma u_j^{n-1}] - \sum_{i=-m}^n b_{n-i} u_j^i + \beta \sin(u_j^n) + f_j^n, & 1 \leq j \leq N, \quad 0 \leq n \leq M-1, \\
u_j^n = \psi(x_j, t_n), & 0 \leq j \leq N+1, \quad -m \leq n \leq 0, \\
u_0^n = 0, & u_N^{n+1} = 0, \quad 1 \leq n \leq M,
\end{cases}
\end{equation}

where \(u_j^n\) is approximation to \(u(x_j, t_n)\) and \(\theta \in [0, 1]\). Note that the coefficient matrix of linear system (2.14) is strictly diagonally dominant. Therefore, method (2.14)–(2.16) is uniquely solvable.

2.3. Convergence analysis of OPFD method

In this section, we study the convergence of the method (2.14)–(2.16). At first, we denote \(V_h = \{v | v = (v_0, v_1, \ldots, v_{N+1})\}\) as the set of grid functions on \(\Omega_h\) and \(\hat{V}_h = \{v \in V_h | v_0 = v_{N+1} = 0\}\). For any \(w, v \in \hat{V}_h\), define their inner product \((w, v) = h \sum_{k=1}^N w_k v_k\) and the corresponding norm \(\|w\| = \sqrt{(w, w)}\). For the convergence analysis, we introduce the following lemmas related to the difference operator \(\delta_2^\gamma\).

**Lemma 2.5 ([18]).** For any two grid functions \(w, v \in \hat{V}_h\), there exists a difference operator \(\Lambda_2^\gamma\) such that
\((\delta_2^\gamma w, v) = -(\Lambda_2^\gamma w, \Lambda_2^\gamma v), \quad \gamma \in (1, 2].\)

**Lemma 2.6.** For any grid function \(w \in \hat{V}_h\), the following inequality holds:
\[\|\Lambda_2^\gamma w\| \leq \frac{2\theta^{(\gamma)}_h}{h^\gamma} \|w\|\]
Proof. According to Lemma 2.5 and the definition of \( \delta^n \), we have

\[
\| \Lambda^n w \|^2 = -h \sum_{j=1}^{N} (\delta^n_j w_j) w_j = h^{1-\gamma} \sum_{j=1}^{N} \sum_{k=j-N}^{j-1} g_k^{(\gamma)} w_{j-k} w_j.
\]

Using the common inequality \( a^2 \leq \frac{a^2 + b^2}{2} \) and Lemma 2.2 leads to

\[
\| \Lambda^n w \|^2 \leq h^{1-\gamma} \sum_{j=1}^{N} \sum_{k=1}^{j} |g_k^{(\gamma)}| |w_{j-k}^2 + w_j^2|/2
\]

\[
= h^{1-\gamma} \sum_{j=1}^{N} \sum_{k=1}^{N} |g_k^{(\gamma)}| w_j^2 + \sum_{j=1}^{N} \sum_{k=1}^{j-1} |g_k^{(\gamma)}| \leq g_0^{(\gamma)} h^{1-\gamma} \sum_{j=1}^{N} w_j^2 + g_0^{(\gamma)} h^{1-\gamma} \sum_{j=1}^{N} w_j^2 = 2g_0^{(\gamma)} h^{1-\gamma} \| u \|^2.
\]

This completes the proof. \( \square \)

Let

\[
U^n = (U_0^n, U_1^n, \ldots, U_N^n)^T, \quad u^n = (u_0^n, u_1^n, \ldots, u_{N+1}^n)^T, \quad e^n = U^n - u^n.
\]

Based on the above lemmas, we derive a convergence theorem below.

**Theorem 2.1.** Assume that the solution \( u(x, t) \) of problem (2.1)–(2.3) belongs to \( C^{5,1}([a, b] \times [-s, T]) \), \( u^*(x, \cdot) \in L^{2+\epsilon}(\mathbb{R}) \) and there exist constants \( c_1 > 1 \) and \( d_1 > 0 \) such that

\[
\tau \leq \left(1 - \frac{1}{c_1} \right) \frac{2d_1}{|\beta| + 3}, \quad d_1 \leq \min \left\{ 1, 1 + 2\kappa g_0^{(\gamma)} \left( \theta - \frac{1}{4} \right) \frac{\tau^2}{h^4} \right\}. \quad (2.17)
\]

Then the global error \( e^n \) of method (2.14)–(2.16) satisfies the following estimate:

\[
\| e^n \| \leq C_1 (\tau^2 + h^2), \quad 1 \leq n \leq M,
\]

where

\[
C_1 = \sqrt{\frac{c_1^2 c_1 (b-a)T^3}{d_1}} \exp \left( c_1 T \left[ (2 + |\beta|)T^2 + |\beta| + 3 \right] \right).
\]

Proof. Writing \( e_j^n = U_j^n - u_j^n \) and subtracting Eq. (2.14) from Eq. (2.13) yield for \( 1 \leq j \leq N \) and \( 0 \leq n \leq M - 1 \) that

\[
\delta_t^2 e_j^n + \sigma \Delta t e_j^n = \kappa \left[ \theta \delta_x^2 e_j^{n+1} + (1 - 2\theta) \delta_x^2 e_j^n + \theta \delta_x^2 e_j^{n-1} \right]
\]

\[
- \sum_{i=n}^{n-m} b_{n-i} \beta \sin(U_j^n) - \beta \sin(u_j^n) + R^n_j. \quad (2.18)
\]

Let \( R^n = (0, R^n_1, \ldots, R^n_N, 0)^T \). Multiplying \( 2h \Delta t e_j^n \) the both sides of equation (2.18) and then summing \( j \) from 1 to \( N \) give that

\[
\frac{\| \delta_t e_j^{n+\frac{1}{2}} \|^2 - \| \delta_t e_j^{n-\frac{1}{2}} \|^2}{\tau} + 2\sigma \| \Delta_t e^n \|^2 + \kappa \theta \| \Lambda_x^2 e_j^{n+1} \|^2 - \| \Lambda_x^2 e_j^{n-1} \|^2 \quad (2.19)
\]
Substituting (2.20)–(2.22) into (2.19) follows for $0 \leq n \leq M - 1$, where we have also used Lemma 2.5. By Cauchy-Schwarz inequality and the common inequality: $2\hat{a}b \leq \hat{a}^2 + \hat{b}^2$ ($\forall \hat{a}, \hat{b} \in \mathbb{R}$), we have that

$$\frac{\kappa(2\theta - 1)}{\tau} \left( \Lambda_2^\gamma e^n - \Lambda_2^\gamma e^{n+1} - \Lambda_2^\gamma e^{n-1} \right) - \sum_{i=n-m}^{n} b_{n-i}(e^i, \delta_1 e^{n+\frac{1}{2}} + \delta_1 e^{n-\frac{1}{2}})$$

and

$$\beta h \sum_{j=1}^{N} (\sin(U^n_j) - \sin(u^n_j))(\delta_1 e^{n+\frac{1}{2}} + \delta_1 e^{n-\frac{1}{2}}) + (R^n, \delta_1 e^{n+\frac{1}{2}} + \delta_1 e^{n-\frac{1}{2}}), \quad 0 \leq n \leq M - 1,$

$$\text{leastsquares}$$

Thus, by the second condition of (2.17) we obtain for all $\theta \in [0, 1]$ that

$$\|\delta_1 e^{n+\frac{1}{2}}\|^2 \leq \frac{1}{d_\theta} E^n, \quad 0 \leq n \leq M - 1.$$
With the above inequality and Cauchy-Schwarz inequality, we further infer for $1 \leq n \leq M$ that
\[
\|e^n\|^2 = \sigma^2 \left( \sum_{k=0}^{n-1} \| \delta_t e^{k+\frac{\sigma}{2}} \| \right)^2 \leq \sigma^2 \left( \sum_{k=0}^{n-1} \| \delta_t e^{k+\frac{\sigma}{2}} \| \right)^2 \leq \sigma^2 n \sum_{k=0}^{n-1} \| \delta_t e^{k+\frac{\sigma}{2}} \|^2 \leq \frac{\sigma^2}{d_1} \sum_{k=0}^{n-1} E^k. \tag{2.25}
\]
Since $\|e^n\| = 0$ for $-m \leq n \leq 0$, the inequality (2.25) also holds for $-m \leq n \leq 0$. This gives that
\[
\|e^n\|^2 \leq \frac{\sigma^2}{d_1} \sum_{k=0}^{n-1} E^k, \quad -m \leq n \leq M. \tag{2.26}
\]
A combination of inequalities (2.4), (2.24), (2.26) derives that
\[
\frac{E^n - E^{n-1}}{\tau} \leq \sum_{i=n-m}^{n} |b_{n-i}| \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + |\beta| \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + \frac{\sum_{i=0}^{m} |b_i| + |\beta| + 1}{2d_1} (E^n + E^{n-1}) + \|R^n\|^2
\]
\[
\leq \sum_{i=n-m}^{n} |b_{n-i}| \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + |\beta| \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + \frac{\sum_{i=0}^{m} |b_i| + |\beta| + 1}{2d_1} (E^n + E^{n-1}) + \|R^n\|^2
\]
\[
\leq \frac{2t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + |\beta| \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + \frac{|\beta| + 3}{2d_1} (E^n + E^{n-1}) + c_0^2 (b - a)(\tau^2 + h^2)^2, \quad 0 \leq n \leq M - 1.
\]
Summing (2.27) for $n$ from 0 to $q$ ($0 \leq q \leq M - 1$) deduces that
\[
\frac{E^q}{\tau} \leq \sum_{n=0}^{q} \frac{2 + |\beta|}{d_1} \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + \frac{|\beta| + 3}{2d_1} \left( \sum_{n=0}^{q} E^n + \sum_{n=0}^{q-1} E^n \right) + \sum_{n=0}^{q} c_0^2 (b - a)(\tau^2 + h^2)^2
\]
\[
\leq \sum_{n=0}^{q} \frac{2 + |\beta|}{d_1} \frac{t_n \tau}{d_1} \sum_{k=0}^{n-1} E^k + \frac{|\beta| + 3}{2d_1} \left( \sum_{n=0}^{q} E^n + \sum_{n=0}^{q-1} E^n \right) + \sum_{n=0}^{q} c_0^2 (b - a)(\tau^2 + h^2)^2
\]
\[
= \frac{(2 + |\beta|) t_q \tau + \frac{q-1}{d_1} \sum_{k=0}^{q-1} E^k + \frac{|\beta| + 3}{2d_1} \left( \sum_{n=0}^{q} E^n + \sum_{n=0}^{q-1} E^n \right) + \sum_{n=0}^{q} c_0^2 (b - a)(\tau^2 + h^2)^2. \tag{2.28}
\]
Subtracting $\frac{|\beta| + 3}{d_1} E^q$ from the both sides of inequality (2.28) and using the first condition in (2.17), we conclude that
\[
E^q \leq c_1 \left[ \frac{(2 + |\beta|) t_{q+1} + |\beta| + 3}{d_1} \sum_{n=0}^{q-1} E^n + c_0^2 c_1 (b - a) t_{q+1} (\tau^2 + h^2)^2 \right]
\]
\[
\leq c_1 \left[ \frac{(2 + |\beta|) T^2 + |\beta| + 3}{d_1} \sum_{n=0}^{q-1} E^n + c_0^2 c_1 (b - a) T (\tau^2 + h^2)^2, \quad 0 \leq q \leq M - 1.
\]
Applying Gronwall inequality (cf. [19]) to the above inequality yields that
\[
E^q \leq c_0^2 c_1 (b - a) T (\tau^2 + h^2)^2 \exp \left[ \frac{c_1 t_q (2 + |\beta|) T^2 + |\beta| + 3}{d_1} \right], \quad 0 \leq q \leq M - 1.
\]
This, together with (2.26), implies that
\[ \| e_n \|^2 \leq t_n \tau c_1 \left( b - a \right) T \left( \tau^2 + h^2 \right)^2 \sum_{q=0}^{n-1} \exp \left[ \frac{c_1 t_q \left( 2 + |\beta| \right) T^2 + |\beta| + 3 }{d_1} \right] \]
\[ \leq t_n^2 \tau^2 c_1 \left( b - a \right) T^2 \left( \tau^2 + h^2 \right)^2 \exp \left[ \frac{c_1 t_{n-1} \left( 2 + |\beta| \right) T^2 + |\beta| + 3 }{d_1} \right] \]
\[ \leq \frac{c_1^2 \left( b - a \right) T^3}{d_1} \exp \left[ \frac{c_1 T \left( 2 + |\beta| \right) T^2 + |\beta| + 3 }{d_1} \right] \left( \tau^2 + h^2 \right)^2, \quad 1 \leq n \leq M. \quad (2.29) \]

Therefore the theorem is proved. \( \square \)

The above theorem shows that OPFD methods (2.14)–(2.16) for 1D IBVPs (2.1)-(2.3) can achieve the second-order convergence both in time and space. However, this result is obtained under the smooth conditions: \( u(x,t) \in C^{5,1}([a,b] \times [-s,T]) \) and \( u^*(x,:) \in \ell^{2+\gamma}(\mathbb{R}) \). In fact, the solutions of nonlocal problems generally have weak singularity near the boundary. Hence, the actual convergence order of the methods may not reach the above theoretical order when the smooth conditions are weakened (see, e.g., Example 4.2). In order to overcome the similar order obstacle, some techniques for the numerical methods solving some space-fractional differential equations without delay have been suggested. For example, Hao & Cao [20] used the extrapolation technique and Chen, Zeng & Karniadakis [21] proposed the technique adding some correction terms into the weighted shifted Grünwald-Letnikov formula. Nevertheless, as the complexity of the solved delay problem, we find it difficult to introduce the existing techniques into OPFD methods to preserve their theoretical convergence order. Hence this issue keeps open at present.

2.4. PCG method

When method (2.14)–(2.16) is applied to IBVPs (2.1)-(2.3), a series of large dense linear systems with Toeplitz structure will be generated. This will result in a large computational cost. Fortunately, we note that several efficient techniques for treating the similar difficult issues have been presented in some references. For solving the two-side space-fractional diffusion equation, Lei & Sun [22] considered the PCG normal residual method with Strang circulant preconditioner. For solving the space-fractional advection-diffusion equation, Qu, Shen & Liang [23] suggested the PCG method with Strang circulant preconditioner. For solving the space-time fractional advection-diffusion equation, Zhao, Jin & Lin [24] and Gu et al. [25] proposed the minimal/normal residual methods with a banded preconditioner and the PCG squared method with Strang circulant preconditioner, respectively. For solving the Riesz distributed-order space-fractional diffusion equations, Huang et al. [26] used the PCG method with Strang circulant preconditioner. These work took use of the weighted shifted Grünwald-Letnikov formula to discretize the space-fractional derivatives and thus derived Toeplitz-like or Toeplitz linear systems. For the derived linear systems, the acquisitions of their acceleration methods benefit from the Toeplitz-like or Toeplitz structure of the systems. While in our paper, the space-fractional derivative was discretized by the fractional central difference formula and hence a series of linear systems with Toeplitz structure were generated. Inspired by the above work, we will combine Strang circulant preconditioner and CG method to accelerate the calculation of the linear systems. In this way, a PCG method for solving IBVPs (2.1)-(2.3) can be followed.
Lemma 2.7. The preconditioner $S_{\gamma,N}$ and coefficient matrix $A_{\gamma,N}$ have following properties:

(1) The smallest eigenvalue of $A_{\gamma,N}$ satisfies that $\lambda_1(A_{\gamma,N}) \geq 1$;

(2) $S_{\gamma,N}$ is invertible and satisfies that

$$
\|S_{\gamma,N}^{-1}\|_2 \leq 1, \quad \|S_{\gamma,N}\|_2 \leq 1 + \frac{\sigma_T}{2} + 2r(\theta, \tau, h)g_{0}(\gamma).$

Proof. It follows from Lemma 2.2 that $G_{\gamma,N}$ is diagonally dominant. Thus, $G_{\gamma,N}$ is symmetric positive semi-definite and

$$
\lambda_1(A_{\gamma,N}) = 1 + \frac{\sigma_T}{2} + r(\theta, \tau, h)\lambda_1(G_{\gamma,N}) \geq 1 + \frac{\sigma_T}{2} \geq 1.
$$
Let \( p_i = \sum_{j=1}^{N} |S(G_{\gamma,N})_{ij}| \). When \( N \) is even, by the definition of \( S(G_{\gamma,N}) \) and Lemma 2.2, we have for \( 1 \leq i \leq N \) that

\[
p_i = 2 \sum_{j=1}^{N} |g_j^{(\gamma)}| = 2 \sum_{k=-\infty}^{\sigma,T} |g_k^{(\gamma)}| = \sum_{k=-\infty}^{\sigma,T} |g_k^{(\gamma)}| = g_0^{(\gamma)}.
\]

(2.31)

Since the eigenvalues of real symmetry matrix \( S(G_{\gamma,N}) \) are all real and the fact: \( S(G_{\gamma,N})_{kk} = g_0^{(\gamma)} \) holds, Gerschgorin circle theorem (see e.g. [15]) derives that

\[
\lambda_i(S(G_{\gamma,N})) \in \bigcup_{k=1}^{N} \{ \lambda \in \mathbb{R} | |\lambda - S(G_{\gamma,N})_{kk}| \leq p_k \} = \bigcup_{k=1}^{N} \{ \lambda \in \mathbb{R} | |\lambda - g_0^{(\gamma)}| \leq p_k \}, \quad 1 \leq i \leq N.
\]

This, together with (2.31), deduces that

\[
\lambda_i(S(G_{\gamma,N})) \in \left\{ \lambda \in \mathbb{R} | |\lambda - g_0^{(\gamma)}| \leq g_0^{(\gamma)} \right\}, \quad 1 \leq i \leq N.
\]

Combining the above relation with equality: \( \lambda_i(S_{\gamma,N}) = 1 + \frac{\sigma,T}{2} + r(\theta, \tau, h)\lambda_i(S(G_{\gamma,N})) \) yields that

\[
\lambda_i(S_{\gamma,N}) \in \left[ 1 + \frac{\sigma,T}{2}, 1 + \frac{\sigma,T}{2} + 2r(\theta, \tau, h)g_0^{(\gamma)} \right], \quad 1 \leq i \leq N.
\]

This implies that \( S_{\gamma,N} \) is invertible and the following estimates are satisfied:

\[
\|S_{\gamma,N}^{-1}\|_2 = \rho(S_{\gamma,N}^{-1}) = \lambda_1^{-1}(S_{\gamma,N}) \leq \frac{1}{1 + \frac{\sigma,T}{2}} \leq 1,
\]

\[
\|S_{\gamma,N}\|_2 = \rho(S_{\gamma,N}) = \lambda_N(S_{\gamma,N}) \leq 1 + \frac{\sigma,T}{2} + 2r(\theta, \tau, h)g_0^{(\gamma)}.
\]

When \( N \) is odd, the proof of the conclusion (2) is similar to that of the case: \( N \) is even and hence we omit it here. This completes the proof.

The following lemma will be applied to study the property of \( S_{\gamma,N}^{-1}A_{\gamma,N} \).

**Lemma 2.8 ([27])**. Assume that the generating function of a real symmetric Toeplitz matrix sequence \( \{T_N\}_{N=1}^{\infty} \) belongs to Wiener class. Then, for any given \( \epsilon > 0 \), there exist integers \( M, N > 0 \) such that

\[
T_N - S(T_N) = \hat{U}_N + \hat{W}_N, \quad \forall N > \hat{N},
\]

where \( \hat{U}_N \) and \( \hat{W}_N \) are two real symmetric matrices satisfying \( \text{rank}(\hat{U}_N) < M \) and \( \|\hat{W}_N\|_2 < \epsilon \), respectively, and \( S(T_N) \) is a circulant matrix having the similar structure as \( S(G_{\gamma,N}) \) with \( \hat{t}_i \) instead of \( g_0^{(\gamma)} \).

A property of matrix \( S_{\gamma,N}^{-1}A_{\gamma,N} \) is stated as follows.

**Theorem 2.2**. Assume that there is a constant \( K > 0 \) such that \( r(\theta, \tau, h) \leq K \). Then, for any given \( \epsilon > 0 \), there exist integers \( M^*, N^* > 0 \) such that matrix \( S_{\gamma,N}^{-1}A_{\gamma,N} \) has at most \( M^* \) eigenvalues outside \((1 - \epsilon, 1 + \epsilon)\) when \( N > N^* \).

**Proof**. The generating function of \( \{G_{\gamma,N}\}_{N=1}^{\infty} \) is given by \( g(x) = \sum_{k=-\infty}^{\infty} g_k^{(\gamma)} e^{ik\theta} \) and, by Lemma 2.2, \( \sum_{k=-\infty}^{\infty} |g_k^{(\gamma)}| = 2g_0^{(\gamma)} < \infty \). Hence \( g(x) \) belongs to Wiener class. According
to Lemma 2.8, there exist integers $M^*, N^* > 0$ and two real symmetric matrices $\hat{U}_{\gamma,N}, \hat{W}_{\gamma,N}$ satisfying $\operatorname{rank}(\hat{U}_{\gamma,N}) < M^*$ and $||\hat{W}_{\gamma,N}||_2 < \epsilon/K$ such that

$$A_{\gamma,N} = S_{\gamma,N} + r(\theta, \tau, h) [G_{\gamma,N} - S(G_{\gamma,N})] = S_{\gamma,N} + r(\theta, \tau, h) \left( \hat{U}_{\gamma,N} + \hat{W}_{\gamma,N} \right), \quad \forall N > N^*. \tag{2.32}$$

Since $S_{\gamma,N}$ is a symmetric positive-definite matrix, there is an orthogonal matrix $Q_{\gamma,N}$ such that $S_{\gamma,N} = Q_{\gamma,N}^T \Lambda_{\gamma,N} Q_{\gamma,N}$, where $\Lambda_{\gamma,N}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $S_{\gamma,N}$. Let

$$U_{\gamma,N} = r(\theta, \tau, h) S_{\gamma,N}^{-1/2} \hat{U}_{\gamma,N} S_{\gamma,N}^{-1/2} \text{ and } W_{\gamma,N} = r(\theta, \tau, h) S_{\gamma,N}^{-1/2} \hat{W}_{\gamma,N} S_{\gamma,N}^{-1/2}.$$ 

Then, by (2.32) we have

$$S_{\gamma,N}^{-1/2} A_{\gamma,N} S_{\gamma,N}^{-1/2} - I_N = U_{\gamma,N} + W_{\gamma,N},$$

and by Lemma 2.7 we have for $N > N^*$ that

$$\operatorname{rank}(U_{\gamma,N}) = \operatorname{rank}(S_{\gamma,N}^{-1/2} \hat{U}_{\gamma,N} S_{\gamma,N}^{-1/2}) = \operatorname{rank}(\hat{U}_{\gamma,N}) < M^*, \tag{2.33}$$

$$||W_{\gamma,N}||_2 \leq K ||S_{\gamma,N}^{-1/2}||_2 ||\hat{W}_{\gamma,N}||_2 = K \lambda_{\gamma}^{-1}(S_{\gamma,N}) ||\hat{W}_{\gamma,N}||_2 \leq K \frac{\epsilon}{K} = \epsilon. \tag{2.34}$$

It follows from (2.33) that $U_{\gamma,N}$ has at most $M^*$ non-zero eigenvalues. Moreover, inequality (2.34) shows that the spectrum of $W_{\gamma,N}$ is contained in $(-\epsilon, \epsilon)$. Hence, we infer from Weyl’s Theorem (see, e.g., [27]) that $U_{\gamma,N} + W_{\gamma,N}$ has at most $M^*$ eigenvalues outside $(-\epsilon, \epsilon)$ when $N > N^*$. According to the fact that $S_{\gamma,N}^{-1} A_{\gamma,N} - I_N$ is similar to $U_{\gamma,N} + W_{\gamma,N}$. Therefore the theorem is proved. \hfill \Box

Theorem 2.2 indicates that the spectrum of $S_{\gamma,N}^{-1} A_{\gamma,N}$ is cluster around 1 for positive integer $N$ large enough. As to the selection of $M^*$ and $N^*$ in Theorem 2.2, in the following, we give a comment. From the proof of Lemma 2.8 in [22], we get to know that, for a given $\epsilon > 0$, $N^*$ should satisfy $\sum_{k=N^*+1}^{\infty} |g_k^{(\gamma)}| < \epsilon/2K$ and $M^* = 2N^*$. Combining this argument with Lemma 2.3, we can select

$$N^* = \left[ \frac{m^*}{2} \epsilon^{-\frac{1}{\gamma}} + \frac{\gamma}{2} - 1 \right] + 1 \text{ with } m^* = 2 \left[ \frac{2K|g_1^{(\gamma)}|(\frac{2}{\gamma} + 2)^{\gamma+1}}{\gamma} \right]^{\frac{1}{\gamma}}$$

and thus

$$M^* = 2N^* = 2 \left[ \frac{m^*}{2} \epsilon^{-\frac{1}{\gamma}} + \frac{\gamma}{2} - 1 \right] + 2 \leq m^* \epsilon^{-\frac{1}{\gamma}} - \frac{\gamma}{2}. \tag{2.35}$$

Moreover, another useful spectral property of matrix $S_{\gamma,N}^{-1} A_{\gamma,N}$ can be stated as follows.

**Theorem 2.3.** Assume that there is a constant $K > 0$ such that $r(\theta, \tau, h) \leq K$. Then, for all integer $N > 0$, the minimum eigenvalue $\lambda_1(S_{\gamma,N}^{-1} A_{\gamma,N})$ of matrix $S_{\gamma,N}^{-1} A_{\gamma,N}$ satisfies that

$$\lambda_1(S_{\gamma,N}^{-1} A_{\gamma,N}) \geq \frac{1}{1 + \frac{4\epsilon}{\gamma} + 2Kg_0^{(\gamma)}}. \tag{2.36}$$
Proof. By the positive definiteness of $S_{y,N}^\frac{\gamma}{2}$ and $A_{y,N}$, $S_{y,N}^\frac{\gamma}{2}A_{y,N}^{-1}S_{y,N}^\frac{\gamma}{2}$ is also positive definite. This implies that the maximum eigenvalue of $S_{y,N}^\frac{\gamma}{2}A_{y,N}^{-1}S_{y,N}^\frac{\gamma}{2}$ has the following estimate:

$$
\lambda_N(S_{y,N}^\frac{\gamma}{2}A_{y,N}^{-1}S_{y,N}^\frac{\gamma}{2}) = \|S_{y,N}^\frac{\gamma}{2}A_{y,N}^{-1}S_{y,N}^\frac{\gamma}{2}\|_2 \leq \|S_{y,N}^\frac{\gamma}{2}\|_2 \|A_{y,N}^{-1}\|_2
$$

(2.37)

Inserting (2.37) to (2.38) and using Lemma 2.7 yield that

$$
\lambda_1(S_{y,N}^{-1}A_{y,N}) = \lambda_1(S_{y,N}^{-\frac{\gamma}{2}}A_{y,N}S_{y,N}^{-\frac{\gamma}{2}}) = \lambda_N(S_{y,N}^\frac{\gamma}{2}A_{y,N}^{-1}S_{y,N}^\frac{\gamma}{2}).
$$

(2.38)

This completes the proof. \(
\square
\)

Let $\epsilon_k$ be the $k$-th iteration error of PCG method applied to linear system (2.30) and $|||\epsilon_k||| = \epsilon_k^TA_{y,N}^{-1}\epsilon_k$. A result characterizing the convergence rate of PCG method can be derived as follows.

**Theorem 2.4.** Assume that there is a constant $K > 0$ such that $r(\theta, \tau, h) \leq K$. Then the PCG method for linear system (2.30) is superlinearly convergent, i.e. there is an integer $M > 0$ such that $|||\epsilon_k||| \leq (\hat{\epsilon}_k)^k|||\epsilon_0|||$ for all $k \geq M$, where $\{\hat{\epsilon}_k\}$ is a sequence satisfying $\hat{\epsilon}_k \to 0$ as $k \to +\infty$.

**Proof.** By Theorems 2.2 and 2.3, it holds that the spectrum of $S_{y,N}^{-1}A_{y,N}$ is uniformly bounded away from zero by $1 - \frac{2\sigma \tau}{1 + 2Kg_0}$ and there are at most $M^*$ eigenvalues of $S_{y,N}^{-1}A_{y,N}$ outside $(1 - \epsilon, 1 + \epsilon)$ when $r(\theta, \tau, h) \leq K$ and $N > N^*$. Combining these results with Corollary 1.11 in [27] yields that

$$
\|\|\epsilon_k\|\| \leq 2[(1 + \epsilon^{-1})\delta]^{M^*} \epsilon^k, \quad \forall k \geq M^*,
$$

(2.39)

where $\delta = 1 + \frac{\sigma \tau}{1 + 2Kg_0}$. Substituting (2.35) into (2.39) and taking $\epsilon = \left(\frac{m^*}{\gamma + k^\frac{1}{\gamma}}\right)^\gamma$ give that

$$
\|\|\epsilon_k\|\| \leq 2[(1 + \epsilon^{-1})\delta]^{M^*} \epsilon^k = 2 \left\{ 1 + \left(\frac{\gamma + k^\frac{1}{\gamma}}{m^*}\right)^\gamma \right\}^{k \frac{1}{\gamma}} \left(\frac{m^*}{\gamma + k^\frac{1}{\gamma}}\right)^k
$$

(2.40)

Moreover, it can be verified that

$$
\lim_{k \to +\infty} 2 \delta^{k \frac{1}{\gamma}} = 1, \quad \lim_{k \to +\infty} \left[ 1 + \left(\frac{\gamma + k^\frac{1}{\gamma}}{m^*}\right)^\gamma \right]^{k \frac{1}{\gamma}} = 1, \quad \lim_{k \to +\infty} \left(\frac{m^*}{\gamma + k^\frac{1}{\gamma}}\right)^\gamma = 0.
$$

which imply that $\lim_{k \to +\infty} \hat{\epsilon}_k = 0$. Therefore the theorem is proved. \(
\square
\)
3. ADI Method and ADI-PCG Method for 2D Problems

3.1. Construction of ADI method and ADI-PCG method

In this section, we extend our studies to the following 2D IBVPs:

\[
\begin{aligned}
&\frac{\partial^2}{\partial t^2} u(x, y, t) + \frac{\partial}{\partial t} u(x, y, t) = \kappa_1 \frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} u(x, y, t) + \kappa_2 \frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} u(x, y, t) - \int_{t-s}^t e^{-(t-\eta)} \frac{\partial}{\partial \eta} u(x, y, \eta) d\eta \\
&\quad + \beta \sin(u(x, y, t)) + f(x, y, t), \\
&u(x, y, t) = \psi(x, y, t), \\
&u(x, y, t) = 0,
\end{aligned}
\]

where \( \sigma, \kappa_1, \kappa_2, \alpha, s, T > 0, a, b, c, d \) and \( \beta \) are some given constants, \( \Omega \) is the closure of set \( \Omega = (a, b) \times (c, d) \), \( \partial \Omega = \Omega \setminus \bar{\Omega} \), \( \psi(x, y, t) \) and \( f(x, y, t) \) are two known functions, and \( \gamma_1, \gamma_2 \in (1, 2) \) are the orders of Riesz fractional derivatives in \( x \) and \( y \), respectively. For the above problems, we assume that each of them has a unique solution \( u(x, y, t) \in C^{5,4}([a, b] \times [c, d] \times [-s, T]) \) and

\[
u^*(x, y, \cdot) := \begin{cases}
\frac{u(x, y, \cdot)}{0,} & (x, y) \in \Omega, \\
\frac{u(x, y, \cdot)}{(x, y) \in \Omega^2 \setminus \Omega,} & \in \ell^{2+\gamma_1+\gamma_2}(\mathbb{R}^2) := \ell^{2+\gamma_1}(\mathbb{R}) \times \ell^{2+\gamma_2}(\mathbb{R}).
\end{cases}
\]

Let \( N_1 \) and \( N_2 \) be positive integers, \( h_1 = \frac{(b-a)}{(N_1 + 1)} \) and \( h_2 = \frac{(d-c)}{(N_2 + 1)} \) are the sizes of spatial grid in \( x \)- and \( y \)-directions, respectively. Define domain \( \Omega_{h_1 h_2 \tau} = \Omega_{h_1 h_2} \times \Omega_{\tau} \) with \( \Omega_{h_1 h_2} = \{(x_i, y_j) | x_i = a + ih_1, y_j = c + jh_2, 0 \leq i \leq N_1 + 1, 0 \leq j \leq N_2 + 1 \} \) and \( \Omega_{\tau} = \{\tau_n | -m \leq n \leq M \} \). On the grid function space \( \mathcal{M} := \{v | v = (v_{ij}^n) \in \ell^{2+\gamma_1+\gamma_2}(\mathbb{R}^2) \} \), we define the following difference operators:

\[
\begin{aligned}
I v_{ij}^n &= v_{ij}^n, \\
\delta^1 v_{ij}^n &= v_{ij}^{n+1} - v_{ij}^n, \\
\delta^2 v_{ij}^n &= v_{ij}^{n+\frac{1}{2}} - v_{ij}^{n-\frac{1}{2}}, \\
D v_{ij}^n &= \frac{v_{ij}^{n+1} - v_{ij}^{n-1}}{2\tau}.
\end{aligned}
\]

Write

\[
U_{ij}^n = u(x_i, y_j, t_n), \quad \delta^1 U_{ij}^n = -h_1^{-\gamma_1} \sum_{k=-N_1}^{N_1} g_k(\gamma_1) U_{i-k,j}^n, \quad \delta^2 U_{ij}^n = -h_2^{-\gamma_2} \sum_{k=-N_2}^{N_2} g_k(\gamma_2) U_{i,j-k}^n.
\]

Then, by Lemma 2.4 we have that

\[
\frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} U_{ij}^n = \delta^1 U_{ij}^n + O(h_1^2), \quad \frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} U_{ij}^n = \delta^2 U_{ij}^n + O(h_2^2).
\]

It follows from Taylor formula that

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} U_{ij}^n &= \delta^2 U_{ij}^n + O(\tau^2), \\
\frac{\partial}{\partial t} U_{ij}^n &= D t U_{ij}^n + O(\tau^2), \\
\frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} U_{ij}^n &= \theta \frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} U_{ij}^{n+1} + (1 - 2\theta) \frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} U_{ij}^n + \theta \frac{\partial^{\gamma_1}}{\partial |x|^{\gamma_1}} U_{ij}^{n-1} + O(\tau^2), \quad \theta \in [0, 1], \\
\frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} U_{ij}^n &= \theta \frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} U_{ij}^{n+1} + (1 - 2\theta) \frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} U_{ij}^n + \theta \frac{\partial^{\gamma_2}}{\partial |y|^{\gamma_2}} U_{ij}^{n-1} + O(\tau^2), \quad \theta \in [0, 1].
\end{aligned}
\]
With a similar proof of Lemma 2.1, we have for a method for 2D IBVPs (3.1)-(3.3) can be derived as follows:

\[
\frac{\partial^{\gamma_1}}{\partial |x|^\gamma_1} U_{ij}^n = \theta \delta_{x}^{\gamma_1} U_{ij}^{n+1} + (1 - 2\theta)\delta_{x}^{\gamma_1} U_{ij}^n + \theta \delta_{x}^{\gamma_1} U_{ij}^{n-1} + \mathcal{O}(\tau^2 + h_1^2),
\]
\[
\frac{\partial^{\gamma_2}}{\partial |y|^\gamma_2} U_{ij}^n = \theta \delta_{y}^{\gamma_2} U_{ij}^{n+1} + (1 - 2\theta)\delta_{y}^{\gamma_2} U_{ij}^n + \theta \delta_{y}^{\gamma_2} U_{ij}^{n-1} + \mathcal{O}(\tau^2 + h_2^2),
\]

(3.10)
(3.11)

With a similar proof of Lemma 2.1, we have for \( u(\cdot, \cdot, t) \in C^2[-s, T] \) that

\[
\int_{t_{n-m}}^{t_n} e^{-\alpha(t_n-\eta)} \frac{\partial}{\partial \eta} u(x, y, \eta) d\eta = \sum_{k=n-m}^{n} b_{n-k} U_{ij}^k + \mathcal{O}(\tau^2).
\]

(3.12)

Substituting equations (3.7) and (3.10)-(3.12) into the equation (3.1), we obtain for \( 1 \leq i \leq N_1, 1 \leq j \leq N_2 \) and \( 0 \leq n \leq M - 1 \) that

\[
\delta_{x}^{\gamma_1} U_{ij}^n + \sigma D_t U_{ij}^n = (\kappa_1 \delta_{x}^{\gamma_1} + \kappa_2 \delta_{y}^{\gamma_2}) \left[ \theta U_{ij}^{n+1} + (1 - 2\theta)U_{ij}^n + \theta U_{ij}^{n-1} \right] \\
- \sum_{k=n-m}^{n} b_{n-k} U_{ij}^k + \beta \sin(u_{ij}^n) + f_{ij}^n,
\]

(3.13)

where \( f_{ij}^n = f(x_i, y_j, t_n) \) and \( R_{ij}^n = \mathcal{O}(\tau^2 + h_1^2 + h_2^2) \). In equality (3.13), when the remainder term \( R_{ij}^n \) is dropped and \( U_{ij}^n \) replaced by the corresponding numerical solution \( u_{ij}^n \), an OPFD method for 2D IBVPs (3.1)-(3.3) can be derived as follows:

\[
\delta_{x}^{\gamma_1} u_{ij}^n + \sigma D_t u_{ij}^n = (\kappa_1 \delta_{x}^{\gamma_1} + \kappa_2 \delta_{y}^{\gamma_2}) \left[ \theta u_{ij}^{n+1} + (1 - 2\theta)u_{ij}^n + \theta u_{ij}^{n-1} \right] \\
- \sum_{k=n-m}^{n} b_{n-k} u_{ij}^k + \beta \sin(u_{ij}^n) + f_{ij}^n,
\]

(3.14)

where \( 1 \leq i \leq N_1, 1 \leq j \leq N_2 \) and \( 0 \leq n \leq M - 1 \).

Nevertheless, the implementation of method (3.14) is very expensive. In order to improve the computational efficiency of the above method, in the following, we consider the ADI technique. Inserting the term \( r_{ij}^n := \tau^2 \alpha_0 \kappa_1 \kappa_2 \delta_{x}^{\gamma_1} \delta_{y}^{\gamma_2} \left[ \theta U_{ij}^{n+1} + (1 - 2\theta)U_{ij}^n + \theta U_{ij}^{n-1} \right] \) (where \( \alpha_0 = \frac{1}{4(2\pi^2)} \)) into the both sides of equality (3.13) yields that

\[
\delta_{x}^{\gamma_1} U_{ij}^n + \sigma D_t U_{ij}^n = (\kappa_1 \delta_{x}^{\gamma_1} + \kappa_2 \delta_{y}^{\gamma_2} - \alpha_1 \kappa_1 \kappa_2 \delta_{x}^{\gamma_1} \delta_{y}^{\gamma_2}) \left[ \theta U_{ij}^{n+1} + (1 - 2\theta)U_{ij}^n + \theta U_{ij}^{n-1} \right] \\
- \sum_{k=n-m}^{n} b_{n-k} U_{ij}^k + \beta \sin(U_{ij}^n) + f_{ij}^n + \tilde{R}_{ij}^n,
\]

(3.15)

where \( \alpha_1 = \tau^2 \alpha_0 \) and \( \tilde{R}_{ij}^n = R_{ij}^n + r_{ij}^n = \mathcal{O}(\tau^2 + h_1^2 + h_2^2) \), which indicates that there is a constant \( \tilde{c}_0 > 0 \) independent of \( \tau, h_1 \) and \( h_2 \) such that \( |\tilde{R}_{ij}^n| \leq \tilde{c}_0(\tau^2 + h_1^2 + h_2^2) \). Omitting the local truncation error \( \tilde{R}_{ij}^n \) in (3.15) and using the initial-boundary conditions (3.2)-(3.3), a numerical method for 2D IBVPs (3.1)-(3.3) can be obtained as follows:

\[
\begin{align*}
\delta_{x}^{\gamma_1} u_{ij}^n + \sigma D_t u_{ij}^n &= (\kappa_1 \delta_{x}^{\gamma_1} + \kappa_2 \delta_{y}^{\gamma_2} - \alpha_1 \kappa_1 \kappa_2 \delta_{x}^{\gamma_1} \delta_{y}^{\gamma_2}) \left[ \theta u_{ij}^{n+1} + (1 - 2\theta)u_{ij}^n + \theta u_{ij}^{n-1} \right] \\
&+ \beta \sin(u_{ij}^n) + f_{ij}^n, \\
1 \leq i \leq N_1, & 1 \leq j \leq N_2, & 0 \leq n \leq M - 1, \\
\end{align*}
\]

\[
\begin{align*}
u_{ij}^n &= \psi(x_i, y_j, t_n), \\
0 \leq i \leq N_1 + 1, & 0 \leq j \leq N_2 + 1, & -m \leq n \leq 0, \\
\end{align*}
\]

(3.17)

\[
\begin{align*}
u_{ij}^n &= 0, \\
i = 0, N_1 + 1 & \text{or } j = 0, N_2 + 1, & 1 \leq n \leq M. \\
\end{align*}
\]

(3.18)
Let
\[ l^{n}_{ij} = - \sum_{k=n-m}^{n} b_{n-k} u_{ij}^{k} + \beta \sin(u_{ij}^{n}) + f_{ij}^{n}, \quad \hat{l}^{n}_{ij} = (1 + \frac{\sigma_{T}}{2} - \theta \sigma_{T}) u_{ij}^{n} + \theta \sigma_{T} u_{ij}^{n-1} + \theta \tau^{2} l_{ij}^{n}. \]

Then method (3.16) can be written in a compact form:
\[ \frac{1}{(1 + \frac{\sigma_{T}}{2} - \theta \sigma_{T})} \left[ (1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y} \right] \left[ (1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y}^{2} \right] \left[ \theta u_{ij}^{n+1} + (1 - 2\theta) u_{ij}^{n} + \theta u_{ij}^{n-1} \right] \]
\[ = \left( 1 + \frac{\sigma_{T}}{2} \right) \hat{l}_{ij}^{n}. \quad (3.19) \]

Write
\[ \tilde{u}_{ij}^{n} = \left[ (1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y} \right] \left[ \theta u_{ij}^{n+1} + (1 - 2\theta) u_{ij}^{n} + \theta u_{ij}^{n-1} \right], \]
\[ \tilde{u}_{ij}^{n} = \theta u_{ij}^{n+1} + (1 - 2\theta) u_{ij}^{n} + \theta u_{ij}^{n-1}. \]

Then method (3.19) can be split into
\[ \frac{1}{(1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y}^{2}} \left[ (1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y} \right] \tilde{u}_{ij}^{n} = (1 + \frac{\sigma_{T}}{2}) \hat{l}_{ij}^{n}, \quad 1 \leq i \leq N_{1}, \quad 1 \leq j \leq N_{2}, \quad 0 \leq n \leq M - 1, \quad (3.20) \]
\[ \frac{1}{(1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y}^{2}} \tilde{u}_{ij}^{n} = \tilde{u}_{ij}^{n}, \quad 1 \leq i \leq N_{1}, \quad 1 \leq j \leq N_{2}, \quad 0 \leq n \leq M - 1, \quad (3.21) \]
and
\[ u_{ij}^{n+1} = \frac{1}{\theta} \left[ \tilde{u}_{ij}^{n} - (1 - 2\theta) u_{ij}^{n} - \theta u_{ij}^{n-1} \right], \quad 1 \leq i \leq N_{1}, \quad 1 \leq j \leq N_{2}, \quad 0 \leq n \leq M - 1, \quad (3.22) \]
in which the initial and boundary approximations are determined respectively by (3.17) and (3.18). In this way, we obtain a class of ADI methods for 2D IBVPs (3.1)–(3.3).

Since the coefficient matrices of linear systems (3.20) and (3.21) can be written as

\[ (1 + \frac{\sigma_{T}}{2}) I - \theta \tau^{2} \kappa_{2} \delta_{y} \]

and (3.22), respectively, their structures are the same as the coefficient matrix \( A_{N \times N} \) of linear system (2.30). This, together with a similar illustration as that in the end of Section 2.2, shows that ADI methods (3.20)–(3.22) are uniquely solvable. Moreover, when we introduce the corresponding Strang circulant preconditioners into linear systems (3.20) and (3.21), respectively, and then apply CG method to the derived linear systems, an accelerated scheme for ADI method can be obtained immediately. In the following, this accelerated scheme will be called ADI-PCG method. With a similar discussion as that in the end of Section 2.4, we can conclude that an ADI-PCG method is superlinear convergent whenever there exist constants \( K_{1}, K_{2} > 0 \) such that \( \kappa_{1} \frac{\theta^{2}}{\kappa_{1}^{2}} \leq K_{1} \) and \( \kappa_{2} \frac{\theta^{2}}{\kappa_{2}^{2}} \leq K_{2} \).

3.2. Convergence analysis of ADI method

Since an ADI method (3.20)–(3.22) is equivalent to the corresponding OPFD method (3.16)–(3.18), we may perform the convergence analysis only for the latter. For this, we first introduce some notations and preliminary results. Denote \( V_{h_{1}h_{2}} = \{ v \, | \, (v, v), 0 \leq i \leq N_{1} + 1, 0 \leq j \leq N_{2} + 1 \} \) as the set of grid functions on \( \Omega_{h_{1}h_{2}} \) and write \( V_{h_{1}h_{2}} = \{ v \in V_{h_{1}h_{2}} | v_{ij} = 0, i = 0, N_{1} + 1, 0 \leq j \leq N_{2} + 1 \} \cap \{ v \in V_{h_{1}h_{2}} | v_{ij} = 0, 0 \leq i \leq N_{1} + 1, j = 0, N_{2} + 1 \} \). For any \( w, v \in \hat{V}_{h_{1}h_{2}} \), we define the following inner products and the corresponding norms:
\[ (w, v)_{\Omega} = \frac{1}{N_{1} N_{2}} \sum_{i=1}^{N_{1}} \sum_{j=1}^{N_{2}} w_{ij} v_{ij}, \quad \| w \|_{\Omega} = (w, w)^{\frac{1}{2}}; \]
\[ (w, v)_{\gamma_{1}\gamma_{2}} = \kappa_{1}(\Lambda_{x}^{2} w, \Lambda_{x}^{2} v) + \kappa_{2}(\Lambda_{y}^{2} w, \Lambda_{y}^{2} v), \quad \| w \|_{\gamma_{1}\gamma_{2}} = (w, w)^{\frac{1}{2}}_{\gamma_{1}\gamma_{2}}. \]
Lemma 3.1. For any two grid functions \( w, v \in \hat{V}_{h_1, h_2} \), there exists two difference operators \( \Lambda_x^{\gamma_1} \) and \( \Lambda_y^{\gamma_2} \) with \( \gamma_1, \gamma_2 \in (1, 2) \) such that

\[
\begin{align*}
[(\delta_x^{\gamma_1} \delta_y^{\gamma_2} w, \Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} w), (\Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} w, \Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} v)] \\
(\kappa_1 \delta_x^{\gamma_1} + \kappa_2 \delta_y^{\gamma_2}) w, v) = -\kappa_1 (\Lambda_x^{\gamma_1} w, \Lambda_y^{\gamma_2} v) - \kappa_2 (\Lambda_y^{\gamma_2} w, \Lambda_x^{\gamma_1} v).
\end{align*}
\]

Proof. According to Lemma 2.5, there exist two linear operators \( \Lambda_x^{\gamma_1} \) and \( \Lambda_y^{\gamma_2} \) such that for any grid functions \( \hat{w}, \hat{v} \in \hat{V}_{h_1, h_2} \),

\[
\begin{align*}
h_1 \sum_{i=1}^{N_1} (\delta_x^{\gamma_1} \hat{w})_{ij} \hat{v}_{ij} &= -h_1 \sum_{i=1}^{N_1} (\Lambda_x^{\gamma_1} \hat{w})_{ij} (\Lambda_x^{\gamma_1} \hat{v})_{ij}, \\
h_2 \sum_{j=1}^{N_2} (\delta_y^{\gamma_2} \hat{w})_{ij} \hat{v}_{ij} &= -h_2 \sum_{j=1}^{N_2} (\Lambda_y^{\gamma_2} \hat{w})_{ij} (\Lambda_y^{\gamma_2} \hat{v})_{ij}.
\end{align*}
\]

By (3.23) and some direct calculation, the following equalities hold:

\[
\begin{align*}
(\delta_x^{\gamma_1} \delta_y^{\gamma_2} w, v) &= h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_x^{\gamma_1} \delta_y^{\gamma_2} w)_{ij} v_{ij} = h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} [- (\Lambda_x^{\gamma_1} \delta_x^{\gamma_1} w)_{ij} (\Lambda_y^{\gamma_2} v)_{ij}] \\
&= (\Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} w, \Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} v).
\end{align*}
\]

\[
\begin{align*}
(\kappa_1 \delta_x^{\gamma_1} + \kappa_2 \delta_y^{\gamma_2}) w, v) &= \kappa_1 h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_x^{\gamma_1} w)_{ij} v_{ij} + \kappa_2 h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_y^{\gamma_2} w)_{ij} v_{ij} \\
&= -\kappa_1 (\Lambda_x^{\gamma_1} w, \Lambda_y^{\gamma_2} v) - \kappa_2 (\Lambda_y^{\gamma_2} w, \Lambda_x^{\gamma_1} v).
\end{align*}
\]

This completes the proof. \( \square \)

Lemma 3.2. For any grid function \( w \in \hat{V}_{h_1, h_2} \), the following inequalities hold.

\[
\|w\|_{\gamma_1, \gamma_2}^2 \leq \frac{(2 \kappa_1 g_0^{(\gamma_1)} + 2 \kappa_2 g_0^{(\gamma_2)})}{h_1^{\gamma_1} + h_2^{\gamma_2}} \|w\|^2, \quad \|\Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} w\|^2 \leq \frac{4 g_0^{(\gamma_1)} g_0^{(\gamma_2)}}{h_1^{\gamma_1} h_2^{\gamma_2}} \|w\|^2.
\]

Proof. A direct computation gives that

\[
\begin{align*}
\|w\|_{\gamma_1, \gamma_2}^2 &= -\kappa_1 (\delta_x^{\gamma_1} w, \Lambda_x^{\gamma_1} \Lambda_y^{\gamma_2} w) - \kappa_2 (\delta_y^{\gamma_2} w, w) \\
&= -\kappa_1 h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_x^{\gamma_1} w)_{ij} w_{ij} - \kappa_2 h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_y^{\gamma_2} w)_{ij} w_{ij}.
\end{align*}
\]

(3.24)
It follows from Lemma 2.6 that
\[
- h_1 \sum_{i=1}^{N_1} (\delta_1^{\gamma_1} w)_{ij} w_{ij} \leq 2 g_0^{(\gamma_1)} h_1^{1-\gamma_1} \sum_{i=1}^{N_1} w_{ij}^2,
\]
\[
- h_2 \sum_{j=1}^{N_2} (\delta_2^{\gamma_2} w)_{ij} w_{ij} \leq 2 g_0^{(\gamma_2)} h_2^{1-\gamma_2} \sum_{j=1}^{N_2} w_{ij}^2.
\]
(3.25)

Inserting (3.25) into (3.24) leads to
\[
\| u \|_{\gamma_1, \gamma_2}^2 \leq \kappa_1 h_1 \sum_{i=1}^{N_1} 2 g_0^{(\gamma_1)} h_1^{1-\gamma_1} \sum_{i=1}^{N_1} w_{ij}^2 + \kappa_2 h_2 \sum_{j=1}^{N_2} 2 g_0^{(\gamma_2)} h_2^{1-\gamma_2} \sum_{j=1}^{N_2} w_{ij}^2
\]
\[
= \left( \frac{2 \kappa_1 g_0^{(\gamma_1)}}{h_1^{1-\gamma_1}} + \frac{2 \kappa_2 g_0^{(\gamma_2)}}{h_2^{1-\gamma_2}} \right) \| u \|_2.
\]
Similarly, it can be deduced that
\[
\| \Lambda_{\gamma_1}^{\gamma_1} \Lambda_{\gamma_2}^{\gamma_2} w \|_2^2 = - h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\delta_1^{\gamma_1} \Lambda_{\gamma_2}^{\gamma_2} w)_{ij} (\Lambda_{\gamma_2}^{\gamma_2} w)_{ij} \leq h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} 2 g_0^{(\gamma_1)} h_1^{1-\gamma_1} \sum_{i=1}^{N_1} (\Lambda_{\gamma_2}^{\gamma_2} w)_{ij}^2
\]
\[
= 2 g_0^{(\gamma_1)} h_1^{1-\gamma_1} \sum_{i=1}^{N_1} \left[ - h_2 \sum_{j=1}^{N_2} (\delta_2^{\gamma_2} w)_{ij} w_{ij} \right] \leq 2 g_0^{(\gamma_1)} h_1^{1-\gamma_1} g_0^{(\gamma_2)} h_2^{1-\gamma_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} w_{ij}^2 = \frac{4 g_0^{(\gamma_1)} g_0^{(\gamma_2)}}{h_1^{1-\gamma_1} h_2^{1-\gamma_2}} \| u \|_2^2.
\]
Hence the lemma is proved. \(\square\)

Let \( U^n = (U^n_{ij}) \in V_{h_1, h_2}, u^n = (u^n_{ij}) \in V_{h_1, h_2} \) and \( e^n = U^n - u^n \). Denote
\[
c_{\theta, \tau, h}^{\gamma_1, \gamma_2} = 2 \kappa_1 g_0^{(\gamma_2)} \tau_2^2 + 2 \kappa_2 g_0^{(\gamma_2)} \tau_2^2 + 4 \alpha_0 \kappa_1 \kappa_2 g_0^{(\gamma_1)} \| g_0^{(\gamma_2)} \|_2 + \tau_4 \frac{g_0^{(\gamma_1)} g_0^{(\gamma_2)}}{h_1^{1-\gamma_1} h_2^{1-\gamma_2}}.
\]
Based on the above arguments, a convergence theorem can be stated as follows.

**Theorem 3.1.** Assume the solution \( u(x, y, t) \) of problem (3.1)–(3.3) belongs to \( C^{5.5.4}([a, b] \times [c, d] \times [-s, T]) \), \( u^*(x, y, \cdot) \in \ell^{2+\gamma_1, 2+\gamma_2}(\mathbb{R}^2) \) and there exist constants \( c_2 > 1 \) and \( d_2 > 0 \) such that
\[
\tau \leq \left( 1 - \frac{1}{\alpha_2} \right) \frac{2d_2}{|\beta| + 3}, \quad d_2 \leq \min \left\{ 1, 1 + \left( \theta - \frac{1}{4} \right) c_{\theta, \tau, h}^{\gamma_1, \gamma_2} \right\}.
\]
(3.26)
Then the global error \( e^n \) of ADI method (3.20)–(3.22) satisfies the following estimate:
\[
\| e^n \| \leq C_2 (\tau^2 + h_1^2 + h_2^2), \quad 1 \leq n \leq M,
\]
where
\[
C_2 = \sqrt{\frac{c_2 c_2 (b-a)(d-c) T^3}{d_2} \exp \left( c_2 T \frac{(2 + |\beta|) T^2 + |\beta| + 3}{d_2} \right)}.
\]
Proof. Writing $e^{n}_{ij} = U^{n}_{ij} - u^{n}_{ij}$ and subtracting equation (3.16) from equation (3.15) yields for $1 \leq i \leq N_1, 1 \leq j \leq N_2$ and $0 \leq n \leq M - 1$ that

$$
\delta_t^2 e^{n}_{ij} + \sigma D_t e^{n}_{ij} = (\kappa_1 \delta_x^2 + \kappa_2 \delta_y^2 - \alpha_1 \kappa_1 \kappa_2 \delta_x^2 \delta_y^2) \left[ \theta e^{n+1}_{ij} + (1 - 2\theta) e^{n}_{ij} + \theta e^{n-1}_{ij} \right]
- \sum_{k=n-m}^{n} b_{n-k} e^{k}_{ij} + \beta \left( \sin(U^n_{ij}) - \sin(u^n_{ij}) \right) + \tilde{R}^n_{ij}.
$$

(3.27)

Let $\tilde{R}^n = (\tilde{R}^n_{ij}) \in \tilde{V}_{h_1 h_2}$. Multiplying $2h_1 h_2 D_t e^{n}_{ij}$ on the both sides of (3.27), summing $i$ (resp. $j$) from 1 to $N_1$ (resp. 1 to $N_2$), and using Lemma 3.1 give that

$$
\| e^{n+\frac{1}{2}} - \| e^{n-\frac{1}{2}} \|^2 + 2\sigma \| D_t e^n \|^2 + \theta \| e^{n+1} \|^2_{\gamma_1 \gamma_2} - \| e^{n-1} \|^2_{\gamma_1 \gamma_2}
$$

(3.28)

$$
= (2\theta - 1) \left( e^n, e^{n+1} - e^{n-1} \right)_{\gamma_1 \gamma_2} + \alpha_1 \kappa_1 \kappa_2 \theta \left( \Lambda_2^2 \Lambda_2^2 e^{n+1} - \| \Lambda_2^2 \Lambda_2^2 e^{n-1} \|^2 \right)_{\gamma_1 \gamma_2}
+ \alpha_1 \kappa_1 \kappa_2 (2\theta - 1) \left( \Lambda_2^2 \Lambda_2^2 e^{n+1} - \Lambda_2^2 \Lambda_2^2 e^{n-1} \right)_{\gamma_1 \gamma_2}
- \sum_{k=n-m}^{n} b_{n-k} (e^k \delta e^{n+\frac{1}{2}} + \delta e^{n-\frac{1}{2}})
+ \beta h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\sin(U^n_{ij}) - \sin(u^n_{ij})) (\delta e^{n+\frac{1}{2}}_{ij} + \delta e^{n-\frac{1}{2}}_{ij}) + (\tilde{R}^n, \delta e^{n+\frac{1}{2}} + \delta e^{n-\frac{1}{2}}), \quad 0 \leq n \leq M - 1.
$$

Similar to the proof of inequalities (2.20)–(2.22), we can derive the following inequalities:

$$
- \sum_{k=n-m}^{n} b_{n-k} (e^k \delta e^{n+\frac{1}{2}} + \delta e^{n-\frac{1}{2}})
\leq \sum_{k=n-m}^{n} |b_{n-k}| \| e^k \|^2 + \sum_{k=n-m}^{n} |b_{n-k}| \left[ \| e^{n+\frac{1}{2}} \|^2 + \| e^{n-\frac{1}{2}} \|^2 \right]
\leq \frac{\beta \| e^{n+\frac{1}{2}} \|^2 + \| e^{n-\frac{1}{2}} \|^2}{2},
$$

(3.29)

$$
\beta h_1 h_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\sin(U^n_{ij}) - \sin(u^n_{ij})) (\delta e^{n+\frac{1}{2}}_{ij} + \delta e^{n-\frac{1}{2}}_{ij})
\leq \beta \| e^{n+\frac{1}{2}} \|^2 + \| e^{n-\frac{1}{2}} \|^2,
$$

(3.30)

$$
(\tilde{R}^n, \delta e^{n+\frac{1}{2}} + \delta e^{n-\frac{1}{2}}) \leq \| \tilde{R}^n \|^2 + \frac{\| e^{n+\frac{1}{2}} \|^2 + \| e^{n-\frac{1}{2}} \|^2}{2}.
$$

(3.31)

Combining (3.29)–(3.31) and (3.28) yields for $0 \leq n \leq M - 1$ that

$$
\| e^{n+\frac{1}{2}} \|^2 - \| e^{n-\frac{1}{2}} \|^2 + \theta \| e^{n+1} \|^2_{\gamma_1 \gamma_2} - \| e^{n} \|^2_{\gamma_1 \gamma_2} + \alpha_1 \kappa_1 \kappa_2 \theta \left( \Lambda_2^2 \Lambda_2^2 e^{n+1} - \| \Lambda_2^2 \Lambda_2^2 e^{n-1} \|^2 \right)
\leq (2\theta - 1) \left( e^n, e^{n+1} - e^{n-1} \right)_{\gamma_1 \gamma_2} + \alpha_1 \kappa_1 \kappa_2 (2\theta - 1) \left( \Lambda_2^2 \Lambda_2^2 e^{n+1} - \Lambda_2^2 \Lambda_2^2 e^{n-1} \right)_{\gamma_1 \gamma_2}
+ \beta \| e^{n+\frac{1}{2}} \|^2 + \| e^{n-\frac{1}{2}} \|^2,
$$

(3.32)

Let

$$
E^n = \frac{\| e^{n} \|^2 + \| e^{n+1} \|^2_{\gamma_1 \gamma_2} + \| e^{n} \|^2_{\gamma_1 \gamma_2}}{2} + \theta \left( \| e^{n} \|^2_{\gamma_1 \gamma_2} + \| e^{n+1} \|^2_{\gamma_1 \gamma_2} \right) + \alpha_1 \kappa_1 \kappa_2 \left( \| \Lambda_2^2 \Lambda_2^2 e^{n+1} \|^2 + \| \Lambda_2^2 \Lambda_2^2 e^{n-1} \|^2 \right)
+ (1 - 2\theta) \left( e^n, e^{n+1} \right)_{\gamma_1 \gamma_2} + (1 - 2\theta) \alpha_1 \kappa_1 \kappa_2 \left( \Lambda_2^2 \Lambda_2^2 e^n, \Lambda_2^2 \Lambda_2^2 e^{n+1} \right).
$$
When $\theta \in [0, 1/4]$, by Lemma 3.2 we have that
\[
E^n \geq \|\delta e^{n+\Delta t/2}\|^2 + \left(\theta - \frac{1}{4}\right) \left[\tau^2 \|\delta e^{n+\Delta t/2}\|^2_{1,2} + \tau^4 \alpha_0 \kappa_1 \kappa_2 \Pi^2 + \tau^4 \alpha_0 \kappa_1 \kappa_2 \Pi^2 \delta e^{n+\Delta t/2}\|^2\right] \\
\geq \left[1 + \left(\theta - \frac{1}{4}\right) \left(2 \kappa_1 \kappa_2 \Pi^2_{1,2} + 2 \kappa_2 \Pi^2_{1,2} + 4 \alpha_0 \kappa_1 \kappa_2 \Pi^2_{1,2}\right)\right] \|\delta e^{n+\Delta t/2}\|^2.
\]
When $\theta \in [1/4, 1]$, by inequality (2.23) we have that $\|\delta e^{n+\Delta t/2}\|^2 \leq E^n$. Hence, the second condition of (3.26) implies for all $\theta \in [0, 1]$ that
\[
\|\delta e^{n+\Delta t/2}\|^2 \leq \frac{1}{d^2} E^n, \quad 0 \leq n \leq M - 1. \tag{3.33}
\]
With a similar proof to (2.25), we can conclude that
\[
\|e^n\|^2 \leq \tau^2 n \sum_{q=0}^{n-1} \|\delta e^{q+\Delta t/2}\|^2 \leq \frac{t_n}{d^2} \sum_{q=0}^{n-1} E^q, \quad -m \leq n \leq M. \tag{3.34}
\]
A combination of (2.4), (3.33), (3.34) and (3.32) infers for $0 \leq n \leq M - 1$ that
\[
\frac{E^n - E^{n-1}}{\tau} \leq \sum_{k=m}^{n} \left|b_k \frac{t_k}{d^2} \sum_{q=0}^{k-1} E^q + |\beta| \frac{t_n}{d^2} \sum_{q=0}^{n-1} E^q\right| + \frac{\sum_{k=0}^{m} |b_k| + |\beta| + 1}{2d^2} (E^n + E^{n-1}) + \|\hat{R}^n\|^2 \\
\leq \sum_{k=m}^{n} \left|b_k \frac{t_k}{d^2} \sum_{q=0}^{k-1} E^q + |\beta| \frac{t_n}{d^2} \sum_{q=0}^{n-1} E^q\right| + \frac{\sum_{k=0}^{m} |b_k| + |\beta| + 1}{2d^2} (E^n + E^{n-1}) + \|\hat{R}^n\|^2 \\
\leq 2 \frac{t_n}{d^2} \sum_{q=0}^{n-1} E^q + |\beta| \frac{t_n}{d^2} \sum_{q=0}^{n-1} E^q + \left|\beta\right| + 3 \frac{2}{2d^2} (E^n + E^{n-1}) + \hat{e}_0^2 (b - a)(d - c)(\tau^2 + h_1^2 + h_2^2)^2.
\]
Using the above inequality and performing the similar proving lines of Theorem 2.1, the following estimate can be obtained:
\[
\|e^n\|^2 \leq \frac{\hat{c}_1 (b - a)(d - c) T^3}{d^2} \exp\left(\frac{c_2 T^2 (2 + |\beta|) T^2 + |\beta| + 3}{d^2}\right) (\tau^2 + h_1^2 + h_2^2)^2, \quad 1 \leq n \leq M.
\]
Therefore the theorem is proved. \thinspace \thinspace \blacksquare

Theorem 3.1 indicates that, under the given conditions, ADI methods (3.20)–(3.22) for 2D IBVPs (3.1)–(3.3) are convergent of order two both in time and space.

4. Numerical Experiments

In this section, we will present some numerical experiments to test the computational effectiveness and accuracy of the following methods:

- **Method I**: OPFD method applied to 1D problem;
- **Method II**: PCG method applied to 1D problem;
- **Method III**: ADI method applied to 2D problem;
- **Method IV**: ADI-PCG method applied to 2D problem,
and give a numerical comparison to the computational efficiency of the above methods.

When a PCG method is used, the numerical solution at previous time-level is chosen as the initial iteration value and the stopping criterion of iteration at \((n + 1)\)-th time-level is given by
\[\| r_k^{(n+1)} \| \leq 10^{-7} \| r_0^{(n+1)} \|,\]
where \(r_k^{(n+1)}\) is the residual quantity after \(k\) iterations at \((n + 1)\)-th time-level. In the following, the symbol “CPU” denotes the average CPU time (in second) that the program is run three times. Moreover, we will use the following formulas:

\[
\text{err}(\tau,h) = \max_{1 \leq k \leq M} \left\{ \frac{1}{N} \sum_{j=1}^{N} (u(x_j,t_k) - u_j^k)^2 \right\},
\]

\[
\hat{\text{err}}(\tau,h_1,h_2) = \max_{1 \leq k \leq M} \left\{ \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (u(x_i,y_j,t_k) - u_{ij}^k)^2 \right\},
\]

\[p = \log_2 \left( \frac{\text{err}(2\tau,2h)}{\text{err}(\tau,h)} \right), \quad \hat{p} = \log_2 \left( \frac{\hat{\text{err}}(2\tau,2h_1,2h_2)}{\hat{\text{err}}(\tau,h_1,h_2)} \right),\]
to characterize the global errors and convergence orders of the used methods for 1D and 2D problems, respectively.

**Example 4.1.** Consider 1D IBVP (2.1)–(2.3) with \(s = b = \kappa = 1, a = -1, \alpha = \beta = 0.5, \sigma = 2, T = 5\) and

\[
f(x,t) = \frac{\sin(\pi t)}{2 \cos(\gamma \pi/2)} f^{(\gamma)}(x) - \frac{1}{2} \sin((1-x)^6(1+x)^6 \sin(\pi t)) + \left\{ \left( \frac{4 \pi^2}{1 + 4 \pi^2} - \pi^2 \right) \sin(\pi t) + \frac{4 \pi e^{-0.5}}{1 + 4 \pi^2} \left[ \frac{1}{2} \cos(\pi t - \pi) + \pi \sin(\pi t - \pi) \right] \right\}(1-x)^6(1+x)^6,
\]

\(L^{(\gamma)}(\nu) = l^{(\gamma)}(\nu,12) - 6l^{(\gamma)}(\nu,11) + 15l^{(\gamma)}(\nu,10) - 20l^{(\gamma)}(\nu,9) + 15l^{(\gamma)}(\nu,8) - 6l^{(\gamma)}(\nu,7) + l^{(\gamma)}(\nu,6),\)

\(l^{(\gamma)}(\nu,p) = 2^{12-p} \frac{\Gamma(p+1)}{\Gamma(p+1-\gamma)} \left[ (1-\nu)^{p-\gamma} + (1+\nu)^{p-\gamma} \right], \quad \psi(x,t) = (1-x)^6(1+x)^6 \sin(\pi t).\)

We write the above problem as *Problem 4.1* and can verify that this problem has an exact solution \(u(x,t) = (1-x)^6(1+x)^6 \sin(\pi t)\) satisfying \(u(x,t) \in C^{5,4}([a,b] \times [-s,T])\) and \(u^*(x,\cdot) \in C^{2+\gamma}(\mathbb{R})\). In terms of Theorem 2.1, Method I for Problem 4.1 is convergent of order two in time and space if there exist constants \(c_1 > 1\) and \(d_1 > 0\) such that condition (2.17) is fulfilled.

In order to give a numerical insight into the computational accuracy and efficiency of Methods I–II, we take \(\gamma = 1.1, 1.5, 2, \theta = k/4 = k/25 (k = 0, 1, \ldots, 4), \tau = 1/(25 \times 2^{i})\) and \(h = 1/(25 \times 2^{i-1})\) \((i = 0, 1, 2, 3)\), respectively, which satisfy condition (2.17) with \(c_1 = 2\) and \(d_1 = 3/4\). Apply Methods I–II with the above parameters to solve problem 4.1, the derived numerical results are reported in Table 4.1 and Table 4.2. These numerical results show that Methods I–II are computationally effective for Problem 4.1 and can arrive at the second-order accuracy in time and space. Moreover, by comparing the CPU times displayed in Tables 4.1–4.2, we know that Method II is superior to Method I in computational efficiency. As a visual example, in Figs. 4.1(a) and 4.1(b), we also plot the numerical solution of Problem 4.1 with \(\gamma = 1.5\) solved by Method I with \(\tau = 1/200, h = 1/100\) and \(\theta = 0.5\) and its error surface, respectively. As to the figures of numerical solution and error surface generated by the corresponding method II, they are similar to Figs. 4.1(a) and 4.1(b) in vision and hence we omit them here.

In the above example, the reason why the convergence order of Method I can arrive at the theoretical one is due to the sufficient smoothness of the exact solution. As stated in the end
Fig. 4.1. (a) Numerical solution of Problem 4.1 with $\gamma = 1.5$ solved by Method I with $\tau = 1/200$, $h = 1/100$ and $\theta = 0.5$; (b) Error surface of Method I with $\tau = 1/200$, $h = 1/100$ and $\theta = 0.5$ for Problem 4.1 with $\gamma = 1.5$.

Table 4.1: Errors, convergence orders and CPU times of Method I for Problem 4.1.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\gamma$</th>
<th>$\tau$</th>
<th>$h$</th>
<th>$\text{err}$</th>
<th>$p$</th>
<th>CPU</th>
<th>$\text{err}$</th>
<th>$p$</th>
<th>CPU</th>
<th>$\text{err}$</th>
<th>$p$</th>
<th>CPU</th>
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<td>2.2901E-03</td>
<td>0.06</td>
<td>4.1101E-03</td>
<td>0.02</td>
<td>6.5040E-03</td>
<td>0.01</td>
<td>1.0457E-02</td>
<td>0.14</td>
<td>1.12E-02</td>
<td>0.14</td>
</tr>
<tr>
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<td>1/25</td>
<td>5.7819E-04</td>
<td>0.05</td>
<td>1.0228E-03</td>
<td>2.0066</td>
<td>0.05</td>
<td>1.6173E-03</td>
<td>2.0077</td>
<td>0.04</td>
<td>2.5924E-03</td>
<td>2.0121</td>
<td>1.06</td>
</tr>
<tr>
<td>1/100</td>
<td>1/50</td>
<td>1.4521E-04</td>
<td>0.19</td>
<td>2.5559E-04</td>
<td>2.0006</td>
<td>0.18</td>
<td>4.0387E-04</td>
<td>2.0017</td>
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<td>6.3915E-05</td>
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<td>0.72</td>
<td>1.0095E-04</td>
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<td>0.74</td>
<td>3.1058E-04</td>
<td>2.0003</td>
<td>70.75</td>
</tr>
</tbody>
</table>

of Section 2.3, however, the actual convergence order of the method may be reduced when the smooth condition is weakened. In the following, we present an example with nonsmoothness in space to show the order-reduction phenomenon.

Example 4.2. Consider 1D IBVP (2.1)–(2.3) with $a = 0$, $b = \kappa = \sigma = \alpha = s = \beta = 1$, $T = 5$, $\psi(x, t) = [x(1 - x)]^2 \exp(-t)$ and

$$f(x, t) = \exp(-t)[\Gamma(\gamma + 1) - x^\gamma (1 - x)^\gamma] - \sin([x(1 - x)]^2 \exp(-t)).$$

We write above problem as Problem 4.2. It can be verified the exact solution of Problem 4.2 is $u(x, t) = [x(1 - x)]^2 \exp(-t)$. This solution is infinitely smooth in time but nonsmooth in
Table 4.2: Errors, convergence orders and CPU-times of Method II for Problem 4.1.

<table>
<thead>
<tr>
<th>γ</th>
<th>θ</th>
<th>τ</th>
<th>h</th>
<th>CPU</th>
<th>err</th>
<th>p</th>
<th>CPU</th>
<th>err</th>
<th>p</th>
<th>CPU</th>
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<tr>
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<td>2.2901E-03</td>
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<td></td>
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<td>1.0228E-03</td>
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<td>0.04</td>
<td>1.6173E-03</td>
<td>2.0077</td>
<td>0.05</td>
</tr>
<tr>
<td>1/100</td>
<td>1/50</td>
<td>1.4521E-04</td>
<td>1.9934</td>
<td>0.17</td>
<td>2.5559E-04</td>
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<td>0.17</td>
<td>4.0387E-04</td>
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<td>0.16</td>
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Table 4.3: Errors and space convergence orders of Method I with τ = 1/200 for Problem 4.2.

<table>
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<th>γ</th>
<th>θ</th>
<th>τ</th>
<th>h</th>
<th>CPU</th>
<th>err</th>
<th>p</th>
<th>CPU</th>
<th>err</th>
<th>p</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
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<td>1.1</td>
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<td>2/25</td>
<td>4.6037E-03</td>
<td>0.03</td>
<td>9.5794E-03</td>
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<td>1.0457E-02</td>
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</tr>
<tr>
<td>1/50</td>
<td>1/25</td>
<td>9.2170E-04</td>
<td>1.9925</td>
<td>0.05</td>
<td>1.9048E-03</td>
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<td>2.5924E-03</td>
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<td>0.04</td>
</tr>
<tr>
<td>1/100</td>
<td>1/50</td>
<td>2.3083E-04</td>
<td>1.9975</td>
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<td>4.7520E-04</td>
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<td>6.4659E-04</td>
<td>2.0033</td>
<td>0.16</td>
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<tr>
<td>1/200</td>
<td>1/100</td>
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<td>0.92</td>
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<td>2.0018</td>
<td>0.90</td>
<td>1.6156E-04</td>
<td>2.0010</td>
<td>0.68</td>
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</table>

space since it has singularities at boundary points x = 0 and x = 1. Hence, Theorem 2.1 is not applicable to this situation and the application of OPFD method (2.14)–(2.16) could suffer from the order-reduction phenomenon in space.

In order to show the order-reduction phenomenon of OPFD method (2.14)–(2.16) in space, we apply Method I with τ = 1/200, h = 1/(25 × 2^i) (i = 0, 1, 2, 3) and θ = k/4 (k = 0, 1, ..., 4) to solve Problem 4.2, whose spatial error will be computed by \( \tilde{p} = \log_2 \left( \frac{\text{err}(\tau, h)}{\text{err}(\tau, 2h)} \right) \). For different space-fractional orders γ = 1.1, 1.5, 1.9, the errors and space convergence orders of Method I are reported in Table 4.3. The derived numerical results indicate that, although Method I is still effective for Problem 4.2, the spatial convergence order is only about one. This shows the order-reduction phenomenon of the method in space when the solved problem lacks sufficient smoothness in space.
Example 4.3. Consider 2D IBVP (3.1)–(3.3) with $s = b = d = \kappa_1 = \kappa_2 = 1$, $a = c = -1$, $\alpha = \beta = 0.5$, $\sigma = 2$, $T = 5$ and

$$
\begin{align*}
f(x, y, t) &= \frac{\exp(-t/2)(y+1)^6(1-y)^6}{2 \cos(\gamma_1 \pi/2)} L^{(\gamma_1)}(x) - \frac{1}{2} \sin((x+1)^6(1-x)^6(1-y)^6(1-y)^6 \exp(-t/2)) \\
&\quad + \frac{\exp(-t/2)(x+1)^6(1-x)^6}{2 \cos(\gamma_2 \pi/2)} L^{(\gamma_2)}(y) - \frac{5}{4} (x+1)^6(1-x)^6(1+y)^6(1+y)^6 \exp(-t/2),
\end{align*}
$$

### Table 4.4: Errors, convergence orders and CPU times of Method IV for Problem 4.3.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\tau$</th>
<th>$h$</th>
<th>$err$</th>
<th>$CPU$</th>
<th>$errr$</th>
<th>$CPU$</th>
<th>$err$</th>
<th>$CPU$</th>
</tr>
</thead>
<tbody>
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<td>4/25</td>
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<td>2/25</td>
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### Table 4.5: Errors, convergence orders and CPU times of Method IV for Problem 4.3.

<table>
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<tr>
<th>$\theta$</th>
<th>$\tau$</th>
<th>$h$</th>
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<td>4/25</td>
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<td>2.2599E-02</td>
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<td>5.2567E-02</td>
<td>0.05</td>
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<tr>
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<tr>
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<td>1/25</td>
<td>8.5839E-04</td>
<td>2.0167</td>
<td>1.04</td>
<td>1.0190E-03</td>
<td>2.0177</td>
<td>1.00</td>
<td>1.2243E-03</td>
</tr>
<tr>
<td>1/100</td>
<td>1/50</td>
<td>2.1369E-04</td>
<td>2.0064</td>
<td>8.75</td>
<td>2.5365E-04</td>
<td>2.0063</td>
<td>8.61</td>
<td>3.0465E-04</td>
</tr>
</tbody>
</table>
For convenience, we write the above problem as Problem 4.3. It can be checked that this problem has an exact solution \( u(x, y, t) = (x+1)^6(y+1)^6(1-y)^6 \exp(-0.5t) \) satisfying \( u(x, y, t) \in C^{5,4}([a,b] \times [c,d] \times [-s,T]) \) and \( u^*(x, y, \cdot) \in \ell^{2+\gamma_1}(\mathbb{R}) \times \ell^{2+\gamma_2}(\mathbb{R}) \). By Theorem 3.1, Method III for Problem 4.3 is convergent of order two in time and space if there exist constants \( c_2 > 1 \) and \( d_2 > 0 \) such that condition (3.26) holds.

In the following, we test the computational accuracy and efficiency of Methods III–IV. For this, we take \( (\gamma_1, \gamma_2) = (1.1, 1.5), (1.5, 1.5), (1.5, 2) \), \( \theta = k/4 \) \( (k = 0, 1, \ldots, 4) \), \( \tau = 1/(25 * 2^{i-1}) \) and \( h = h_1 = h_2 = 1/(25 * 2^{i-3}) \) \( (i = 0, 1, 2, 3) \), respectively, which satisfy condition (3.26) with \( c_2 = 3 \) and \( d_2 = 1/4 \). Apply Methods III–IV with the above parameters to solve Problem 4.3, the derived numerical results are shown in Table 4.4 and Table 4.5. These numerical results indicate that Methods III–IV are computationally effective for Problem 4.3 and can arrive at the second-order accuracy in time and space. Moreover, by comparing the CPU times presented in Tables 4.4-4.5, we conclude that Method IV is better than Method III in computational efficiency. In order to give an intuitive example, in Figs. 4.2(a) and 4.2(b), we plot the numerical solution of Problem 4.3 with \( \gamma_1 = \gamma_2 = 1.5 \) solved by Method III with \( \theta = 0.5, \tau = 1/100 \) and \( h_1 = h_2 = 1/50 \) and its error surface, respectively. As to the figures of numerical solution and error surface generated by the corresponding method III, they are similar to Figs. 4.2(a) and 4.2(b) in vision and thus omitted here.

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[24] Z. Zhao, X. Jin, M. M. Lin, Preconditioned iterative methods for space-time fractional advection-

