East Asian Journal on Applied Mathematics Vol. **xx**, No. x, pp. 1-22
doi: 10.4208/eajam.2021-275.210722 xxx 2023

A Mixed-Type Circulant Preconditioner for a Nonlocal Elastic Model

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Received 16 September 2021; Accepted (in revised version) 21 July 2022.

Abstract. A linear system from finite difference discretization of a generalized nonlocal elastic model was studied, where the model is composed of a Riesz potential operator with a fractional differential operator. Some properties of the coefficient matrix are proven theoretically and it is found that the linear system is very ill-conditioned when the parameter in the long-range hydrodynamic interactions is close to zero. Therefore, the usual Krylov subspace method with the Strang-Strang circulant preconditioner loses the power of preconditioning so that the iterative method converges slowly. Here the problem is fixed by utilizing a mixed-type circulant preconditioner which is obtained by both Strang's and Chan's circulant approximations. The invertibility of the preconditioner and a small-norm-low-rank decomposition of the difference matrix of the coefficient matrix and the preconditioner are shown theoretically under certain conditions. Numerical examples are given to illustrate the efficiency of the proposed fast solver.

AMS subject classifications: 65M10, 78A48

Key words: Generalized nonlocal elastic model, peridynamic, fractional differential operator, Toeplitz linear system, circulant preconditioner.

1. Introduction

Continuum elastic models have been widely used to study the dynamics of real physical systems in such applications as flexible polymers, growing interfaces, and membranes [7, 17, 29]. Recently, a generalized elastic model was developed in [25] to study a generalization and relations of these elastic models. This model is expressed as a composition of a Riesz-like potential operator with a fractional differential operator [1, 23]. In a sense, the generalized elastic model can be viewed as a composition of a peridynamic model with a fractional differential equation model. Due to the nonlocal nature and complexity of these nonlocal models, the corresponding numerical methods typically generate dense or full coefficient matrices, which require $\mathcal{O}(N^3)$ computational complexity and $\mathcal{O}(N^2)$ storage by direct methods, where N signifies the number of grid points. Extensive research has been

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conducted on the development of fast and accurate numerical methods for nonlocal models [6,9,26]. In [8], the authors decomposed the nonlocal elastic model as two systems and used the Meerschaert-Tadjeran finite difference and a collocation method to discretize them respectively. Strang's preconditioner was proposed to solve the linear system by preconditioned fast Krylov subspace method. However, they did not give theory analysis. In this paper, we study the linear system from the discretization of the generalized elastic model developed in [8]. The coefficient matrix of the linear system can be written as a product of two matrices AB where the dimensions of A and B are $N\times(N+2)$ and $(N+2)\times N$, respectively. As pointed out in [8], matrix A is related to the potential operator and its central square part $\tilde{A}$ is a symmetric Toeplitz matrix. The matrix B is related to the fractional differential operator and its central square part $\tilde{B}$ is generally nonsymmetric and possesses a Toeplitz-like structure. Here we further prove that the square matrix $\tilde{A}$ is positive definite by making use of an appropriate congruent transformation. Besides, it is noticed that $AB=\tilde{A}\tilde{B}+R$, where R is a matrix with rank being equal to 2. When the diffusivity coefficients are positive constant, it is shown that the real part of every eigenvalue of $\tilde{A}\tilde{B}$ is negative. Besides, all eigenvalues of R are real and nonpositive under certain conditions. Hence, it is conjectured that the spectrum of the coefficient matrix AB is included in the set $\{z\in\mathbb{C}\,|\,\mathrm{Re}(z)<0\}$, where $\mathbb{C}$ is the set of all complex numbers and $\mathrm{Re}(z)$ denotes the real part of z. Besides the properties of AB, an effective preconditioner is proposed for solving linear systems with coefficient matrix AB. By approximating $\tilde{A}$ and $\tilde{B}$ by using Chan's circulant matrix [5, 13, 19] and Strang's circulant matrix [3, 4, 13] respectively, a mixed-type circulant preconditioner is obtained. Since circulant matrices can be diagonalized by Fourier matrix so that the proposed preconditioner can be inverted efficiently via fast Fourier transform (FFT) [8]. It is also found that the matrix AB is very ill-conditioned when the parameter in the long-range hydrodynamic interactions is close to zero. The proposed mixed-type circulant preconditioner performs very well in the ill-conditioned case while the preconditioner proposed in [8] fails to reduce the number of iterations in the preconditioned GMRES method. Besides, we point out band-Toeplitz preconditioners are also known to be a good choice for the ill-conditioned Toeplitz systems [2,11,12,16,20,21,24].

The paper is organized as follows. In Section 2, the elastic model and the discretized linear system are reviewed. Some properties of the coefficient matrix are given in Section 3 and a well-defined mixed-type preconditioner is proposed in Section 4. In Section 5, numerical results are reported to demonstrate the efficiency of the proposed preconditioner. Conclusions are drawn in Section 6.

2. Numerical Scheme for Model Problem

We consider the following nonconventional Dirichlet boundary-value problem of the generalized fractional elastic model:

$$\begin{aligned}&\int_0^1 \frac{1}{|x-y|^\alpha}\left(d^+(y)\frac{\partial^\beta u(y)}{\partial_+ y^\beta}+d^-(y)\frac{\partial^\beta u(y)}{\partial_- y^\beta}\right)dy\\ &=f(x),\quad x\in(0,1),\quad 0<\alpha<1,\end{aligned}\tag{2.1a}$$

$$u(x) = 0, \quad x \notin (0,1). \tag{2.1b}$$

Here $d^+(x)$ and $d^-(x)$ are left- and right-sided diffusivity coefficients, $\partial^\beta u(y)/\partial_+ y^\beta$ and $\partial^\beta u(y)/\partial_- y^\beta$ are the left-sided and right-sided (Grünwald-Letnikov) fractional derivatives of order β, $1 < \beta < 2$, i.e.

$$\begin{aligned} \frac{\partial^\beta u}{\partial_+ x^\beta} &= \lim_{h \to 0} \frac{1}{h^\beta} \sum_{l=0}^{\lfloor x/h \rfloor} g_l^{(\beta)} u(x - lh), \\ \frac{\partial^\beta u}{\partial_- x^\beta} &= \lim_{h \to 0} \frac{1}{h^\beta} \sum_{l=0}^{\lfloor (1-x)/h \rfloor} g_l^{(\beta)} u(x + lh), \end{aligned} \tag{2.2}$$

where $\lfloor x \rfloor$ represents the floor of x, and $g_k^{(\beta)} = (-1)^k \binom{\beta}{k}$ with $\binom{\beta}{k}$ being the alternating fractional binomial coefficients [22, 23].

Let N be a positive integer, and $h = 1/(N+1)$ be the size of spatial grid. We define a spatial partition $x_i = ih$ for $i = 0, 1, \ldots, N+1$. Let u_i and v_i be the approximations to $u(x_i)$ and $v(x_i)$, respectively, and denote $d_i^\pm = d^\pm(x_i)$, $f_i = f(x_i)$.

To develop an efficient and accurate numerical approximation, the model problem was decomposed as the following two systems [8]:

$$\begin{aligned} &d^+(x) \frac{\partial^\beta u(x)}{\partial_+ x^\beta} + d^-(x) \frac{\partial^\beta u(x)}{\partial_- x^\beta} = v(x), \quad && x \in (0,1), \\ &u = 0, && x \notin (0,1), \end{aligned} \tag{2.3}$$

and

$$\int_0^1 \frac{1}{|x-y|^\alpha} v(y) dy = f(x), \quad x \in (0,1), \tag{2.4}$$

and then the Meerschaert-Tadjeran finite difference method and the collocation method were applied to (2.3) and (2.4) respectively [8]. More precisely, a shifted Grünwald approximation was used to approximate the fractional differential operator in (2.3) and the resulting Meerschaert-Tadjeran scheme

$$\begin{aligned} &\frac{d_i^+}{h^\beta} \sum_{s=0}^{i} g_s^{(\beta)} u_{i-s+1} + \frac{d_i^-}{h^\beta} \sum_{s=0}^{N-i+1} g_s^{(\beta)} u_{i+s-1} = v_i, \quad 1 \le i \le N, \\ &u_0 = u_{N+1} = 0 \end{aligned} \tag{2.5}$$

is proven to be unconditionally stable and convergent [18]. Due to the nonlocal nature of fractional differential operators, the system (2.3)-(2.4) needs to be solved in a coupled fashion instead of in a sequential way. By direct truncation of (2.2), v_0 and v_{N+1} can be expressed in terms of u_i for $i = 1, 2, \ldots, N$ in the following form:

$$v_0 = \frac{d_0^-}{h^\beta} \sum_{l=1}^{N} g_l^{(\beta)} u_l, \quad v_{N+1} = \frac{d_{N+1}^-}{h^\beta} \sum_{l=1}^{N} g_l^{(\beta)} u_{N+1-l}. \tag{2.6}$$

Then (2.5) and (2.6) can be rewritten into a matrix form

$$Bu = v, \tag{2.7}$$

where B is an $(N+2)\times N$ matrix, and u and v are column vectors

$$u = (u_1, u_2, \ldots, u_N)^T, \quad v = (v_0, v_1, \ldots, v_{N+1})^T.$$

For the integral part (2.4), the collocation method was used to derive the following collocation scheme:

$$\sum_{j=0}^{N+1} v_j \int_0^1 \frac{1}{|x_i - y|^\alpha} \phi_j(y) dy = f(x_i), \quad i = 1, 2, \ldots, N, \tag{2.8}$$

where

$$\phi_j(y) = \begin{cases} \dfrac{y - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} \le y \le x_j, \\ \dfrac{x_{j+1} - y}{x_{j+1} - x_j}, & x_j \le y \le x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The Eq. (2.8) can be written in a matrix form

$$Av = f, \tag{2.9}$$

where the entries of the N-dimensional vector $f = (f_1, f_2, \ldots, f_N)^T$ and the $N \times (N+2)$ matrix $A = [a_{i,j}]$ are given by

$$\begin{aligned} f_i &= f(x_i), && 1 \le i \le N, \\ a_{i,j} &= \int_0^1 \frac{1}{|x_i - y|^\alpha} \phi_j(y) dy, && 1 \le i \le N, \quad 0 \le j \le N+1. \end{aligned}$$

By combining the linear systems (2.7) and (2.9), the following linear system can be obtained:

$$ABu = f. \tag{2.10}$$

The matrices A and B have been proven to possess the following structure in [8]:

$$A = (a, \tilde{A}, \tilde{a})$$

and

$$B = D^+ B_L + D^- B_R = D^+ \begin{pmatrix} \mathbf{0} \\ G \\ \tilde{r} \end{pmatrix} + D^- \begin{pmatrix} r \\ G^T \\ \mathbf{0} \end{pmatrix},$$

where

$$
\begin{aligned}
&a=(a_{10},a_{20},\dots,a_{N0})^T,\\
&\tilde{a}=(a_{N0},\dots,a_{20},a_{10})^T,\\
&\tilde{A}=\begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{N-1}\\
a_1 & a_0 & a_1 & \ddots & \vdots\\
a_2 & a_1 & \ddots & \ddots & a_2\\
\vdots & \ddots & \ddots & a_0 & a_1\\
a_{N-1} & \cdots & a_2 & a_1 & a_0
\end{pmatrix},
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&a_0=\frac{2h^{1-\alpha}}{(1-\alpha)(2-\alpha)},\\
&a_i=\frac{h^{1-\alpha}\left[(i-1)^{2-\alpha}-2i^{2-\alpha}+(i+1)^{2-\alpha}\right]}{(1-\alpha)(2-\alpha)},\qquad i\geq 1,\\
&a_{j0}=\frac{h^{1-\alpha}\left[(j-1)^{2-\alpha}-j^{2-\alpha}+(2-\alpha)j^{1-\alpha}\right]}{(1-\alpha)(2-\alpha)},\quad j\geq 1,\\
&D^+=\frac{1}{h^\beta}\operatorname{diag}\left(d_0^+,d_1^+,\dots,d_{N+1}^+\right),\quad D^-=\frac{1}{h^\beta}\operatorname{diag}\left(d_0^-,d_1^-,\dots,d_{N+1}^-\right),\\
&\mathbf{0}=(0,0,\dots,0),\quad r=\left(g_1^{(\beta)},g_2^{(\beta)},\dots,g_N^{(\beta)}\right),\quad \tilde{r}=\left(g_N^{(\beta)},\dots,g_2^{(\beta)},g_1^{(\beta)}\right),\\
&G=\begin{pmatrix}
g_1^{(\beta)} & g_0^{(\beta)} & 0 & \cdots & 0\\
g_2^{(\beta)} & g_1^{(\beta)} & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & 0\\
g_{N-1}^{(\beta)} & \ddots & \ddots & g_1^{(\beta)} & g_0^{(\beta)}\\
g_N^{(\beta)} & g_{N-1}^{(\beta)} & \cdots & g_2^{(\beta)} & g_1^{(\beta)}
\end{pmatrix}_{N\times N}.
\end{aligned}
$$

3. Properties of Coefficient Matrix

By direct expansion, we have

$$
\begin{aligned}
AB&=\begin{pmatrix} a & \tilde{A} & \tilde{a}\end{pmatrix}\cdot\left(d^+\begin{pmatrix}\mathbf{0}\\ G\\ \tilde{r}\end{pmatrix}+d^-\begin{pmatrix} r\\ G^T\\ \mathbf{0}\end{pmatrix}\right)\\
&=d^+\left(\tilde{a}\tilde{r}+\tilde{A}G\right)+d^-\left(ar+\tilde{A}G^T\right)\\
&=\tilde{A}\tilde{B}+R,
\end{aligned}
$$

where $\tilde{B}=d^+G+d^-G^T$ and $R=d^+\tilde{a}\tilde{r}+d^-ar$. In this section, spectral properties of the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{A}\tilde{B}$, R, and AB are analysed one by one.

3.1. $\tilde{A}$ is symmetric positive definite

We first introduce some lemmas, which are the key to prove the positive definiteness of $\tilde{A}$.

Lemma 3.1. *Let $\varphi_\gamma(x) = (x+1)^\gamma - x^\gamma$, $\psi_\gamma(x) = (x-1)^\gamma - 2x^\gamma + (x+1)^\gamma$, and $\xi_\gamma(x) = \psi_\gamma(x) - \psi_\gamma(x+1)$, $x \geq 1$. Then we have*

(i) $\varphi_\gamma(x)$ is increasing if $\gamma \in (1,2)$;

(ii) $\psi_\gamma(x)$ is increasing if $\gamma \in (0,1)$ and decreasing if $\gamma \in (1,2)$;

(iii) $\xi_\gamma(x)$ is decreasing if $\gamma \in (1,2)$.

Proof. For (i), the result is obvious by taking derivative of $\varphi_\gamma(x)$.

For (ii), when $x > 1$, by using mean value theorem,

$$\begin{aligned}\psi'_\gamma(x) &= \gamma\Big\{\big[(x+1)^{\gamma-1} - x^{\gamma-1}\big] - \big[x^{\gamma-1} - (x-1)^{\gamma-1}\big]\Big\}\\ &= \gamma(\gamma-1)\Big(\epsilon_1^{\gamma-2} - \epsilon_2^{\gamma-2}\Big),\end{aligned}$$

where $\epsilon_1 \in (x, x+1)$ and $\epsilon_2 \in (x-1, x)$. Therefore, $\psi'_\gamma(x) > 0$ if $\gamma \in (0,1)$ and $\psi'_\gamma(x) < 0$ if $\gamma \in (1,2)$.

For (iii), by taking derivative,

$$\begin{aligned}\xi'_\gamma(x) &= \psi'_\gamma(x) - \psi'_\gamma(x+1)\\ &= \gamma\Big\{\big[(x-1)^{\gamma-1} - 2x^{\gamma-1} + (x+1)^{\gamma-1}\big] - \big[x^{\gamma-1} - 2(x+1)^{\gamma-1} + (x+2)^{\gamma-1}\big]\Big\}.\end{aligned}$$

By using the result of (ii), the result follows. □

With Lemma 3.1, we can continue to prove Lemmas 3.2 and 3.3.

Lemma 3.2. *For any $\alpha \in (0,1)$, the sequence $\{a_k\}_{k=0}^\infty$ is decreasing and positive.*

Proof. Let

$$\theta = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} > 0.$$

Then $a_0 = 2\theta$ and

$$a_k = \theta\big[(k-1)^{2-\alpha} - 2\cdot k^{2-\alpha} + (k+1)^{2-\alpha}\big] = \theta\psi_\gamma(k)$$

for $\gamma = 2-\alpha \in (1,2)$, $k \geq 1$. By part (ii) of Lemma 3.1, $\psi_\gamma(x)$ is decreasing for all $k = 1, 2, \ldots$, and $a_0 > a_1$ is obvious, we get that $\{a_k\}_{k=0}^\infty$ is a decreasing sequence.

Now, let us prove that $a_k > 0$, for all $k = 0,1,2,\dots,$ and $\alpha \in (0,1)$. Obviously, $a_0 > 0$ and $a_1 > 0$. For $k \geq 2$,

$$\begin{aligned} a_k &= \theta\left[(k-1)^{2-\alpha} - 2\cdot k^{2-\alpha} + (k+1)^{2-\alpha}\right] \\ &= \theta\left\{\left[(k+1)^{2-\alpha} - k^{2-\alpha}\right] - \left[k^{2-\alpha} - (k-1)^{2-\alpha}\right]\right\} \\ &= \theta\left[\varphi_\gamma(k) - \varphi_\gamma(k-1)\right] \end{aligned}$$

with $\gamma = 2-\alpha \in (1,2)$. Then by part (i) of Lemma 3.1, we obtain $a_k > 0$. □

Lemma 3.3. *For any $\alpha \in (0,1)$, the sequence $\{a_k - a_{k+1}\}_{k=0}^{\infty}$ is decreasing and positive.*

Proof. Firstly, $a_k - a_{k+1} > 0$ for all $k = 0,1,\dots$ is a direct consequence of Lemma 3.2. Now note that $a_0 - a_1 > a_1 - a_2$ and

$$a_k - a_{k+1} = \theta\xi_\gamma(k) \quad \text{with} \quad \gamma = 2-\alpha \in (1,2), \quad k \geq 1.$$

By using part (iii) of Lemma 3.1, we get that $\{a_k - a_{k+1}\}_{k=0}^{\infty}$ is a positive decreasing sequence. □

Using Lemmas 3.2 and 3.3, we can prove the positive definiteness of the symmetric Toeplitz matrix $\tilde{A}$.

Theorem 3.1. *For any integer N, the matrix $\tilde{A}$ in* (2.11) *is symmetric and positive definite.*

Proof. Let $C = U^T\tilde{A}U$, where

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

Note that C and $\tilde{A}$ are congruent symmetric matrices with real entries and hence they have the same number of positive, negative, and zero eigenvalues. Now we would like to prove the positive definiteness of the symmetric matrix C by using the Gershgorin disc theorem. In fact, the (i,j)-th entry of C is given as

$$C_{ij} = \begin{cases} -a_{|i-j|-1} + 2a_{|i-j|} - a_{|i-j|+1}, & i \neq j, \quad 1 \leq i, \quad j \leq N-1, \\ 2a_0 - 2a_1, & i = j, \quad 1 \leq i, \quad j \leq N-1, \\ a_0, & i = j = N, \\ a_{|i-j|} - a_{|i-j|-1}, & \text{otherwise}, \end{cases}$$

i.e.,

$$C=\begin{pmatrix} 2a_0-2a_1 & -a_0+2a_1-a_2 & \cdots & -a_{N-3}+2a_{N-2}-a_{N-1} & a_{N-1}-a_{N-2} \\ -a_0+2a_1-a_2 & 2a_0-2a_1 & \cdots & -a_{N-4}+2a_{N-3}-a_{N-2} & a_{N-2}-a_{N-3} \\ -a_1+2a_2-a_3 & -a_0+2a_1-a_2 & \cdots & -a_{N-5}+2a_{N-4}-a_{N-3} & a_{N-3}-a_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{N-3}+2a_{N-2}-a_{N-1} & -a_{N-4}+2a_{N-3}-a_{N-2} & \cdots & 2a_0-2a_1 & a_1-a_0 \\ a_{N-1}-a_{N-2} & a_{N-2}-a_{N-3} & \cdots & a_1-a_0 & a_0 \end{pmatrix}.$$

Using Lemmas 3.2 and 3.3, we note that $C_{ij}>0$ for all $i=j$ and $C_{ij}<0$ for all $i\neq j$. Now we consider the Gershgorin disc of the i-th row in matrix C.

When $i=1$, the Gershgorin disc is centered at $2(a_0-a_1)$ with radius

$$\begin{aligned} r_1 &= \sum_{j=2}^{N-1}|-a_{j-2}+2a_{j-1}-a_j|+|a_{N-1}-a_{N-2}| \\ &= \sum_{j=2}^{N-1}(a_{j-2}-2a_{j-1}+a_j)+(a_{N-2}-a_{N-1}) \\ &= a_0-a_1<2(a_0-a_1). \end{aligned}$$

When $i=2,3,\ldots,N-2$, the Gershgorin disc is centered at $2(a_0-a_1)$ with radius

$$\begin{aligned} r_i &= \sum_{j=2}^{i}|-a_{j-2}+2a_{j-1}-a_j|+\sum_{j=2}^{N-i}|-a_{j-2}+2a_{j-1}-a_j|+|a_{N-i}-a_{N-i-1}| \\ &= \sum_{j=2}^{i}(a_{j-2}-2a_{j-1}+a_j)+\sum_{j=2}^{N-i}(a_{j-2}-2a_{j-1}+a_j)+(a_{N-i-1}-a_{N-i}) \\ &= 2(a_0-a_1)-(a_{i-1}-a_i)<2(a_0-a_1). \end{aligned}$$

When $i=N-1$, the Gershgorin disc is centered at $2(a_0-a_1)$ with radius

$$\begin{aligned} r_{N-1} &= \sum_{j=2}^{N-1}|-a_{j-2}+2a_{j-1}-a_j|+|a_1-a_0| \\ &= \sum_{j=2}^{N}(a_{j-2}-2a_{j-1}+a_j)+(a_0-a_1) \\ &= 2(a_0-a_1)-(a_{N-1}-a_N)<2(a_0-a_1). \end{aligned}$$

When $i=N$, the Gershgorin disc is centered at a_0 with radius

$$r_N=\sum_{j=1}^{N-1}|a_j-a_{j-1}|=\sum_{j=1}^{N-1}(-a_j+a_{j-1})=a_0-a_{N-1}<a_0.$$

Therefore, all the eigenvalues of C are real and positive, so that $\tilde{A}$ is symmetric positive definite. $\square$

3.2. Eigenvalues of $\tilde{B}$ and $\tilde{A}\tilde{B}$

According to the properties of $g_i^{(\beta)}$ (see [15, Proposition 1]), it is easy to know $\mathrm{Re}(\lambda(\tilde{B})) < 0$ for positive diffusivity coefficients $d^{\pm}$ by the Gershgorin disc theorem. Particularly, when $d^+ = d^-$, $\tilde{B}$ is negative definite. Similar argument can be found in [27].

The remaining part of this subsection is to show $\mathrm{Re}(\lambda(\tilde{A}\tilde{B})) < 0$ when the diffusivity coefficients are positive constant. In fact, suppose (λ, x) is an eigenpair of $\tilde{A}\tilde{B}$, then

$$\mathrm{Re}(x^*\tilde{B}x) = \mathrm{Re}\left(x^*\tilde{A}^{-1}(\tilde{A}\tilde{B})x\right) = \mathrm{Re}\left(x^*\tilde{A}^{-1}(\lambda x)\right) = \mathrm{Re}(\lambda)(x^*\tilde{A}^{-1}x).$$

Since the symmetric Toeplitz matrix $\tilde{B} + \tilde{B}^T$ is negative definite [14] and $\tilde{A}$ is symmetric positive definite, then $\mathrm{Re}(x^*\tilde{B}x) < 0$ [14] and $x^*\tilde{A}^{-1}x > 0$, which means $\mathrm{Re}(\lambda) < 0$. In particular, when $d^+ = d^-$, λ is real and negative.

3.3. Eigenvalues of the rank 2 matrix R

Now we analyse the eigenvalues of the rank 2 matrix $R = d^+\tilde{a}\tilde{r} + d^- ar$ for $d^+ = d^- = 1$. Before that, let us give some important lemmas.

Lemma 3.4. *Let* $\eta_\gamma(x) = (x-1)^\gamma - x^\gamma + \gamma \cdot x^{\gamma-1}$, $x \geq 1$, *then*

(i) $\eta_\gamma(x)$ *is increasing if* $\gamma \in (0,1)$ *and decreasing if* $\gamma \in (1,2)$*;*

(ii) $\eta_\gamma(x) > 0$ *if* $\gamma \in (1,2)$.

Proof. For (i), when $x > 1$, by using mean value theorem,

$$\begin{aligned}\eta'_\gamma(x) &= \gamma\left[(x-1)^{\gamma-1} - x^{\gamma-1} + (\gamma-1)x^{\gamma-2}\right] \\ &= -\gamma(\gamma-1)\left(\epsilon_3^{\gamma-2} - x^{\gamma-2}\right),\end{aligned}$$

where $\epsilon_3 \in (x-1, x)$. It can be seen that $\eta'_\gamma(x) > 0$ when $\gamma \in (0,1)$ and $\eta'_\gamma(x) < 0$ when $\gamma \in (1,2)$. Therefore, the result follows.

For (ii), by using mean value theorem, we have

$$\begin{aligned}\eta_\gamma(x) &= (x-1)^\gamma - x^\gamma + \gamma x^{\gamma-1} \\ &= -\gamma\epsilon_4^{\gamma-1} + \gamma x^{\gamma-1}\end{aligned}$$

for some $\epsilon_4 \in (x-1, x)$. Therefore, when $\gamma \in (1,2)$, we have $\epsilon^{\gamma-1} < x^{\gamma-1}$, so that $\eta_\gamma(x) > -\gamma x^{\gamma-1} + \gamma x^{\gamma-1} = 0$. □

Lemma 3.4 is useful for proving the following properties of the sequence $\{a_{k0}\}_{k=1}^{\infty}$.

Lemma 3.5. *For any* $\alpha \in (0,1)$*, the sequence* $\{a_{k0}\}_{k=1}^{\infty}$ *is decreasing and positive.*

Proof. Writing

$$a_{k0} = \theta\left[(k-1)^{2-\alpha} - k^{2-\alpha} + (2-\alpha)\cdot k^{1-\alpha}\right] = \theta\eta_\gamma(k)$$

with $\gamma = 2 - \alpha \in (1,2)$ and $k \geq 1$, we note that the result follows from Lemma 3.4. □

Lemma 3.6. *For any $\alpha \in (0,1)$, the sequence $\{a_{k0} - a_{k+1,0}\}_{k=1}^{\infty}$ is decreasing and positive.*

Proof. The positivity of the sequence follows directly from Lemma 3.5, so let us show that it is decreasing. Observe that for $k \geq 1$,

$$\begin{aligned} a_{k0} - a_{(k+1)0} = \theta\Big\{ & \big[(k-1)^{2-\alpha} - k^{2-\alpha} + (2-\alpha)k^{1-\alpha}\big] \\ & - \big[k^{2-\alpha} - (k+1)^{2-\alpha} + (2-\alpha)(k+1)^{1-\alpha}\big]\Big\}. \end{aligned}$$

Treat k as a continuous variable and the derivative of the above function with respect to k is given as

$$\begin{aligned} & (2-\alpha)\theta\Big\{\big[(k-1)^{1-\alpha} - k^{1-\alpha} + (1-\alpha)k^{-\alpha}\big] \\ & \qquad - \big[k^{1-\alpha} - (k+1)^{1-\alpha} + (1-\alpha)(k+1)^{-\alpha}\big]\Big\} \\ & = (2-\alpha)\theta\big[\eta_\gamma(k) - \eta_\gamma(k+1)\big] \end{aligned}$$

with $\gamma = 1-\alpha \in (0,1)$. Therefore, by part (i) of Lemma 3.4, the derivative above is less than zero and hence the sequence $\{a_{k0} - a_{k+1,0}\}_{k=1}^{\infty}$ is decreasing and positive. $\square$

Now, let J be the counter identity matrix whose elements are all equal to zero except those on the counter diagonal, which are all equal to 1.

Definition 3.1 (cf. El-Mikkawy [10]). *A matrix H is called centrosymmetric if $JHJ = H$.*

It is easy to check that the rank 2 matrix

$$R = \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \\ \vdots \\ a_{N0} \end{pmatrix} \begin{pmatrix} g_1^{(\beta)} & g_2^{(\beta)} & \cdots & g_N^{(\beta)} \end{pmatrix} + \begin{pmatrix} a_{N0} \\ \vdots \\ a_{30} \\ a_{20} \\ a_{10} \end{pmatrix} \begin{pmatrix} g_N^{(\beta)} & g_{N-1}^{(\beta)} & \cdots & g_1^{(\beta)} \end{pmatrix}$$

is centrosymmetric, so it has at most two nonzero eigenvalues. The following lemma is the main tool for proving the spectral property of R.

Lemma 3.7 (cf. Weaver [28]). *(i) If*

$$P = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

is an $n \times n$ centrosymmetric matrix with $n = 2s$ and T_1, T_2, T_3, and T_4 are $s \times s$ matrices, then P is orthogonally similar to

$$\begin{pmatrix} T_1 + JT_3 & 0 \\ 0 & T_1 - JT_3 \end{pmatrix}.$$

(ii) If

$$P = \begin{pmatrix} T_1 & x & T_2 \\ y & q & yJ \\ T_3 & Jx & T_4 \end{pmatrix}$$

is an $n \times n$ centrosymmetric matrix with $n = 2s+1$, T_1, T_2, T_3, and T_4 are $s \times s$ matrices, x is an $s \times 1$ matrix, y is a $1 \times s$ matrix, and q is a scalar, then P is orthogonally similar to

$$\begin{pmatrix} T_1 + JT_3 & \sqrt{2}x & 0 \\ \sqrt{2}y & q & 0 \\ 0 & 0 & T_1 - JT_3 \end{pmatrix}.$$

Now, it is ready to prove the following theorem.

Theorem 3.2. *All the eigenvalues of the rank 2 matrix $R = \tilde{a}\tilde{r} + ar$ are zero except two are real and negative.*

Proof. When N is even, let $N = 2k$, then by part (i) of Lemma 3.7, R is orthogonally similar to the 2×2 block diagonal matrix with blocks

$$H_1 = \begin{pmatrix} (a_{10}+a_{2k,0})(g_1^{(\beta)}+g_{2k}^{(\beta)}) & (a_{10}+a_{2k,0})(g_2^{(\beta)}+g_{2k-1}^{(\beta)}) & \cdots & (a_{10}+a_{2k,0})(g_k^{(\beta)}+g_{k+1}^{(\beta)}) \\ (a_{20}+a_{2k-1,0})(g_1^{(\beta)}+g_{2k}^{(\beta)}) & (a_{20}+a_{2k-1,0})(g_2^{(\beta)}+g_{2k-1}^{(\beta)}) & \cdots & (a_{20}+a_{2k-1,0})(g_k^{(\beta)}+g_{k+1}^{(\beta)}) \\ \vdots & \vdots & \cdots & \vdots \\ (a_{k0}+a_{k+1,0})(g_1^{(\beta)}+g_{2k}^{(\beta)}) & (a_{k0}+a_{k+1,0})(g_2^{(\beta)}+g_{2k-1}^{(\beta)}) & \cdots & (a_{k0}+a_{k+1,0})(g_k^{(\beta)}+g_{k+1}^{(\beta)}) \end{pmatrix}$$

$$H_2 = \begin{pmatrix} (a_{10}-a_{2k,0})(g_1^{(\beta)}-g_{2k}^{(\beta)}) & (a_{10}-a_{2k,0})(g_2^{(\beta)}-g_{2k-1}^{(\beta)}) & \cdots & (a_{10}-a_{2k,0})(g_k^{(\beta)}-g_{k+1}^{(\beta)}) \\ (a_{20}-a_{2k-1,0})(g_1^{(\beta)}-g_{2k}^{(\beta)}) & (a_{20}-a_{2k-1,0})(g_2^{(\beta)}-g_{2k-1}^{(\beta)}) & \cdots & (a_{20}-a_{2k-1,0})(g_k^{(\beta)}-g_{k+1}^{(\beta)}) \\ \vdots & \vdots & \cdots & \vdots \\ (a_{k0}-a_{k+1,0})(g_1^{(\beta)}-g_{2k}^{(\beta)}) & (a_{k0}-a_{k+1,0})(g_2^{(\beta)}-g_{2k-1}^{(\beta)}) & \cdots & (a_{k0}-a_{k+1,0})(g_k^{(\beta)}-g_{k+1}^{(\beta)}) \end{pmatrix}.$$

By observing that each column vectors of H_1 and H_2 are parallel and hence they are of rank 1 matrices, so that they have at most one nonzero eigenvalue which equals to the trace of the matrix. Using the properties of the sequences $\{g_k^{(\beta)}\}$, cf. [15, Proposition 1], and $\{a_{k0}\}$ in Lemmas 3.5 and 3.6, we estimate the traces as the follows:

$$\begin{aligned} Tr(H_1) &= (a_{10}+a_{2k,0})\left(g_1^{(\beta)}+g_{2k}^{(\beta)}\right) + (a_{20}+a_{2k-1,0})\left(g_2^{(\beta)}+g_{2k-1}^{(\beta)}\right) + \cdots \\ &\quad + (a_{k0}+a_{k+1,0})\left(g_k^{(\beta)}+g_{k+1}^{(\beta)}\right) \\ &< (a_{10}+a_{2k,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_{2k}^{(\beta)}\right) < 0, \\ Tr(H_2) &= (a_{10}-a_{2k,0})\left(g_1^{(\beta)}-g_{2k}^{(\beta)}\right) + (a_{20}-a_{2k-1,0})\left(g_2^{(\beta)}-g_{2k-1}^{(\beta)}\right) + \cdots \\ &\quad + (a_{k0}-a_{k+1,0})\left(g_k^{(\beta)}-g_{k+1}^{(\beta)}\right) \\ &< (a_{10}-a_{2k,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_k^{(\beta)}-g_{k+1}^{(\beta)}-g_{k+2}^{(\beta)}-\cdots-g_{2k}^{(\beta)}\right) \\ &< (a_{10}-a_{2k,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_k^{(\beta)}+g_{k+1}^{(\beta)}+g_{k+2}^{(\beta)}+\cdots+g_{2k}^{(\beta)}\right) < 0. \end{aligned}$$

When N is odd, let $N = 2k+1$, then R is orthogonally similar to the 2-by-2 block diagonal matrix with blocks

$$H_3 = \begin{pmatrix} H_3' & \dagger \\ \ddagger & 2a_{k+1,0}g_{k+1}^{(\beta)} \end{pmatrix},$$

where

$$H_3' = \begin{pmatrix} (a_{10}+a_{2k+1,0})(g_1^{(\beta)}+g_{2k+1}^{(\beta)}) & (a_{10}+a_{2k+1,0})(g_2^{(\beta)}+g_{2k}^{(\beta)}) & \cdots & (a_{10}+a_{2k+1,0})(g_k^{(\beta)}+g_{k+2}^{(\beta)}) \\ (a_{20}+a_{2k,0})(g_1^{(\beta)}+g_{2k+1}^{(\beta)}) & (a_{20}+a_{2k,0})(g_2^{(\beta)}+g_{2k}^{(\beta)}) & \cdots & (a_{20}+a_{2k,0})(g_k^{(\beta)}+g_{k+2}^{(\beta)}) \\ \vdots & \vdots & \cdots & \vdots \\ (a_{k0}+a_{k+2,0})(g_1^{(\beta)}+g_{2k+1}^{(\beta)}) & (a_{k0}+a_{k+2,0})(g_2^{(\beta)}+g_{2k}^{(\beta)}) & \cdots & (a_{k0}+a_{k+2,0})(g_k^{(\beta)}+g_{k+2}^{(\beta)}) \end{pmatrix},$$

$$H_4 = \begin{pmatrix} (a_{10}-a_{2k+1,0})(g_1^{(\beta)}-g_{2k+1}^{(\beta)}) & (a_{10}-a_{2k+1,0})(g_2^{(\beta)}-g_{2k}^{(\beta)}) & \cdots & (a_{10}-a_{2k+1,0})(g_k^{(\beta)}-g_{k+2}^{(\beta)}) \\ (a_{20}-a_{2k,0})(g_1^{(\beta)}-g_{2k+1}^{(\beta)}) & (a_{20}-a_{2k,0})(g_2^{(\beta)}-g_{2k}^{(\beta)}) & \cdots & (a_{20}-a_{2k,0})(g_k^{(\beta)}-g_{k+2}^{(\beta)}) \\ \vdots & \vdots & \cdots & \vdots \\ (a_{k0}-a_{k+2,0})(g_1^{(\beta)}-g_{2k+1}^{(\beta)}) & (a_{k0}-a_{k+2,0})(g_2^{(\beta)}-g_{2k}^{(\beta)}) & \cdots & (a_{k0}-a_{k+2,0})(g_k^{(\beta)}-g_{k+2}^{(\beta)}) \end{pmatrix}.$$

Note that H_4 is a rank 1 matrix and hence it has at most one nonzero eigenvalue which equals to the trace of H_4 satisfying

$$\begin{aligned} Tr(H_4) &= (a_{10}-a_{2k+1,0})\left(g_1^{(\beta)}-g_{2k+1}^{(\beta)}\right)+(a_{20}-a_{2k,0})\left(g_2^{(\beta)}-g_{2k}^{(\beta)}\right)+\cdots \\ &\quad +(a_{k0}-a_{k+2,0})\left(g_k^{(\beta)}-g_{k+2}^{(\beta)}\right) \\ &< (a_{10}-a_{2k+1,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_k^{(\beta)}-g_{k+2}^{(\beta)}-g_{k+3}^{(\beta)}-\cdots-g_{2k+1}^{(\beta)}\right) \\ &< (a_{10}-a_{2k+1,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_k^{(\beta)}+g_{k+1}^{(\beta)}+g_{k+2}^{(\beta)}+\cdots+g_{2k+1}^{(\beta)}\right)<0. \end{aligned}$$

On the other hand, since R is a rank 2 matrix so that H_3 has at most one nonzero eigenvalue which equals to the trace of H_3 which satisfies

$$\begin{aligned} Tr(H_3) &= (a_{10}+a_{2k+1,0})\left(g_1^{(\beta)}+g_{2k+1}^{(\beta)}\right)+\cdots+(a_{k0}+a_{k+2,0})\left(g_k^{(\beta)}+g_{k+2}^{(\beta)}\right)+2a_{k+1,0}g_{k+1}^{(\beta)} \\ &< (a_{10}+a_{2k+1,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_k^{(\beta)}+g_{k+1}^{(\beta)}+g_{k+2}^{(\beta)}+\cdots+g_{2k+1}^{(\beta)}\right) \\ &< (a_{10}+a_{2k+1,0})\left(g_1^{(\beta)}+g_2^{(\beta)}+\cdots+g_{2k+1}^{(\beta)}\right)<0. \end{aligned}$$

To sum up, the eigenvalues of the rank 2 matrix $R = \tilde{a}\tilde{r} + ar$ are all zero except two are real and negative. □

For the spectrum of matrix AB with positive constant diffusion coefficients $d^{\pm}$, by intensive numerical experiments, we found that when $d^+ = d^-$, all the eigenvalues are real and negative, and when $d^+ \neq d^-$, the real parts of all eigenvalue are negative, see Fig. 1 for examples.

4. Mixed-Type Circulant Preconditioner

For solving (2.10) by using Krylov subspace method, it is well-known that, the main operation cost depends on the matrix-vector multiplication ABw for any vector w in each

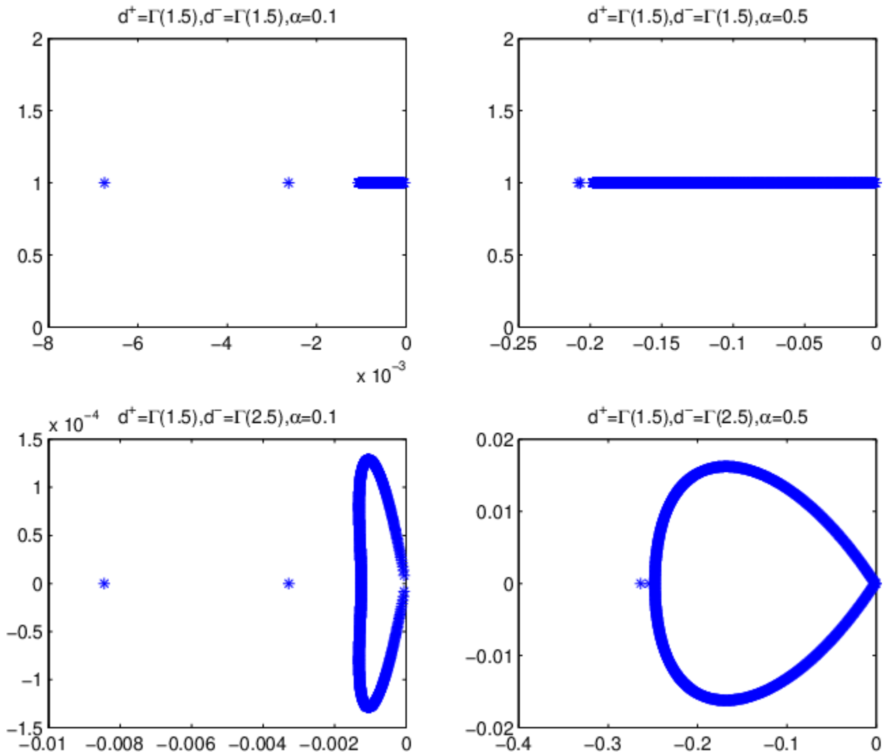

Figure 1: Spectra of the coefficient matrices AB for $\beta = 1.5$, $N = 2^{10}$.

iteration. Thanks to the Toeplitz-like structure of matrices A and B, this multiplication can be done in $\mathcal{O}(N \log N)$ operations via the fast Fourier transform (FFT) [8].

However, the coefficient matrix AB is ill-conditioned so that the Krylov subspace method requires a large number of iteration for convergence. To reduce the number of iteration, the following mixed-type circulant preconditioner:

$$M = c(\tilde{A})s(\bar{B}) \tag{4.1}$$

is proposed where $c(\tilde{A})$ is the Chan's circulant approximation [5, 13, 19] to the symmetric positive definite Toeplitz matrix $\tilde{A}$, and $s(\bar{B})$ is the Strang's circulant approximation [3,4,13] to the Toeplitz matrix

$$\bar{B} = \bar{d}^+ G + \bar{d}^- G^T$$

with

$$\bar{d}^+ = \frac{1}{Nh^\beta} \sum_{i=1}^{N} d_i^+, \quad \bar{d}^- = \frac{1}{Nh^\beta} \sum_{i=1}^{N} d_i^+.$$

In fact, the first column of the circulant matrix $c(\tilde{A})$ is given as $p = (p_1, p_2, \ldots, p_N)^T$, where

$$p_k = \frac{(N-k+1)a_{k-1} + (k-1)a_{N-k+1}}{N}, \quad 1 \le k \le N,$$

and the first column of the circulant matrix $s(\bar{B})$ is given as

$$\bar{d}^{+}\begin{pmatrix} g_1^{(\beta)} \\ g_2^{(\beta)} \\ \vdots \\ g_{\lfloor \frac{N+1}{2} \rfloor}^{(\beta)} \\ 0 \\ \vdots \\ 0 \\ g_0^{(\beta)} \end{pmatrix} + d^{-}\begin{pmatrix} g_1^{(\beta)} \\ g_0^{(\beta)} \\ 0 \\ \vdots \\ 0 \\ g_{\lfloor \frac{N+1}{2} \rfloor}^{(\beta)} \\ \vdots \\ g_2^{(\beta)} \end{pmatrix}.$$

Now let us show the invertibility of the proposed mixed-type circulant preconditioner.

Theorem 4.1. *The preconditioner $M = c(\tilde{A})s(\bar{B})$ is invertible.*

Proof. Since $\tilde{A}$ is symmetric positive definite, from [19], we know that $c(\tilde{A})$ is also symmetric positive definite and their spectra satisfy the inequality

$$0 < \lambda_{\min}(\tilde{A}) \le \lambda_{\min}\big(c(\tilde{A})\big) \le \lambda_{\max}\big(c(\tilde{A})\big) \le \lambda_{\max}(\tilde{A}). \tag{4.2}$$

The invertibility of $s(\bar{B}) = \bar{d}^{+}s(G) + \bar{d}^{-}s(G^T)$ can be obtained by using Gershgorin disc theorem. In fact, all eigenvalues of $s(G)$ and $s(G^T)$ fall inside the open disc [15] $\{z \in \mathbb{C} : |z+\beta| < \beta\}$. Therefore, the preconditioner $M = c(\tilde{A})s(\bar{B})$ is invertible. ☐

It is worth to mention that the invertibility of Du and Wang's preconditioner [8], which is defined as

$$M' = s(\tilde{A})s(\bar{B}), \tag{4.3}$$

cannot be derived similarly, since the Strang's circulant matrix $s(\tilde{A})$ in M' cannot preserve the positiveness of all eigenvalues, i.e., inequality (4.2) is invalid for $s(\tilde{A})$. In fact, through intensive numerical tests, it is found that $\tilde{A}$ becomes very ill-conditioned when α is close to 0, and at the same time, some eigenvalues of $s(\tilde{A})$ are negative. This observation agrees with the comment in [3] on the spectrum of Strang's circulant preconditioner for ill-conditioned Toeplitz matrix.

By noticing that all circulant matrices can be diagonalized by the Fourier matrix, the product $M^{-1}w$, for any vector w, can be done in $\mathcal{O}(N\log N)$ operations via FFT. Therefore, the complexity of the preconditioned Krylov subspace method is $\mathcal{O}(N\log N)$ in each iteration, and from the numerical tests in the next section, one can see that the number of iterations can be significantly reduced after utilizing the proposed preconditioner.

For the convergence rate of the preconditioned iterative method, it will be expected to be fast provided that the proposed preconditioner M is close to the coefficient matrix AB. In order to study the spectrum of the difference matrix $M - AB$, we add a subscript N to each matrix to denote the matrix size. Then the matrix $\tilde{A}$ in (2.11) becomes $\tilde{A}_N = h^{1-\alpha}\hat{A}_N$ and the generating function $p(\theta)$ of the sequence of Toeplitz matrices $\{\hat{A}_N\}_{N=1}^{\infty}$ is $p(\theta) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}$ with b_k being the k-th diagonal of $\hat{A}_N$.

Theorem 4.2. *If $p(\theta)$, the generating function of $\hat{A}$ is 2π-periodic continuous, and $d^+(x) = d^+, d^-(x) = d^-$ are constant, then $M - AB$ has a small-norm-low-rank decomposition.*

Proof. Firstly, we have

$$\begin{aligned} M - AB &= c(\tilde{A})s(\bar{B}) - \tilde{A}\tilde{B} - R \\ &= c(\tilde{A})s(\bar{B}) - \tilde{A}s(\bar{B}) + \tilde{A}s(\bar{B}) - \tilde{A}\tilde{B} - R \\ &= \big(c(\tilde{A}) - \tilde{A}\big)s(\bar{B}) + \tilde{A}\big(s(\bar{B}) - \tilde{B}\big) - R, \end{aligned}$$

where R is a rank 2 matrix. Now let us show that the remaining two terms above are of low-rank plus small-norm, so that $M - AB$ is of low-rank plus small-norm.

For the term $\tilde{A}(s(\bar{B}) - \tilde{B})$, by noticing that $s(\bar{B}) - \tilde{B}$ is of low-rank plus small-norm [15], if the two-norm of the matrix $\tilde{A}$ is bounded above, the desired result follows. In fact, for $N = 2k + 1$, we have

$$\begin{aligned} \|\tilde{A}\|_2 \le \|\tilde{A}\|_1 &= a_0 + 2\sum_{i=1}^{k} a_i \\ &= \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)}\Big[2 + 2\big[(-2 + 2^{2-\alpha}) + (1 - 2\cdot 2^{2-\alpha} + 3^{2-\alpha}) + \cdots \\ &\qquad + (k-1)^{2-\alpha} - 2\cdot k^{2-\alpha} + (k+1)^{2-\alpha}\big]\Big] \\ &= \frac{2h^{1-\alpha}}{(1-\alpha)(2-\alpha)}\big[(k+1)^{2-\alpha} - k^{2-\alpha}\big] \\ &= \frac{2}{1-\alpha}\cdot\frac{1}{(N+1)^{1-\alpha}}\epsilon^{1-\alpha} \end{aligned}$$

with an $\epsilon \in [k, k+1]$. Consequently,

$$\|\tilde{A}\|_2 < \frac{2}{1-\alpha}.$$

For even N, the above inequality can be proved similarly.

For the term $(c(\tilde{A}) - \tilde{A})s(\bar{B})$, note that the two-norm of the matrix $s(\bar{B})$ is bounded above [15]. Since the matrix $c(\hat{A}) - \hat{A}$ has a small-norm-low-rank decomposition [19] under the condition that p is 2π-periodic continuous, then $c(\tilde{A}) - \tilde{A} = h^{1-\alpha}(c(\hat{A}) - \hat{A})$ is of low-rank plus small-norm. Thus, $M - AB$ is the sum of a matrix with small-norm and a matrix with low-rank, which completes the proof. □

Remark 4.1. In fact, numerical test as shown in Table 1 illustrates that the number of eigenvalues of $c(\tilde{A}) - \tilde{A}$ outside the interval $[-0.01, 0.01]$ remains almost a constant.

5. Numerical Experiments

Numerical results of solving the generalized elastic model problem (2.1) is presented in this section. The discretized linear system (2.10) is solved by using preconditioned GMRES

Table 1: Numbers of eigenvalues of $c(\tilde{A})-\tilde{A}$ outside of $[-0.01, 0.01]$.

N \ α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2^8	3	5	7	7	7	8	7	7	6
2^9	3	5	7	7	7	8	7	7	7
2^{10}	3	5	7	7	8	8	7	7	7
2^{11}	3	5	7	7	8	9	7	7	7
2^{12}	3	5	7	7	8	9	8	9	8

(PGMRES) method with the mixed-type circulant preconditioner (4.1) which consists of T. Chan's and Strang's circulant approximations and it is denoted as PGMRES (T+S). All numerical experiments are carried out in MATLAB (R2012b) on a Laptop with configuration: Intel(R) Core(TM) i5-3337U CPU @ 1.80GHz and 8.0GB RAM.

To show the advantage of our mixed-type circulant preconditioner, the numerical results of utilizing the direct method (denoted as Direct), the GMRES method (denoted as GMRES), and the PGMRES method with the Strang's circulant preconditioner (4.3) are also presented for comparison. Since the preconditioner (4.3) consists of two Strang's circulant approximations, it is denoted as PGMRES (S+S).

In the implementations, the direct method for solving (2.10) is carried out by using the MATLAB left division. The PGMRES method is carried out by using the MATLAB built-in function `gmres` with restart being equal to 30. The stopping criterion is given as $\|r_k\|/\|r_0\| \leq 10^{-12}$, where r_k denotes the residual vector of the linear system after k iterations, and the initial guess is chosen as the zero vector. Note that every matrix-vector multiplication is done via FFT in all GMRES implementations no matter with or without preconditioner.

In the following tables, "CPU" denotes the execution time in seconds, "Iter" denotes the number of iterations for the GMRES method, "-" denotes not convergent or out of memory, and "Rate" denotes $\log_2(E_{2h}/E_h)$ where $E_h = \|u_h - u\| = \max_{1\leq i\leq N} |u_i - u(x_i)|$.

Example 5.1 (Constant coefficients)**.** Consider problem (2.1) with $\beta = 1.5$, $d^+ = \Gamma(1.5)$, $d^- = \Gamma(2.5)$. For $\alpha = 0.5$, the source term is given as

$$
\begin{aligned}
f(x) = & \left(4x^3 - 6x^2 + 2x\right)\ln\frac{1+\sqrt{1-x}}{\sqrt{x}} \\
& + \left(6(1-x)^3 - 9(1-x)^2 + 3(1-x)\right)\ln\frac{1+\sqrt{x}}{\sqrt{1-x}} \\
& + \left(4x^2 - \frac{10}{3}x + \frac{2}{15}\right)\sqrt{1-x} + \left(6(1-x)^2 - 5(1-x) + \frac{1}{5}\right)\sqrt{x} \\
& + \pi\left(2x^3 + 3(1-x)^3 - 3x^2 - 4.5(1-x)^2 - 0.5x + 1.5\right),
\end{aligned}
$$

and the source term is computed by using numerical integration for $\alpha = 0.1$. The exact solution to this problem is $u(x) = x^2(1-x)^2$.

Errors of different numerical solvers and convergence rate of the proposed solver PGMRES (T+S) are presented in Table 2, and the corresponding CPU time and the number of iterations for iterative methods are presented in Table 3. When $\alpha = 0.5$, the performance of PGMRES (T+S) and PGMRES (S+S) are the best and they have almost the same accuracy, iteration number, and CPU time. However, for $\alpha = 0.1$, PGMRES (T+S) keeps the best performance while PGMRES (S+S) performs the worst. It can be observed from Fig. 2 that the spectra of the preconditioned matrices $M^{-1}AB$ and $M'^{-1}AB$ are quite clustered around 1 when $\alpha = 0.5$, and when $\alpha = 0.1$, the spectrum of $M'^{-1}AB$ becomes not clustered while $M^{-1}AB$ preserves similar clustered spectrum.

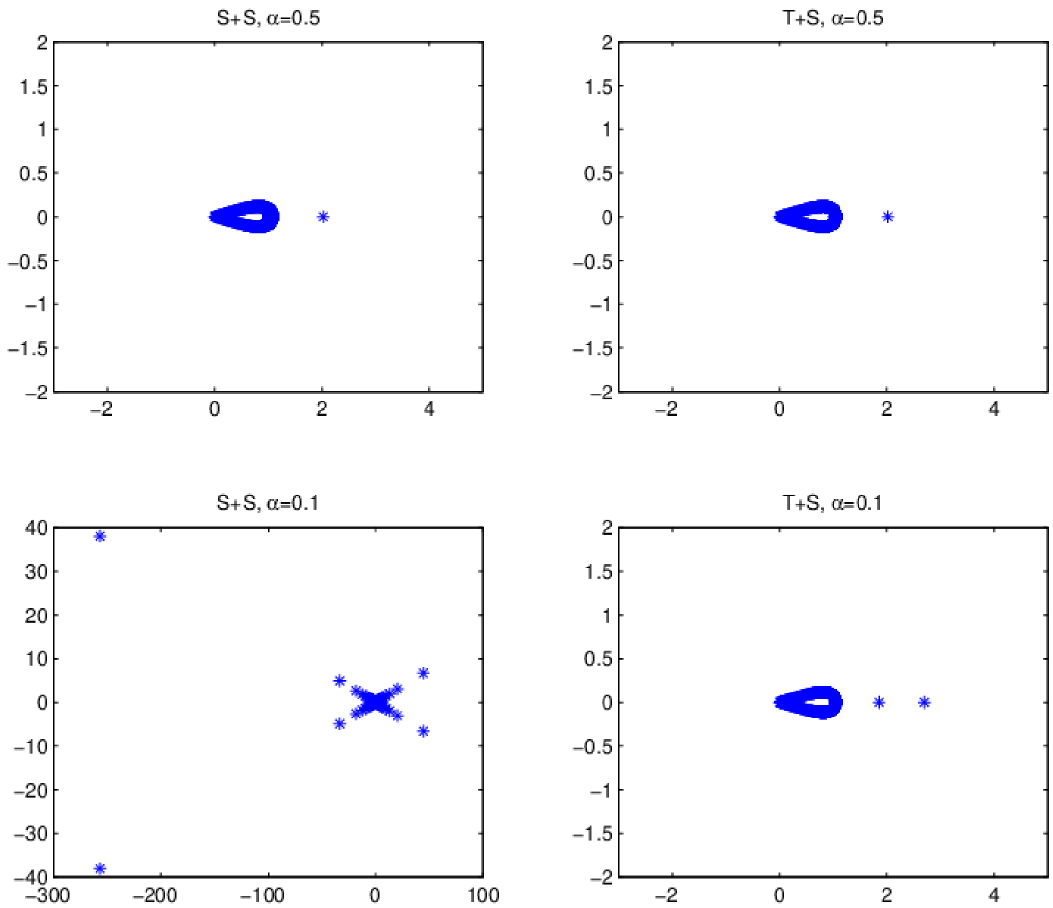

Figure 2: Spectra of different preconditioned matrices for $\beta = 1.5$, $N = 2^{10}$, $d^+ = \Gamma(1.5)$, $d^- = \Gamma(2.5)$.

Example 5.2 (Variable coefficients)**.** Consider problem (2.1) with $\beta = 1.8$, variable coefficients $d^+ = \Gamma(1.2)x^{0.8}$, $d^- = \Gamma(1.2)(1-x)^{0.8}$, and source term

$$f(x) = \frac{100}{11}\Big[\frac{6}{(4-\alpha)(3-\alpha)(2-\alpha)(1-\alpha)}\big(x^{4-\alpha} + (1-x)^{4-\alpha}\big) + \frac{1}{1-\alpha}(1-x)^{1-\alpha}x^3$$

$$
\begin{aligned}
&\quad + \frac{3}{2-\alpha}(1-x)^{2-\alpha}x^2 + \frac{3}{3-\alpha}(1-x)^{3-\alpha}x + \frac{1}{4-\alpha}(1-x)^{4-\alpha} \\
&\quad + \frac{1}{1-\alpha}(1-x)^3 x^{1-\alpha} + \frac{3}{2-\alpha}(1-x)^2 x^{2-\alpha} + \frac{3}{3-\alpha}(1-x)x^{3-\alpha} + \frac{1}{4-\alpha}x^{4-\alpha}\Bigg] \\
&-10\Bigg[\frac{2}{(3-\alpha)(2-\alpha)(1-\alpha)}\left(x^{3-\alpha}+(1-x)^{3-\alpha}\right) + \frac{1}{1-\alpha}(1-x)^{1-\alpha}x^2 \\
&\quad + \frac{2}{2-\alpha}(1-x)^{2-\alpha}x + \frac{1}{3-\alpha}(1-x)^{3-\alpha} \\
&\quad + \frac{1}{1-\alpha}(1-x)^2 x^{1-\alpha} + \frac{2}{2-\alpha}(1-x)x^{2-\alpha} + \frac{1}{3-\alpha}x^{3-\alpha}\Bigg] \\
&+2\Bigg[\frac{1}{(1-\alpha)(2-\alpha)}\left(x^{2-\alpha}+(1-x)^{2-\alpha}\right) + \frac{1}{1-\alpha}(1-x)^{1-\alpha}x \\
&\quad + \frac{1}{2-\alpha}(1-x)^{2-\alpha} + \frac{1}{1-\alpha}(1-x)x^{1-\alpha} + \frac{1}{2-\alpha}x^{2-\alpha}\Bigg].
\end{aligned}
$$

The exact solution to this problem is $u(x) = x^2(1-x)^2$.

Errors of different numerical solvers and convergence rate of the PGRMES (T+S) are presented in Table 4, and the corresponding CPU time and the number of iterations for iterative methods are reported in Table 5. Similar to the case of constant coefficients (Example 5.1), the proposed preconditioner M still performs the best no matter $\alpha = 0.1$ or $\alpha = 0.5$.

6. Conclusions

In this paper, the preconditioned GMRES method with a mixed-type circulant preconditioner is employed for solving the discretized system of a generalized elastic model. By direct expansion, the coefficient matrix AB can be written as the sum of $\tilde{A}\tilde{B}$ and a rank 2 matrix R, where $\tilde{A}$ is proven to be positive definite by appropriate congruent transformation and $\mathrm{Re}(x^*\tilde{B}x) < 0$. Then we obtain $\mathrm{Re}(\lambda(\tilde{A}\tilde{B})) < 0$. When the diffusion coefficients of the model are the same positive constant, the matrix R is found to possess a centrosymmetric structure and it is orthogonally similar to a 2-by-2 block diagonal matrix. After analyzing the spectrum of each block, it is found that all eigenvalues of R are zero except two are real and negative. We further notice that, from intensive numerical tests, the spectrum of the matrix AB is on the left half of the complex plane. Besides, for α is close to 0, $\tilde{A}$ becomes ill-conditioned, and the corresponding Strang's preconditioner $s(\tilde{A})$ cannot preserve the positiveness of all eigenvalues. Therefore, a new mixed-type circulant preconditioner is proposed by substituting $s(\tilde{A})$ in [8] by Chan's preconditioner $c(\tilde{A})$, which can preserve the positiveness of eigenvalues. Our proposed preconditioner is shown to be well-defined and numerical examples are provided showing the efficiency of the preconditioner for both constant and variable coefficients cases.

Table 2: Errors and convergence rate for Example 5.1.

α	N	Direct	GMRES	PGMRES (S+S)	PGMRES (T+S)	Rate
		$\|u-u_h\|$	$\|u-u_h\|$	$\|u-u_h\|$	$\|u-u_h\|$	
0.5	2^8	2.7231×10^{-4}	2.7231×10^{-4}	2.7231×10^{-4}	2.7231×10^{-4}	1.0134
	2^9	1.3519×10^{-4}	1.3519×10^{-4}	1.3519×10^{-4}	1.3519×10^{-4}	1.0102
	2^{10}	6.7253×10^{-5}	6.7253×10^{-5}	6.7253×10^{-5}	6.7253×10^{-5}	1.0074
	2^{11}	3.3507×10^{-5}	3.3507×10^{-5}	3.3507×10^{-5}	3.3507×10^{-5}	1.0051
	2^{12}	1.6713×10^{-5}	1.6713×10^{-5}	1.6713×10^{-5}	1.6713×10^{-5}	1.0035
	2^{13}	-	8.3429×10^{-6}	8.3429×10^{-6}	8.3429×10^{-6}	1.0024
	2^{14}	-	4.1669×10^{-6}	4.1669×10^{-6}	4.1669×10^{-6}	1.0016
	2^{15}	-	2.0819×10^{-6}	2.0819×10^{-6}	2.0819×10^{-6}	1.0011
	2^{16}	-	1.0405×10^{-6}	1.0405×10^{-6}	1.0404×10^{-6}	1.0008
0.1	2^8	2.6787×10^{-4}	2.6787×10^{-4}	2.6787×10^{-4}	2.6787×10^{-4}	1.0043
	2^9	1.3365×10^{-4}	1.3365×10^{-4}	1.3365×10^{-4}	1.3365×10^{-4}	1.0031
	2^{10}	6.6721×10^{-5}	6.6721×10^{-5}	6.6721×10^{-5}	6.6721×10^{-5}	1.0022
	2^{11}	3.3323×10^{-5}	3.3323×10^{-5}	3.3903×10^{-5}	3.3323×10^{-5}	1.0016
	2^{12}	1.6648×10^{-5}	1.6648×10^{-5}	1.6643×10^{-5}	1.6648×10^{-5}	1.0012
	2^{13}	-	8.3178×10^{-6}	-	8.3178×10^{-6}	1.0010
	2^{14}	-	4.1557×10^{-6}	-	4.1557×10^{-6}	1.0011
	2^{15}	-	2.0755×10^{-6}	-	2.0755×10^{-6}	1.0016
	2^{16}	-	1.0356×10^{-6}	-	1.0356×10^{-6}	1.0029

Table 3: The consumed CPU time and number of iteration for Example 5.1.

α	N	Direct	GMRES		PGMRES (S+S)		PGMRES (T+S)	
		CPU	CPU	Iter	CPU	Iter	CPU	Iter
0.5	2^8	3.0×10^{-3}	8.8×10^{-2}	177	1.1×10^{-2}	10	8.0×10^{-3}	10
	2^9	1.3×10^{-2}	1.7×10^{-1}	257	8.0×10^{-3}	10	9.0×10^{-3}	10
	2^{10}	6.1×10^{-2}	4.2×10^{-1}	422	1.1×10^{-2}	10	1.1×10^{-2}	10
	2^{11}	4.1×10^{-1}	1.1×10^{0}	680	2.1×10^{-2}	11	2.0×10^{-2}	11
	2^{12}	2.6×10^{0}	4.7×10^{0}	1268	4.4×10^{-2}	11	4.3×10^{-2}	11
	2^{13}	-	1.2×10^{1}	1857	8.2×10^{-2}	12	8.0×10^{-2}	10
	2^{14}	-	5.3×10^{1}	3924	1.7×10^{-1}	11	1.7×10^{-1}	11
	2^{15}	-	2.4×10^{2}	7428	3.7×10^{-1}	12	3.7×10^{-1}	11
	2^{16}	-	1.5×10^{3}	13130	9.0×10^{-1}	11	9.9×10^{-1}	11
0.1	2^8	3.0×10^{-3}	3.4×10^{-2}	51	4.2×10^{-1}	660	6.0×10^{-3}	10
	2^9	1.1×10^{-2}	6.1×10^{-2}	70	1.2×10^{0}	1705	8.0×10^{-3}	10
	2^{10}	6.7×10^{-2}	1.0×10^{-1}	88	5.4×10^{0}	5036	1.5×10^{-2}	11
	2^{11}	5.2×10^{-1}	2.3×10^{-1}	110	1.1×10^{2}	14040	4.4×10^{-2}	11
	2^{12}	3.9×10^{0}	6.3×10^{-1}	145	4.8×10^{2}	57330	5.4×10^{-2}	11
	2^{13}	-	1.4×10^{0}	189	-	-	1.1×10^{-1}	11
	2^{14}	-	3.5×10^{0}	239	-	-	1.0×10^{-1}	11
	2^{15}	-	1.0×10^{1}	306	-	-	5.5×10^{-1}	10
	2^{16}	-	4.4×10^{1}	395	-	-	8.6×10^{-1}	11

Table 4: Errors and convergence rate for Example 5.2.

α	N	Direct	GMRES	PGMRES (S+S)	PGMRES (T+S)	Rate
		$\Vert u-u_h\Vert$	$\Vert u-u_h\Vert$	$\Vert u-u_h\Vert$	$\Vert u-u_h\Vert$	
0.5	2^8	5.5145×10^{-5}	5.5145×10^{-5}	5.5145×10^{-5}	5.5145×10^{-5}	0.9931
	2^9	2.7491×10^{-5}	2.7491×10^{-5}	2.7491×10^{-5}	2.7491×10^{-5}	1.0043
	2^{10}	1.3693×10^{-5}	1.3693×10^{-5}	1.3693×10^{-5}	1.3693×10^{-5}	1.0055
	2^{11}	6.8267×10^{-6}	6.8267×10^{-6}	6.8267×10^{-6}	6.8267×10^{-6}	1.0042
	2^{12}	3.4070×10^{-6}	3.4070×10^{-6}	3.4069×10^{-6}	3.4070×10^{-6}	1.0027
	2^{13}	-	1.7016×10^{-6}	1.7015×10^{-6}	1.7024×10^{-6}	1.0009
	2^{14}	-	-	8.4874×10^{-7}	8.5167×10^{-7}	0.9992
	2^{15}	-	-	4.2342×10^{-7}	4.3198×10^{-7}	0.9793
	2^{16}	-	-	2.3059×10^{-7}	2.2735×10^{-7}	0.9261
0.1	2^8	5.6169×10^{-5}	5.6196×10^{-5}	5.6196×10^{-5}	5.6196×10^{-5}	1.0162
	2^9	2.7755×10^{-5}	2.7755×10^{-5}	2.7755×10^{-5}	2.7755×10^{-5}	1.0177
	2^{10}	1.3756×10^{-5}	1.3756×10^{-5}	1.4652×10^{-5}	1.3756×10^{-5}	1.0126
	2^{11}	6.8415×10^{-6}	6.8415×10^{-6}	9.4488×10^{-6}	6.8415×10^{-6}	1.0077
	2^{12}	3.4103×10^{-6}	3.4103×10^{-6}	8.3302×10^{-6}	3.4104×10^{-6}	1.0044
	2^{13}	-	1.7024×10^{-6}	-	1.7024×10^{-6}	1.0024
	2^{14}	-	8.5041×10^{-7}	-	8.5067×10^{-7}	1.0009
	2^{15}	-	4.2510×10^{-7}	-	4.2462×10^{-7}	1.0024
	2^{16}	-	2.1273×10^{-7}	-	2.1197×10^{-7}	1.0023

Table 5: The consumed CPU time and number of iteration for Example 5.2.

α	N	Direct	GMRES		PGMRES (S+S)		PGMRES (T+S)	
		CPU	CPU	Iter	CPU	Iter	CPU	Iter
0.5	2^8	2.0×10^{-3}	1.7×10^{-1}	269	7.0×10^{-3}	12	7.0×10^{-3}	12
	2^9	1.0×10^{-2}	4.4×10^{-1}	512	1.2×10^{-2}	12	1.0×10^{-2}	12
	2^{10}	6.8×10^{-2}	1.1×10^{0}	980	1.4×10^{-2}	12	1.4×10^{-2}	12
	2^{11}	4.0×10^{-1}	3.0×10^{0}	1858	2.6×10^{-2}	12	2.6×10^{-2}	12
	2^{12}	2.5×10^{0}	1.6×10^{1}	4543	5.2×10^{-2}	13	5.5×10^{-2}	12
	2^{13}	-	7.5×10^{1}	11676	1.0×10^{-1}	13	9.7×10^{-2}	12
	2^{14}	-	-	-	2.2×10^{-1}	13	2.1×10^{-1}	13
	2^{15}	-	-	-	4.9×10^{-1}	13	4.4×10^{-1}	13
	2^{16}	-	-	-	1.2×10^{0}	14	1.1×10^{0}	13
0.1	2^8	3.0×10^{-3}	3.2×10^{-2}	62	3.1×10^{-1}	527	7.0×10^{-3}	13
	2^9	1.3×10^{-2}	6.6×10^{-2}	101	2.4×10^{0}	3369	9.0×10^{-3}	13
	2^{10}	5.5×10^{-2}	1.6×10^{-1}	157	3.5×10^{1}	27960	1.5×10^{-2}	13
	2^{11}	4.2×10^{-1}	4.8×10^{-1}	252	1.3×10^{2}	21090	3.2×10^{-2}	14
	2^{12}	2.5×10^{0}	1.4×10^{0}	399	4.8×10^{2}	122880	5.6×10^{-2}	14
	2^{13}	-	4.3×10^{0}	658	-	-	1.0×10^{-1}	14
	2^{14}	-	1.4×10^{1}	1144	-	-	2.2×10^{-1}	14
	2^{15}	-	3.9×10^{1}	1567	-	-	4.4×10^{-1}	14
	2^{16}	-	2.1×10^{2}	3107	-	-	1.1×10^{0}	14

Acknowledgments

The authors would like to express their sincere thanks to Hai-Wei Sun for his support and encouragement, Yunchi Huang for his inspiring discussion about exploring the properties of the elements in the coefficient matrix. In particular authors would like to thank Lot-Kei Chou for his helpful suggestions and kind assistance on the spectrum analysis of preconditioned matrix. The authors also would like to thank the reviewers for their thoughtful comments and suggestions that helped to improve the paper.

The first author was supported by FDCT (Grant 0118/2018/A3). The second author was supported by University of Macau (Grants MYRG2022-00262-FST, MYRG2020-00208-FST).

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