INVARIRANTS-PRESERVING DU FORT-FRANKEL SCHEMES AND THEIR ANALYSES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH WAVE OPERATOR

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Abstract

Du Fort-Frankel finite difference method (FDM) was firstly proposed for linear diffusion equations with periodic boundary conditions by Du Fort and Frankel in 1953. It is an explicit and unconditionally von Neumann stable scheme. However, there has been no research work on numerical solutions of nonlinear Schrödinger equations with wave operator by using Du Fort-Frankel-type finite difference methods (FDMs). In this study, a class of invariants-preserving Du Fort-Frankel-type FDMs are firstly proposed for one-dimensional (1D) and two-dimensional (2D) nonlinear Schrödinger equations with wave operator. By using the discrete energy method, it is shown that their solutions possess the discrete energy and mass conservative laws, and conditionally converge to exact solutions with an order of $O(\tau^2 + h_x^2 + (\tau/h_x)^2)$ for 1D problem and an order of $O(\tau^2 + h_x^2 + h_y^2 + (\tau/h_x)^2 + (\tau/h_y)^2)$ for 2D problem in $H^1$-norm. Here, $\tau$ denotes time-step size, while, $h_x$ and $h_y$ represent spatial meshsizes in $x$- and $y$-directions, respectively. Then, by introducing a stabilized term, a type of stabilized invariants-preserving Du Fort-Frankel-type FDMs are devised. They not only preserve the discrete energies and masses, but also own much better stability than original schemes. Finally, numerical results demonstrate the theoretical analyses.

Mathematics subject classification: 26A33, 34A08, 65M06, 65M12.
Key words: Nonlinear Schrödinger equations with wave operator, Du Fort-Frankel finite difference methods, Discrete energy and mass conservative laws, Numerical convergence.

1. Introduction

The nonlinear Schrödinger equations have been extensively applied in various mathematical and physical fields, such as, plasma physics, nonlinear optics and bimolecular dynamics. A nonlinear Schrödinger equation with wave operator, which was firstly proposed by Matsutchi [31], was used to describe the nonlinear interactions between two waves travelling in opposite directions. In [14], the author studied the existence and uniqueness of the weak and strong solutions of the nonlinear Schrödinger equations with wave operator by means of Galerkin method, the regularity of the solutions, and the existence of the smooth solutions of 1D nonlinear Schrödinger...
equations with wave operator under weaker assumption. In [15], the author has devoted to researching the existence and nonexistence for this equation in the case of possessing different signs in nonlinear term under some conditions.

In this paper, we consider the numerical solutions of the initial-boundary value problem for the nonlinear Schrödinger equations with wave operator as follows:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + iu_t + |u|^2u + f(x)u &= 0, \quad x \in \Omega, \quad t \in (0, T), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\
u(x, t) &= 0, \quad x \in \partial \Omega, \quad t \in [0, T],
\end{align*}
\]

(1.1a)-(1.1c)

by using invariants-preserving Du Fort-Frankel-type FDMs. Here \(u(x, t)\) and \(f(x)\) are an unknown complex function and a given real-valued function, respectively, \(i = \sqrt{-1}\). For 1D case, we set \(\Omega = (X_l, X_r)\) and denote \(x = x\). For 2D case, we write \(\Omega = (X_l, X_r) \times (Y_l, Y_r)\) and \(x = (x, y)\). The conjugate complex number of \(u\) is denoted by \(\overline{u}\).

**Proposition 1.1 ([19, 32, 35]).** Let the mass and energy conservative laws for the problem (1.1a)-(1.1c) be defined as follows:

\[
\begin{align*}
Q(t) &= 2\text{Im}\langle u_t, u \rangle + \|u\|^2 + c, \\
E(t) &= \|u_t\|^2_{L^2} + |u|^2_{H^1} + \frac{1}{2}\|u\|_{L^4}^4 + \langle f, |u|^2 \rangle + c,
\end{align*}
\]

(1.2a)-(1.2b)

respectively. Then we have that \(Q(t) = Q(0)\) and \(E(t) = E(0)\). Here, \(c\) is an arbitrary constant.

**Proof.** Acting the inner product of (1.1a) with \(2u\), taking the imaginary part, applying Green formula and noting homogeneous Dirichlet boundary conditions, we obtain

\[
\text{Im}\langle u_{tt}, 2u \rangle + \text{Re}\langle u_t, 2u \rangle = 0.
\]

(1.3)

Besides, by simple computation, we have

\[
\frac{d}{dt}(u_t\overline{u}) = u_{tt}\overline{u} + u_t\overline{u}_t,
\]

which shows that

\[
\frac{d}{dt}[\text{Im}(2u_t\overline{u})] = \text{Im}(2u_{tt}\overline{u}).
\]

(1.4)

Furthermore, by simple computation, we obtain

\[
\frac{d}{dt}(u\overline{u}) = u_t\overline{u} + u\overline{u}_t = 2\text{Re}(u_t\overline{u}).
\]

(1.5)

Applying (1.4) and (1.5) to (1.3) shows that

\[
\frac{d}{dt}Q(t) = 0, \quad Q(t) = Q(0).
\]

(1.6)

Acting the inner product of (1.1a) with \(2u_t\), applying the Green formula and noting homogeneous Dirichlet boundary conditions, taking the real part, using (1.5) and
\[
\frac{d}{dt}(|u_t|^2) = 2\text{Re}(u_{tt} \overline{u}), \quad -2\text{Re}(\Delta u, 2u_t) = \frac{d}{dt}|u_t|^2, \quad \frac{d}{dt}\left(\frac{1}{2}|u|^4\right) = 2|u|^2\text{Re}(u\overline{u}),
\]  
\( (1.7) \)

we can directly derive that
\[
\frac{d}{dt} E(t) = 0, \quad E(t) = E(0).
\]  
\( (1.8) \)

The proof is complete. \(\Box\)

Over the years, much attention has been paid on the development and analyses of energy-preserving FDMs for 1D and 2D nonlinear Schrödinger equations with wave operator. For example, a three-level linearized and implicit energy-preserving FDM, which was extended to the construction of the linearly energy-preserving compact finite difference method for solving 1D nonlinear Schrödinger equations with wave operator in [25], was devised for 1D nonlinear Schrödinger equations with wave operator in [43]. An explicit energy-preserving FDM with stable condition \((\tau/h_x) \leq 1/2\), which was generalized to the development of an explicit, conditionally stable energy-preserving FDM with fourth-order space accuracy for 1D nonlinear Schrödinger equations with wave operator in [27], was established for 1D nonlinear Schrödinger equations with wave operator in [42]. Several nonlinear energy-preserving FDMs and energy-preserving compact FDMs for 1D and 2D nonlinear Schrödinger equations with wave operator can be found in [5, 17, 26, 27, 35, 38]. Besides, other energy-preserving numerical algorithms including energy-preserving Fourier pseudo-spectral methods [2, 19, 23], energy conserving local discontinuous Galerkin methods [16], energy-conservative finite element method [3] have been developed for problem (1.1a)-(1.1c). More recently, the invariant energy-quadratization approaches (cf. [41]) and scalar auxiliary variable methods (cf. [24]) have been used to the developments of the energy-conserving numerical algorithms for nonlinear Schrödinger equations with wave operator, which can not preserve the discrete masses. To the best of our knowledge, most of references have not referred to the discrete mass-conserving law except the references [19, 32, 35]. Most of the existent energy-conserving methods are implicitly linear schemes, or fully implicitly nonlinear schemes, thus the corresponding computations are complex, and may be very time-consuming. Explicit algorithms have an important advantage of computational simplicity. However, the existent explicitly energy-preserving FDMs suffer from the strong limitations of conditional stability (see [27, 42]). In this study, we aim at the development and analyses of the explicitly mass- and energy-preserving numerical methods with good stability and simple implementation for problem (1.1a)-(1.1c).

Recently, there has been growing interest in the applications of the Du Fort-Frankel-type FDMs to solve various partial differential equations. For example, two Du Fort-Frankel-type FDMs were developed for solving time-fractional subdiffusion equations in [1, 28]. A Du Fort-Frankel scheme was established for solving 1D uncertain heat equation in [40]. As we know, Du Fort-Frankel FDM was firstly proposed by Du Fort and Frankel to solve 1D linear diffusion with periodic boundary condition in [10]. It is an explicit and unconditionally Neumann stable scheme. Besides, the stable analyses of the Du Fort-Frankel-type methods can be found in [6, 10–13]. Numerical solutions to Schrödinger equation by using Du Fort-Frankel-type FDMs have attracted much attention. For example, a Du Fort-Frankel-type FDM was suggested for solving 1D linear Schrödinger equations with a variable coefficient in [7]. A energy-preserving Du Fort-Frankel-type FDM was developed for solving 1D linear and nonlinear Schrödinger equations in [39]. However, the discrete mass-conserving law and convergent result were not given in [39]. In [18], the development and convergence of a mass- and energy-preserving Du Fort-Frankel-type FDM, which has been generalized to numerical solutions of multi-dimensional Schrödinger
equations on different geometries in [20], were derived for 1D Schrödinger equations in detail. As for more details, please refer to the references [7,8,18,20,29,30,39] and the related references therein. However, although those Du Fort-Frankel-type FDMs for Schrödinger equations can preserve the discrete mass and energy conservative laws, and provide reliable solutions, they are still conditionally stable, and suffer from strong grid restrictions.

Moreover, nonlinear Schrödinger equations with wave operator is also viewed as a combination of the complex Klein-Gordon equations with $i u_t$. Also, there have been a lot of invariants-preserving numerical algorithms for Klein-Gordon equations, (see [4,9,21,22,34] and the related references therein). However, as the majority of them are also fully implicit nonlinear schemes, or linearly implicit schemes, the corresponding computations are complex and time-consuming. To the best of our knowledge, the invariants-preserving Du Fort-Frankel-type methods for Klein-Gordon equations have not been studied, either. Moreover, little attention has been devoted to the numerical solutions of the problem (1.1a)-(1.1c) by explicitly invariants-preserving numerical algorithms with good stability. Therefore, to avoid numerical solutions of the system of the linear or nonlinear algebraic equations, it is very necessary to develop explicitly invariants-preserving numerical algorithms with good stability for problem (1.1a)-(1.1c).

Motivating by the existent works on invariants-preserving numerical algorithms for Klein-Gordon equations and Schrödinger equations, the purpose of this study is to design explicitly invariants-preserving numerical algorithms with good stability for problem (1.1a)-(1.1c). At first, using second-order centered difference methods to discrete temporal derivatives, applying Du Fort-Frankel-type FDMs to deal with spatial derivatives and utilizing \[ |u(x_j, t_n)|^2 |u(x_j, t_{n+1}) + u(x_j, t_{n-1})|/2 \] to approximate nonlinear term, an invariants-preserving Du Fort-Frankel-type FDM, which can preserve the discrete mass and energy conservative laws, is derived for 1D problem (1.1a)-(1.1c). As we can not make sure that the discrete energies of numerical solutions obtained by this invariants-preserving Du Fort-Frankel-type FDM are non-negative, a class of stabilized invariants-preserving Du Fort-Frankel-type FDMs are devised by adding a stabilized term to this invariants-preserving Du Fort-Frankel-type FDM. Numerical solutions obtained by the stabilized invariants-preserving Du Fort-Frankel-type FDMs satisfy the discrete mass and energy conservative laws, are stable in the $L^\infty$-norm under conditions that $\tau = O(h_x)$ and the parameter of stabilized term is greater than or equal to 0.5$\|f\|_{\infty}$, and unconditionally stable in the $L^2$-norm as long as the parameter of stabilized term is greater than or equal to 0.5$\|f\|_{\infty}$. Secondly, by generalizing the techniques developed in the constructions of invariants-preserving Du Fort-Frankel-type FDMs for 1D problem, invariants-preserving Du Fort-Frankel-type FDMs are established for 2D problems. Thirdly, the theoretical analyses of the invariants-preserving Du Fort-Frankel-type FDMs are all derived using the discrete energy method, rigorously. Numerical results confirm the accuracy and efficiency of our methods. It is worth mentioning that they can be successfully used to the long-term simulations because of the good stability, convergence and easy implementation.

The remainder of this paper is organized as follows. In Section 2, some notations and auxiliary lemmas are introduced. The constructions and theoretical investigations of the invariants-preserving Du Fort-Frankel schemes for 1D problem (1.1a)-(1.1c) are given in Section 3. Section 4 is devoted to the establishments and analyses of invariants-preserving Du Fort-Frankel schemes for 2D problem (1.1a)-(1.1c). Numerical results demonstrate the correctness of our theoretical findings, and the efficiency of the proposed methods in Section 5. Finally, some comments are given in Section 6.
2. Notations and Auxiliary Lemmas

In what follows, we give some notations and lemmas used later. To begin with, denote temporal domain by \([0, T]\). Setting temporal meshsize \(\tau = T/N\) \((N \in \mathbb{Z}^+)\) and \(t_n = n\tau\), temporal interval \([0, T]\) is covered by \(\Omega_\tau = \{ t_n \mid t_n = n\tau, 0 \leq n \leq N \}\). On \(\Omega_\tau\), notations are introduced as follows:

\[
\delta_t U^n = \frac{1}{2\tau} (U^{n+1} - U^{n-1}), \quad U^n = \frac{1}{2} (U^{n+1} + U^{n-1}),
\]

\[
\delta_t^2 U^n = \frac{1}{\tau} (U^{n+1} - 2U^n + U^{n-1}), \quad U^{n+\frac{1}{2}} = \frac{1}{2} (U^{n+1} + U^n),
\]

\[
\delta_t U^{n+\frac{1}{2}} = \frac{1}{\tau}(U^{n+1} - U^n).
\]

For 1D case, let \(h_x = (X_r - X_l)/M_x\) \((M_x \in \mathbb{Z}^+)\) be the spatial meshsize. Denote \(x_j = X_l + jh_x\) \((0 \leq j \leq M_x\), \(\Omega_h = \{ x_j \mid 0 \leq j \leq M_x \}\). On \(\Omega_h\), difference operators are introduced as follows:

\[
\delta_x U_{j+\frac{1}{2}} = \frac{1}{h_x}(U_{j+1} - U_j), \quad \delta_x^2 U_j = \frac{1}{h_x}(U_{j+1} - 2U_j + U_{j-1}).
\]

Define grid function space as

\[
T_h^0 = \{ V \mid V = \{ V_j \mid 0 \leq j \leq M_x \}, \text{and } V_0 = V_{M_x} = 0 \}.
\]

For any \(U, V \in T_h^0\), the following inner products and norms are defined as:

\[
(U, V) = h_x \sum_{j=1}^{M_x-1} V_j \delta_x V_j, \quad (\delta_x^2 U, \delta_x V) = h_x \sum_{j=1}^{M_x} \delta_x U_j \sqrt{\delta_x V_j}, \quad \| U \| = \sqrt{(U, U)},
\]

\[
|U|_{H^1} = \sqrt{(\delta_x U, \delta_x V)}, \quad \| U \|_{H^1} = \sqrt{|U|^2 + |U|_{H^1}^2}, \quad \| U \|_\infty = \max_{0 \leq j \leq M_x} |U_j|.
\]

For 2D case, accordingly, denote spatial domain by \(\Omega = [X_l, X_r] \times [Y_l, Y_r]\). Let \(h_x = (X_r - X_l)/M_x\) and \(h_y = (Y_r - Y_l)/M_y\) \((M_x, M_y \in \mathbb{Z}^+)\) be spatial meshizes in \(x\) - and \(y\)-directions, respectively. Denote \(x_j = X_l + jh_x\) \((0 \leq j \leq M_x\), \(y_k = Y_l + kh_y\) \((0 \leq k \leq M_y\)) and \(x_{j,k} = (x_j, y_k)\) \((0 \leq j \leq M_x, 0 \leq k \leq M_y)\), which form \(\Omega_h = \{ x_{j,k} \mid 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1 \}\), \(\partial\Omega_h = \{ x_{j,k} \mid j = 0 \text{ or } j = M_x \text{ or } k = 0 \text{ or } k = M_y \}\) and \(\Omega_h = \Omega_h \cup \partial\Omega_h\).

Define grid function space as

\[
S_h^0 = \{ V \mid V = \{ V_{j,k} \mid 0 \leq j \leq M_x, 0 \leq k \leq M_y \} \text{ and } V_{j,k} = 0 \text{ as } x_{j,k} \in \partial\Omega_h \}.
\]

On \(\Omega_h\), we introduce the following difference operators:

\[
\delta_x^2 V_{j,k} = \frac{1}{h_x^2}(V_{j+1,k} - 2V_{j,k} + V_{j-1,k}), \quad \delta_y^2 V_{j,k} = \frac{1}{h_y^2}(V_{j,k+1} - 2V_{j,k} + V_{j,k-1}),
\]

\[
\delta_x V_{j-\frac{1}{2},k} = \frac{1}{h_x}(V_{j,k} - V_{j-1,k}), \quad \delta_y V_{j,k-\frac{1}{2}} = \frac{1}{h_y}(V_{j,k} - V_{j,k-1}).
\]

\[
\Delta_h V_{j,k} = (\delta_x^2 + \delta_y^2)V_{j,k}.
\]

Besides, for any \(U, V \in S_h^0\), the inner products and norms are defined as follows:

\[
(U, V) = h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} U_{j,k} V_{j,k},
\]
\[
\langle \delta_x U, \delta_x V \rangle = h_x h_y \sum_{j=1}^{M_x} \sum_{k=1}^{M_y-1} \delta_x U_{j-\frac{1}{2},k} \delta_x V_{j-\frac{1}{2},k},
\]
\[
\langle \delta_y U, \delta_y V \rangle = h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y} \delta_y U_{j,k-\frac{1}{2}} \delta_y V_{j,k-\frac{1}{2}},
\]
\[
\|U\| = \sqrt{\langle U, U \rangle}, \quad \|U\|_{\infty} = \max_{0 \leq j \leq M_x} |U_{j,k}|.
\]
\[
\|\delta_x U\| = \sqrt{\langle \delta_x U, \delta_x U \rangle}, \quad \|\delta_y U\| = \sqrt{\langle \delta_y U, \delta_y U \rangle}.
\]
\[
|U|_{H^1} = \sqrt{\|\delta_x U\|^2 + \|\delta_y U\|^2}, \quad \|U\|_{H^1} = \sqrt{\|U\|^2 + |U|_{H^1}^2},
\]
\[
\|U\|_p = \left( h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U_{j,k}|^p \right)^{\frac{1}{p}}.
\]

**Lemma 2.1** ([36, Gronwall Inequality]). Let the series \( \{F^n \mid n \geq 0 \} \) be nonnegative and satisfy the inequality
\[
F^N \leq A + \tau \sum_{n=1}^{N} B_n F^n,
\]
where \( A \) and \( B_n, k = 1, 2, \ldots, N \) are nonnegative constants. Then
\[
\max_{0 \leq n \leq N} |F^n| \leq A \exp \left( 2 \sum_{n=1}^{N} B_n \tau \right),
\]
where \( \tau \) is sufficiently small, such that
\[
\tau \max_{1 \leq n \leq N} B_n \leq \frac{1}{2}.
\]

**Lemma 2.2.** Let \( V \in T_h^0 \). Then we have
\[
|V^{n+1}|_{H^1}^2 \leq 2 |V^{n+\frac{1}{2}}|_{H^1}^2 + 2r_x^2 \left| |\delta V^{n+\frac{1}{2}}| \right|^2,
\]
where \( r_x = \tau / h_x \).

**Proof.** By
\[
V^{n+1}_{j+\frac{1}{2}} = \frac{1}{2} \left( V^{n+\frac{1}{2}}_{j+\frac{1}{2}} + V^*_j + V^{n+1}_j - V^n_j \right) = V^{n+\frac{1}{2}}_j + \frac{1}{2} \left( \tau \delta V^{n+\frac{1}{2}}_j \right),
\]
we derive that
\[
\delta V^{n+1}_{j+\frac{1}{2}} = \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} + \frac{1}{2} \tau \delta \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} + \frac{1}{2} \tau \left( \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} - \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} \right),
\]
which is applied along with the inequality \( |a \pm b| \leq 2|a|^2 + |b|^2 \) to infer that
\[
\left| \delta_v V^{n+1}_{j+\frac{1}{2}} \right|^2 \leq 2 \left| \delta_v V^{n+\frac{1}{2}}_{j+\frac{1}{2}} \right|^2 + r_x^2 \left[ \left| \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} \right|^2 + \left| \delta V^{n+\frac{1}{2}}_{j+\frac{1}{2}} \right|^2 \right]. \tag{2.1}
\]
Multiplying \( h_x \) to both sides of (2.1) and summing in \( j \) from 0 to \( M_x - 1 \) obtain the claimed result. The proof is complete. \( \square \)

By the Lemma 2.2, we can get the Corollary 2.1 as follows.
Using the Taylor expansion with Lagrange remainder, we have
\[ |V^{n+1}|_{H^1}^2 \leq 2|V^n|^2_{H^1} + 2(r^2 + r^n_x)\|\delta_t V^n + \frac{\delta_t}{2}\|^2, \]
where \( r_x = \tau/h_x \) and \( r_y = \tau/h_y \).

**Lemma 3.2** ([33]). Let \( V \in T^3_h \). Then it holds that
\[ \|V\| \leq \frac{\sqrt{X_r - X_l}}{2}\|V\|_{H^1}, \quad \|V\| \leq \frac{X_r - X_l}{\sqrt{6}}\|V\|_{H^1}, \quad h^2\|\delta_x V\|^2 \leq 4\|V\|^2. \]

**Lemma 2.4** ([33]). Let \( V \in S^3_h \). Then it holds that
\[ h^2\|\delta_x V\|^2 \leq 4\|V\|^2, \quad h^2\|\delta_y V\|^2 \leq 4\|V\|^2, \quad \|V\|^2 \leq \bar{c}\|V\|^2_{H^1}, \quad \|V\|^2_{H^1} \leq \|V\|^2 \leq \bar{c}\|V\|^2_{H^1}, \]
where \( \delta = |\Omega|/12, |\Omega| = (X_r - X_l)(Y_r - Y_l) \) and \( \bar{c} = 1 + \delta \).

**Lemma 2.5.** Let the series be \( \{V^n|n \geq 0\} \). Then it holds that
\[ (V^n)^2 \leq 2(V^0)^2 + 2\tau T \sum_{l=1}^{n} (\delta_l V^{l-\frac{2}{3}})^2. \]

**Proof.** The application of the Cauchy-Schwarz inequality yields
\[ (V^n - V^0)^2 = \left[ \tau \sum_{l=1}^{n} \delta_l V^{l-\frac{2}{3}} \right]^2 \leq \tau T \sum_{l=1}^{n} (\delta_l V^{l-\frac{2}{3}})^2. \]
Substituting the above equation into \( (V^n)^2 \leq 2(V^n - V^0)^2 + 2(V^0)^2 \) infers the claimed results. The proof is complete. \(\square\)

3. Invariants-Preserving Du Fort-Frankel Schemes for 1D Nonlinear Schrödinger Equations with Wave Operator

3.1. The derivations of the invariants-preserving numerical methods

This section is devoted to the establishments of the invariants-preserving Du Fort-Frankel-type FDMs for 1D nonlinear Schrödinger equations with wave operator.

Let \( u^n_j = u(x_j, t_n), (x_j, t_n) \in \Omega_h \times \Omega_T \). Then the approximation of \( u^n_j \) is denoted by \( U^n_j \).

Using the Taylor expansion with Lagrange remainder, we have
\[ u_t(x_j, t_n) = \delta^n_i u^n_j - \frac{\tau^2}{12} \frac{\partial^4 u}{\partial t^4}(x_j, (\xi_1)^n_j), \quad t_{n-1} \leq (\xi_1)^n_j \leq t_{n+1}, \quad (3.1a) \]
\[ u_{xx}(x_j, t_n) = -r_x^2 \delta_x^2 u^n_j + \delta_x^2 u^n_j + \frac{\tau^2}{12} \frac{\partial^4 u}{\partial x^4}(x_j, (\xi_2)^n_j) \]
\[ - \frac{h^2}{r_x^2} \frac{\partial^4 u}{\partial x^4}(x_j, (\xi_2)^n_j), \quad x_{j-1} \leq (\xi_2)^n_j \leq x_{j+1}, \quad t_{n-1} \leq (\xi_2)^n_j \leq t_{n+1}, \quad (3.1b) \]
\[ u_t(x_j, t_n) = \delta^n_i u^n_j - \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3}(x_j, (\xi_2)^n_j), \quad t_{n-1} \leq (\xi_2)^n_j \leq t_{n+1}, \quad (3.1c) \]
\[ u(x_j, t_n) = u^n_j - \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, (\xi_2)^n_j), \quad t_{n-1} \leq (\xi_2)^n_j \leq t_{n+1}. \quad (3.1d) \]
Thus, using difference approximations (3.1a)-(3.1d) to approximate IBVP (1.1a)-(1.1c) at \((x_j, t_n)\), we can obtain that

\[
(1 + r^2_j) \delta^2_{x} u^n_j - 2 \delta_x u^n_j + i \delta_t u^n_j + |u^n_j|^2 u^n_j + f(x_j)u^n_j = (R^n_j)_j, \quad 1 \leq n \leq N - 1, \tag{3.2}
\]

where

\[
(R^n_j)_j = \frac{\tau^2}{12} \frac{\partial^4 u}{\partial x^4}(x_j, (\xi_1)_j) + \frac{\tau^2}{h^2} \frac{\partial^2 u}{\partial x^2}(x_j, (\xi_2)_j) - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}((\xi_3)_j, t_n) + \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3}(x_j, (\xi_4)_j)
\]

\[
+ |u^n_j|^2 \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, (\xi_5)_j) + f(x_j) \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_j, (\xi_6)_j). \tag{3.3}
\]

Besides, for starting computation, \(U^n_j\) can be firstly solved by another scheme. Utilizing Taylor expansion formula with Lagrange remainder, one gets that

\[
u^n_j = u_0(x_j) + \tau u_1(x_j) + \frac{\tau^2}{2} [(u_0(x_j)) - i u_1(x_j) - |u_0(x_j)|^2 u_0(x_j) - f(x_j) u_0(x_j)]
\]

\[
+ \frac{\tau^3}{3!} \frac{\partial^3 u}{\partial t^3}(x_j, (\xi)_j), \quad x_j \in \Omega, \quad t_0 \leq (\xi)_j \leq t_1. \tag{3.4}
\]

Thus, omitting the small quantity \((R^n_j)_j\), and replacing \(u^n_j\) with its approximation \(U^n_j\), an explicit invariants-preserving Du Fort-Frankel-type FDM, which is denoted by IP-DFFT-FDM-I, is derived as follows:

\[
(1 + r^2_j) \delta^2_{x} U^n_j - 2 \delta_x U^n_j + i \delta_t U^n_j + |U^n_j|^2 U^n_j
\]

\[
+ f(x_j)U^n_j = 0, \tag{3.5a}
\]

\[
U^n_0 = u_0(x_j), \quad 1 \leq j \leq M_x - 1, \tag{3.5b}
\]

\[
U^n_j = U^n_0 + \tau u_1(x_j) + \frac{\tau^2}{2} [(U^n_0)_{xx} - i u_1(x_j) - |U^n_0|^2 U^n_0 - f(x_j) U^n_0], \quad 1 \leq j \leq M_x - 1, \tag{3.5c}
\]

\[
U^n_0 = U^n_{M_x} = 0, \quad 0 \leq n \leq N. \tag{3.5d}
\]

Adding small term \(\lambda r^2 \delta^2_{x} U^n_j\) to the left of (3.5a), (here, \(\lambda\) is a real constant), then a class of stabilized invariants-preserving Du Fort-Frankel-type FDMs, which are represented by IP-DFFT-FDM-II, are derived for 1D problem (1.1a)-(1.1c) as follows:

\[
(1 + r^2_j) \delta^2_{x} U^n_j - 2 \delta_x U^n_j + i \delta_t U^n_j + |U^n_j|^2 U^n_j
\]

\[
+ f(x_j)U^n_j + \lambda r^2 \delta^2_{x} U^n_j = 0, \quad 1 \leq n \leq N - 1, \quad 1 \leq j \leq M_x - 1, \tag{3.6a}
\]

\[
U^n_0 = u_0(x_j), \quad 1 \leq j \leq M_x - 1, \tag{3.6b}
\]

\[
U^n_j = U^n_0 + \tau u_1(x_j) + \frac{\tau^2}{2} [(U^n_0)_{xx} - i u_1(x_j) - |U^n_0|^2 U^n_0 - f(x_j) U^n_0], \quad 1 \leq j \leq M_x - 1, \tag{3.6c}
\]

\[
U^n_0 = U^n_{M_x} = 0, \quad 0 \leq n \leq N. \tag{3.6d}
\]

Also, this type of stabilized invariants-preserving Du Fort-Frankel-type FDMs have a truncation error in the form of \(\mathcal{O}(r^2 + h^2 + (\tau/h)^2)\).

### 3.2. The discrete conservation laws

To discuss the discrete conservation laws, we firstly give Lemma 3.1.
Lemma 3.1. Let grid functions $U^n, U^{n+1} \in T^0_h$. Then the equalities
\begin{align*}
\text{Im} \left\{ \frac{\tau}{h_x^2} \left[ (U_{j+1}^n + U_j^n)U_{j+1}^{n+1} \right] \right\} \\
= \tau \text{Im} \left[ (\delta^2 U^n_j)U_j^{n+1} \right] - 2 \left( \frac{\tau}{h_x^2} \right)^2 \text{Im} \left[ (\delta U_j^{n+\frac{1}{2}})U_j^{n+1} \right], \tag{3.7a}
\end{align*}

\begin{align*}
\frac{1}{\tau} \text{Im}(U_j^{n+1}U^n_j) = \text{Im} \left[ (\delta U_j^{n+\frac{1}{2}})U_j^{n+\frac{1}{2}} \right] 
\end{align*}

hold.

Proof. From $\text{Im}(U_j^{n+1}U^n_j) = 0$, it follows that
\begin{align*}
\text{Im} \left\{ \frac{\tau}{h_x^2} \left[ (U_{j+1}^n + U_j^n)U_{j+1}^{n+1} \right] \right\} \\
= \text{Im} \left\{ \frac{\tau}{h_x^2} \left[ (U_{j+1}^n - 2U_j^n + U_{j-1}^n) - 2(U_j^{n+1} - U_j^n) \right]U_j^{n+1} \right\} \\
= \tau \text{Im} \left[ (\delta^2 U^n_j)U_j^{n+1} \right] - 2 \left( \frac{\tau}{h_x^2} \right)^2 \text{Im} \left[ (\delta U_j^{n+\frac{1}{2}})U_j^{n+1} \right].
\end{align*}
The proof of (3.7a) is finished. Furthermore, using
\begin{align*}
\frac{1}{\tau} \text{Im} \left( (U_j^{n+1})^2 + U_j^{n+1}U^{n+1}_j - U_j^nU_j^{n+1} - |U_j^n|^2 \right) = \text{Im} \left[ (\delta U_j^{n+\frac{1}{2}})U_j^{n+\frac{1}{2}} \right]
\end{align*}
directly derives that (3.7b) holds. \hfill \Box

\begin{align*}
\text{Theorem 3.1. Let the discrete mass and energy of the IP-DFFT-FDM-I (3.5a)-(3.5d) be defined as follows:}
\end{align*}
\begin{align*}
Q^n_l &= \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} - \frac{\tau}{h_x^2} \text{Im} \left\{ h_x \sum_{j=1}^{M_x-1} \left[ (U^n_{j+1} + U^n_{j-1})U^{n+1}_j \right] \right\} \\
&+ \frac{2}{\tau} \text{Im}(U^n, U^{n+1}) + c, \tag{3.8a}
\end{align*}
\begin{align*}
E^n_l &= (1 + r_2^2)\|\delta U^{n+\frac{1}{2}}\|^2 + \text{Re}(\delta U^n, \delta U^{n+1}) + \frac{1}{2}(\|U^n\|^2\|U^{n+1}\|^2, 1) \\
&+ \left\langle f, \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} \right\rangle + c \tag{3.9a}
\end{align*}
\begin{align*}
&= \|\delta U^{n+\frac{1}{2}}\|^2 + \|U^{n+\frac{1}{2}}\|^2_{H^1} + \frac{1}{2} \left\langle \left( \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} \right)^2, 1 \right\rangle \\
&+ \left\langle f, \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} \right\rangle + \left( \frac{\tau}{h_x^2} \right)^2 \|\delta U^{n+\frac{1}{2}}\|^2 - \frac{\tau^2}{4} \|\delta U^{n+\frac{1}{2}}\|^2_{H^1} \\
&- \frac{\tau^2}{8} \left\langle \left( \frac{\|U^{n+1}\|^2 - \|U^n\|^2}{\tau} \right)^2, 1 \right\rangle + c, \quad n = 0, 1, 2, \ldots, N - 1, \tag{3.9b}
\end{align*}
respectively. Then they are both conservative. Namely, \( Q_1^n = Q_1^{n-1} = \cdots = Q_1^0 \) and \( E_1^n = E_1^{n-1} = \cdots = E_1^0 \). Here, \( \epsilon \) is an arbitrary constant.

**Proof.** Multiplying \( h_x U_j^n \) to both sides of (3.5a), and summing in \( j \) from 1 to \( M_x - 1 \), and taking the imaginary part, then using the discrete Green formula, we have

\[
\text{Im}(\delta^2 U^n, U^n) - \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \frac{U^n_{j+1} - (U^n_{j-1} + U^n_{j+1}) + U^n_{j-1} U^n_{j+1}}{h_x^2} \right] + \text{Re}(\delta U^n, U^n) = 0. \tag{3.10}
\]

By direct computations and noting \( \text{Im}(\tilde{a}) = -\text{Im}(\alpha) \), we have

\[
\text{Re}(\delta U^n, U^n) = \frac{1}{2\tau} \left( \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} - \frac{\|U^n\|^2 + \|U^{n-1}\|^2}{2} \right), \tag{3.11}
\]

\[
\text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \frac{U^n_{j+1} - (U^n_{j-1} + U^n_{j+1}) + U^n_{j-1} U^n_{j+1}}{h_x^2} \right] = \text{Im} \left[ \frac{1}{2h_x} \sum_{j=1}^{M_x-1} \left( U^n_{j+1} U^n_{j+1} + U^n_{j-1} U^n_{j+1} + U^n_{j+1} U^n_{j-1} + U^n_{j-1} U^n_{j-1} \right) \right] \]

\[
= \frac{1}{2h_x} \text{Im} \left[ \sum_{j=1}^{M_x-1} \left( U^n_{j+1} U^n_{j+1} + U^n_{j-1} U^n_{j+1} \right) \right] + \frac{1}{2h_x} \text{Im} \left[ \sum_{j=1}^{M_x-1} \left( U^n_{j+1} U^n_{j-1} + U^n_{j+1} U^n_{j+1} \right) \right] \]

\[
= \frac{1}{2h_x} \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \left( U^n_{j+1} U^n_{j+1} + U^n_{j-1} U^n_{j+1} \right) \right] - \frac{1}{2h_x} \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \left( U^n_{j+1} U^n_{j+1} + U^n_{j-1} U^n_{j+1} \right) \right], \tag{3.12}
\]

and

\[
\text{Im}(\delta^2 U^n, U^n) = \frac{1}{2\tau} \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \left( U^{n+1} - 2U^n + U^{n-1} \right) \left( U^n + \frac{1}{U^n} \right) \right] \]

\[
= -\frac{1}{\tau} \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \left( U^n U^n + U^n U^n \right) \right] \]

\[
= \frac{1}{\tau} \text{Im} \left[ h_x \sum_{j=1}^{M_x-1} \left( U^n U^n - U^n U^n \right) \right]. \tag{3.13}
\]

Therefore, by the substitutions of (3.11)-(3.13) into (3.10), we have 0.5 \( \delta_1 Q_1^{n-1/2} = 0 \), and \( Q_1^n = Q_1^{n-1} = \cdots = Q_1^0 \), in which \( Q_1^n \) is defined by (3.8a). By using Lemma 3.1, it is easily shown that \( Q_1^n \) can be determined by (3.8b) as well.

Multiplying \( h_x \delta U_j^n \) to both sides of (3.5a), summing in \( j \) from 1 to \( M_x - 1 \), taking the real part, then using the discrete Green formula, one has that

\[
\frac{1}{2\tau} \left( 1 + h_x^2 \right) \left( \|\delta U^{n+\frac{1}{2}}\|^2 - \|\delta U^{n-\frac{1}{2}}\|^2 \right) + \frac{1}{2\tau} \left[ \text{Re}(\delta U^n, \delta U^{n+1}) - \text{Re}(\delta U^{n-1}, \delta U^n) \right] \]

\[
+ \frac{1}{4\tau} \left[ h_x \sum_{j=1}^{M_x-1} \left( U_j^n \right)^2 |U_j^{n+1}|^2 - h_x \sum_{j=1}^{M_x-1} \left| U_j^{n-1} \right|^2 |U_j^n|^2 \right] \]

\[
+ \frac{1}{4\tau} \left[ h_x \sum_{j=1}^{M_x-1} f(x_j) \left( |U_j^n|^2 + |U_j^{n+1}|^2 \right) - h_x \sum_{j=1}^{M_x-1} f(x_j) \left( |U_j^{n-1}|^2 + |U_j^n|^2 \right) \right] = 0. \tag{3.14}
\]
From (3.14) and the definition of $E^n_1$ (3.9a), it follows that $\delta_t E^n_1/2 = 0$ and $E^n_1 = E^n_{-1} = \cdots = E^n_0$. Besides, the equality $\text{Re}(a\bar{b}) = |a + b|^2 - |a - b|^2)/4$ shows that $E^n_0$ is also defined by (3.9b).

**Theorem 3.2.** Let the discrete mass and energy of the IP-DFFT-FDM-II (3.6a)-(3.6d) be defined as follows:

$$Q^n_2 = \frac{||U^{n+1}||^2 + ||U^n||^2}{2} - \frac{\tau}{h_x^2} \text{Im} \left\{ h_x \sum_{j=1}^{M_x-1} \left[ \frac{1}{2} (U^n_{j} + U^n_{j+1}) \right] \right\} + \frac{2}{\tau} (1 + \lambda \tau^2) \text{Im}(U^{n+1}, U^n) + c$$

$$= \frac{||U^{n+1}||^2 + ||U^n||^2}{2} + 2(1 + \lambda \tau^2) \text{Im}(\delta_t U^n, U^n) - \tau \text{Im}(\delta^2 U^n, U^{n+1}) + 2 \left( \frac{\tau}{h_x^2} \right)^2 \text{Im}(\delta_t U^n, U^n) + c, \quad n = 0, 1, 2, \ldots, N - 1,$$

$$E^n_2 = (1 + r^2 + \lambda \tau^2) ||\delta_t U^n + \frac{1}{2}||^2 + \text{Re}(\delta_t U^n, \delta_t U^{n+1}) + \frac{1}{2} h_x \sum_{j=1}^{M_x-1} |U^n_j|^2 |U^{n+1}_j|^2$$

$$+ \frac{1}{2} h_x \sum_{j=1}^{M_x-1} f(x_j) \left[ ||U^{n+1}_j||^2 + ||U^n_j||^2 \right] + c$$

$$= \left[ 1 + \lambda \tau^2 + \left( \frac{\tau}{h_x} \right)^2 \right] ||\delta_t U^n + \frac{1}{2}||^2 + \frac{1}{2} \left( \frac{||U^{n+1}_j||^2 + ||U^n_j||^2}{2} \right)^2 + 1$$

$$\left( \frac{f, \frac{||U^{n+1}_j||^2 + ||U^n_j||^2}{2}}{2} \right) - \frac{\tau}{4} \delta_t U^n + \frac{1}{4} ||\delta_t U^n + \frac{1}{2}||^2$$

$$- \frac{\tau}{8} \left( \left( \frac{||U^{n+1}_j||^2 - ||U^n_j||^2}{2} \right)^2 \right), \quad n = 0, 1, 2, \ldots, N - 1,$$

respectively. Then the discrete mass $Q^n_2$ and energy $E^n_2$ are both invariant. Namely, $Q^n_2 = Q^n_{-1} = \cdots = Q^n_0$ and $E^n_2 = E^{n-1}_2 = \cdots = E^0_2$. Here, $c$ is an arbitrary constant. Furthermore, as $\lambda \geq 0.5 \|f\|_{\infty}$ and $c \geq 2\lambda^2 (X_0 - X_1)$, we have that

$$E^0_2 = E^n_2 \geq ||\delta_t U^n + \frac{1}{2}||^2 + ||U^n + \frac{1}{2}||^2_{H^1} + \frac{1}{2} ||U^n||^{2}_{H^1} - 2\lambda ||U^n|| \geq 0, \quad n = 0, 1, 2, \ldots, N - 1.$$

**Proof.** Similar to the proof of Theorem 3.1, multiplying $h_x \overline{U^n_j}$ to both sides of (3.6a), summing in $j$ from 1 to $M_x - 1$, taking the imaginary part, then using the discrete Green formula, one has $Q^n_2 = Q^n_{-1} = \cdots = Q^n_0$, $n = 1, 2, \ldots, N - 1$ as well, in which $Q^n_2$ is defined by (3.15a). Beside, by Lemma 3.1, it is easy to find that $Q^n_2$ is also given by (3.15b).

Also, similar to the proof of Theorem 3.1, multiplying both sides of (3.6a) by $h_x \overline{U^n_j}$, summing in $j$ from 1 to $M_x - 1$, and taking the real part, then using the discrete Green formula, we can deduce that

$$E^n_0 = E^n_{-1} = \cdots = E^n_2, \quad n = 1, 2, \ldots, N - 1,$$

where $E^n_2$ is defined by (3.16a). Also, by using $\text{Re}(a\bar{b}) = |a + b|^2 - |a - b|^2)/4$, we can derive that (3.16a) is equivalent to (3.16b).
By using \( \text{Re}(ab) = |a + b|^2 - |a - b|^2/4 \) and

\[
|U_j^{n} U_j^{n+1}|^2 - 4 \lambda \text{Re}(U_j^{n} \overline{U_j^{n+1}}) + 4 \lambda^2 = |U_j^{n} U_j^{n+1} - 2\lambda|^2,
\]

we have

\[
E_2^0 = E_2^n = (1 + r_x^2) \| \delta U^{n+\frac{1}{2}} \|^2 + |U^{n+\frac{1}{2}}|_{H^1}^2 - \frac{\tau^2}{4} \| \delta U^{n+\frac{1}{2}} \|^2_{H^1} + \lambda \left( \frac{\tau}{h_x} \right)^2 \left( |U_j^{n+1}|^2 + |U_j^n|^2 \right) + \frac{1}{2} \| U_j^{n} U_j^{n+1} - 2\lambda \|^2
\]

\[
- 2\lambda^2 (M_x - 1) h_x + c.
\]

Besides, by using Lemma 2.4, we have

\[
r_x^2 \| \delta U^{n+\frac{1}{2}} \|^2 - \frac{\tau^2}{4} \| \delta U^{n+\frac{1}{2}} \|^2_{H^1} = \frac{\tau}{h_x} \left( |U_j^{n+1}|^2 + |U_j^n|^2 \right) + \frac{1}{2} \| U_j^{n} U_j^{n+1} - 2\lambda \|^2.
\]

Thus, using (3.19), \( \lambda \geq 0.5 ||f||_{\infty} \) and \( c \geq 2\lambda^2 (X_r - X_l) \) to (3.18) shows that (3.17) is valid. \( \square \)

**Remark 3.1.** From (3.8b), (3.15b), (3.9b) and (3.16b), it follows that \( Q_l^n = 0 \) and \( E_l^n \), \( l = 1, 2 \) are the approximations of the exact mass \( Q(t) \) and energy \( E(t) \) at \( t_{n+1/2} \), \( n = 0, 1, 2, \ldots, N - 1 \) as \( \tau, h_x, h_y, \tau/h_x \) and \( \tau/h_y \) tend to zero.

### 3.3 Nonlinearly stable analyses

**Theorem 3.3.** IP-DFFT-FDM-II (3.6a)-(3.6d) are stable in the \( H^1 \)- and \( L^\infty \)-norms provided the parameter \( \lambda \geq 0.5 ||f||_{\infty} \) and \( \tau = O(h_x) \). Besides, IP-DFFT-FDM-II (3.6a)-(3.6d) are unconditionally stable in the \( L^2 \)-norm as long as the parameter \( \lambda \geq 0.5 ||f||_{\infty} \). Furthermore, as \( \lambda > 0.5 ||f||_{\infty} \) and \( c \geq 2\lambda^2 (X_r - X_l) \), an estimation in the \( L^2 \)-norm is derived as

\[
\max_{0 \leq n \leq N} \| U^n \|_{H^1} \leq \min \left\{ \sqrt{2 (\| U^n \|^2 + E_2^n T^2)}, \sqrt{E_2^n / \lambda - 0.5 \| f \|_{\infty}} \right\}, \quad (3.20)
\]

**Proof.** In the proof of this theorem, the \( c \) in \( E_2^n \) satisfies \( c \geq 2\lambda^2 (X_r - X_l) \). By using Lemma 2.2 and (3.17), we have that

\[
\| U^{n+1} \|^2_{H^1} \leq 2 \| U^{n+\frac{1}{2}} \|^2_{H^1} + 2 \| \delta U^{n+\frac{1}{2}} \|^2_{H^1} \leq \max \left\{ 2, 2 r_x^2 \right\} \left( |U_j^{n+1}|^2_{H^1} + \| \delta U^{n+\frac{1}{2}} \|^2_{H^1} \right) \leq \max \left\{ 2, 2 r_x^2 \right\} E_2^n, \quad n = 0, 1, \ldots, N - 1.
\]

Thus, by using Lemma 2.3 and (3.21), we obtain

\[
\max_{0 \leq n \leq N} \{ \| U^n \|_{H^1}, \| U^n \|_{H^1} \} \leq c_0.
\]
in which
\[ c_0 = \max \left\{ \|U^0\|_{H^s}, \|U^0\|_\infty, \max \left( 1, \frac{\sqrt{X_r - X_l}}{2} \right) \sqrt{\max(2, 2r_x^2)} \right\}. \]

Namely, as \( \lambda \geq 0.5\|f\|_\infty \) and \( \tau = \mathcal{O}(h_x) \), IP-DFFT-FDM-II (3.6a)-(3.6d) are stable in the \( H^1 \) and \( L^\infty \)-norms.

Besides, using Lemma 2.5 and (3.17), we have that
\[
\|U^n\|^2 \leq 2\|U^0\|^2 + 2\tau \sum_{i=1}^{n} \|\delta_i U^i \|^2 \leq 2\|U^0\|^2 + 2\tau \sum_{i=1}^{n} E^0_i \leq 2\left(\|U^0\|^2 + E^0_0 T^2\right), \quad n = 1, 2, \ldots, N,
\]
which implies that
\[
\max_{0 \leq n \leq N} \|U^n\| \leq \sqrt{2\left(\|U^0\|^2 + E^0_0 T^2\right)}. \tag{3.22}
\]
Hence, IP-DFFT-FDM-II (3.6a)-(3.6d) are unconditionally stable in the \( L^2 \)-norm as long as \( \lambda \geq 0.5\|f\|_\infty \). By using (3.19), \( \lambda > 0.5\|f\|_\infty \) and \( c \geq 2\lambda^2(X_r - X_l) \) in (3.18), we have
\[
\|U^{n+1}\|^2 + \|U^n\|^2 \leq \frac{E^0_2}{\lambda - 0.5\|f\|_\infty} = \frac{E^0_2}{\lambda - 0.5\|f\|_\infty}. \tag{3.23}
\]
A combination of (3.22) and (3.23) yields (3.20).

### 3.4. Convergence analyses

In this section, we concentrate on the convergence analyses of the numerical schemes. Let
\[
(e_1)^n_j = u^n_j - U^n_j, \quad 0 \leq j \leq M_x, \quad 0 \leq n \leq N.
\]
Then subtracting (3.5a) and (3.5c) from (3.2) and (3.4) yields the error equations as follows:
\[
(1 + r_x^2)\delta_j^2(e_1)^n_j - \delta_j^2(e_1)^n_j + i\delta_j^2(e_1)^n_j + |u^n_j|^2 u^n_j - |U^n_j|^2 U^n_j + f(x_j)(e_1)^n_j = (R_1)^n_j, \quad 1 \leq j \leq M_x - 1, \quad 1 \leq n \leq N - 1, \tag{3.24a}
\]
\[
(e_1)^0_j = 0, \quad 0 \leq j \leq M_x, \tag{3.24b}
\]
\[
(e_1)^n_j = \mathcal{O}(\tau^2), \quad 0 \leq j \leq M_x, \tag{3.24c}
\]
\[
(e_1)^n_{M_x} = (e_1)^n_{M_x} = 0, \quad 0 \leq n \leq N. \tag{3.24d}
\]

In what follows, we introduce some lemmas used in the subsequent proofs.

**Lemma 3.2.** Denote
\[
G^n = (1 + r_x^2)\|\delta_j(e_1)^{n+\frac{1}{2}}\|^2 + \text{Re}<\delta_j(e_1)^n, \delta_j(e_1)^{n+1} >.
\]
Then it holds that
\[
\|\delta_j(e_1)^{n+\frac{1}{2}}\|^2 + \|\delta_j(e_1)^{n+\frac{1}{2}}\|^2_{H^1} \leq G^n, \tag{3.25a}
\]
\[
\|\delta_j(e_1)^{n+\frac{1}{2}}\|^2_{H^1} + \|\delta_j(e_1)^{n+\frac{1}{2}}\|^2 \leq \max(2, 1 + 2r_x^2)G^n. \tag{3.25b}
\]
Proof. Applying the equality \( \text{Re}(ab) = (|a| + |b|^2 - |a - b|^2)/4 \) and Lemma 2.3, one gets
\[
G^n = (1 + r_x^2) \| \delta(x) e^{it} \|_H^2 + \| e^{it} \|_H^2 - \frac{\tau^2}{4} \| \delta(x) e^{it} \|_H^2,
\]
\[
= \| \delta(x) e^{it} \|_H^2 + \| e^{it} \|_H^2 - \frac{\tau^2}{4} \| \delta(x) e^{it} \|_H^2,
\]
\[
\geq \| \delta(x) e^{it} \|_H^2.
\]
By using Lemma 2.2, we can derive that
\[
\| (e^{it})^{n+1} \|_H^2 + \| \delta(x) e^{it} \|_H^2 \leq 2 \| (e^{it})^{n+1} \|_H^2 + (1 + 2r_x^2) \| \delta(x) e^{it} \|_H^2
\]
\[
\leq \max \{ 2, 1 + 2r_x^2 \} \left[ \| (e^{it})^{n+1} \|_H^2 + \| \delta(x) e^{it} \|_H^2 \right]
\]
\[
\leq \max \{ 2, 1 + 2r_x^2 \} G^n.
\]
The proof is complete. \( \square \)

Suppose that the exact solutions of the 1D problem (1.1a)-(1.1c) \( u(x, t) \in C^{4,4}([X_l, X_r] \times \{0, T\}) \) \( f(x) \in C([X_l, X_r]). \) Then, we can assume that there exist positive numbers \( c_1, c_2 \) and \( M_1 \) such that
\[
\max_{0 \leq n \leq N-1} \| (R_1^n) \| \leq c_1 \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right],
\]
\[
G_0 \leq c_2 \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right],
\]
\[
\| u \|_\infty \leq M_1.
\]

**Theorem 3.4.** Assume that \( u(x, t) \in C^{4,4}([X_l, X_r] \times \{0, T\}) \) is the exact solutions to the 1D problem (1.1a)-(1.1c). Then, under the conditions of the assumptions (3.26a)-(3.26c),
\[
\tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \leq \frac{2(M_1^n - M_1)}{4 + \tau \sqrt{X_r - X_l}}
\]
and \( \tau \leq 1/(2c_3) \) the error estimations of the IP-DFFT-FDM-I (3.5a)-(3.5d) for 1D problem (1.1a)-(1.1c) are given as follows:
\[
\max_{0 \leq n \leq N} \| (e^{it})^{n+1} \|_H \leq c_4 \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right],
\]
\[
\max_{0 \leq n \leq N} \| U^n \|_\infty \leq M_1^*,
\]
\[
\max_{0 \leq n \leq N} \| (e^{it})^{n+1} \|_\infty \leq \frac{c_4 \sqrt{X_r - X_l}}{2} \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right],
\]
in which \( M_1^* \) is a positive constant, bigger than \( M_1 \) and independent of grid parameters \( \tau \) and \( h_x \),
\[
c_3 = 2 \max \{ 2, 1 + 2r_x^2 \} \max \left\{ \frac{1}{2} \left[ M_1^2 + M_1 M_1^* + (M_1^*)^2 + \| f \|_\infty \right], \frac{1}{2} \left[ M_1^2 + M_1 M_1^* + (M_1^*)^2 + \| f \|_\infty + 1 \right] \right\},
\]
\[
c_4 = \sqrt{\max \{ 2, 1 + 2r_x^2 \} (c_1 T + c_2) \exp(2c_3 T)}.
\]
In what follows, each term at the right of (3.29) is estimated step by step. Using Cauchy-Schwarz Lemma 2.3, we have that

\[ \| U^n \|_\infty \leq \| u^n \|_\infty + \| (e_1)^n \|_\infty \leq M_1 + \frac{\sqrt{X_r - X_f}}{2} |(e_1)^n|_{H^1} \]
\[ \leq M_1 + \frac{\sqrt{X_r - X_f}}{2} c_4 \left( \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right) \leq M'_1, \quad n = 0, 1, 2, \ldots, K - 1. \] (3.28)

Multiplying \( h_x \delta_1^n \) to both sides of (3.24a), summing in \( j \) from 1 to \( M_x - 1 \), taking the real part, using the discrete Green formula and Lemma 3.2 yield that

\[ \frac{G^n - G^{n-1}}{2\tau} = -\text{Re}(\| u^n \|^2 u^n - |U^n|^2 U^n, \delta_1^n) - \text{Re}(f(e_1)^n, \delta_1^n) \]
\[ + \text{Re}((R_1)^n, \delta_1^n). \] (3.29)

In what follows, each term at the right of (3.29) is estimated step by step. Using Cauchy-Schwarz inequality and the inequality \( \text{Re}(ab) \leq (|a|^2 + |b|^2)/2 \), we can find that

\[-\text{Re}(\| u^n \|^2 u^n - |U^n|^2 U^n, \delta_1^n) \]
\[ = -\text{Re}(\langle u^n \rangle^2 u^n + U^n u^n(1) + |U^n|^2 (e_1)^n, \delta_1^n) \]
\[ \leq M_1^2 \| (e_1)^n \|_1^2 + \| (e_1)^n-1 \|_1^2 \]
\[ \leq \left( \frac{M_1^2}{2} + \frac{\| f \|_\infty}{2} \right) \left[ \| (e_1)^n \|_1^2 + \| (e_1)^n-1 \|_1^2 \right]. \] (3.30)

\[ -\text{Re}(f(e_1)^n, \delta_1^n) \leq \| f \|_\infty \langle (e_1)^n, \delta_1^n \rangle \]
\[ \leq \| f \|_\infty \left[ \frac{1}{2} \left( \| (e_1)^n+1 \| + \| (e_1)^n-1 \| \right) \right]. \] (3.31)

\[ \text{Re}((R_1)^n, \delta_1^n) \]
\[ \leq \left[ \frac{1}{2} \left( \| (e_1)^n \|_2^2 + \| (e_1)^n-1 \|_2^2 \right) \right]. \] (3.32)

Substituting (3.30)-(3.32) into (3.29) yields that

\[ \frac{G^n - G^{n-1}}{2\tau} \leq \frac{M_1^2 + M_1 M_1^2}{2} \| (e_1)^n \|_2^2 \]
\[ + \frac{1}{2} \left[ \frac{M_1^2}{2} + \frac{\| f \|_\infty}{2} \right] \left[ \| (e_1)^n+1 \|_2^2 + \| (e_1)^n-1 \|_2^2 \right] \]
\[ + \frac{1}{4} \left[ M_1^2 + M_1 M_1^2 + (M_1^2) \right] \left[ \| (e_1)^n+\frac{1}{2} \|_2^2 + \| (e_1)^n-\frac{1}{2} \|_2^2 \right] \]
\[ + \frac{1}{2} \left[ (R_1)^n \right]^2, \quad n = 0, 1, 2, \ldots, K - 1. \] (3.33)
Multiplying $2\tau\max\{2, 1 + 2r_x^2\}$ to both sides of (3.33), replacing $n$ with $l$, summing in $l$ from 1 to $n$, applying Lemmas 2.3 and 3.2 gives that

$$
\|(e_1)^{n+1}\|_{H^1} + \|\delta_t(e_1)^{n+\frac{1}{2}}\|^{2} \leq \max \{2, 1 + 2r_x^2\}G^n
$$

$$
\leq \max \{2, 1 + 2r_x^2\}G^0 + \tau c_3 \sum_{i=1}^{n+1} \|((e_1)^i_{H^1} + \|\delta_t(e_1)^{i-\frac{1}{2}}\|^{2} + \max \{2, 1 + 2r_x^2\}\tau \sum_{i=1}^{n} \|((R_1)^i\|^2],
$$

(3.34)

which is used together with Lemma 2.1 to infer that

$$
\|(e_1)^{n+1}\|_{H^1} + \|\delta_t(e_1)^{n+\frac{1}{2}}\|^{2} \leq \frac{c_4\tau^2 + h_x^2 + (\frac{\tau}{h_x})^2}{c_4\sqrt{X_r - X_l}},
$$

(3.35)

Thus, we can prove that (3.27a) is valid for $n = K$ as $n = K - 1$ is taken in (3.35).

Besides, similar to (3.28), applying

$$
\tau^2 + h_x^2 + (\frac{\tau}{h_x})^2 \leq \frac{2(M_1^+ - M_1)}{c_4\sqrt{X_r - X_l}},
$$

and Lemma 2.3, we can obtain that

$$
\|U^K\|_{\infty} \leq \|u^K\|_{\infty} + \|(e_1)^K\|_{\infty} \leq M_1 + \frac{\sqrt{X_r - X_l}}{2}(e_1)^K_{H^1},
$$

$$
\leq M_1 + \frac{\sqrt{X_r - X_l}}{2} c_4 \left[\tau^2 + h_x^2 + (\frac{\tau}{h_x})^2\right] \leq M_1^+,
$$

(3.36)

which shows (3.27b) holds for $n = K$. Therefore, (3.27a)-(3.27b) are available by mathematical induction. Finally, (3.27c) is directly obtained by the use of Lemma 2.3 to (3.27a).

Let $(e_1)^n_j = u^n_j - U^n_j$, $0 \leq j \leq M_x$, $0 \leq n \leq N$. Then using the analytical method similar to (3.24a)-(3.24d), the error equations of IP-DFFT-FDM-II (3.6a)-(3.6d) are derived as follows:

$$
(1 + r_x^2 + \lambda \tau^2)\delta_t^2(e_1)^n_j - \delta_x^2(e_1)^n_j + i\delta_t(e_1)^n_j + |u^n_j|^2 u^n_j - |U^n_j|^2 U^n_j
$$

$$
+ f(x_j)(e_1)^n_j = (R_2)^n_j, \quad 1 \leq j \leq M_x - 1, \quad 1 \leq n \leq N - 1,
$$

(3.37a)

$$
(e_1)^0_j = 0, \quad 0 \leq j \leq M_x,
$$

(3.37b)

$$
(e_1)^1_j = (R_2)^0_j, \quad 0 \leq j \leq M_x,
$$

(3.37c)

$$
(e_1)^n_j = (e_1)^n_{M_x} = 0, \quad 0 \leq n \leq N,
$$

(3.37d)

in which

$$
(R_2)^0_j = O(\tau^2), \quad (R_2)^n_j = O\left(\tau^2 + h_x^2 + \left(\frac{\tau}{h_x}\right)^2\right), \quad 1 \leq n \leq N - 1.
$$

Using the same analytical method as Lemma 3.2 yields the following Lemma 3.3.

**Lemma 3.3.** Denoting

$$
\hat{G}^n = (1 + r_x^2 + \lambda \tau^2)\|\delta_t(e_1)^n + \frac{1}{2}\|^2 + \Re\{\delta_x(e_1)^n, \delta_x(e_1)^{n+1}\},
$$

as $\lambda \geq 0$, it also holds that

$$
\|(e_1)^{n+1}\|_{H^1} + \|\delta_t(e_1)^{n+\frac{1}{2}}\|^{2} \leq \max \{2, 1 + 2r_x^2\}\hat{G}^n.
$$
Suppose that the exact solution of the 1D problem (1.1a)-(1.1c) \( u(x,t) \in C^{4,4}([X_1, X_r] \times [0, T]) \) and \( f(x) \in C([X_1, X_r]) \). Then, we can assume that there exist positive constants \( c_5 \) and \( c_6 \), such that

\[
\max_{0 \leq n \leq N-1} \| (R_2)^n \|^2 \leq c_5 \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right]^2, \tag{3.39a}
\]

\[
\tilde{G}^0 \leq c_6 \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right]^2. \tag{3.38b}
\]

**Theorem 3.5.** Assume that the exact solution of the 1D problem (1.1a)-(1.1c) is the exact solutions to the 1D problem (1.1a)-(1.1c). Then, under the conditions of the assumptions (3.38a)-(3.38b), (3.26c), \( \lambda \geq 0.5 \| f \|_\infty \) and \( \tau \leq 1/(2c_7) \), the following error estimations of the IP-DFFT-FDM-II (3.6a)-(3.6d) for 1D problem (1.1a)-(1.1c)

\[
\max_{0 \leq n \leq N} \| (e_1)^n \|_{H^1} \leq c_8 \sqrt{\frac{X_r - X_l}{2}} \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right]^{3/2}, \tag{3.39a}
\]

\[
\max_{0 \leq n \leq N} \| (e_1)^n \|_\infty \leq \frac{c_7 \sqrt{X_r - X_l}}{2} \left[ \tau^2 + h_x^2 + \left( \frac{\tau}{h_x} \right)^2 \right]^{3/2}. \tag{3.39b}
\]

hold. Here,

\[
c_7 = 2 \max \{ 2, 1 + 2r_x^2 \} \max \left\{ 1 + \frac{1}{2}, M^2_1 + M^2_1c_0 + c_0^2 + \| f \|_\infty \frac{(X_r - X_l)^2}{6}, \right. \]

\[
\left. \frac{1}{2}M^2_1 + M_1c_0 + (c_0)^2 + \| f \|_\infty + 1 \right\},
\]

\[
c_8 = \sqrt{2} \max \{ 2, 1 + 2r_x^2 \} (c_5T + c_6) \exp(2c_7T).
\]

**Proof.** As Theorem 3.3 implies that the numerical solutions of IP-DFFT-FDM-II (3.6a)-(3.6d) are bounded as \( \tau = O(h_x) \) and \( \lambda \geq 0.5 \| f \|_\infty \), the convergent results claimed in Theorem 3.4 can be derived without the use of mathematical induction.

Similar to (3.30)-(3.32), multiplying \( h_x \delta_t(e_1)^n \) to both sides of (3.37a), summing in \( j \) from 1 to \( M_x - 1 \), taking the real part, using the discrete Green formula and Lemma 3.3, then, multiplying \( 2r \max \{ 2, 1 + 2r_x^2 \} \) to both sides of the obtained result, replacing \( n \) with \( l \), summing in \( l \) from 1 to \( n \), applying Lemmas 2.3 and 3.3, we can deduce that

\[
\| (e_1)^{n+1} \|_{H^1} + \| \delta_t(e_1)^{n+\frac{1}{2}} \|_\infty^2 \leq \max \{ 2, 1 + 2r_x^2 \} \tilde{G}^0
\]

\[
\leq \max \{ 2, 1 + 2r_x^2 \} \tilde{G}^0 + 2r c_7 \sum_{i=1}^{n+1} \| (e_1)^i \|_{H^1} + \| \delta_t(e_1)^{i-\frac{1}{2}} \|_\infty^2 \]

\[
+ \max \{ 2, 1 + 2r_x^2 \} \tau \sum_{i=1}^{n} \| (R_2)^i \|^2. \tag{3.40}
\]

The use of Lemma 2.1 to (3.40) gives (3.39a). A combination of (3.39a) with Lemma 2.3 derives (3.39b). \( \square \)
4. Invariants-Preserving Du Fort-Frankel Schemes for 2D Nonlinear Schrödinger Equations with Wave Operator

This section is devoted to the developments and analyses of the invariants-preserving Du Fort-Frankel schemes for 2D problem (1.1a)-(1.1c).

4.1. The developments of the invariants-preserving Du Fort-Frankel FDMs

To begin with, we introduce \( u_{j,k}^n = u(x_{j,k}, t_n) \) and \( f_{j,k} = f(x_{j,k}) \) \((x_{j,k}, t_n) \in \Omega_h \times \Omega_t\). Besides, \( U_{j,k}^n \) represents the approximation to \( u_{j,k}^n \).

Set \( r_x = \tau/h_x \) and \( r_y = \tau/h_y \). Similar to (3.2) and (3.4), a fully discrete FDM for 2D problem (1.1a)-(1.1c) is devised as follows:

\[
\begin{align*}
\frac{(1 + r_x^2 + r_y^2) \delta_t^2 U_{j,k}^n - \Delta_x U_{j,k}^n + i \delta_t U_{j,k}^n + |U_{j,k}^n|^2 U_{j,k}^n}{\tau^2} & + f(x_{j,k}) U_{j,k}^n = (R_3)_{j,k}^n, \\
\quad & 1 \leq n \leq N - 1, \quad x_{j,k} \in \Omega_h, \quad (4.1a) \\
U_{j,k}^0 & = u_0(x_{j,k}), \quad x_{j,k} \in \Omega_h, \quad (4.1b) \\
U_{j,k}^1 & = u_0(x_{j,k}) + \tau u_1(x_{j,k}) \\
& \quad + \frac{\tau^2}{2} [\Delta u_0(x_{j,k}) - i u_1(x_{j,k}) - |u_0(x_{j,k})|^2 u_0(x_{j,k})] - f(x_{j,k}) u_0(x_{j,k}) \quad x_{j,k} \in \Omega_h, \quad (4.1c) \\
U(x_{j,k}, t_n) & = 0, \quad 0 \leq n \leq N, \quad x_{j,k} \in \partial\Omega_h. \quad (4.1d)
\end{align*}
\]

Here, \((R_3)_{j,k}^n\), which is very similar to \((R_1)_{j,k}^n\), can be obtained by applying the Taylor formula with Lagrange remainder. Thus their expressions are omitted to avoid length.

Dropping small terms \((R_3)_{j,k}^n\) in (4.1a)-(4.1d), and replacing \( u_{j,k}^n \) with \( U_{j,k}^n \), then an invariants-preserving Du Fort-Frankel-type FDM, which is denoted by IP-DFFT-FDM-III, is devised as follows:

\[
\begin{align*}
\frac{(1 + r_x^2 + r_y^2) \delta_t^2 U_{j,k}^n - \Delta_x U_{j,k}^n + i \delta_t U_{j,k}^n + |U_{j,k}^n|^2 U_{j,k}^n}{\tau^2} & + f(x_{j,k}) U_{j,k}^n = 0, \\
\quad & 1 \leq n \leq N - 1, \quad x_{j,k} \in \Omega_h, \quad (4.2a) \\
U_{j,k}^0 & = u_0(x_{j,k}), \quad x_{j,k} \in \Omega_h, \quad (4.2b) \\
U_{j,k}^1 & = u_0(x_{j,k}) + \tau u_1(x_{j,k}) \\
& \quad + \frac{\tau^2}{2} [\Delta u_0(x_{j,k}) - i u_1(x_{j,k}) - |u_0(x_{j,k})|^2 u_0(x_{j,k})] - f(x_{j,k}) u_0(x_{j,k}) \quad x_{j,k} \in \Omega_h, \quad (4.2c) \\
U_{j,k}^n & = 0, \quad 0 \leq n \leq N, \quad x_{j,k} \in \partial\Omega_h. \quad (4.2d)
\end{align*}
\]

Besides, adding small term \( \theta \tau^2 \delta_t^2 U_{j,k}^n \) to the left of (4.2a), (here, \( \theta \) is a real constant), then a family of stabilized invariants-preserving Du Fort-Frankel-type FDMs, which are represented by IP-DFFT-FDM-IV, are devised as follows:

\[
\begin{align*}
\frac{(1 + r_x^2 + r_y^2 + \theta \tau^2) \delta_t^2 U_{j,k}^n - \Delta_x U_{j,k}^n + i \delta_t U_{j,k}^n + |U_{j,k}^n|^2 U_{j,k}^n}{\tau^2} & + f(x_{j,k}) U_{j,k}^n = 0, \\
\quad & 1 \leq n \leq N - 1, \quad x_{j,k} \in \Omega_h, \quad (4.3a) \\
U_{j,k}^0 & = u_0(x_{j,k}), \quad x_{j,k} \in \Omega_h, \quad (4.3b) \\
U_{j,k}^1 & = u_0(x_{j,k}) + \tau u_1(x_{j,k})
\end{align*}
\]
Invariants-Preserving Du Fort-Frankel Schemes for Nonlinear Schrödinger Equations with Wave Operator

\[ U^n_{j,k} = 0, \quad 0 \leq n \leq N, \quad x_{j,k} \in \partial \Omega_h. \]  

4.2. The discrete conservation laws

Using the same analytical methods as Lemma 3.1 directly derives Lemma 4.1.

Lemma 4.1. Let grid functions \( U^n, U^{n+1} \in \mathcal{D}_h^0 \). Then the relations

\[
\text{Im} \left\{ \frac{\tau}{h_x^2} \left[ (U^n_{j+1,k} + U^n_{j-1,k})U^{n+1}_{j,k} \right] \right\} = \tau \text{Im} \left[ (\delta_x U^n_{j,k})U^{n+1}_{j,k} \right] - 2 \left( \frac{\tau}{h_x} \right)^2 \text{Im} \left[ (\delta_u U^n_{j,k})U^{n+1}_{j,k} \right],
\]

\[
\text{Im} \left\{ \frac{\tau}{h_y} \left[ (U^n_{j,k+1} + U^n_{j,k-1})U^{n+1}_{j,k} \right] \right\} = \tau \text{Im} \left[ (\delta_y U^n_{j,k})U^{n+1}_{j,k} \right] - 2 \left( \frac{\tau}{h_y} \right)^2 \text{Im} \left[ (\delta_u U^n_{j,k})U^{n+1}_{j,k} \right],
\]

\[
\frac{1}{\tau} \text{Im} \left[ (U^{n+1} U^n_{j,k}) \right] = \text{Im} \left[ (\delta^U U^n_{j,k})U^{n+1}_{j,k} \right]
\]

hold.

Theorem 4.1. The discrete mass \( Q^m_3 \) and energy \( E^n_3 \) of the IP-DFFT-FDM-III (4.2a)-(4.2d) are defined as follows:

\[
Q^m_3 = \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} - \frac{\tau}{h_x^2} \text{Im} \left\{ \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left[ (U^n_{j+1,k} + U^n_{j-1,k})U^{n+1}_{j,k} \right] \right\}
\]

\[
- \frac{\tau}{h_y} \text{Im} \left\{ \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left[ (U^n_{j,k+1} + U^n_{j,k-1})U^{n+1}_{j,k} \right] \right\} + \frac{2}{\tau} \text{Im}(U^{n+1}, U^n) + c
\]

\[
E^n_3 = \left( 1 + r_x^4 + r_y^4 \right) \|\delta U^{n+1}\|^2 + \text{Re} \left( \delta_x U^n, \delta_x U^{n+1} \right) + \text{Re} \left( \delta_y U^n, \delta_y U^{n+1} \right)
\]

\[
+ \frac{1}{2} \left( \|U^n\|^2 \right) + \text{Re} \left( \left( \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} \right) \right)
\]

\[
= \|\delta^x U^{n+1}\|^2 + \|U^{n+1}\|^2 + \frac{1}{2} \left( \left( \|U^{n+1}\|^2 + \|U^n\|^2 \right)^2, 1 \right) + \text{Re} \left( \left( \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} \right) \right)
\]

\[
+ \left( \frac{\tau}{h_x^2} \right)^2 + \left( \frac{\tau}{h_y^2} \right)^2 \|\delta^y U^{n+1}\|^2 - \frac{\tau}{4} \|\delta U^{n+1}\|^2 \right)
\]

\[
- \frac{\tau}{4} \|\delta U^{n+1}\|^2 \right) + c, \quad n = 0, 1, 2, \ldots, N - 1,
\]

respectively. Then they are both invariant as well. Namely, \( Q^m_3 = Q^{m-1}_3 = \cdots = Q^0_3 \) and \( E^n_3 = E^{n-1}_3 = \cdots = E^0_3 \). Here, \( c \) is an arbitrary constant.
Proof. Multiplying $h_x h_y \overline{U^n_{j,k}}$ to both sides of (4.2a), summing in $j$ from 1 to $M_x - 1$ and $k$ from 1 to $M_y - 1$, taking the imaginary part, and using the discrete Green formula, we have

$$\text{Im} \langle \delta_t^2 U^n, U^n \rangle - \text{Im} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \frac{U^n_{j+1,k} - (U^n_{j,k} + U^n_{j+1,k} + U^n_{j-1,k})}{h_x^2} \right]$$

By direct computation, we have

$$\text{Re} \langle \delta_t U^n, U^n \rangle = \frac{1}{2\tau} \left( \frac{\|U^{n+1}\|^2 + \|U^n\|^2 - \|U^n\|^2}{2} - \frac{\|U^n\|^2}{2} \right).$$

$$\text{Re} \langle \delta_t U^n, U^n \rangle = \frac{1}{2}\text{Im} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \frac{U^n_{j+1,k} - (U^n_{j,k} + U^n_{j+1,k} + U^n_{j-1,k})}{h_x^2} \right]$$

$$= \frac{1}{2h_x^2} \text{Im} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left( U^n_{j+1,k} \overline{U^n_{j,k}} + U^n_{j-1,k} \overline{U^n_{j,k}} \right) \right].$$

$$\text{Im} \langle \delta_t^2 U^n, U^n \rangle = \frac{1}{2\tau} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left( U^n_{j+1,k} - 2U^n_{j,k} + U^n_{j-1,k} \right) \left( U^n_{j,k} + U^n_{j-1,k} \right) \right]$$

Therefore, the substitutions of (4.7)-(4.10) into (4.6) yields $0.5 \delta_t Q_3^{n+1/2} = 0$, and $Q_3^n = Q_3^{n-1} = \cdots = Q_3^0$, in which $Q_3^n$ is defined by (4.4a). Using Lemma 4.1, it is easy to find that (4.4a) is equal to (4.4b).

Multiplying $h_x h_y \delta_t U^n_{j,k}$ to both sides of (4.2a), summing in $j$ from 1 to $M_x - 1$, summing in $k$ from 1 to $M_y - 1$, using the discrete Green formula, taking the real part and noting the definition of $E_3^n$ (4.5a), one gets that

$$\frac{1 + \tau_x^2 + \tau_y^2}{2\tau} \left( \|\delta_t U^{n+1}\|^2 - \|\delta_t U^n\|^2 \right) + \frac{1}{2\tau} \left( \text{Re} \langle \delta_t U^n, \delta_x U^n \rangle - \text{Re} \langle \delta_t U^{n+1}, \delta_x U^n \rangle \right)$$
\[ + \frac{1}{2\tau} \left[ \text{Re}(\delta_y U^n, \delta_y U^{n+1}) - \text{Re}(\delta_y U^{n-1}, \delta_y U^n) \right] \\
+ \frac{1}{4\tau} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U^n_{j,k}|^2 |U^{n+1}_{j,k}|^2 - h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U^{n-1}_{j,k}|^2 |U^n_{j,k}|^2 \right] \\
+ \frac{1}{4\tau} \left[ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} f_{j,k} (|U^n_{j,k}|^2 + |U^{n+1}_{j,k}|^2) - h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} f_{j,k} (|U^{n-1}_{j,k}|^2 + |U^n_{j,k}|^2) \right] = 0, \]

which shows that \( 0.5 \delta_t E_3^{n-1/2} = 0 \), and \( E_3^n = E_3^{n-1} = \cdots = E_3^0 \). From the equality \( \text{Re}(ab) = ||a + b||^2 - ||a - b||^2)/4 \), it follows that (4.5a) is equivalent to (4.5b).

**Theorem 4.2.** The discrete mass \( Q_4^n \) and energy \( E_4^n \) of the IP-DFFT-FDM-IV (4.3a)-(4.3d) are defined as follows:

\[
Q_4^n = \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} - \frac{\tau}{h_x} \text{Im} \left\{ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left[ (U^n_{j+1,k} + U^n_{j-1,k}) \overline{U^{n+1}_{j,k}} \right] \right\} \\
- \frac{\tau}{h_y} \text{Im} \left\{ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left[ (U^n_{j,k+1} + U^n_{j,k-1}) \overline{U^{n+1}_{j,k}} \right] \right\} \\
+ \frac{2}{\tau} (1 + \theta^2) \|U^{n+1}, U^n\| + c \quad (4.11a)
\]

\[
= \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} + 2(1 + \theta^2) \text{Im} \langle \delta_t U^{n+1/2}, U^{n+1/2} \rangle - \tau \text{Im} \langle \Delta h U^n, U^{n+1} \rangle \\
+ 2 \left( \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right) \text{Im} \langle \delta_t U^{n+1/2}, U^{n+1} \rangle + c, \quad n = 0, 1, 2, \ldots, N - 1, \quad (4.11b)
\]

\[
E_4^n = (1 + r_x^2 + r_y^2 + \theta^2) \|\delta_t U^{n+1/2}\|^2 + \text{Re}(\delta_x U^n, \delta_x U^{n+1}) + \text{Re}(\delta_y U^n, \delta_y U^{n+1}) \\
+ \frac{1}{2} \left( \|U^n\|^2 \|U^{n+1}\|^2, 1 \right) + \left\langle f, \frac{\|U^n\|^2 + \|U^{n+1}\|^2}{2} \right\rangle + c \quad (4.12a)
\]

\[
= \|\delta_t U^{n+1/2}\|^2 + \|U^{n+1/2}\|^2_{H^1} + \frac{1}{2} \left\langle \left( \frac{\|U^n\|^2 + \|U^{n+1}\|^2}{2} \right), 1 \right\rangle + \left\langle f, \frac{\|U^n\|^2 + \|U^{n+1}\|^2}{2} \right\rangle \\
+ \left( \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 + \theta^2 \right) \|\delta_t U^{n+1/2}\|^2 - \frac{\tau^2}{4} |\delta_t U^{n+1/2}|^2_{H^1} \\
- \frac{\tau^2}{8} \left\langle \left( \frac{\|U^n\|^2 - \|U^{n+1}\|^2}{\tau} \right)^2, 1 \right\rangle + c, \quad n = 0, 1, 2, \ldots, N - 1, \quad (4.12b)
\]

respectively. Then they are both conservative as well. Namely, \( Q_4^n = Q_4^{n-1} = \cdots = Q_4^0 \) and \( E_4^n = E_4^{n-1} = \cdots = E_4^0 \). Here, \( c \) is an arbitrary constant. Furthermore, as the parameter \( \theta \geq 0.5 \|f\|_\infty \) and \( c \geq 2\theta^2 \Omega \), we have that

\[
E_4^n = E_4^0 \geq \|\delta_t U^{n+1/2}\|^2 + \|U^{n+1/2}\|^2_{H^1} + \frac{1}{2} \|U^n U^{n+1} - 2\| \geq 0. \quad (4.13)
\]

**Proof.** Using the techniques similar to those proposed in the proofs of Theorems 3.1 and 4.1, multiplying \( h_x h_y U^n_{j,k} \) to both sides of (4.3a), summing in \( j \) from 1 to \( M_x - 1 \) and \( k \) from
1 to $M_y - 1$, using the discrete Green formula, and taking the imaginary part and noting the definition of $Q^n_{\theta}$ (4.11a), we can derive that $0.5 \delta Q_{\theta}^{n-1/2} = 0$ and $Q^n_{\theta} = Q_{\theta}^{n-1} = \cdots = Q_1^{\theta}$.

Besides, by using Lemma 4.1, we can show that (4.11a) is equivalent to (4.11b).

Similar to the proofs of Theorems 3.1 and 4.1, multiplying both sides of (4.3a) by $h_x h_y \overline{U}_{\theta}^{n}$, summing in $j$ from 1 to $M_x - 1$ and $k$ from 1 to $M_y - 1$, using the discrete Green formula, taking the real part and noting the definition of $E^n_{\theta}$ (4.12a), we can obtain that $E^n_{\theta} = E_{\theta}^{n-1} = \cdots = E^n_{\theta}$. Furthermore, using the equality $\text{Re}(ab) = \|a + b|^2 - |a - b|^2|/4$ shows that (4.12a) is equivalent to (4.12b).

By applying $\text{Re}(ab) = \|a + b|^2 - |a - b|^2|/4$ and

$$
\left| U_{j,k}^{n+1} \right|^2 - 4 \theta \text{Re} \left( U_{j,k}^{n} \overline{U_{j,k}^{n+1}} \right) + 4 \theta^2 = \left| U_{j,k}^{n} \overline{U_{j,k}^{n+1}} - 2 \theta \right|^2,
$$

we have

$$
E^n_{\theta} = E^n_{\theta} = \left( 1 + r_x^2 + r_y^2 \right) \| \delta \theta U^{n+1/2} \|^2 + \left| U^{n+1/2} \right|_{H^1}^2 - \frac{\tau^2}{4} \| \delta \theta U^{n+1/2} \|^2_{H^1} + \frac{1}{2} \| U^{n+1} - \theta \|^2
$$

$$
+ h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} \left( \theta + \frac{f(x_j,k)}{2} \right) \left( \left| U_{j,k}^{n+1} \right|^2 + \left| U_{j,k}^{n} \right|^2 \right)
$$

$$
+ c - 2 \theta^2 (M_x - 1)(M_y - 1) h_x h_y.
$$

(4.14)

Similar to (3.19), using Lemma 2.4 gives

$$
(r_x^2 + r_y^2) \| \delta \theta U^{n+1/2} \|^2 - \frac{\tau^2}{4} \| \delta \theta U^{n+1/2} \|^2_{H^1} \geq 0.
$$

(4.15)

Using (4.15), $\theta \geq 0.5 \| f \|_{\infty}$ and $c \geq 2 \theta^2 |\Omega|$ to (4.14) derives (4.13). □

**Remark 4.1.** From (4.4b), (4.11b), (4.5b) and (4.12b), we can find that $Q^n_{\theta}$ and $E^n_{\theta}$, $l = 1, 3, 4$ are the approximations of the exact mass $Q(t)$ and energy $E(t)$ at $t_{n+1/2}$, $n = 0, 1, 2, \ldots, N - 1$ as $\tau, h_x, h_y, \tau/h_x$ and $\tau/h_y$ tend to zero.

### 4.3 Nonlinearly stable analyses

**Theorem 4.3.** IP-DFFT-FDM-IV (4.3a)-(4.3d) are stable in the $H^1$-norm provided the parameter $\theta \geq 0.5 \| f \|_{\infty}$, $\tau = O(h_x)$ and $\tau = O(h_y)$. Besides, IP-DFFT-FDM-IV (4.3a)-(4.3d) are unconditionally stable in the $L^2$-norm as long as the parameter $\theta \geq 0.5 \| f \|_{\infty}$. Furthermore, as $\theta > 0.5 \| f \|_{\infty}$ and $c \geq 2 \theta^2 |\Omega|$, an estimation in the $L^2$-norm is derived as

$$
\| U^n \| \leq \max \left\{ \| U^0 \|, \min \left\{ \sqrt{2(\| U^0 \|^2 + E^n_{\theta} T^2)}, \sqrt{E_0^n / (\theta - 0.5 \| f \|_{\infty})} \right\} \right\}.
$$

(4.16)

**Proof.** In the proof of this theorem, the $c$ in $E^n_{\theta}$ satisfies $c \geq 2 \theta^2 |\Omega|$. By using Corollary 2.1 and (13.1), we obtain

$$
\left| U^{n+1} \right|_{H^1}^2 \leq 2 \left| U^{n+1/2} \right|_{H^1}^2 + 2 (r_x^2 + r_y^2) \| \delta \theta U^{n+1/2} \|^2
$$

$$
\leq 2 \max \left\{ 1, r_x^2 + r_y^2 \right\} \left( \left| U^{n+1/2} \right|_{H^1}^2 + \| \delta \theta U^{n+1/2} \|^2 \right)
$$

$$
\leq 2 \max \left\{ 1, r_x^2 + r_y^2 \right\} E^n_{\theta}, \quad n = 0, 1, 2, \ldots, N - 1,
$$
which shows that \( \max_{0 \leq n \leq N} \{ |U^n|_{H^1} \} \leq d_0 \), where
\[
d_0 = \max \left\{ |U^0|_{H^1}^2, \sqrt{2} \max \{ 1, r_2^2 + r_4^2 \} E^0_4 \right\}.
\]
This exactly illustrates that the IP-DFFT-FDM-IV (4.3a)-(4.3d) is stable in the \( H^1 \)-norm provided the parameter \( \theta \geq 0.5 \| f \|_\infty \), \( \tau = O(h_x) \) and \( \tau = O(h_y) \).

Finally, using Lemma 2.5 and (4.13), we have that
\[
\| U^n \|^2 \leq 2 \| U^0 \|^2 + 2 \tau T \sum_{i=1}^n \| \delta_t U^{i-1} \|^2
\leq 2 \| U^0 \|^2 + 2 \tau T \sum_{i=1}^n E^0_i \leq 2 (\| U^0 \|^2 + E^0_4 T^2),
\]
which exactly shows that the IP-DFFT-FDM-IV (4.3a)-(4.3d) is unconditionally stable in the \( L^2 \)-norm.

By using (4.15), \( \theta > 0.5 \| f \|_\infty \) and \( c \geq 2 \theta^2 |\Omega| \) in (4.14), we have
\[
\| U^{n+1} \|^2 + \| U^n \|^2 \leq \frac{E^0_4}{\theta - 0.5 \| f \|_\infty} = \frac{E^0_4}{\theta - 0.5 \| f \|_\infty}.
\]
A combination of (4.17) and (4.18) derives (4.16). \( \square \)

### 4.4. Convergence analyses

This section is concerned about the convergence analyses of the numerical schemes. Denote \( h = \max \{ h_x, h_y \} \), assume there exist positive constants \( \mu, \gamma \) and \( \varepsilon \), such that \( \mu h \leq h_x, h_y \leq h \) and \( \tau \leq \gamma h^{3/2} + \varepsilon \).

Let \( (e_2)^n_{j,k} = u^n_{j,k} - U^n_{j,k}, x_{j,k} \in \Omega_h, 0 \leq n \leq N \). Then subtracting (4.2a)-(4.2d) from (4.1a)-(4.1d) gives that
\[
(1 + r^2_2 + r^2_4) \delta_t^2 (e_2)^n_{j,k} - \Delta_h (e_2)^n_{j,k} + i \delta_t (e_2)^n_{j,k} + |u^n_{j,k}|^2 u^n_{j,k} - (U^n_{j,k})^2 U^n_{j,k} + f_{j,k} (e_2)^n_{j,k} = (R_3)^n_{j,k}, \quad x_{j,k} \in \Omega_h, \quad 1 \leq n \leq N - 1,
\]
\[
(e_2)^0_{j,k} = 0, \quad x_{j,k} \in \Omega_h, \quad (e_2)^1_{j,k} = O(\tau^2), \quad x_{j,k} \in \Omega_h,
\]
\[
(e_2)^n_{j,k} = 0, \quad x_{j,k} \in \partial \Omega_h, \quad 0 \leq n \leq N.
\]

**Lemma 4.2.** Denoting
\[
H^n = (1 + r^2_2 + r^2_4) \| \delta_t (e_2)^n \|^2 + \Re \langle \delta_x (e_2)^n, \delta_x (e_2)^n \rangle + \Re \langle \delta_y (e_2)^n, \delta_y (e_2)^n \rangle,
\]
then it holds that
\[
\| \delta_t (e_2)^n \|^2 + \| (e_2)^n \|^2_{H^1} \leq H^n,
\]
\[
| (e_2)^n |^2_{H^1} + \| \delta_t (e_2)^n \|^2_{H^1} \leq \max \left( 2, 1 + 2r^2_2 + 2r^2_4 \right) H^n.
\]

**Proof.** Similar to the proof of (3.25a), applying the equality \( \Re (ab) = (|a| + |b|^2 - |a - b|^2)/4 \) and
\[
(r^2_2 + r^2_4) \| \delta_t U^n \|^2 - \frac{r^2_4}{4} \| \delta_t U^n \|^2_{H^1} \geq 0,
\]

one gets
\[
H^n = \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 + \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 + \left[ \left( r_x^2 + r_y^2 \right) \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 - \frac{r^2}{4} \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 \right].
\]
By using Corollary 2.1, we can derive that
\[
\|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 + \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 \leq 2 \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 + (1 + 2r_x^2 + 2r_y^2) \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 \leq \max (2, 1 + 2r_x^2 + 2r_y^2) \left[ \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 + \|\delta_t(e_2)^{n+\frac{1}{2}}\|_H^2 \right] \leq \max (2, 1 + 2r_x^2 + 2r_y^2) H^n.
\]
The proof is complete.

Suppose that the exact solutions of 2D problem (1.1a)-(1.1c) \(u(x, t) \in C^{4,4}(\Omega \times [0, T])\). Then, we can assume that there exist positive constants \(d_1, d_2\) and \(M_2\), such that
\[
\max \limits_{0 \leq n \leq N-1} \| (R_3)^n \|_2 \leq d_1 \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right]^2, \quad (4.21a)
\]
\[
H^0 \leq d_2 \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right]^2, \quad (4.21b)
\]
\[
\|u\|_\infty \leq M_2. \quad (4.21c)
\]

**Theorem 4.4.** Assume that \(u(x, t) \in C^{4,4}(\Omega \times [0, T])\) is the exact solutions to the 2D problem (1.1a)-(1.1c). Then under the conditions of the assumptions (4.21a)-(4.21c),
\[
\mu h \leq h_x, \quad h_y \leq h, \quad \frac{\gamma^2}{\mu} h^{2+3\epsilon} + \frac{2h}{\sqrt{\mu}} + \frac{2\gamma^2 h^{2\epsilon}}{\mu^3} \leq \frac{M_2^2 - M_2}{\sqrt{\epsilon} d_4}, \quad \tau \leq \gamma h^{\frac{3}{2} + \epsilon}, \quad \tau \leq \frac{1}{2d_3},
\]
the following error estimations of the IP-DFFT-FDM-III (4.2a)-(4.2d) for 2D problem (1.1a)-(1.1c) are given as follows:
\[
\max \limits_{0 \leq n \leq N} |e_2|^n |_{H^1} \leq d_4 \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right], \quad (4.22a)
\]
\[
\max \limits_{0 \leq n \leq N} ||U^n||_{\infty} \leq M_2^* \quad (4.22b)
\]
\[
\max \limits_{0 \leq n \leq N} ||e_2^n||_{\infty} \leq d_4 \sqrt{\epsilon} \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right], \quad (4.22c)
\]
in which \(M_2^*\) is a positive constant, bigger than \(M_2\) and independent of grid parameters \(\tau, h_x\) and \(h_y\), and
\[
d_3 = 2 \max \{2, 1 + 2r_x^2 + 2r_y^2\} \max \left\{ \frac{1}{2} \left[ M_2^2 + M_2 M_2^* + (M_2^*)^2 + \|f\|_{\infty}\right], \quad \frac{1}{2} \left[ M_2^2 + M_2 M_2^* + (M_2^*)^2 + \|f\|_{\infty} + 1\right] \right\},
\]
\[
d_4 = \sqrt{\max \{2, 1 + 2r_x^2 + 2r_y^2\}(d_1 T + d_2) \exp(2d_3 T)}.
\]
Proof. By (4.19b)-(4.19c), it is clear that (4.22a)-(4.22b) are valid for $n = 0, 1$. Suppose that (4.22a)-(4.22b) hold for $n = 0, 1, \ldots, K - 1$, $(\forall K \in \mathbb{Z}^+ \text{ and } 1 \leq n \leq K \leq N)$. Then using

$$\mu h \leq h_x, \quad h_y \leq h, \quad \frac{\gamma^2}{\mu} h^{2+3\varepsilon} + \frac{2h}{\sqrt{\mu}} + \frac{2\gamma^2 h^{2\varepsilon}}{\mu^3} \leq \frac{M_2^2 - M_2}{\sqrt{c d_4}}, \quad \tau \leq \gamma h^{\frac{5}{2}+\varepsilon},$$

and Lemma 2.4, we have

$$(e_2)^n_{j,k} \leq h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} \| (e_2)^n \| \leq h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} \sqrt{c} |(e_2)^n|_{H^1},$$

$$\leq \sqrt{c d_4} \left( \tau^2 h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} + h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} + \tau^2 h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} + \tau^2 h_x^{-\frac{5}{2}} h_y^{-\frac{5}{2}} \right) + \sqrt{c d_4} \left( \frac{\gamma^2}{\mu} h^{2+2\varepsilon} + \frac{2h}{\sqrt{\mu}} + \frac{2\gamma^2 h^{2\varepsilon}}{\mu^3} \right) \leq M_2^2 - M_2,$$

which implies that

$$\| U^n \|_\infty \leq \| u^n \|_\infty + \| (e_2)^n \|_\infty \leq M_2^2, \quad n = 0, 1, \ldots, K - 1. \quad (4.23)$$

Multiplying $h_x h_y \delta_l(e_2)^n_{j,k}$ to both sides of (4.19a), summing in $j$ from 1 to $M_x - 1$, summing in $k$ from 1 to $M_y - 1$, taking the real part, using the discrete Green formula and Cauchy-Schwarz inequality and Lemma 2.4, then multiplying $2\tau \max \{ 2, 1 + 3^2 + r_y^2 \}$ to both sides of the obtained results, replacing $n$ with $l$, summing in $l$ from 1 to $n$, finally utilizing Lemma 4.2, we can obtain that

$$\| (e_2)^{n+1} \|_{H^1}^2 + \| \delta_l (e_2)^{n+1} \|_2^2 \leq \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} H^n$$

$$\leq \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} H_0^n + \tau d_3 \sum_{l=1}^{n+1} \left( |(e_2)^{l} \|_{H^1} + \| \delta_l (e_2)^{l} \|_2 \right)^2$$

$$+ \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \tau \sum_{l=1}^{n} \left( \| (R_0)^l \| \right)^2,$$  \quad (4.24)

which is used together with Lemma 2.1 to infer that

$$\| (e_2)^{n+1} \|_{H^1}^2 + \| \delta_l (e_2)^{n+1} \|_2^2 \leq d_2^2 \left( \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right)^2, \quad n = 0, 1, 2, \ldots, K - 1. \quad (4.25)$$

Thus, we can prove that (4.22a) is valid for $n = K$ as $n = K - 1$ in (4.25) is taken. Besides, using

$$\mu h \leq h_x, \quad h_y \leq h, \quad \frac{\gamma^2}{\mu} h^{2+3\varepsilon} + \frac{2h}{\sqrt{\mu}} + \frac{2\gamma^2 h^{2\varepsilon}}{\mu^3} \leq \frac{M_2^2 - M_2}{\sqrt{c d_4}}, \quad \tau \leq \gamma h^{\frac{5}{2}+\varepsilon},$$

Lemma 2.4 and (4.22a), we have that

$$\| U^K \|_\infty \leq \| u^K \|_\infty + \| (e_2)^K \|_\infty \leq M_2 + \sqrt{c d_4} \left( \frac{\gamma^2}{\mu} h^{2+3\varepsilon} + \frac{2h}{\sqrt{\mu}} + \frac{2\gamma^2 h^{2\varepsilon}}{\mu^3} \right) \leq M_2^2. \quad (4.26)$$

Namely, (4.22b) is valid for $n = K$. Therefore, it follows from mathematical induction that (4.22a)-(4.22b) hold. Finally, a combination of (4.22a) and Lemma 2.4 derives (4.22c). \qed
Let \((e_2)^n_{j,k} = u^n_{j,k} - U^n_{j,k} \), \(x_{j,k} \in \Omega_h, 0 \leq n \leq N \). Using the analytical technique similar to (4.19a)-(4.19d), the error equations of IP-DFFT-FDM-IV (4.3a)-(4.3d) are given as

\[
(1 + r_x^2 + r_y^2 + \theta \tau^2)\delta_t^2(e_2)^n_{j,k} - \Delta_h(e_2)^n_{j,k} + \theta r \delta_t(e_2)^n_{j,k} + |u^n_{j,k}|^2 u^n_{j,k} - \left| U^n_{j,h} \right|^2 U^n_{j,h} + f_{j,k}(e_2)^n_{j,k} = (R_4)^n_{j,k},
\]

\(x_{j,k} \in \Omega_h, \quad 1 \leq n \leq N - 1, \quad (4.27a)\)

\[(e_2)^1_{j,k} = \mathcal{O}(\tau^2), \quad x_{j,k} \in \Omega_h, \quad (4.27c)\]

\[(e_2)^n_{j,k} = 0, \quad x_{j,k} \in \partial \Omega_h, \quad 0 \leq n \leq N, \quad (4.27d)\]

in which

\[(R_4)^n_{j,k} = \mathcal{O}\left(\tau^2 + h_x^2 + h_y^2 + \left(\frac{\tau}{h_x}\right)^2 + \left(\frac{\tau}{h_y}\right)^2\right).
\]

**Lemma 4.3 ([37, 44]).** For any grid function \(U \) defined in \( \Omega_h \), it holds that

\[
||U||_p \leq ||U||_\infty \left(C_p ||U||_{H^1} + \frac{1}{L} ||U||\right)^{1 - \frac{2}{p}}, \quad 2 \leq p \leq +\infty,
\]

where \(C_p = \max\{2\sqrt{2}, p/\sqrt{2}\} \) and \(L = \min\{X_c - X_i, Y_c - Y_i\} \).

Using the same analytical method as Lemma 4.2 yields the following Lemma 4.4.

**Lemma 4.4.** Let

\[
\tilde{H}^n = (1 + r_x^2 + r_y^2 + \theta \tau^2)||\delta_t(e_2)^n + \frac{1}{2}||^2 + \text{Re}(\delta_x(e_2)^n, \delta_x(e_2)^{n+1}) + \text{Re}(\delta_y(e_2)^n, \delta_y(e_2)^{n+1}).
\]

Then as \(\theta \geq 0\), it holds that

\[
||\delta_t(e_2)^{n+\frac{1}{2}}||^2 + ||(e_2)^{n+\frac{1}{2}}||_{H^1}^2 \leq \tilde{H}^n, \quad (4.28a)
\]

\[
||e_2||_{H^1} + ||\delta_t(e_2)^{n+\frac{1}{2}}||^2 \leq \max\{2, 1 + 2r_x^2 + 2r_y^2\} \tilde{H}^n. \quad (4.28b)
\]

**Lemma 4.5.** Let

\[
S^n = \langle |u^n|^2 u^n - |U^n|^2 U^n, \delta_t(e_2)^n \rangle.
\]

Then we have that

\[
S^n \leq d_5 ||(e_2)^n||_{H^1} + \frac{d_6}{2} \left[ ||(e_2)^{n-\frac{1}{2}}||_{H^1}^2 + ||(e_2)^{n-\frac{1}{2}}||_{H^1} \right] + \frac{d_7}{2} \left[ ||\delta_t(e_2)^{n+\frac{1}{2}}||^2 + ||\delta_t(e_2)^{n-\frac{1}{2}}||^2 \right],
\]

where

\[
d_5 = \frac{M_2^2}{2} + \frac{M_2}{2} d_6 \sqrt{c}(2\sqrt{2} + L^{-1})(2\sqrt{2} + L^{-1}\sqrt{c}),
\]

\[
d_6 = \sqrt{c} d_6^2 (4\sqrt{2} + L^{-1})^3(2\sqrt{2} + L^{-1}\sqrt{c}),
\]

\[
d_7 = \frac{1}{2}(M_2^2 + M_2 + 1).
\]

**Proof.** Using Cauchy-Schwarz inequality and the assumption (4.21c) derives that

\[
||\langle |u^n|^2 u^n, \delta_t(e_2)^n \rangle || \leq M_2^2 \left\{ \frac{c}{2} ||(e_2)^n||_{H^1}^2 + \frac{1}{2} \left[ ||\delta_t(e_2)^{n+\frac{1}{2}}||^2 + ||\delta_t(e_2)^{n-\frac{1}{2}}||^2 \right] \right\}. \quad (4.29)
\]
Using Cauchy-Schwarz inequality, Theorem 4.3, Lemmas 2.4, 4.3 and (4.21c) infers that

\[
\frac{M_3}{2} \left\{ \left\| U^n(e_2) \right\|^2 + \frac{1}{2} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right] \right\}
\leq \frac{M_2}{2} \left\{ \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{2} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right] \right\}
\leq \frac{M_2}{2} \left\{ \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{2} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right] \right\}
\leq \frac{M_2}{2} \left\{ \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{2} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right] \right\}.
\]

Applying Cauchy-Schwarz inequality, Theorem 4.3, Lemmas 2.4 and 4.3 yields that

\[
\left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2
\leq \frac{1}{2} \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{4} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right]
\leq \frac{1}{2} \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{4} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right]
\leq \left\| U^n \right\|^2 \left\| \bar{U}(e_2)^n \right\|^2 + \frac{1}{4} \left[ \left\| \delta t(e_2)^n + \frac{1}{2} \right\|^2 + \left\| \delta t(e_2)^n - \frac{1}{2} \right\|^2 \right]
\leq \frac{d_0^2}{2} \left[ \left\| \delta t(e_2)^n \right\|^2 + \left\| \delta t(e_2)^n \right\|^2 \right].
\]

Thus, by

\[
|u^n_{j,k}|^2 u^n_{j,k} - U^n_{j,k} = \frac{u^n_{j,k}}{u^n_{j,k}} \delta t(e_2)^n + U^n_{j,k} \delta t(e_2)^n + |u^n_{j,k}|^2 \delta t(e_2)^n,
\]
we have

\[
S^n \leq \left| \left( u^n \right)^2 \delta t(e_2)^n \right| + \left| \left( u^n \right)^2 \delta t(e_2)^n \right| + \left| \left( u^n \right)^2 \delta t(e_2)^n \right|.
\]

Finally, the substitutions of (4.29)-(4.31) into (4.32) yields the claimed result. \( \square \)

Let \( u(x, t) \in C^{1,1}(\Omega \times [0, T]) \) be the exact solution to 2D problem (1.1a)-(1.1c). Thus, we can suppose that there are two positive constants \( d_0 \) and \( d_1 \) such that

\[
\max_{0 \leq n \leq N-1} \left\| (R_4)^n \right\|^2 \leq d_0 \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right]^2.
\]

\[
\tilde{H}^0 \leq \frac{d_1}{2} \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right]^2.
\]
Theorem 4.5. Assume that \( u(x, t) \in C^{4,4}(\Omega \times [0, T]) \) is the exact solution to the 2D problem (1.1a)-(1.1c). Then, under conditions of assumptions (4.21c), (4.33a)-(4.33b), and \( \tau \leq 1/(2d_{10}) \), the error estimations of the IP-DFFT-FDM-IV (4.3a)-(4.3d) with \( \theta \geq 0.5 \|f\|_{\infty} \) are derived as follows:

\[
\max_{0 \leq n \leq N} |(e_2)^n|_{H^1} \leq d_{11} \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right], \tag{4.34a}
\]

\[
\max_{0 \leq n \leq N} \| (e_2)^n \| \leq d_{11} \sqrt{c} \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right], \tag{4.34b}
\]

in which

\[
d_{10} = 2 \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \max \{ d_5 + d_6 \} \|f\|_{\infty}, d_7 + 0.5 \|f\|_{\infty} + 0.5, \]

\[
d_{11} = \sqrt{\max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \exp(2d_{10}T)}.
\]

Proof. Multiplying \( h_x h_y \theta_j (e_2)^n \) to both sides of (4.27a), summing in \( j \) from 1 to \( M_x - 1 \) and \( k \) from 1 to \( M_y - 1 \), using the discrete Green formula, Cauchy-Schwarz inequality and Lemma 4.5, we have

\[
\frac{\tilde{H}^n - \tilde{H}^{n-1}}{2\tau} \leq d_{10} \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \|\delta_t e_2\|_{H^1} + 0.5 \|f\|_{\infty} \|e_2\|_{H^1} + (d_7 + 0.5 \|f\|_{\infty} + 0.5) \left[ \sum_{l=1}^{n+1} \left\| \delta_t (e_2)^{l-\frac{1}{2}} \right\|^2 + \left\| (e_2)^l \right\|_{H^1}^2 \right] + \left\| (R_4)^l \right\|^2. \tag{4.35}
\]

Multiplying \( 2\tau \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \) to both sides of (4.35), replacing \( n \) by \( l \), summing in \( l \) from 1 to \( n \), we obtain

\[
\| (e_2)^{n+1} \|_{H^1} + \left\| \delta_t (e_2)^{n+\frac{1}{2}} \right\|^2 \leq \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \tilde{H}^n \]

\[
\leq \max \left\{ 2, 1 + 2r_x^2 + 2r_y^2 \right\} \hat{H}^0 + 0.5 \tau d_{10} \sum_{l=1}^{n+1} \left[ \left\| \delta_t (e_2)^{l-\frac{1}{2}} \right\|^2 + \left\| (e_2)^l \right\|_{H^1}^2 \right] + \left\| (R_4)^l \right\|^2. \tag{4.36}
\]

The use of Lemma 2.1 to (4.36) shows

\[
\| (e_2)^{n+1} \|_{H^1} + \left\| \delta_t (e_2)^{n+\frac{1}{2}} \right\|^2 \leq d_{11}^2 \left[ \tau^2 + h_x^2 + h_y^2 + \left( \frac{\tau}{h_x} \right)^2 + \left( \frac{\tau}{h_y} \right)^2 \right]^2, \ n = 0, 1, \ldots, N - 1. \tag{4.37}
\]

A combination of (4.37) with (4.27b) shows (4.34a). Also, the combination of (4.34a) with Lemma 2.4 gives (4.34b).

5. Numerical Experiments

In this section, three numerical experiments are carried out to verify the correctness of the theoretical results, and the efficiency of the algorithms. The 1st and 3rd numerical examples,
whose exact solutions are known, are introduced to confirm the accuracy of the schemes; while, the exact solution of the 2nd numerical example is unknown.

Besides, we use EC-FDM-H to denote the energy-conserving FDM derived in [17], which is a nonlinear three-level scheme. As iterative method is used to solve the corresponding nonlinear system of algebraic equations, the iterations are terminated when the $L^\infty$-norm of the relative residual is reduced by a factor of $10^{-14}$.

To exhibit the performance of the numerical algorithms for 1D case, $L^\infty$, $L^2$, and $H^1$-norms errors at $t_n$ which are denoted by $ME(h_x, \tau)$, $LE(h_x, \tau)$, and $HE(h_x, \tau)$, respectively, and CPU time (in second) are reported in Tables 5.1-5.3. The corresponding convergent rates, which are defined as follows:

$$r_\infty = \log_2 \left( \frac{ME(2h_x)}{ME(h_x)} \right), \quad r_{L^2} = \log_2 \left( \frac{LE(2h_x)}{LE(h_x)} \right), \quad r_{H^1} = \log_2 \left( \frac{HE(2h_x)}{HE(h_x)} \right)$$

in the case of $\tau = h_x^2$ or $\tau = h_x$, respectively, are also shown.

For 2D case, uniform spatial grids (i.e. $h_x = h_y = h$) with different mesh size $h$ are adopted. In Tables 5.4-5.6, similar to 1D case, denote $L^\infty$, $L^2$, and $H^1$-norms errors at $t_n$ by $ME(h, \tau)$, $LE(h, \tau)$ and $HE(h, \tau)$, respectively; while, corresponding convergent orders are represented by $r_\infty$, $r_{L^2}$, and $r_{H^1}$ without confusion, respectively.

Moreover, the discrete energies $E^n_\kappa$ and the discrete masses $Q^n_\kappa$, $\kappa = 1, 2, 3, 4$ with $c = 0$ are employed to test the ability to preserve invariants. Furthermore, the discrete energy at $t_n$ obtained by EC-FDM-H is denoted by $E^n$. All computer programs are carried out by Matlab R2016b.

**Example 5.1.** We consider the numerical solutions of the following 1D problem:

$$u_{tt} - u_{xx} + au + |u|^2u + \left( \pi^2 - \sqrt{2}\pi - \sin^2(\pi x) \right)u = 0, \quad (x, t) \in [0, 1] \times [0, T],$$

$$u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = -\sqrt{2}\pi i \sin(\pi x), \quad x \in [0, 1],$$

$$u(0, t) = u(1, t) = 0, \quad t \in [0, T],$$

by using IP-DFFT-FDM-I and IP-DFFT-FDM-II. The exact solution of the problem is $u(x, t) = \sin(\pi x) \exp(-\sqrt{2}\pi it)$.

Tables 5.1 and 5.2 show that both of IP-DFFT-FDM-I and IP-DFFT-FDM-II have a convergent rate of $O(h_x^2)$ with respect to $L^\infty$, $L^2$, and $H^1$-norms as $\tau = h_x^2$. Table 5.3 indicates that EC-FDM-H is temporally and spatially second-order convergent in $L^\infty$, $L^2$, and $H^1$-norms. Although the accuracy of EC-FDM-H is higher than that of the current methods, the current methods are slightly more efficient. For example, IP-DFFT-FDM-I with $h_x = 1/2^7$ costs 0.6426 s to obtain $ME(h_x, \tau) = 1.8712e-04$; IP-DFFT-FDM-II with $h_x = 1/2^7$ costs 0.6255 s for yielding $ME(h_x, \tau) = 1.8714e-04$; however, EC-FDM-H with $h_x = 1/2^8$ consumes 0.7058 s to give $ME(h_x, \tau) = 2.1283e-04$.

**Table 5.1:** Computational results at $t = 1$ for Example 5.1 using IP-DFFT-FDM-I ($\tau = h_x^2$).

<table>
<thead>
<tr>
<th>$h_x$</th>
<th>$ME(h_x, \tau)$</th>
<th>$r_\infty$</th>
<th>$LE(h_x, \tau)$</th>
<th>$r_{L^2}$</th>
<th>$HE(h_x, \tau)$</th>
<th>$r_{H^1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^6$</td>
<td>2.9918e-03</td>
<td>–</td>
<td>2.1118e-03</td>
<td>–</td>
<td>6.9600e-03</td>
<td>–</td>
<td>0.0113</td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>7.4838e-04</td>
<td>1.9992</td>
<td>5.2827e-04</td>
<td>1.9991</td>
<td>1.7415e-03</td>
<td>1.9987</td>
<td>0.0782</td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>1.8712e-04</td>
<td>1.9998</td>
<td>1.3209e-04</td>
<td>1.9998</td>
<td>4.3548e-04</td>
<td>1.9997</td>
<td>0.6426</td>
</tr>
<tr>
<td>$1/2^3$</td>
<td>4.6776e-05</td>
<td>2.0001</td>
<td>3.3019e-05</td>
<td>2.0001</td>
<td>1.0886e-04</td>
<td>2.0001</td>
<td>4.8500</td>
</tr>
</tbody>
</table>
Table 5.2: Computational results at $t = 1$ for Example 5.1 using IP-DFFT-FDM-II ($\tau = h_x^2$).

<table>
<thead>
<tr>
<th>$h_x$</th>
<th>$M E(h_x, \tau)$</th>
<th>$r_{\infty}$</th>
<th>$L E(h_x, \tau)$</th>
<th>$r_{L2}$</th>
<th>$H E(h_x, \tau)$</th>
<th>$r_{H1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^5$</td>
<td>2.9974e-03</td>
<td>—</td>
<td>2.1158e-03</td>
<td>—</td>
<td>6.9731e-03</td>
<td>—</td>
<td>0.0191</td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>7.4873e-04</td>
<td>2.0012</td>
<td>5.2852e-04</td>
<td>2.0012</td>
<td>1.7423e-03</td>
<td>2.0008</td>
<td>0.0815</td>
</tr>
<tr>
<td>$1/2^7$</td>
<td>1.8714e-04</td>
<td>2.0003</td>
<td>1.3210e-04</td>
<td>2.0003</td>
<td>4.3553e-04</td>
<td>2.0002</td>
<td>0.6255</td>
</tr>
<tr>
<td>$1/2^8$</td>
<td>4.6778e-05</td>
<td>2.0002</td>
<td>3.3020e-05</td>
<td>2.0002</td>
<td>1.0886e-04</td>
<td>2.0002</td>
<td>4.7790</td>
</tr>
</tbody>
</table>

Table 5.3: Computational results at $t = 1$ for Example 5.1 using EC-FDM-H ($\tau = h_x$).

<table>
<thead>
<tr>
<th>$h_x$</th>
<th>$M E(h_x, \tau)$</th>
<th>$r_{\infty}$</th>
<th>$L E(h_x, \tau)$</th>
<th>$r_{L2}$</th>
<th>$H E(h_x, \tau)$</th>
<th>$r_{H1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^5$</td>
<td>1.3172e-02</td>
<td>—</td>
<td>9.2802e-03</td>
<td>—</td>
<td>3.0586e-02</td>
<td>—</td>
<td>0.0144</td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>3.3610e-03</td>
<td>1.9706</td>
<td>2.3682e-03</td>
<td>1.9704</td>
<td>7.8074e-03</td>
<td>1.9700</td>
<td>0.0481</td>
</tr>
<tr>
<td>$1/2^7$</td>
<td>8.4781e-04</td>
<td>1.9871</td>
<td>5.9742e-04</td>
<td>1.9870</td>
<td>1.9697e-03</td>
<td>1.9869</td>
<td>0.1653</td>
</tr>
<tr>
<td>$1/2^8$</td>
<td>2.1283e-04</td>
<td>1.9940</td>
<td>1.4998e-04</td>
<td>1.9940</td>
<td>4.9449e-04</td>
<td>1.9939</td>
<td>0.7058</td>
</tr>
</tbody>
</table>

Using Matlab R2016b gives that $E(0) = 17.330267333099535$, $Q(0) = -3.942882938158366$. From Figs. 5.1 and 5.2, we can observe that $E_n^\kappa - E(0)$ and $Q_n^\kappa - Q(0)$, $\kappa = 1, 2$ approximately reach to $1.0e^{-12}$. These exactly confirm that the discrete energies and masses obtained by IP-DFFT-FDM-I and IP-DFFT-FDM-II are both conservative. Similarly, Fig. 5.3 illustrates that the discrete energies yielded by EC-FDM-H are also invariant. However, from these figures, we can find that both IP-DFFT-FDM-I and IP-DFFT-FDM-II are slightly superior to EC-FDM-H in terms of the ability preserving the discrete energies; however, the accuracies of the discrete energies obtained by EC-FDM-H are slightly better than those provided by IP-DFFT-FDM-I and IP-DFFT-FDM-II, respectively.

Fig. 5.1. Example 5.1 (solved by IP-DFFT-FDM-I with $h_x = 0.1, \tau = h_x^2$): $E_n^1$, $E_n^0 - E_n^1$, and $|E_n^1 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ listed in the 1st row; $Q_n^1$, $Q_n^0 - Q_n^1$, and $|Q_n^1 - Q(0)|/|Q(0)|$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ exhibited in the 2nd row.
Example 5.1. To further confirm the theoretical results, we use IP-DFFT-FDM-I and IP-
DFFT-FDM-II to solve the following 1D problem:

\[ u_{tt} - u_{xx} + iu_t + |u|^2u = 0, \quad t \in (0, 1), \quad x \in (-40, 40), \]
\[ u(x, 0) = (1 + i)x \exp \left(-10(1-x^2)\right), \quad u_t(x, 0) = 0, \quad x \in [-40, 40], \]
\[ u(-40, t) = u(40, t) = 0, \]

whose exact solutions are unknown. Besides, with the help of Matlab R2016b, the exact energy
and mass are computed as follows: \( E(0) = 9.123952979578970 \) and \( Q(0) = 0.812482096009232 \).

Figs. 5.4 and 5.5 illustrate that the discrete energies and masses yielded by IP-DFFT-FDM-I
and IP-DFFT-FDM-II are both conservative. The surface and some curve lines of numerical
solutions are exhibited in Fig. 5.6, from which we can see some dynamical behaviors of numerical
solutions, such as, collision, propagation and oscillation.

Example 5.2. To further confirm the theoretical results, we use IP-DFFT-FDM-I and IP-
DFFT-FDM-II to solve the following 1D problem:

\[ u_{tt} - u_{xx} + iu_t + |u|^2u = 0, \quad t \in (0, 1), \quad x \in (-40, 40), \]
\[ u(x, 0) = (1 + i)x \exp \left(-10(1-x^2)\right), \quad u_t(x, 0) = 0, \quad x \in [-40, 40], \]
\[ u(-40, t) = u(40, t) = 0, \]

whose exact solutions are unknown. Besides, with the help of Matlab R2016b, the exact energy
and mass are computed as follows: \( E(0) = 9.123952979578970 \) and \( Q(0) = 0.812482096009232 \).

Figs. 5.4 and 5.5 illustrate that the discrete energies and masses yielded by IP-DFFT-FDM-I
and IP-DFFT-FDM-II are both conservative. The surface and some curve lines of numerical
solutions are exhibited in Fig. 5.6, from which we can see some dynamical behaviors of numerical
solutions, such as, collision, propagation and oscillation.

Example 5.3. Set \( \Omega = [0, 1] \times [0, 1] \). On \( \Omega \times [0, T] \), we consider the numerical solutions of the
2D problem (1.1a)-(1.1c) with \( f(x, y) = -[\sqrt{2}\pi + \sin^2(\pi x) \sin^2(\pi y)] \) by using IP-DFFT-FDM-
III and IP-DFFT-FDM-IV. Initial-boundary-value conditions of this problem are taken by their
exact solution \( u(x, y, t) = \sin(\pi x) \sin(\pi y) \exp(-\sqrt{2}\pi i t) \).
Fig. 5.4. Example 5.2 (solved by IP-DFFT-FDM-I with $h_x = 0.05, \tau = h_x^2$): $E^n_1$, $E^n_1 - E^0_1$, and $|E^n_1 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ listed in the 1st row; $Q^n_1$, $Q^n_1 - Q^0_1$, and $|Q^n_1 - Q(0)|/Q(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ exhibited in the 2nd row.

Fig. 5.5. Example 5.2 (solved by IP-DFFT-FDM-II with $\lambda = 0.1$, $h_x = 0.05, \tau = h_x^2$): $E^n_2$, $E^n_2 - E^0_2$, and $|E^n_2 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ listed in the 1st row; $Q^n_2$, $Q^n_2 - Q^0_2$, and $|Q^n_2 - Q(0)|/Q(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ exhibited in the 2nd row.

Fig. 5.6. Example 5.2 (solved by IP-DFFT-FDM-I with $h_x = 0.05$, $\tau = h_x^2$): The curve surfaces and lines of the discrete solutions $|U^n|$. 
Tables 5.4 and 5.5 imply that IP-DFFT-FDM-III and IP-DFFT-FDM-IV are both convergent with an order of $O(h^2)$ in $L^\infty$, $L^2$, and $H^1$-norms as $\tau = h^2$, respectively. Also, Table 5.6 confirms EC-FDM-H is second-order accurate in both time and space with respect to $L^\infty$, $L^2$, and $H^1$-norms. Although EC-FDM-H is more accurate in time than our methods from theoretical point of view, it is less efficient than our methods. Moreover, with the same temporal and spatial meshes, IP-DFFT-FDM-III is slightly more accurate than IP-DFFT-FDM-IV. For instance, $ME(h, \tau) = 3.7618e-04$ yielded by IP-DFFT-FDM-III with $h = 1/2^7$ and $\tau = h^2$ costs 11.7170 s; $ME(h, \tau) = 3.7620e-04$ obtained by IP-DFFT-FDM-IV with $h = 1/2^7$ and $\tau = h^2$ costs 11.6224 s; however, $ME(h, \tau) = 9.0992e-04$ given by EC-FDM-H with $\tau = h = 1/2^7$ costs 13.0570 s.

Using Matlab R2016b gives that $E(0) = 8.688571166549767$ and $Q(0) = -1.971441469079183$. Figs. 5.7 and 5.8 confirm that the discrete energies and masses yielded by IP-DFFT-FDM-III and IP-DFFT-FDM-IV are both conservative, respectively. Also, Fig. 5.9 shows that the discrete energies provided by EC-FDM-H are invariant as time increases. Besides, from Figs. 5.7-5.9, we can find that both IP-DFFT-FDM-III and IP-DFFT-FDM-IV are slightly superior to EC-FDM-H in terms of the ability to preserve the discrete energies; however, the accuracies of the discrete energies obtained by EC-FDM-H are slightly better than those of IP-DFFT-FDM-III and IP-DFFT-FDM-IV, respectively.

Table 5.4: Computational results at $t = 1$ for Example 5.3 using IP-DFFT-FDM-III ($\tau = h^2$).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$ME(h, \tau)$</th>
<th>$E_{\infty}$</th>
<th>$LE(h, \tau)$</th>
<th>$E_{L^2}$</th>
<th>$HE(h, \tau)$</th>
<th>$E_{H^1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^7$</td>
<td>5.9997e-03</td>
<td>2.9923e-03</td>
<td>1.3622e-02</td>
<td>1.0002</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>1.5038e-03</td>
<td>1.9963</td>
<td>3.4154e-03</td>
<td>1.0267</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>3.7618e-04</td>
<td>1.9991</td>
<td>8.5446e-04</td>
<td>11.7170</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>9.4059e-05</td>
<td>1.9998</td>
<td>2.1365e-04</td>
<td>165.6426</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.5: Computational results at $t = 1$ for Example 5.3 using IP-DFFT-FDM-IV ($\tau = h^2$).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$ME(h, \tau)$</th>
<th>$E_{\infty}$</th>
<th>$LE(h, \tau)$</th>
<th>$E_{L^2}$</th>
<th>$HE(h, \tau)$</th>
<th>$E_{H^1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^7$</td>
<td>6.0053e-03</td>
<td>2.9951e-03</td>
<td>1.3635e-02</td>
<td>0.1079</td>
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</tr>
<tr>
<td>$1/2^6$</td>
<td>1.5041e-03</td>
<td>1.9973</td>
<td>3.4162e-03</td>
<td>1.0375</td>
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<td></td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>3.7620e-04</td>
<td>1.9993</td>
<td>8.5451e-04</td>
<td>11.6224</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>9.4060e-05</td>
<td>1.9998</td>
<td>2.1365e-04</td>
<td>165.7672</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.6: Computational results at $t = 1$ for Example 5.3 using EC-FDM-H ($\tau = h$).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$ME(h, \tau)$</th>
<th>$E_{\infty}$</th>
<th>$LE(h, \tau)$</th>
<th>$E_{L^2}$</th>
<th>$HE(h, \tau)$</th>
<th>$E_{H^1}$</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^7$</td>
<td>1.4153e-02</td>
<td>7.0392e-03</td>
<td>3.2045e-02</td>
<td>0.4267</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>3.6084e-03</td>
<td>1.7953e-03</td>
<td>8.1753e-03</td>
<td>2.2601</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>9.0992e-04</td>
<td>4.5277e-04</td>
<td>2.0619e-03</td>
<td>13.0570</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>2.2839e-04</td>
<td>1.1365e-04</td>
<td>5.1758e-04</td>
<td>95.5010</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 5.7. Example 5.3 (solved by IP-DFFT-FDM-III with $h = 0.1, \tau = h^2$): $E_n^3, E_n^3 - E_0^3$, and $|E_n^3 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ listed in the 1st row; $Q_n^3, Q_n^3 - Q_0^3$, and $|Q_n^3 - Q(0)|/|Q(0)|$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ exhibited in the 2nd row.

Fig. 5.8. Example 5.3 (solved by IP-DFFT-FDM-IV with $\theta = 0.5\|f\|_\infty, h = 0.1, \tau = h^2$): $E_n^4, E_n^4 - E_0^4$, and $|E_n^4 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ listed in the 1st row; $Q_n^4, Q_n^4 - Q_0^4$, and $|Q_n^4 - Q(0)|/|Q(0)|$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$ exhibited in the 2nd row.

Fig. 5.9. Example 5.3 (solved by EC-FDM-II with $h = 0.1, \tau = h^2$): $E_n^2, E_n^2 - E_0^2$, and $|E_n^2 - E(0)|/E(0)$ between $t_0$ and $t_n$ from $t = 0$ to $t = 1000$. 
6. Conclusion

In this study, invariants-preserving Du Fort-Frankel-type FDMs and stabilized invariants-preserving Du Fort-Frankel-type FDMs have been developed for solving 1D and 2D problems (1.1a)-(1.1c). The discrete energies and masses obtained by them are all invariant as time increases. Using the discrete energy methods, it has been shown that they possess a convergent order of $O(\tau^2 + h_x^2 + (\tau/h_x)^2)$ for 1D and a convergent order of $O(\tau^2 + h_x^2 + h_y^2 + (\tau/h_x)^2 + (\tau/h_y)^2)$ for 2D in $H^1$-norm under conditions of certain assumptions. To achieve to optimally convergent rates, the optimal relationships between temporal and spatial grids are $\tau = O(h_x^2)$ or $\tau = O(h_y^2)$ for 1D and 2D problems, respectively, according to the consistent requirements and truncation errors. As the optimal relationships between temporal and spatial grids are taken, IP-DFFT-FDM-I and IP-DFFT-FDM-III are slightly superior IP-DFFT-FDM-II and IP-DFFT-FDM-IV in terms of accuracy, respectively. This meets our anticipation because the stabilized terms bring in additional errors. However, from the theoretical point of the view, numerical solutions obtained by IP-DFFT-FDM-II with $\lambda \geq 0.5\|f\|_{\infty}$ and IP-DFFT-FDM-IV with suitable $\theta \geq 0.5\|f\|_{\infty}$ are both bounded in $L^2$- and $H^1$-norms as $\tau = O(h_x)$ and $\tau = O(h_y)$, respectively. Especially, numerical solutions obtained by IP-DFFT-FDM-II with $\lambda \geq 0.5\|f\|_{\infty}$ and IP-DFFT-FDM-IV with $\theta \geq 0.5\|f\|_{\infty}$ are both uniformly bounded without any grids requirement in $L^2$-norm. Namely, IP-DFFT-FDM-II and IP-DFFT-FDM-IV are superior to IP-DFFT-FDM-I and IP-DFFT-FDM-III in terms of stability, respectively.

Besides, in comparison with the existent studies, our study has the following merits and novelties:

(1) This study focuses on the numerical solutions of the Schrödinger equation with wave operator by invariants-preserving Du Fort-Frankel-type FDMs for the first time.

(2) Comparing with invariants-preserving Du Fort-Frankel-type FDMs for Schrödinger equations in [7, 8, 18, 20, 30, 39], the current algorithms possess the following distinguishing features:

   (i) Numerical solutions provided by our methods are convergent with much weaker grid conditions.

   (ii) The discrete energies obtained by the stabilized invariants-preserving Du Fort-Frankel-type FDMs are non-negative and bounded as grid conditions and the parameter of the stabilized term are suitably given. Especially, the stabilized invariants-preserving Du Fort-Frankel-type FDMs with suitably stabilized parameter are unconditionally stable in $L^2$-norm.

   (iii) Some concisely analytical skills, which are completely different from those developed in [31], have been proposed to analyze our methods.

(3) Our methods with good stability are fully explicit, and very easy to be implemented.

(4) Our methods can be successfully used to the simulations for long-term because of the good stability, convergence and easy implementation.

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References


