Arbitrary High-Order Structure-Preserving Schemes for Generalized Rosenau-Type Equations

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Abstract. Arbitrary high-order numerical schemes conserving the momentum and energy of the generalized Rosenau-type equation are studied. Derivation of momentum-preserving schemes is made within the symplectic Runge-Kutta method coupled with the standard Fourier pseudo-spectral method in space. Combining quadratic auxiliary variable approach, symplectic Runge-Kutta method, and standard Fourier pseudo-spectral method, we introduce a class of high-order mass- and energy-preserving schemes for the Rosenau equation. Various numerical tests illustrate the performance of the proposed schemes.

AMS subject classifications: 65M06, 65M70

Key words: Momentum-preserving, energy-preserving, high-order, symplectic Runge-Kutta method, Rosenau equation.

1. Introduction

We consider the following generalized Rosenau-type equations:

\begin{align*}
\partial_t u(x, t) + \kappa \partial_x u(x, t) - \delta \partial_{xxx} u(x, t) + b \partial_{xxxx} u(x, t) \\
+ \alpha \partial_{xxxxx} u(x, t) + \beta \partial_x (u(x, t)^p) &= 0, & x \in \Omega \subset \mathbb{R}, \quad t > 0, \\
u(x, 0) &= u_0(x), & x \in \Omega \subset \mathbb{R},
\end{align*}

(1.1)

where \( t \) and \( x \) are respectively time and spatial variables, \( \kappa, \delta > 0, b, \alpha > 0 \) and \( \beta \) given real constants, \( u := u(x, t) \) is a real-valued wave function, \( p \) a given positive integer, \( u_0(x) \) an initial condition, and \( \Omega = [x_l, x_r] \) a bounded domain. In what follows, the Rosenau equation (1.1) will be also supplemented by periodic boundary conditions.

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Initially, the Eq. (1.1) has been used to describe the dynamics of dense discrete systems [34]. Nowadays, it plays an important role in fluid mechanics of the atmosphere and ocean. Moreover, when \( u \) is assumed to be smooth, the Eq. (1.1) satisfies the following Hamiltonian formulation:

\[
\frac{\partial u}{\partial t} + \delta \frac{\partial \mathcal{H}}{\delta u} = 0,
\]

(1.2)

where \( \delta = (1 - \delta \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial^4}{\partial x^4})^{-1} \frac{\partial}{\partial x} \) is a Hamiltonian operator, and \( \mathcal{H} \) the Hamiltonian functional — i.e.

\[
\mathcal{H}(t) = \int_{\Omega} \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{p + 1} u^{p+1} \right) dx, \quad t \geq 0.
\]

(1.3)

In addition to the Hamiltonian energy (1.3), the Eq. (1.1) also conserves the mass

\[
\mathcal{M}(t) = \int_{\Omega} u dx \equiv \mathcal{M}(0), \quad t \geq 0,
\]

(1.4)

and the momentum

\[
\mathcal{I}(t) = \int_{\Omega} \left( \frac{1}{2} u_x^2 + \frac{\delta}{2} u_{xx}^2 + \frac{\alpha}{2} u_{xxx}^2 \right) dx \equiv \mathcal{I}(0), \quad t \geq 0.
\]

(1.5)

In order to solve such Hamiltonian partial differential equations, it is often preferable to use special numerical schemes, which would inherit one or more intrinsic properties of the original system exactly in a discrete sense. Note that such a kind schemes are called structure-preserving — cf. [13, 14, 21]. Chung [10] proposed an implicit finite difference (IFD) scheme, which can satisfy the discrete analogue of momentum (1.5) and proved that the scheme has the second-order accuracy both in time and space. Subsequently, Omrani et al. [30] developed a linearly implicit momentum-preserving finite difference scheme for the classical Rosenau equation, in which a linear system is to be solved at every time step. Thus, it is computationally much cheaper than that of the IFD scheme. Over the years, various momentum-preserving schemes for the Eq. (1.1) have been proposed and analyzed — cf. Refs [1, 2, 22, 28, 31, 42, 43, 45–47]. However, to the best of our knowledge, all of existing momentum-preserving schemes have at most second-order accuracy in time. It has been shown in [18,26] that, compared with the second-order schemes, the high-order ones not only provide smaller numerical errors as a large time step chosen, but also are more advantages in the robustness. Consequently, one of the goals of this work is to present a novel paradigm for developing arbitrary high-order momentum-preserving schemes for the Eq. (1.1).

In addition to the momentum conservation law (1.5), the Eq. (1.1) satisfies the Hamiltonian energy (1.3), which is one of the most important first integrals of the Hamiltonian system. Cai et al. [3] proposed a second-order energy-preserving scheme based on the averaged vector field method [32] and two fourth-order energy-preserving schemes based on composition ideas [21]. Nevertheless, it is shown in [21] that the high-order schemes obtained by the composition method will be at the price of a terrible zig-zag of the step points — cf. [21, Fig. 4.2], which may be tedious and time consuming. Thus, the construction of high-order energy-preserving schemes for the Rosenau equation (1.1) seems to be still at
its beginning stage. Therefore, the other goal of this work is to present a new strategy for developing arbitrary high-order energy-preserving schemes for the Rosenau equation (1.1) based on the quadratic auxiliary variable (QAV) approach [6, 16, 41], which is inspired by the idea of the energy quadratization (EQ) [39, 40, 48].

The rest of this paper is organized as follows. In Section 2, the high-order momentum-preserving and energy-preserving schemes for the Eq. (1.1) are proposed and the structure-preserving properties of such schemes are analysed. In Section 3, an efficient fixed-point iteration solver for the nonlinear equations of the proposed schemes is presented. Various numerical examples, considered in Section 4, illustrate the performance of the proposed schemes. We draw conclusions in Section 5.

2. High-Order Structure-Preserving Schemes

In this section, we propose high-order momentum-preserving schemes and energy-preserving schemes for the Eq. (1.1).

Choose a spatial step size \( h = \frac{x_r - x_l}{N} \) with an even positive integer \( N \) and consider grid points \( x_j = jh, j = 0, 1, 2, \ldots, N \). Let \( u_j \) be an approximation of \( u(x_j, t) \), \( j = 0, 1, \ldots, N \), and \( U := (u_0, u_1, \cdots, u_{N-1})^T \) the solution vector. We also define a discrete inner product, \( l^2 \)- and \( l^{\infty} \)-norms as follows:

\[
\langle U, V \rangle_h = h \sum_{j=0}^{N-1} u_j v_j, \quad \|U\|_{h^2}^2 = \langle U, U \rangle_h, \quad \|U\|_{h,\infty} = \max_{0 \leq j < N-1} |u_j|.
\]

Besides, in what follows we use the element product of vectors \( U \) and \( V \) defined as

\[
U \cdot V = (u_0 v_0, u_1 v_1, \cdots, u_{N-1} v_{N-1})^T
\]

and write \( U^2 \) for \( U \cdot U \).

As achieving high-order accuracy in time, the spatial accuracy shall be comparable to that of the time-discrete discretization. Actually, we consider the periodic boundary condition in this paper, so that the Fourier pseudo-spectral method is a very good choice because of the spectral accuracy and the fast Fourier transform (FFT) algorithm. Thus, we first expounded the Fourier pseudo-spectral method, as follows.

Consider the interpolation space

\[
\mathcal{S}_h = \text{span} \{l_j(x), \ 0 \leq j \leq N-1\},
\]

where

\[
l_j(x) = \frac{1}{N} \sum_{k=-N/2}^{N/2} \frac{1}{c_k} e^{i k \mu (x-x_j)},
\]

\[
\mu = \frac{2\pi}{x_r - x_l}, \quad c_k = \begin{cases} 1, & |k| < N/2, \\ 2, & |k| = N/2, \end{cases}
\]

are the trigonometric polynomials of the degree \( N/2 \).
Following [5, 8], we consider the interpolation operator $\mathcal{S}_N : C(\Omega) \rightarrow \mathcal{S}_h$, 

$$\mathcal{S}_N u(x, t) = \sum_{k=0}^{N-1} u_k(t) l_k(x),$$

where $u_k(t) = u(x_k, t)$, $k = 0, 1, 2, \ldots, N - 1$. Computing partial derivatives in $x$ at the collocation points $x_j$ yields

$$\frac{\partial^r \mathcal{S}_N u(x_j, t)}{\partial x^r} = \sum_{k=0}^{N-1} u_k(t) \frac{d^r I_k(x_j)}{d x^r} = \sum_{k=0}^{N-1} (D_r)_{j,k} u_k(t), \quad j = 0, 1, \ldots, N - 1,$$

where $D_r$ represents the spectral differential matrix with the entries

$$(D_r)_{j,k} = \frac{d^r I_k(x_j)}{d x^r}, \quad j, k = 0, 1, \ldots, N - 1.$$

In particular, we have — cf. [24, 37],

$$(D_1)_{j,k} = \begin{cases} \frac{1}{2} \mu (-1)^{j+k} \cot \left( \frac{x_j - x_k}{2} \right), & j \neq k, \\ 0, & j = k, \end{cases}$$

$$(D_2)_{j,k} = \begin{cases} \frac{1}{2} \mu^2 (-1)^{j+k+1} \csc^2 \left( \frac{x_j - x_k}{2} \right), & j \neq k, \\ -\mu_2 \frac{N^2 + 2}{12}, & j = k, \end{cases}$$

$$(D_3)_{j,k} = \begin{cases} \frac{3 \mu^3}{4} (-1)^{j+k} \cos \left( \frac{x_j - x_k}{2} \right) \csc^3 \left( \frac{x_j - x_k}{2} \right), & j \neq k, \\ 0, & j = k, \end{cases}$$

$$(D_4)_{j,k} = \begin{cases} \mu^4 (-1)^{j+k} \csc^2 \left( \frac{x_j - x_k}{2} \right) \left( \frac{N^2}{4} - \frac{1}{2} - \frac{3}{2} \cot^2 \left( \frac{x_j - x_k}{2} \right) \right), & j \neq k, \\ \mu^4 \left( \frac{N^4}{80} + \frac{N^2}{12} + \frac{1}{30} \right), & j = k. \end{cases}$$

Remark 2.1. We should note that

$$D_r = \begin{cases} \mathcal{F}_N^{H} \tilde{\Lambda}^r \mathcal{F}_N, & \text{if } r \text{ is an odd integer}, \\ \mathcal{F}_N^{H} \Lambda^r \mathcal{F}_N, & \text{if } r \text{ is an even integer}, \end{cases}$$

where $\tilde{\Lambda}$ and $\Lambda$ are

$$\tilde{\Lambda} = \text{diag} \left( i \mu \left[ 0, 1, \cdots, \frac{N}{2}, -1, 0, -\frac{N}{2} + 1, \cdots, -2, -1 \right] \right),$$

$$\Lambda = \text{diag} \left( i \mu \left[ 0, 1, \cdots, \frac{N}{2}, -1, \frac{N}{2} - 1, \frac{N}{2} + 1, \cdots, -2, -1 \right] \right),$$
and $\mathcal{F}_N$ is the discrete Fourier transform (DFT) and $\mathcal{F}_N^H$ represents the conjugate transpose of $\mathcal{F}_N$ \cite{15,37}.

We set $t_n = n\tau$, and $t_{ni} = t_n + c_i\tau$, $i = 1, 2, \ldots, s$, $n = 0, 1, 2, \ldots$, where $\tau$ is the time step size and $c_1, c_2, \ldots, c_s$ are distinct real numbers (usually $0 \leq c_i \leq 1$). The approximations of the function $u(x, t)$ at points $(x_j, t_n)$ and $(x_j, t_{ni})$ are denoted by $u_{ij}^n$ and $u_{ij}^{ni}$, respectively.

### 2.1. High-order momentum-preserving schemes

In this section, we propose a class of high-order momentum-preserving schemes for the Eq. (1.1). Rewrite (1.1) as

$$\mathcal{A}_1 u = \mathcal{F}(u) U, \quad \mathcal{F}(u) = -\left[\kappa \partial_x bx_{xxx} + \frac{p\beta}{p+1} (u^{p-1} \partial_x \beta x (u^{p-1}))\right],$$

(2.1)

where $\mathcal{A}_1 = 1 - \delta \partial_x x + \alpha \partial_x xxx$ is a self-adjoint operator, $\mathcal{F}(u)$ is an anti-adjoint operator, and

$$\mathcal{F}(u) U := -\left[\kappa \partial_x u + b \partial_x xxx u + \frac{p\beta}{p+1} (u^{p-1} \partial_x \beta x u + \partial_x (u^p))\right].$$

The Fourier pseudo-spectral method is employed to solve the Eq. (2.1) in space, so that

$$\begin{cases}
\mathcal{A}_h \frac{d}{dt} U = \mathcal{F}_h(U) U, \\
\mathcal{F}_h(U) = -\left[\kappa D_1 + b D_3 + \frac{p\beta}{p+1} (\text{diag}(U^{p-1}) D_1 + D_1 \text{diag}(U^{p-1}))\right],
\end{cases}$$

(2.2)

where $\mathcal{A}_h = I - \delta D_2 + \alpha D_4$ is a symmetric matrix, $\mathcal{F}_h(U)$ is anti-symmetric for $U$, and $\mathcal{F}_h(U) U$ is defined by

$$\mathcal{F}_h(U) U = -\left[\kappa D_1 U + b D_3 U + \frac{p\beta}{p+1} (U^{p-1} \cdot D_1 U + D_1 (U^p))\right].$$

(2.3)

**Theorem 2.1.** The semi-discrete system (2.2) preserves the following semi-discrete momentum conservation law:

$$\frac{d}{dt} I_h(t) = 0, \quad I_h(t) = \frac{1}{2} \langle \mathcal{A}_h U, U \rangle_h, \quad t \geq 0.$$

*Proof.* The symmetric property of $\mathcal{A}_h$ and anti-symmetric property of $\mathcal{F}_h(U)$ for $U$ gives

$$\frac{d}{dt} I_h(t) = \left\langle \mathcal{A}_h \frac{d}{dt} U, U \right\rangle_h = \left\langle \mathcal{F}_h(U) U, U \right\rangle_h = 0,$$

which completes the proof.

**Theorem 2.2.** If $p = 2$, the semi-discrete system (2.2) preserves the following semi-discrete mass:

$$\frac{d}{dt} M_h(t) = 0, \quad M_h(t) = \langle U, 1 \rangle_h, \quad t \geq 0.$$  

(2.4)
Proof. It follows from (2.2) that

\[ \frac{d}{dt} M_h(t) = \langle \mathcal{A}_h^{-1} \mathcal{F}_h(U), 1 \rangle_h = \langle \mathcal{F}_h(U), \mathcal{A}_h^{-1} 1 \rangle_h = \langle \mathcal{F}_h(U), 1 \rangle_h. \]

Using (2.3) gives

\[ \langle \mathcal{F}_h(U), 1 \rangle_h = - \left( \kappa D_1 U + b D_3 U + \frac{p \beta}{p+1} \left( U^{p-1} \cdot D_1 U + D_1(U^p) \right), 1 \right)_h \]

As \( p = 2 \), we obtain from the above equation

\[ \langle \mathcal{F}_h(U), 1 \rangle_h = 0, \]

which implies that

\[ \frac{d}{dt} M_h(t) = 0. \]

This completes the proof. \( \square \)

Remark 2.2. If \( p > 2 \), we can deduce that the Fourier spectral differential matrix \( D_1 \) cannot satisfy the discrete equation

\[ \langle D_1 U, U^{p-1} \rangle_h = 0 \quad \text{for all} \quad U. \]

Thus, the system (2.2) cannot conserve the semi-discrete mass (2.4), as \( p > 2 \).

We then apply an RK method to the system (2.2) in time to give a class of fully discrete schemes for the Eq. (1.1), as follows:

**Scheme 2.1.** Let \( b_i, a_{ij}, i, j = 1, \ldots, s \) be real numbers and let \( c_i = \sum_{j=1}^{s} a_{ij} \). For a given \( U^n \), an \( s \)-stage Runge-Kutta method has the form

\[ \mathcal{A}_h K^n_i = \mathcal{F}_h(U^n), \quad U^{ni} = U^n + \tau \sum_{j=1}^{s} a_{ij} K^n_j, \quad i = 1, 2, \ldots, s, \]

\[ U^{n+1} = U^n + \tau \sum_{i=1}^{s} b_i K^n_i. \]  

(2.5)

**Theorem 2.3.** If the coefficients of the RK method satisfy the condition

\[ b_i a_{ij} + b_j a_{ji} = b_i b_j \quad \text{for all} \quad i, j = 1, \ldots, s, \]

then Scheme 2.1 conserves the following discrete momentum:

\[ I_h^{n+1} = I_h^n, \quad I_h^n = \frac{1}{2} \langle U^n, \mathcal{A}_h U^n \rangle_h, \quad n = 0, 1, 2, \ldots. \]  

(2.7)
Proof: It follows from the second equation in (2.5) that

\[ I^{n+1}_h - I^n_h = \frac{1}{2} \langle U^{n+1}, \alpha_h U^{n+1} \rangle_h - \frac{1}{2} \langle U^n, \alpha_h U^n \rangle_h \]

\[ = \frac{1}{2} \left( U^n + \tau \sum_{i=1}^s b_i K^n_i, \alpha_h \left( U^n + \tau \sum_{j=1}^s b_j K^n_j \right) \right)_h - \frac{1}{2} \langle U^n, \alpha_h U^n \rangle_h \]

\[ = \frac{\tau}{2} \sum_{i=1}^s b_i \langle U^n, \alpha_h K^n_i \rangle_h + \frac{\tau}{2} \sum_{i=1}^s b_i \langle K^n_i, \alpha_h U^n \rangle_h \]

\[ + \frac{\tau^2}{2} \sum_{i,j=1}^s b_i b_j \langle K^n_i, \alpha_h K^n_j \rangle_h. \] (2.8)

Noting that

\[ \frac{\tau}{2} \sum_{i=1}^s b_i \langle U^n, \alpha_h K^n_i \rangle_h = \frac{\tau}{2} \sum_{i=1}^s b_i \left( U^n - \tau \sum_{j=1}^s a_{ij} K^n_j, \alpha_h K^n_i \right)_h \]

\[ = \frac{\tau}{2} \sum_{i=1}^s b_i \langle U^n, \alpha_h K^n_i \rangle_h - \frac{\tau^2}{2} \sum_{i,j=1}^s b_i a_{ij} \langle K^n_i, \alpha_h K^n_j \rangle_h, \] (2.9)

we use similar transformation to get

\[ \frac{\tau}{2} \sum_{i=1}^s b_i \langle K^n_i, \alpha_h U^n \rangle_h = \frac{\tau}{2} \sum_{i=1}^s b_i \langle K^n_i, \alpha_h U^n \rangle_h - \frac{\tau^2}{2} \sum_{i,j=1}^s b_i a_{ij} \langle K^n_i, \alpha_h K^n_j \rangle_h. \] (2.10)

Substituting (2.9), (2.10) into (2.8) and using the symmetry of \( \alpha_h \) yields

\[ I^{n+1}_h - I^n_h = \frac{\tau}{2} \sum_{i=1}^s b_i \left[ \langle U^n, \alpha_h K^n_i \rangle_h + \langle K^n_i, \alpha_h U^n \rangle_h \right] \]

\[ + \frac{\tau^2}{2} \sum_{i,j=1}^s (b_i b_j - b_i a_{ij} - b_j a_{ij}) \langle K^n_i, \alpha_h K^n_j \rangle_h \]

\[ = \tau \sum_{i=1}^s b_i \langle U^n, \alpha_h K^n_i \rangle_h + \frac{\tau^2}{2} \sum_{i,j=1}^s (b_i b_j - b_i a_{ij} - b_j a_{ij}) \langle K^n_i, \alpha_h K^n_j \rangle_h. \]

The condition (2.6) and the equation

\[ \langle U^n, \alpha_h K^n_i \rangle_h = \langle U^n, \alpha_h (U^n) \rangle_h = 0 \]

give \( I^{n+1}_h = I^n_h \). This completes the proof. \( \square \)

Theorem 2.4. If \( p = 2 \), Scheme 2.1 conserves the discrete mass — viz.

\[ M^{n+1}_h = M^n_h, \quad M^n_h = \langle U^n, 1 \rangle_h, \quad n = 0, 1, 2, \ldots \] (2.11)
Proof. It follows from (2.5) that
\[
M_{n+1}^h - M_n^h = \tau \sum_{i=1}^s b_i \langle \mathcal{K}_i^n, 1 \rangle_h
\]
\[
= \tau \sum_{i=1}^s b_i \langle\alpha_h^{-1}\mathcal{S}_h(U^{ni})U^{ni}, 1 \rangle_h = \tau \sum_{i=1}^s b_i \langle\mathcal{S}_h(U^{ni})U^{ni}, 1 \rangle_h
\]
\[
= -\tau \sum_{i=1}^s b_i \left(\kappa D_1 U^{ni} + b D_3 U^{ni} + \frac{p\beta}{p+1} ((U^{ni})^{p-1} \cdot D_1 U^{ni} + D_1 ((U^{ni})^p)), 1 \right)_h
\]
\[
= -\frac{p\beta}{p+1} \langle D_1 U^{ni}, (U^{ni})^{p-1} \rangle_h.
\]
If \(p = 2\), the antisymmetry of \(D_1\) gives
\[
\langle D_1 U^{ni}, (U^{ni})^{p-1} \rangle_h = 0,
\]
and (2.11) follows. \(\square\)

Remark 2.3. If the initial condition \(u_0(x)\) is sufficiently smooth, then it follows from (2.7) that the numerical solution of Scheme 2.1 satisfies
\[
\sqrt{\|U^n\|^2_h + \delta \langle -D_2 U^n, U^n \rangle_h + \alpha \langle D_4 U^n, U^n \rangle_h}
= \sqrt{\|U^0\|^2_h + \delta \langle -D_2 U^0, U^0 \rangle_h + \alpha \langle D_4 U^0, U^0 \rangle_h} \leq C.
\]
Since \(\alpha > 0\) and \(\delta > 0\), this implies
\[
\|U^n\|_h \leq C, \quad \langle -D_2 U^n, U^n \rangle_h \leq C, \quad \langle D_4 U^n, U^n \rangle_h \leq C,
\]
i.e. the above terms are uniformly bounded. Thus, Scheme 2.1 is unconditionally stable.

Remark 2.4. If we take \(c_1, c_2, \cdots, c_s\) as the zeros of the \(s\)-th shifted Legendre polynomial \(d^s/dx^s(x^s(x-1)^s)\), the RK (or collocation) method based on these nodes has the order \(2s\) and satisfies the condition (2.6), cf. [21, 35, 36] and references therein. In particular, the RK coefficients for \(s = 2\) and \(s = 3\), respectively denoted by 4-th and 6-th-Gauss methods, are given in Table 1, cf. Ref. [21]. In addition, we introduce the following notations:

- 4-th MPS: Using the 4-th Gauss method in Scheme 2.1.
- 6-th MPS: Using the 6-th Gauss method in Scheme 2.1.

2.2. High-order energy-preserving schemes

In this section, we consider a class of high-order energy-preserving schemes for the Eq. (1.1). Inspired by [16, 27, 41], we first introduce appropriate quadratic auxiliary variables to reformulate the Hamiltonian energy into a quadratic form. For clarity, we take \(p = 2, 3\) and 5 as examples to expound this procedure — viz.
Table 1: Gauss methods of orders 4 and 6.

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Case I. If \( p = 2 \), we set

\[ q := q(x, t) = u^2. \]  

(2.12)

Then, the Hamiltonian energy (1.3) is rewritten into

\[ \mathcal{H}(t) = \int_{\Omega} \left( \frac{\kappa}{2} u_x^2 - \frac{b}{2} u_x^2 + \frac{\beta}{3} u q \right) dx, \quad t \geq 0, \]

and according to the energy variational principle, we obtain the following reformulated system from (1.2):

\[ \partial_t u = \mathcal{J}(\kappa u + bu_{xx} + \frac{\beta}{3} u q + \frac{2\beta}{3} u^2), \]

\[ \partial_t q = 2u \cdot \partial_x u \]

(2.13)

with the consistent initial conditions

\[ u(x, 0) = u_0(x), \quad q(x, 0) = (u_0(x))^2. \]  

(2.14)

Case II. If \( p = 3 \), the quadratic auxiliary variable is introduced as (2.12), and the quadratic energy is given by

\[ \mathcal{H}(t) = \int_{\Omega} \left( \frac{\kappa}{2} u_x^2 - \frac{b}{2} u_x^2 + \frac{\beta}{4} u^2 q \right) dx, \quad t \geq 0, \]

which implies that the reformulated system is given by

\[ \partial_t u = \mathcal{J}(\kappa u + bu_{xx} + \beta u q), \]

\[ \partial_t q = 2u \cdot \partial_x u \]

with the consistent initial conditions (2.14).
Case III. If $p = 5$, we introduce auxiliary variables

$$q_1 := q_1(x, t) = u^2, \quad q_2 := q_2(x, t) = uq_1,$$

and then rewrite the Hamiltonian energy (1.3) as

$$\mathcal{H}(t) = \int_\Omega \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{6} q_2^2 \right) dx, \quad t \geq 0.$$ 

Similarly, we obtain reformulated system from (1.2), as follows:

$$\partial_t u = \mathcal{G} \left( \kappa u + bu_{xx} + \frac{\beta}{3} q_2 (q_1 + 2u^2) \right),$$

$$\partial_t q_1 = 2u \cdot \partial_t u,$$

$$\partial_t q_2 = \partial_t u \cdot q_1 + u \cdot \partial_t q_1$$

with the consistent initial conditions

$$u(x, 0) = u_0(x), \quad q_1(x, 0) = \left( u_0(x) \right)^2, \quad q_2(x, 0) = u_0(x, 0) \cdot q_1(x, 0).$$

For simplicity, in the following construction of the energy-preserving schemes, we consider the parameter $p = 2$ for the Eq. (1.1). Note that the extensions to the parameter $p > 2$ are straightforward.

**Theorem 2.5.** Under periodic boundary conditions, the reformulated system (2.13) conserves the mass (1.4).

**Proof.** It follows from the first equation in (2.13) and the periodicity of boundary condition that

$$\frac{d}{dt} \mathcal{M}(t) = (\partial_t u, 1) = \left( \mathcal{G} \left( \kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2 \right), 1 \right)$$

$$= -\left( \partial_x \left( \kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2 \right), (1 - \delta \partial_{xx} + \alpha \partial^2_{xxxx})^{-1} \right)$$

$$= -\left( \partial_x \left( \kappa u + bu_{xx} + \frac{\beta}{3} q + \frac{2\beta}{3} u^2 \right), 1 \right) = 0.$$

This completes the proof. \qed

**Theorem 2.6.** Under periodic boundary conditions, the reformulation (2.13) preserves the following invariants:

$$\mathcal{H}_{1,1}(x, t) = q - u^2 = 0, \quad x \in \Omega, \quad t \geq 0,$$

$$\mathcal{H}(t) = \int_\Omega \left( \frac{\kappa}{2} u^2 - \frac{b}{2} u_x^2 + \frac{\beta}{3} uq \right) dx, \quad t \geq 0.$$
Proof: It follows from the second equation of (2.13) that
\[ \partial_t \mathcal{H}_{1,1}(x, t) = \partial_t q - 2u \cdot \partial_q u = 0. \]
The relations (2.13) and the anti-adjoint property of \( \mathcal{J} \) give
\[
\frac{d}{dt} \mathcal{H}(t) = \int_{\Omega} \left( \kappa u \partial_t u - b \partial_x u \partial_{xx} u + \frac{\beta}{3} (q \partial_t u + u \partial_q q) \right) dx \\
= \int_{\Omega} \left( \kappa u \partial_t u + b \partial_{xx} u \partial_t u + \frac{\beta}{3} (q + 2u^2) \partial_t u \right) dx \\
= \int_{\Omega} \left( \kappa u + \partial_{xx}u + \frac{\beta}{3} (q + 2u^2) \right) \mathcal{J} \left( \kappa u + bu_{xx} + \frac{\beta}{3} (q + 2u^2) \right) dx = 0.
\]
This completes the proof. \( \square \)

Next, we apply an RK method to the system (2.13) in time and the Fourier pseudo-spectral method in space, thus obtaining a class of the following fully discrete schemes for (2.13).

**Scheme 2.2.** Let \( b_i, a_{ij}, i, j = 1, \ldots, s \) be real numbers and let \( c_i = \sum_{j=1}^{s} a_{ij} \). For a given \((U^n, Q^n)\), the \( s \)-stage Runge-Kutta method has the form
\[
K_i^n = \mathcal{J} \left( \kappa U^{ni} + b D_2 U^{ni} + \frac{\beta}{3} (Q^{ni} + 2(U^{ni})^2) \right), \\
U^{ni} = U^n + \tau \sum_{j=1}^{s} a_{ij} K_i^n, \quad \mathcal{J} = -(I - \delta D_2 + \alpha D_4)^{-1} D_1, \\
Q^{ni} = Q^n + \tau \sum_{j=1}^{s} a_{ij} L_i^n, \quad L_i^n = 2U^{ni} \cdot K_i^n, \quad i = 1, 2, \ldots, s,
\]
and \((U^{n+1}, Q^{n+1})\) is then updated by
\[
U^{n+1} = U^n + \tau \sum_{i=1}^{s} b_i K_i^n, \quad Q^{n+1} = Q^n + \tau \sum_{i=1}^{s} b_i L_i^n. \tag{2.16}
\]

**Theorem 2.7.** If the coefficients of (2.15) and (2.16) satisfy (2.6), Scheme 2.2 preserves the following discrete invariants:
\[
H_{1,1}^{n+1} = H_{1,1}^n = 0, \quad H_{1,1}^n = Q^n - (U^n)^2, \tag{2.17}
\]
\[
\mathcal{E}_h^{n+1} = \mathcal{E}_h^n, \quad \mathcal{E}_h^n = \frac{K}{2} \|U^n\|^2 + \frac{b}{2} \langle D_2 U^n, U^n \rangle_h + \frac{\beta}{3} (U^n, Q^n)_h, \quad n = 0, 1, \ldots. \tag{2.18}
\]

Proof: It follows from (2.16) that
\[
H_{1,1}^{n+1} - H_{1,1}^n = Q^{n+1} - Q^n - (U^{n+1})^2 + (U^n)^2
\]
\[
= \tau \sum_{j=1}^{s} b_j L_i^n - \tau \sum_{i=1}^{s} b_i K_i^n \cdot U^n - \tau \sum_{i=1}^{s} b_i U^n \cdot K_i^n - \tau^2 \sum_{i,j=1}^{s} b_i b_j K_i^n \cdot K_j^n.
\]
Using the equality \( U^n = U^{ni} - \tau \sum_{j=1}^{n} a_{ij} K^n \) and \( L_i^n = 2U^{ni} \cdot K^n_i \), together with (2.6), we can obtain from the above equation

\[
H_{1,1}^{n+1} - H_{1,1}^n = \tau \sum_{j=1}^{s} b_j L_i^n - 2\tau \sum_{i=1}^{s} b_i K_i^n \cdot U^{ni} - \tau^2 \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) K_i^n \cdot K_j^n = 0.
\]

(2.19)

With noting the initial condition \( Q^0 - (U^0)^2 = 0 \), we obtain (2.17) from (2.19). According to (2.15) and (2.16), we have

\[
\epsilon_{\theta}^{n+1} - \epsilon_{\theta}^n = \frac{K}{2} \langle U^{n+1}, U^{n+1} \rangle_h + \frac{b}{2} \langle D_2 U^{n+1}, U^{n+1} \rangle_h + \frac{\beta}{3} \langle U^{n+1}, Q^{n+1} \rangle_h
\]

\[- \frac{K}{2} \langle U^n, U^n \rangle_h - \frac{b}{2} \langle D_2 U^n, U^n \rangle_h - \frac{\beta}{3} \langle U^n, Q^n \rangle_h
\]

\[= \frac{\tau K}{2} \sum_{i=1}^{s} b_i \left( \langle U^n, K^n_i \rangle_h + \langle K^n_i, U^n \rangle_h \right) + \frac{\tau^2 K}{2} \sum_{i,j=1}^{s} b_i b_j \langle K^n_i, K^n_j \rangle_h
\]

\[+ \frac{\tau b}{2} \sum_{i=1}^{s} b_i \left( \langle D_2 U^n, K^n_i \rangle_h + \langle D_2 K^n_i, U^n \rangle_h \right) + \frac{\tau^2 b}{2} \sum_{i,j=1}^{s} b_i b_j \langle D_2 K^n_i, K^n_j \rangle_h
\]

\[+ \frac{\tau \beta}{3} \sum_{i=1}^{s} b_i \left( \langle L_i^n, K^n_i \rangle_h + \langle Q^n, K^n_i \rangle_h \right) + \frac{\tau^2 \beta}{3} \sum_{i,j=1}^{s} b_i b_j \langle K^n_i, L^n_j \rangle_h
\]

\[= \tau \sum_{i=1}^{s} b_i \left( \kappa \langle U^{ni}, K^n_i \rangle_h + b \langle D_2 U^{ni}, K^n_i \rangle_h + \frac{\beta}{3} \langle U^{ni}, L_i^n \rangle_h + \frac{\beta}{3} \langle Q^{ni}, K^n_i \rangle_h \right)
\]

\[+ \frac{\tau^2 \kappa}{2} \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle K^n_i, K^n_j \rangle_h
\]

\[+ \frac{\tau^2 b}{2} \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle D_2 K^n_i, K^n_j \rangle_h
\]

\[+ \frac{\tau^2 \beta}{3} \sum_{i,j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle K^n_i, L^n_j \rangle_h.
\]

With the condition (2.6) and the antisymmetry of \( \mathcal{J}_h \), we have

\[
\kappa \langle U^{ni}, K^n_i \rangle_h + b \langle D_2 U^{ni}, K^n_i \rangle_h + \frac{\beta}{3} \langle U^{ni}, L_i^n \rangle_h + \frac{\beta}{3} \langle Q^{ni}, K^n_i \rangle_h
\]

\[= \kappa \langle U^n, K^n_i \rangle_h + b \langle D_2 U^n, K^n_i \rangle_h + \frac{2\beta}{3} \langle (U^{ni})^2, K^n_i \rangle_h + \frac{\beta}{3} \langle Q^{ni}, K^n_i \rangle_h
\]

\[= \left( \kappa U^n + b D_2 U^n + \frac{2\beta}{3} (U^{ni})^2 + \frac{\beta}{3} Q^{ni} \right) \langle K^n_i, \mathcal{J}_h K^n_i \rangle_h = 0,
\]

which implies \( \epsilon_{\theta}^{n+1} = \epsilon_{\theta}^n \). This completes the proof. \( \square \)
Remark 2.5. It follows from (2.17) and (2.18) that under the condition (2.6), Scheme 2.2 can conserve the following discrete Hamiltonian energy:

\[ H^n_h = \frac{\kappa}{2} \| U^n \|^2_h + \frac{b}{2} (D_2 U^n, U^n)_h + \frac{b}{3} \langle (U^n)^3, 1 \rangle_h, \quad n = 0, 1, 2, \ldots. \]

Theorem 2.8. For any RK method, Scheme 2.2 conserves the following discrete mass:

\[ M_{n+1}^n = M^n_n, \quad M^n_n = \langle U^n, 1 \rangle_h, \quad n = 0, 1, 2, \ldots. \]

Proof. It follows from the first equation of (2.16) that

\[ M_{n+1}^n = M^n_n + \tau \sum_{i=1}^s b_i \langle K^n_i, 1 \rangle_h. \]

Therefore, using the first equation in (2.15), we obtain

\[ \langle K^n_i, 1 \rangle_h = -\langle D_1 \left( \kappa U^{ni} + b D_2 U^{ni} + \frac{b}{3} (Q^{ni} + 2(U^{ni})^2) \right), K^{-1}_h \rangle_h \]

and the proof is complete. \qed

Remark 2.6. Here we introduce the following notations:

- 4-th EPS: Using the 4-th Gauss method in Scheme 2.2.
- 6-th EPS: Using the 6-th Gauss method in Scheme 2.2.

3. Implementation of Numerical Schemes

It turns out, there is no existing explicit RK methods that satisfy the condition (2.6), cf. [13, Proposition 7.1.1]. Therefore, motivated by [11, 50], we propose an efficient fixed-point iteration solver for the nonlinear equations in the numerical schemes under consideration. For simplicity, we consider the 4-th MP scheme with the RK coefficients given in Table 1. Note that the extensions to \( s > 2 \) are straightforward.

According to Scheme 2.1, for a given \( U^n \), the 4-th MP scheme is equivalent to

\[ .s_h K^n_1 = -\left[ \kappa D_1 U^{n1} + b D_2 U^{n1} + \frac{p b}{p + 1} (\langle (U^{n1})^{p-1} \cdot D_1 U^{n1} + D_1 ((U^{n1})^p) \rangle) \right], \quad (3.1) \]

\[ .s_h K^n_2 = -\left[ \kappa D_1 U^{n2} + b D_2 U^{n2} + \frac{p b}{p + 1} (\langle (U^{n2})^{p-1} \cdot D_1 U^{n2} + D_1 ((U^{n2})^p) \rangle) \right], \quad (3.2) \]

\[ U^{n1} = U^n + \tau a_{11} K^n_1 + \tau a_{12} K^n_2, \quad U^{n2} = U^n + \tau a_{21} K^n_1 + \tau a_{22} K^n_2, \quad (3.3) \]

\[ U^{n+1} = U^n + \tau b_1 K^n_1 + \tau b_2 K^n_2. \]
It follows from (3.1)-(3.3) that

\[ \phi h K_1^n = -\left[ \tau \kappa a_{11} D_1 K_1^n + \tau \kappa a_{12} D_1 K_2^n + \tau b a_{11} D_3 K_1^n + \tau b a_{12} D_3 K_2^n + F_1 \right], \]
\[ \phi h K_2^n = -\left[ \tau \kappa a_{21} D_1 K_1^n + \tau \kappa a_{22} D_1 K_2^n + \tau b a_{21} D_3 K_1^n + \tau b a_{22} D_3 K_2^n + F_2 \right], \]

where

\[ F_1 = \kappa D_1 U^n + b D_3 U^n + \frac{p \beta}{p + 1} ((U_1^{n+1} - 1) D_1 U^{n+1} + D_1 ((U_1^{n+1})^p)), \]
\[ F_2 = \kappa D_1 U^n + b D_3 U^n + \frac{p \beta}{p + 1} ((U_2^{n+1} - 1) D_1 U^{n+1} + D_1 ((U_2^{n+1})^p)). \]

Using Remark 2.1, we obtain from (3.4)-(3.5) that

\[ A_h K_1^n = -\left[ \tau \kappa a_{11} \tilde{A}_h K_1^n + \tau \kappa a_{12} \tilde{A}_h K_2^n + \tau b a_{11} \tilde{A}_h K_1^n + \tau b a_{12} \tilde{A}_h K_2^n + \tilde{W}_1 \right], \]
\[ A_h K_2^n = -\left[ \tau \kappa a_{21} \tilde{A}_h K_1^n + \tau \kappa a_{22} \tilde{A}_h K_2^n + \tau b a_{21} \tilde{A}_h K_1^n + \tau b a_{22} \tilde{A}_h K_2^n + \tilde{W}_2 \right], \]

where \( A_h = I - \delta \Lambda^2 + \alpha \Lambda^4 \) and \( \tilde{W} = \tilde{\mathcal{S}}_h W \).

For nonlinear algebraic equations as above, we apply the following fixed-point iteration strategy, for \( l = 0, 1, 2, \ldots, M - 1 \):

\[
\begin{bmatrix}
A_h + \tau \kappa a_{11} \tilde{A}_h + \tau b a_{11} \tilde{A}_h^3 \\
\tau \kappa a_{21} \tilde{A}_h + \tau b a_{21} \tilde{A}_h^3
\end{bmatrix}
\begin{bmatrix}
\frac{K_1^{n+1}}{K_1^n} \\
\frac{K_2^{n+1}}{K_2^n}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{F_1}{F_2} \\
\frac{F_2}{F_2}
\end{bmatrix},
\]

which implies that

\[
\begin{bmatrix}
(A_h)_{j} + \tau \kappa a_{11} \tilde{A}_h_{j} + \tau b a_{11} \tilde{A}_h^3_{j} \\
\tau \kappa a_{21} \tilde{A}_h_{j} + \tau b a_{21} \tilde{A}_h^3_{j}
\end{bmatrix}
\begin{bmatrix}
\frac{K_1^{n+1}}{K_1^n} \\
\frac{K_2^{n+1}}{K_2^n}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{F_1}{F_2} \\
\frac{F_2}{F_2}
\end{bmatrix},
\]

where \( j = 0, 1, \ldots, N - 1 \).

Solving the above equations, we obtain \( K_1^{n+1} \) and \( K_2^{n+1} \). Then \( K_1^{n+1} \) and \( K_2^{n+1} \) are given by \( K_1^{n+1} = \tilde{F}_N^{n+1} \) and \( K_2^{n+1} = \tilde{F}_N^{n+1} \), respectively. In our computations, we take the iterative initial value \( K_1^{0, n} = U^n \) and \( K_2^{0, n} = U^n \). The iteration terminates when the number of maximum iterative step \( M = 30 \) is reached or the infinity norm of the error between two adjacent iterative steps is less than \( 10^{-14} \), i.e. if

\[
\max_{1 \leq i \leq 2} \left\{ \| K_1^{n+1} - K_1^n \|_{\infty, h} \right\} < 10^{-14}.
\]

Finally, we have \( U^{n+1} = U^n + \tau b_1 K_1^{n+1} + \tau b_2 K_2^{n+1} \).

**Remark 3.1.** Similarly, the efficient iteration solver for the resulting nonlinear equations of the 4-th EP scheme (see Scheme 2.2) is given by, as follows:

\[
\begin{bmatrix}
1 - \tau \kappa a_{11} J_h - \tau b a_{11} J_h \Lambda^2 \\
-\tau \kappa a_{21} J_h - \tau b a_{21} J_h \Lambda^2
\end{bmatrix}
\begin{bmatrix}
\frac{K_1^{n+1}}{K_1^n} \\
\frac{K_2^{n+1}}{K_2^n}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{F_1}{F_2} \\
\frac{F_2}{F_2}
\end{bmatrix},
\]
where

\[
F_1 = \mathcal{J}_h \left( \kappa U^n + b D^n U^n + \frac{\beta}{3} (Q^{n1} + 2(U^{n1})^2) \right),
\]

\[
F_2 = \mathcal{J}_h \left( \kappa U^n + b D^n U^n + \frac{\beta}{3} (Q^{n2} + 2(U^{n2})^2) \right),
\]

\[
U^{ni} = U^n + \tau a_{1i} K_1^n + \tau a_{2i} K_2^n, \quad J_h = -(I - \delta \Lambda^2 + a \Lambda^4)^{-1} \Lambda,
\]

\[
Q^{ni} = (U^n)^2 + \tau \sum_{j=1}^s a_{ij} L_j^n, \quad L_i^n = 2U_i^n \cdot K_i^n, \quad i = 1, 2.
\]

In particular, the above iteration equations can be rewritten as the following subsystems:

\[
\begin{bmatrix}
1 - \tau \kappa a_{11} (J_h)_j - \tau b a_{11} (J_h)_j A_j^2 - \tau \kappa a_{12} (J_h)_j - \tau b a_{12} (J_h)_j A_j^2

- \tau \kappa a_{21} (J_h)_j - \tau b a_{21} (J_h)_j A_j^2

1 - \tau \kappa a_{22} (J_h)_j - \tau b a_{22} (J_h)_j A_j^2
\end{bmatrix}
\begin{bmatrix}
(\mathcal{J}_h^{n,l+1})_j

(\mathcal{J}_h^{n,l+1})_j

(\mathcal{J}_h^{n,l+1})_j
\end{bmatrix}
= \begin{bmatrix}
(\mathcal{T}_h^j)

(\mathcal{T}_h^j)

(\mathcal{T}_h^j)
\end{bmatrix},
\]

where \( j = 0, 1, 2, \ldots, N - 1 \). Proceeding similar to 4-th MP scheme, we obtain \( U^{n+1} \).

### 4. Numerical Results

This section is devoted to the study of accuracy and invariant-preservation of the above schemes for the Eq. (1.1). In what follows, 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS are only used for demonstration purposes. In order to quantify the numerical solution, we introduce the following \( l^2 \)- and \( l^\infty \)-norm error functions:

\[
e_2(t_n) = ||u(\cdot, t_n) - u^n||_h, \quad e_\infty(t_n) = ||u(\cdot, t_n) - u^n||_{h, \infty}.
\]

We also investigate the residuals for the mass, the momentum, and the Hamiltonian energy, defined by

\[
Error_{1}^{n} = |M^n_h - M^0_h|, \quad Error_{2}^{n} = |I^n_h - I^0_h|, \quad Error_{3}^{n} = |H^n_h - H^0_h|.
\]

All simulations are performed on a Win10 machine with Intel Core i7 and 32GB using MATLAB R2015b.

#### 4.1. Rosenau-RLW equation

Choosing the parameters \( \kappa = \delta = \alpha = \beta = 1 \) and \( b = 0 \), we reduce the Eq. (1.1) to a generalized Rosenau-RLW equation, which has the exact solution

\[
u(x, t) = \text{Asech}^{\frac{1}{2}}(B(x - C t - x_0)),
\]

where

\[
A = \exp \left( \frac{\ln((p + 3)(3p + 1)(p + 1))/(2(p^2 + 3)(p^2 + 4p + 7))}{p - 1} \right),
\]

\[
B = \frac{p - 1}{\sqrt{4p^2 + 8p + 20}}, \quad C = \frac{p^4 + 4p^3 + 14p^2 + 20p + 25}{p^4 + 4p^3 + 10p^2 + 12p + 21}.
\]
and \( x_0 \) represents the initial phase of the solution and the periodic boundary condition is considered \([4,31]\).

We first verify the convergence order in time. Considering the computational domain \( \Omega = [-200, 200] \), we take the initial value

\[
u(x, 0) = \text{Asech}^{-1} \left( B(x - x_0) \right), \quad x \in \Omega
\]

with the initial phase \( x_0 = 0 \).

The temporal convergence are investigated by fixing the Fourier node 2048. We compute the numerical solutions at \( t = 10 \) using 4-th MPS and 4-th EPS with time step sizes \( \tau = 2^{-k}, \quad k = 2, 3, 4, 5, 6 \) and 6-th MPS and 6-th EPS with time step sizes \( \tau = 2^{-k}, \quad k = 0, 1, 2, 3, 4 \). The relations between the \( l^2 \)- and \( l^\infty \)-norm errors and time step sizes are summarized in Fig. 1, where the up picture corresponds to the parameter \( p = 2 \), the middle to \( p = 3 \), and the bottom to \( p = 5 \). The fourth-order temporal accuracy for 4-th MPS and 4-th EPS and the sixth-order for 6-th MPS and 6-th EPS are observed for different parameters \( p \), as desired.

Next we choose the initial value

\[
u(x, 0) = \exp \left( -0.05(x - 40)^2 \right), \quad x \in \Omega
\]

where \( \Omega = [-50, 250] \) is the computational domain with the periodic boundary condition. We take the uniform spatial size \( h = 1 \) and time step \( \tau = 0.1 \). Applying 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS, we compute the profile of \( u \) at \( T = 100 \) by using different parameters \( p \), the residuals of discrete mass, momentum and Hamiltonian energy. Fig. 2 shows the profile of \( u \) determined by 4-th MPS at \( t = 100 \). The influence of the parameter \( p \) on the dispersion wave propagation is similar to \([12]\). We should note that the profiles of \( u \) at \( t = 100 \) computed by other schemes are similar to Fig. 2. Therefore, they are not presented here. Fig. 3 shows the evolutions of the residuals on the discrete mass, momentum and Hamiltonian energy of numerical solutions determined by 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS, respectively. We can reveal the following observations:

(1) For \( p = 2 \), all schemes preserve the discrete mass exactly, while the momentum-preserving schemes do not conserve the discrete mass for \( p = 3 \) and \( p = 5 \).

(2) 4-th MPS and 6-th MPS can preserve the discrete momentum up to the machine precision.

(3) For all parameters \( p \), 4-th EPS and 6-th EPS conserve both the discrete mass and the Hamiltonian energy.

4.2. Rosenau-KdV equation

As the parameters \( \kappa = b = \alpha = 1 \) and \( \delta = 0 \) are chosen, the Eq. (1.1) reduces to the generalized Rosenau-KdV equation \([51]\). For simplicity, we also consider the parameters \( p = 2, \quad p = 3 \) and \( p = 5 \), respectively, where the exact solution can be given respectively, by
Figure 1: The $l^2$ and $l^\infty$-norm errors vs. the time step sizes provided by the proposed 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS for the Rosenau-RLW equation with different parameters $p$, respectively.

Figure 2: The profile of $u$ provided by 4-th MPS with different parameters $p$ at $t = 100$, where the uniform spatial and time step size is chosen as $\tau = h = 0.1$ for the Rosenau-RLW equation.
Figure 3: The residuals on the discrete mass, momentum and Hamiltonian energy from $t = 0$ to $t = 1000$ provided by the proposed 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS for the Rosenau-RLW equation with different parameters $p$, respectively.

**Case I.** As $\beta = 1/2$ and $p = 2$, the Rosenau-KdV equation has an exact solution [23]

$$u(x, t) = k_{11}\text{sech}^4(k_{12}(x - k_{13}t)),$$

where $k_{11} = -35/24 + 35/312\sqrt{313}$, $k_{12} = 1/24\sqrt{-26 + 2\sqrt{313}}$, $k_{13} = 1/2 + \sqrt{313}/26$.

**Case II.** If $\beta = 1$ and $p = 3$, the Rosenau-KdV equation admits an exact solution [33]

$$u(x, t) = k_{21}\text{sech}^2(k_{22}(x - k_{23}t)),$$

where $k_{21} = 1/4\sqrt{-15 + 3\sqrt{41}}$, $k_{22} = 1/4\sqrt{(-5 + \sqrt{41})/2}$, $k_{23} = 1/10(5 + \sqrt{41})$.

**Case III.** When $\beta = 1$ and $p = 5$, the exact solution of the Rosenau-KdV equation is given by [51]

$$u(x, t) = k_{31}\text{sech}^2(k_{32}(x - k_{33}t)),$$

where $k_{31} = 4\sqrt{4/15(-5 + \sqrt{34})}$, $k_{32} = 1/3\sqrt{-5 + \sqrt{34}}$, $k_{33} = 1/10(5 + \sqrt{34})$.
First of all, we verify the convergence order in time for the selected four structure-preserving schemes. Let us set the computational domain $\Omega = [-200, 200]$ and take the initial value as the exact solution at $t = 0$ for the parameters $p = 2$, $p = 3$ and $p = 5$. Temporal convergence tests are conducted by fixing the Fourier node 2048. Fig. 4 shows the $l^2$- and $l^\infty$-norm errors with various time step sizes at $t = 10$, where we take time steps $\tau = 2^{-k}, k = 2, 3, 4, 5, 6$ for 4-th MPS and 4-th EPS, while for 6-th MPS and 6-th EPS, we choose time steps $\tau = 2^{-k}, k = 0, 1, 2, 3, 4$. It is clear that 4-th MPS and 4-th EPS are of fourth-order temporal accuracy, and 6-th MPS and 6-th EPS can achieve sixth-order accuracy in time.

Then, we set the computational domain $\Omega = [-100, 100]$ and take the uniform spatial step $h = 200/512$ and time step $\tau = 0.1$. Fig. 5 shows the residuals for the discrete mass,
Figure 5: The residuals on the mass, momentum and Hamiltonian energy from \( t = 0 \) to \( t = 10000 \) provided by the proposed 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS for the Rosenau-RdV equation with different parameters \( p \), respectively.

momentum and Hamiltonian energy determined by 4-th MPS, 4-th EPS, 6-th MPS and 6-th EPS. They are similar to the ones in Fig. 3. We should note that the residuals for the discrete mass provided by 4-th MPS and 6-th MPS are also up to the machine precision because of the fine spatial mesh.

4.3. Comparison with other methods

The following numerical experiments are aimed to compare numerical errors in time and the computational efficiency of the proposed structure-preserving schemes, the linearized Crank-Nicolson momentum-preserving scheme (LCN-MPS) [25], and fourth-order energy-preserving schemes YC-EPS and SC-EPS [3].

We consider the Rosenau-KdV equation, but only with the parameters \( \beta = 1/2 \) and \( p = 2 \) (Case I in Section 4.2) for the sake of simplicity. We choose \( \Omega = [-100, 100] \) as the computational domain and the uniform spatial mesh \( h = 200/1024 \). Table 2 shows numerical errors and convergence order in time for the different schemes with various time steps at \( t = 1 \). Note that:
Table 2: Numerical errors and convergence order for the different schemes with various time steps at $t = 1$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$\tau$</th>
<th>$e_2(t_n = 1)$</th>
<th>order</th>
<th>$e_\infty(t_n = 1)$</th>
<th>order</th>
</tr>
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<tr>
<td>4-th EPS</td>
<td>1/10</td>
<td>1.585e-09</td>
<td>-</td>
<td>6.292e-10</td>
<td>-</td>
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<td></td>
<td>1/20</td>
<td>9.905e-11</td>
<td>4.000</td>
<td>3.933e-11</td>
<td>4.000</td>
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<td></td>
<td>1/40</td>
<td>6.191e-12</td>
<td>4.000</td>
<td>2.458e-12</td>
<td>4.000</td>
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<tr>
<td></td>
<td>1/80</td>
<td>3.867e-13</td>
<td>4.001</td>
<td>1.540e-13</td>
<td>3.997</td>
</tr>
<tr>
<td>4-th MPS</td>
<td>1/10</td>
<td>1.518e-09</td>
<td>-</td>
<td>5.984e-10</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1/20</td>
<td>9.490e-11</td>
<td>4.000</td>
<td>3.740e-11</td>
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<td></td>
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<td>4.000</td>
<td>2.338e-12</td>
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<tr>
<td></td>
<td>1/80</td>
<td>3.706e-13</td>
<td>4.000</td>
<td>1.462e-13</td>
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<td>LCN-MPS [25]</td>
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<td>2.800e-05</td>
<td>-</td>
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<tr>
<td></td>
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<td>1.813e-05</td>
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<td>6.950e-06</td>
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<td>1.731e-06</td>
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<tr>
<td></td>
<td>1/80</td>
<td>1.128e-06</td>
<td>2.002</td>
<td>4.319e-07</td>
<td>2.003</td>
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<td>YC-EPS [3]</td>
<td>1/10</td>
<td>8.024e-08</td>
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<td>3.400e-08</td>
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<td>5.024e-09</td>
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<td>SC-EPS [3]</td>
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<td>1.081e-09</td>
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</tbody>
</table>

(i) LCN-MPS is second-order accurate in time and its numerical errors are largest.

(ii) All 4-th MPS, 4-th EPS, YC-EPS and SC-EPS are of fourth-order accuracy in time, and the numerical error provided by SC-EPS is smallest, while the ones provided by YC-EPS are much larger than other fourth-order schemes.

Finally, choosing the computational domain $\Omega = [-300, 300]$ and the Fourier node $2^{13}$, we then investigate the global $l^2$- and $l^\infty$- errors of $u$ versus the CPU times for the five selected structure-preserving schemes with various time steps at $t = 30$. The results are summarized in Fig. 6. For a given global error, we observe that

1. The cost of the LCN-MPS is the most expensive because of the low-order accuracy in time.
2. The cost of 4-th EPS is the cheapest.
3. The cost of SC-EPS is much cheaper than the one provided by the 4-th MPS, and the cost of 4-th MPS is much cheaper than the one provided by the YC-EPS.
4. If $p$ increases, more QAV variables should be introduced. As the result, the computational cost of the high-order energy-preserving schemes also increase.
Figure 6: The $l^2$ and $l^\infty$-norm errors vs. the CPU times provided by LCN-MPS, 4-th MPS, 4-th EPS, YC-EPS and SC-EPS for the Rosenau-KdV equation.

5. Conclusions

We propose two types of high-order structure-preserving schemes for the generalized Rosenau-type equation (1.1). In particular, there are schemes, which conserve the discrete momentum conservation law, which is based on the use of the symplectic RK method in time and the standard Fourier pseudo-spectral method in space, respectively. The others conserve the discrete Hamiltonian energy and mass, where the main idea is based on the combination of the QAV approach with the symplectic RK method in time, together with the standard Fourier pseudo-spectral method in space. Extensive numerical tests and comparisons are also addressed to verify the performance of the proposed schemes.

We conclude this paper with two main remarks. First, we note that the construction of the energy-preserving schemes should be discussed case by case, and the proposed momentum-preserving schemes are not mass-conserving if the parameter $p > 2$. Thus, in practical computations, such trade-offs between two classes of schemes shall be treated carefully. Second, as far as we know, there are some works on optimal error estimates of EQ schemes [9, 29, 38, 44] and Fourier pseudo-spectral methods [7, 17, 19, 49], but to the best of our knowledge, the error estimates of high-order structure-preserving Fourier pseudo-spectral schemes are still not available. Therefore, optimal error estimates for the proposed schemes is an interesting topic for future studies. In fact, the uniformly bounded of numerical solutions in $l^\infty$-norm can be obtained by using the discrete momentum (2.7) and Sobolev imbedding theorems [20], thus our first attempt will focus on the high-order momentum-preserving schemes, followed by the energy-preserving schemes.

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