Journal of Computational Mathematics
Vol.xx, No.x, 2023, 1–31.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2304-m2022-0243

# EFFICIENT SPECTRAL METHODS FOR EIGENVALUE PROBLEMS OF THE INTEGRAL FRACTIONAL LAPLACIAN ON A BALL OF ANY DIMENSION*

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**Abstract**

An efficient spectral-Galerkin method for eigenvalue problems of the integral fractional Laplacian on a unit ball of any dimension is proposed in this paper. The symmetric positive definite linear system is retained explicitly which plays an important role in the numerical analysis. And a sharp estimate on the algebraic system's condition number is established which behaves as $N^{4s}$ with respect to the polynomial degree $N$, where $2s$ is the fractional derivative order. The regularity estimate of solutions to source problems of the fractional Laplacian in arbitrary dimensions is firstly investigated in weighted Sobolev spaces. Then the regularity of eigenfunctions of the fractional Laplacian eigenvalue problem is readily derived. Meanwhile, rigorous error estimates of the eigenvalues and eigenvectors are obtained. Numerical experiments are presented to demonstrate the accuracy and efficiency and to validate the theoretical results.

*Mathematics subject classification:* 65N35, 65N25.
*Key words:* Integral fractional Laplacian, Spectral method, Eigenvalue problem, Regularity analysis, Error estimate.

## 1. Introduction

Nonlocal operators have been an active area of research in different branches of mathematics. These operators arise in many applications such as image processing, finance, electromagnetic fluids, peridynamics, and porous media flow [7, 13, 23, 24, 33, 36], among which the fractional Laplace operator is of common interests of mathematicians and physicists.

* Received October 31, 2022 / Revised version received January 11, 2023 / Accepted April 3, 2023 / Published online August 28, 2023 /
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In this work, we consider eigenvalue problems concerning the integral fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ on the unit ball of any dimension

$$\begin{cases}(-\Delta)^s u = \lambda u, & \boldsymbol{x} \in \mathbb{B}^d,\\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^d \setminus \mathbb{B}^d.\end{cases} \tag{1.1}$$

Here the fractional Laplacian is defined in singular integral [4]

$$(-\Delta)^s u(\boldsymbol{x}) = C(d,s)\int_{\mathbb{R}^d} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}}\mathrm{d}\boldsymbol{y}, \quad C(d,s) = \frac{2^{2s}s\Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1.2}$$

It can also be equivalently defined via a pseudodifferential operator of symbol $\|\boldsymbol{\xi}\|^{2s}$ in the Fourier space. Indeed, for a function $u$ of the Schwartz class,

$$(-\Delta)^s u(\boldsymbol{x}) = \big[\mathscr{F}^{-1}\big(\|\boldsymbol{\xi}\|^{2s}\widehat{u}(\boldsymbol{\xi})\big)\big](\boldsymbol{x}), \tag{1.3}$$

where we denote by $\mathscr{F}f$ or simply by $\widehat{f}$ the Fourier transform of any function $f(\boldsymbol{x}) \in L^2(\mathbb{R}^d)$, and denote by $\mathscr{F}^{-1}\widehat{f}$ the inversion of the Fourier transform

$$\widehat{f}(\boldsymbol{\xi}) = [\mathscr{F}f](\boldsymbol{\xi}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x},$$
$$f(\boldsymbol{x}) = \big[\mathscr{F}^{-1}\widehat{f}\,\big](\boldsymbol{x}) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{\xi}.$$

This eigenvalue problem is closely related to fractional quantum mechanics such as the fractional Schrödinger equation. In this regard, eigenfunctions of the fractional Laplacian correspond to the energy states of the system being modeled [31]. Many researchers have shown their interest in this kind of fractional problem from the physical, mathematical and computational point of view. Most of existing studies focus on the theoretical research [9,15,19,22,26]. Kwaśnicki [30] introduced the Weyl's asymptotic law for the eigenvalues of the one-dimensional fractional Laplace operator $(-\Delta)^s$ on the interval $(-1,1)$ with the zeros exterior boundary conditions: The $n$-th eigenvalue $\lambda_n$ is equal to $(n\pi/2 - (2-2s)\pi/8)^{2s} + \mathcal{O}(1/n)$. Chen *et al.* [14] and DeBlassie [17] have derived the estimate for the $n$-th eigenvalue $\lambda_n$ on a bounded convex domain in $\mathbb{R}^d$ is $(n\pi/2)^{2s}/2 \le \lambda_n \le (n\pi/2)^{2s}$. Owing to the non-locality of the fractional Laplacian, it is usually impossible to obtain analytically a closed expression for the eigenfunctions, and it is also hard to precisely specify the behavior of an eigenfunction near the boundary of the unit ball. This motivates researchers to carry out numerical studies on eigenvalue problems of the fractional Laplacian. Borthagaray *et al.* [10] have studied the finite element approximation for one- and two-dimensional eigenvalue problems of the fractional Laplacian in which it showed the eigenfuncions belonged to $H^{s+1/2-\varepsilon}$ for any $\varepsilon > 0$, and the conforming finite element method exhibited a convergence rate of order $1-\varepsilon$. As mentioned above, the eigenfunctions of the fractional Laplacian operator have only a limited regularity measured in usual Sobolev space, and eigensolutions obtained by ordinary numerical methods have a very poor accuracy. Thus, high order methods may be required to conquer this difficulty.

Some recent advances have been gained on related theoretical analysis and numerical computations on the fractional differential equations [1,18,28,29,34,38,40,42,43]. However, the regularity estimates of solutions to the high-dimensional fractional diffusion-reaction equation are not yet available. Acosta *et al.* [2] presented the regularity of $(1-x^2)^{-s}u(x)$ was $r+2s$ when the regularity index for the right hand side function was $r$ in weighted Sobolev spaces with $2s$ being the order of the fractional Laplacian, which yielded a higher convergence order of the spectral-Galerkin method for the fractional diffusion equation in one-dimensional space. But the analysis in [2] is for the fractional diffusion equation and cannot be extended to the fractional elliptic equation with a reaction term. Zhang in [43] used the Fourier-Jacobi analysis and regularity bootstrap to show that the regularity index for $(1-x^2)^{-s}u(x)$ was $4s+1$. For the fractional diffusion equation on a disk, an exponential convergence rate was reported in [12] when the righthand function was analytic. Hao *et al.* [27] proved the regularity index in radial direction for $(1-\|\boldsymbol{x}\|^2)^{-s}u(\boldsymbol{x})$ was $5s+1-\varepsilon$ with $\varepsilon>0$ arbitrary small when the righthand function was smooth enough for the two-dimensional fractional diffusion-reaction equation.

The intrinsic singularity of the fractional Laplace operator is one of the challenges of efficiently computing the fractional Laplace problem on bounded domains. Fortunately, evidences showed that in general the eigenfunctions of the fractional Laplacian scale as $(1-x^2)^s$ as $x\to\pm1$ [11,44]. Using the ball polynomials $\{P_k^{(s,n+d/2-1)}(2\|\boldsymbol{x}\|^2-1)Y_\ell^n(\boldsymbol{x})\}$ together with the multiplier $(1-\|\boldsymbol{x}\|^2)^s$ to mimic the singular behavior of the eigenfunctions, one may naturally expect a spectrally high order of convergence rate of the spectral method for the eigenvalue problems.

Our aim is to propose an efficient spectral Galerkin method for solving the eigenvalue problems of the fractional Laplacian on the ball of any dimension, and then to conduct a comprehensive numerical analysis. We start with the introduction to some Sobolev spaces, in which the fractional Laplacian is proved to be self-adjoint and positive definite. Thus, eigenvalue problems of the fractional Laplacian can be equivalently written in a symmetric weak formulation, which plays an important role in the numerical analysis. Moreover, by adopting the ball functions as the basis functions, an efficient implementation is given for this spectral Galerkin method. Indeed, the stiffness matrix is identity owing to orthonormality of the basis function with respect to the inner product induced by the fractional Laplace operators; while all entries of the mass matrix can be explicitly evaluated via their analytical formula. Meanwhile, an elaborative analysis shows that the smallest numerical eigenvalue of the fractional Laplace operator of order $2s$ behaves as $\mathcal{O}(1)$ while the largest one behaves as $\mathcal{O}(N^{4s})$. This indicates that the condition number of the mass matrix increases at a rate of $\mathcal{O}(N^{4s})$, the result of which is in consistent with eigenvalue problems of the Laplace operator. Then, the regularity estimate of solutions to source problems of the fractional Laplacian in arbitrary dimensions is firstly investigated in weighted Sobolev spaces which helps establish the regularity of eigenfunctions of the fractional Laplacian eigenvalue problem. In what follows, the orthogonal polynomial approximation on the unit ball is studied. Following the approximation theory of Babuška and Osborn on the Ritz method for self-adjoint and positive-definite eigenvalue problems, rigorous error estimates for the eigenvalue problems of the fractional Laplacian (1.1) for both the eigenvalues and eigenfunctions are presented.

The main features of this paper are:

- The resulting linear system is symmetric positive definite, which helps estimate the error of the Spectral-Galerkin approximation for the eigenvalues and eigenvectors. Moreover, all entries of the mass matrix and stiffness matrix can be explicitly evaluated via their analytical formula.
- The estimate on the algebraic system's condition number is established which behaves as $N^{4s}$, where $2s$ is the fractional derivative order.
- The regularity of eigenfunctions of the fractional Laplacian eigenvalue problem is analyzed

thanks to the regularity of solutions to source problems of the fractional Laplacian in weighted Sobolev spaces.

The remainder of this paper is organized as follows. We introduce necessary notations and Sobolev spaces and review relevant polynomials/functions which especially include the spherical harmonics and ball polynomials/functions in Section 2. In Section 3, we propose the spectral approximation methods for eigenvalue problems of the fractional Laplacian on the unit ball of any dimension. In Section 4, we consider the regularity theory and error estimates of the spectral-Galerkin method for the fractional eigenvalue problems. Numerical results are shown in Section 5 to verify the condition number and the theoretical convergence order.

## 2. Preliminaries

In this section, we first introduce some notations. Let $\mathbb{R}^d$ denote $d$-dimensional Euclidean space. For $\boldsymbol{x} \in \mathbb{R}^d$, we write $\boldsymbol{x} = (x_1, \cdots, x_d)^{\mathrm{t}}$. The inner product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ is denoted by $\boldsymbol{x} \cdot \boldsymbol{y}$ or $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^d x_i y_i$, and the norm of $\boldsymbol{x}$ is denoted by $\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}\boldsymbol{x}^{\mathrm{t}}}$, where $(\cdot)^{\mathrm{t}}$ denotes matrix or vector transpose. The unit sphere $\mathbb{S}^{d-1}$ and the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$ are respectively defined by

$$\mathbb{S}^{d-1} := \big\{ \hat{\boldsymbol{x}} \in \mathbb{R}^d : \|\hat{\boldsymbol{x}}\| = 1 \big\}, \quad \mathbb{B}^d := \big\{ \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \leq 1 \big\}. \tag{2.1}$$

Let $D \subseteq \mathbb{R}^d$ be an arbitrary domain and $\omega$ be a generic positive weight function which is not necessary in $L^1(D)$. Denote by

$$(u, v)_{\omega, D} := \int_D u(\boldsymbol{x}) v(\boldsymbol{x}) \omega(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

the inner product of $L^2_\omega(D)$ with the norm $\|\cdot\|_{\omega, D}$. Whenever no confusion would arise, we shall drop the subscripts $\omega$ if $\omega = 1$ and drop the subscript $D$ if $D = \mathbb{B}^d$.

### 2.1. Sobolev spaces

For $s \in \mathbb{R}$, it is well known that $H^s(\mathbb{R}^d)$ can be defined through the Fourier transform [25, 32, 39]

$$H^s(\mathbb{R}^d) = \big\{ u \in \big(\mathscr{D}(\mathbb{R}^d)\big)' : \|u\|_{s, \mathbb{R}^d} < \infty \big\}.$$

Its norm and seminorm for $s \geq 0$ are defined by

$$\|u\|_{s, \mathbb{R}^d} = \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^s |\widehat{u}(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi} \right)^{1/2},$$
$$|u|_{s, \mathbb{R}^d} = \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} |\widehat{u}(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi} \right)^{1/2}.$$

And $H^s(\Omega)$ can be derived from $H^s(\mathbb{R}^d)$ by extension whenever $\Omega$ is a bounded Lipschitz domain [25, 32, 39]

$$H^s(\Omega) = \big\{ u = U|_\Omega : U \in H^s(\mathbb{R}^d) \big\}, \quad \|u\|_s = \inf_{\substack{U \in H^s(\mathbb{R}^d) \\ U|_\Omega = u}} \|U\|_{s, \mathbb{R}^d}.$$

The zero extension $\tilde{u}$ of $u$ defined on $\Omega$ is of particular interest [25, 32]

$$\tilde{u}|_\Omega = u \quad \text{and} \quad \tilde{u}|_{\mathbb{R}^d \setminus \Omega} = 0.$$

Besides the zero extension induces a special type of Sobolev spaces

$$H_*^s(\Omega) = \big\{u \in \big(\mathscr{D}(\Omega)\big)' : \tilde{u} \in H^s(\mathbb{R}^d)\big\},$$

its norm and seminorm are defined by

$$\|u\|_{s,*} = \|\tilde{u}\|_{s,\mathbb{R}^d}, \quad |u|_{s,*} = |\tilde{u}|_{s,\mathbb{R}^d}.$$

Actually, $\mathscr{D}(\Omega)$ is dense in $H_*^s(\Omega)$ for $s \geq 0$ [39, Theorem 3.2.4/1], thus the completion of $\mathscr{D}(\Omega)$ in $H_*^s(\Omega)$ is $H_*^s(\Omega)$ itself. As a result, one can obtain by Parseval's theorem that

$$\big((-\Delta)^s u, v\big)_{\mathbb{R}^d} = \big((-\Delta)^{s/2} u, (-\Delta)^{s/2} v\big)_{\mathbb{R}^d} = \big(v, (-\Delta)^s u\big)_{\mathbb{R}^d}, \quad u, v \in H_*^s(\Omega). \tag{2.2}$$

Moreover, it gets from (1.3) together with Parseval's theorem that

$$|v|_{s,*} = \|(-\Delta)^{s/2} v\|_{\mathbb{R}^d} := |v|_s. \tag{2.3}$$

**Lemma 2.1 ([25, Lemma 1.3.2.6]).** *Let $\varpi^s := \varpi^s(\boldsymbol{x}) = (1 - \|\boldsymbol{x}\|^2)^s$. For all $u \in H_*^s(\Omega)$, $0 \leq s < 1$, it holds that*

$$\|u\|_{s,*}^2 \asymp |u|_{s,*}^2 \asymp |u|_s^2 + \|u\|_{\varpi^{-2s}}^2, \quad u \in H_*^s(\Omega). \tag{2.4}$$

Hereafter, we denote by $C_1$ and $C_2$ some generic positive constants which are independent of any function, discretization parameter but possibly dependent on the geometry of $\Omega$. We abbreviate $a \lesssim b$ for $a \leq C_1 b$ and $a \asymp b$ for $C_1 b \leq a \leq C_2 b$.

Owing to the fact that $H^s(\Omega)$ with $s > 0$ is compactly embedded in $L^2(\Omega)$ and $H_*^s(\Omega) \subseteq H^s(\Omega)$, it is concluded with the following embedding theorem.

**Theorem 2.1.** *$H_*^s(\Omega)$ with $s > 0$ is compactly embedded in $L^2(\Omega)$.*

### 2.2. Jacobi polynomials

For parameters $\alpha, \beta > -1$, the Jacobi weight function is defined by

$$\omega^{\alpha,\beta}(z) = (1-z)^\alpha (1+z)^\beta, \quad z \in I := (-1, 1).$$

The Jacobi polynomials, denoted by $\{P_k^{(\alpha,\beta)}(z),\ k \geq 0\}$, admit the following hypergeometric representation on $I$:

$$P_k^{(\alpha,\beta)}(z) = \binom{k+\alpha}{k} {}_2F_1\Big(-k, k+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\Big), \quad -k-\alpha-\beta \notin \{1, 2, \cdots, k\}, \tag{2.5}$$

where the hypergeometric function

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

They are orthogonal to each other with respect to the weight function $\omega^{\alpha,\beta}$

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(\eta) P_m^{(\alpha,\beta)}(\eta) \omega^{\alpha,\beta}(\eta) \mathrm{d}\eta = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \delta_{nm}. \tag{2.6}$$

Alternatively, Jacobi polynomials can be defined via the following Rodrigues' formula:

$$P_k^{(\alpha,\beta)}(z) = \frac{(-1)^k}{2^k k!}(1-z)^{-\alpha}(1+z)^{-\beta}\partial_z^k\big[(1-z)^{\alpha+k}(1+z)^{\beta+k}\big]. \tag{2.7}$$

Moreover, replacing $z$ by $-z$ in (2.7) immediately leads to the symmetric relation

$$P_k^{(\alpha,\beta)}(-z) = (-1)^k P_k^{(\beta,\alpha)}(z). \tag{2.8}$$

Then, we state some necessary results about Jacobi polynomials.

**Lemma 2.2 ([5, p. 304]).**

$$\frac{\mathrm{d}}{\mathrm{d}z}P_k^{(\alpha,\beta)}(z) = \frac{1}{2}(k+\alpha+\beta+1)P_{k-1}^{(\alpha+1,\beta+1)}(z), \tag{2.9}$$

$$(2k+\alpha+\beta+1)P_k^{(\alpha,\beta)}(z) = (k+\alpha+\beta+1)P_k^{(\alpha+1,\beta)}(z) - (k+\beta)P_{k-1}^{(\alpha+1,\beta)}(z), \tag{2.10}$$

$$\left(k+\frac{\alpha+\beta}{2}+1\right)(1-z)P_k^{(\alpha+1,\beta)}(z) = (k+\alpha+1)P_k^{(\alpha,\beta)}(z) - (k+1)P_{k+1}^{(\alpha,\beta)}(z). \tag{2.11}$$

**Lemma 2.3.** *For $\mu \geq 0$, it holds that*

$$\begin{aligned}&\int_{-1}^{1} P_k^{(\alpha,\beta)}(z)(1-z)^{\alpha}(1+z)^{\mu+\beta}\mathrm{d}z \\ &= 2^{\mu+\alpha+\beta+1}\frac{(\mu-k+1)_k\,\Gamma(k+\alpha+1)\,\Gamma(\mu+\beta+1)}{k!\,\Gamma(k+\mu+\alpha+\beta+2)},\end{aligned} \tag{2.12}$$

*where the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$ for any $a \in \mathbb{R}$ and $n \in \mathbb{N}_0$.*

*Proof.* By the Rodrigues' formula (2.7) and integration by parts, one obtains

$$\begin{aligned}&\int_{-1}^{1} P_k^{(\alpha,\beta)}(z)(1-z)^{\alpha}(1+z)^{\mu+\beta}\mathrm{d}z \\ &= \frac{(-1)^k}{2^k k!}\int_{-1}^{1}\partial_z^k\big[(1-z)^{k+\alpha}(1+z)^{k+\beta}\big](1+z)^{\mu}\mathrm{d}z \\ &= \frac{1}{2^k k!}\int_{-1}^{1}(1-z)^{k+\alpha}(1+z)^{k+\beta}\partial_z^k(1+z)^{\mu}\mathrm{d}z \\ &= \frac{(\mu-k+1)_k}{2^k k!}\int_{-1}^{1}(1-z)^{k+\alpha}(1+z)^{\mu+\beta}\mathrm{d}z \\ &= 2^{\mu+\alpha+\beta+1}\frac{(\mu-k+1)_k\,\Gamma(k+\alpha+1)\,\Gamma(\mu+\beta+1)}{k!\,\Gamma(k+\alpha+\mu+\beta+2)},\end{aligned}$$

where we use (2.6) with $m = n = 0$ for the fourth equality sign. This completes the proof. □

### 2.3. Spherical harmonic

Let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree $n$ in $d$ variables, i.e.

$$\mathcal{P}_n^d = \big\{\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d} : |\boldsymbol{k}| = k_1+k_2+\cdots+k_d = n\big\}.$$

Define $\mathcal{H}_n^d$ be the space of all harmonic polynomials of degree $n$

$$\mathcal{H}_n^d := \big\{p \in \mathcal{P}_n^d : \Delta P(\boldsymbol{x}) = 0\big\}. \tag{2.13}$$

It is known that (cf. [16])

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n} \quad \text{and} \quad a_n^d := \dim \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

The spherical harmonics are the restriction of elements in $\mathcal{H}_n^d$ on the unit sphere. Note that for any $Y \in \mathcal{H}_n^d$, we have

$$Y(\boldsymbol{x}) = r^n Y(\hat{\boldsymbol{x}}), \quad \boldsymbol{x} = r\hat{\boldsymbol{x}}, \quad r = \|\boldsymbol{x}\|, \quad \hat{\boldsymbol{x}} \in \mathbb{S}^{d-1} \tag{2.14}$$

in spherical-polar coordinates. It is evident that $Y(\boldsymbol{x})$ is uniquely determined by its restriction $Y(\hat{\boldsymbol{x}}) \in \mathcal{H}_n^d$ on the sphere. With a little abuse of notation, we still use $\mathcal{H}_n^d$ to denote the set of spherical harmonics of degree $n$ on the unite sphere $\mathbb{S}^{d-1}$.

Spherical harmonics of different degrees are orthogonal with respect to the inner product

$$(f,g)_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\hat{\boldsymbol{x}}) g(\hat{\boldsymbol{x}}) \mathrm{d}\sigma(\hat{\boldsymbol{x}}),$$

where $\mathrm{d}\sigma$ is the surface measure. Further, let $\{Y_\ell^n : 1 \le \ell \le a_n^d\}$ be the orthonormal (real) basis of $\mathcal{H}_n^d$, $n \in \mathbb{N}_0$ such that

$$(Y_\ell^n, Y_\iota^m)_{\mathbb{S}^{d-1}} = \delta_{nm}\delta_{\ell\iota}, \quad 1 \le \ell \le a_n^d, \quad 1 \le \iota \le a_m^d, \quad m \ge 0, \quad n \ge 0. \tag{2.15}$$

**Remark 2.1.**

- For $d = 1$, there exist only two orthonormal harmonic polynomials: $Y_1^0 = 1/\sqrt{2}$ and $Y_1^1 = x/\sqrt{2}$.
- For $d = 2$, the space $\mathcal{H}_n^2$ has dimension $a_n^2 = 2 - \delta_{n,0}$ and the orthogonal basis of $\mathcal{H}_n^2$ can be given by the real and imaginary parts of $(x_1 + \mathrm{i}x_2)^n$. Thus, in polar coordinates $\boldsymbol{x} = (r\cos\theta, r\sin\theta)^t \in \mathbb{R}^2$, we simply take
$$Y_1^0(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}, \quad Y_1^1(\boldsymbol{x}) = \frac{r^n}{\sqrt{\pi}}\cos(n\theta), \quad Y_1^1(\boldsymbol{x}) = \frac{r^n}{\sqrt{\pi}}\sin(n\theta), \quad n \ge 1.$$
- For $d = 3$, the dimensionality of the harmonic polynomial space of degree $n$ is $a_n^3 = 2n+1$. In spherical coordinates $\boldsymbol{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)^t \in \mathbb{R}^3$, the orthonormal basis can be taken as
$$\begin{aligned} Y_1^n(\boldsymbol{x}) &= \frac{1}{\sqrt{8\pi}} P_n^{(0,0)}(\cos\theta), Y_{2k}^n(\boldsymbol{x}) \\ &= \frac{r^n}{2^{k+1}\sqrt{\pi}} (\sin\theta)^k P_{n-k}^{(k,k)}(\cos\theta)\cos(k\phi), \qquad 1 \le k \le n, \\ Y_{2k+1}^n(\boldsymbol{x}) &= \frac{r^n}{2^{k+1}\sqrt{\pi}} (\sin\theta)^k P_{n-k}^{(k,k)}(\cos\theta)\sin(k\phi), \quad 1 \le k \le n. \end{aligned}$$

In spherical polar coordinates, the Laplace operator can be written as

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\Delta_0, \tag{2.16}$$

where $r = \|\boldsymbol{x}\|$. The Laplace-Beltrami operator $\Delta_0$ has spherical harmonics as eigenfunctions, more precisely, for $n = 0, 1, 2, \ldots$,

$$\Delta_0 Y(\hat{\boldsymbol{x}}) = -n(n+d-2)Y(\hat{\boldsymbol{x}}), \quad Y(\hat{\boldsymbol{x}}) \in \mathcal{H}_n^d, \quad \hat{\boldsymbol{x}} \in \mathbb{S}^{d-1}. \tag{2.17}$$

The following lemma shows that the spherical component of the Fourier transform of spherical harmonics is related to the Bessel functions.

**Lemma 2.4 ([5, Lemma 9.10.2]).** *For any* $\hat{\boldsymbol{x}}, \hat{\boldsymbol{\xi}} \in \mathbb{S}^{d-1}$ *and* $w \geq 0$, *it holds that*

$$\int_{\mathbb{S}^{d-1}} \mathrm{e}^{-\mathrm{i}w\langle\hat{\boldsymbol{\xi}},\hat{\boldsymbol{x}}\rangle} Y_\ell^n(\hat{\boldsymbol{x}}) \mathrm{d}\sigma(\hat{\boldsymbol{x}}) = \frac{(2\pi)^{d/2}(-\mathrm{i})^n}{w^{(d-2)/2}} J_{n+(d-2)/2}(w) Y_\ell^n(\hat{\boldsymbol{\xi}}), \tag{2.18}$$

*where* $J_\nu(z)$ *is the Bessel function of the first kind of order* $\nu > -1/2$.

### 2.4. Ball polynomials/functions

For any $\alpha > -1$, ball polynomials which are orthogonal polynomials on $\mathbb{B}^d$ are defined by

$$P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) = P_k^{(\alpha,n+d/2-1)}(2\|\boldsymbol{x}\|^2 - 1) Y_\ell^n(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{B}^d, \quad n, k \in \mathbb{N}_0, \quad 1 \leq \ell \leq a_n^d. \tag{2.19}$$

Note that the total degree of $P_{k,\ell}^{\alpha,n}(\boldsymbol{x})$ is $n + 2k$. They are orthogonal with respect to the weight function $\varpi^\alpha(\boldsymbol{x}) = (1 - \|\boldsymbol{x}\|^2)^\alpha$ [16, Proposition 11.1.13]

$$\begin{aligned} &\int_{\mathbb{B}^d} P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) P_{j,\iota}^{\alpha,m}(\boldsymbol{x}) \varpi^\alpha(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &= \chi_{k,n}^\alpha \delta_{m,n} \delta_{k,j} \delta_{\ell,\iota}, \quad k, j, m, n \in \mathbb{N}_0, \quad 1 \leq \ell \leq a_n^d, \quad 1 \leq \iota \leq a_m^d, \end{aligned} \tag{2.20}$$

where

$$\chi_{k,n}^\alpha = \frac{\Gamma(k+\alpha+1)\Gamma(k+n+d/2)}{2(2k+\alpha+n+d/2)\Gamma(k+1)\Gamma(k+\alpha+n+d/2)}.$$

**Theorem 2.2 ([16, Theorem 11.1.5]).** *The ball polynomials are the eigenfunctions of the differential operator*

$$\mathcal{K}_{\boldsymbol{x}}^\alpha P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) := \big( -\Delta + \nabla \cdot \boldsymbol{x}(2\alpha + \boldsymbol{x} \cdot \nabla) - 2\alpha d \big) P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) = \gamma_m^{(\alpha)} P_{k,\ell}^{\alpha,n}(\boldsymbol{x}),$$

*where* $\gamma_m^{(\alpha)} = m(m + 2\alpha + d)$.

The following theorem indicates that the Sturm-Liouville operator $\mathcal{K}_{\boldsymbol{x}}^\alpha$ has an equivalent form.

**Theorem 2.3 ([41, Theorem 2.2]).** *For* $\alpha > -1$, *it holds that*

$$\mathcal{K}_{\boldsymbol{x}}^\alpha = -(1 - \|\boldsymbol{x}\|^2)^{-\alpha} \nabla \cdot (1 - \|\boldsymbol{x}\|^2)^{\alpha+1} \nabla - \Delta_0.$$

**Lemma 2.5.** *For any integer* $\nu \in \mathbb{N}_0$, *it holds that*

$$(-\Delta)^\nu P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) = (-4)^\nu \left(k + n + \alpha + \frac{d}{2}\right)_\nu \left(k + n + \frac{d}{2} - \nu\right)_\nu P_{k-\nu,\ell}^{\alpha+2\nu,n}(\boldsymbol{x}). \tag{2.21}$$

To be undistracted from the main results, we postpone the proof of Lemma 2.5 to Appendix A.

For any $\alpha > -1$, ball functions on $\mathbb{B}^d$ are defined by

$$Q_{k,\ell}^{-\alpha,n}(\boldsymbol{x}) := \begin{cases} \varrho_{k,n}^\alpha \big(1 - \|\boldsymbol{x}\|^2\big)^\alpha P_{k,\ell}^{\alpha,n}(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{B}^d, \\ 0, & \boldsymbol{x} \notin \mathbb{B}^d, \end{cases} \quad k, n \in \mathbb{N}_0, \quad 1 \leq \ell \leq a_n^d, \tag{2.22}$$

where

$$\varrho_{k,n}^\alpha = \frac{\Gamma(k+1)\Gamma(k+n+\alpha+d/2)}{\Gamma(k+\alpha+1)\Gamma(k+n+d/2)}.$$

**Lemma 2.6.** *For $\alpha > -1$ and $n, k \in \mathbb{N}_0, 1 \le \ell \le a_n^d$, it holds that*

$$
\begin{aligned}
&\int_{\mathbb{R}^d} Q_{k,\ell}^{-\alpha,n}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x} \\
&= \frac{(-\mathrm{i})^{n+2k} 2^{\alpha} (2\pi)^{d/2} \Gamma(k+n+\alpha+d/2)}{\rho^{d/2+\alpha}\Gamma(k+n+d/2)} J_{n+2k+d/2+\alpha}(\rho)\, Y_\ell^n(\hat{\boldsymbol{\xi}}),
\end{aligned} \tag{2.23}
$$

*where $\boldsymbol{\xi} = \rho\hat{\boldsymbol{\xi}}$ with $\rho = \|\boldsymbol{\xi}\|, \hat{\boldsymbol{\xi}} \in \mathbb{S}^{d-1}$.*

The proof of this lemma is given in Appendix A.

Based on the definition of the fractional Laplacian (1.3), we can obtain the following theorem.

**Theorem 2.4.** *For $s \ge 0$ and any integer $\nu \in [0, s + d/2)$, it holds that*

$$
(-\Delta)^{s-\nu} Q_{k,\ell}^{-s,n}(\boldsymbol{x}) := \sigma_{k,n}^{s,\nu} P_{k+\nu,\ell}^{s-2\nu,n}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{2.24}
$$

*where*

$$
\sigma_{k,n}^{s,\nu} = \frac{\Gamma(k+n+s+d/2)\Gamma(n+k+d/2+s-\nu)}{(-1)^{\nu} 2^{2\nu-2s}\Gamma(k+n+d/2+\nu)\Gamma(k+n+d/2)}. \tag{2.25}
$$

The proof of Theorem 2.4 is in Appendix A. Combining (2.2), (2.20) and Theorem 2.4 yields the following theorem.

**Theorem 2.5.** *For $s > 0$ and $n, k \in \mathbb{N}_0, 1 \le \ell \le a_n^d$, $Q_{k,\ell}^{-s,n}$ form a complete orthogonal system in $H_*^{s/2}(\mathbb{B}^d)$. More precisely,*

$$
\big((-\Delta)^{s/2} Q_{k,\ell}^{-s,n}, (-\Delta)^{s/2} Q_{j,l}^{-s,m}\big)_{\mathbb{R}^d} = \frac{2^{2s-1}\Gamma^2(k+n+s+d/2)}{(2k+s+n+d/2)\Gamma^2(k+n+d/2)} \delta_{\ell,l}\delta_{k,j}\delta_{m,n}. \tag{2.26}
$$

## 3. Spectral Approximation Methods

### 3.1. Approximation scheme and implementation of the spectral Galerkin method

The variational form of (1.1) reads: To find nontrivial $(\lambda, u) \in \mathbb{R} \times H_*^{s/2}(\mathbb{B}^d)$ such that

$$
a(u, v) = \lambda\, b(u, v), \quad v \in H_*^{s/2}(\mathbb{B}^d), \tag{3.1}
$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the bilinear forms defined by

$$
\begin{aligned}
a(u, v) &= \big((-\Delta)^{s/2} u, (-\Delta)^{s/2} v\big)_{\mathbb{R}^d}, \quad && u, v \in H_*^{s/2}(\mathbb{B}^d), \\
b(u, v) &= (u, v), && u, v \in L^2(\mathbb{B}^d).
\end{aligned}
$$

It is obvious that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric, positive definite, continuous and coercive on $H_*^{s/2}(\mathbb{B}^d) \times H_*^{s/2}(\mathbb{B}^d)$ and $L^2(\mathbb{B}^d) \times L^2(\mathbb{B}^d)$ respectively.

**Remark 3.1.** Problem (1.1) has an infinite sequence of eigensolutions $\{(\lambda_i, \psi_i)\}_{i=1}^{\infty}$ with eigenvalues being ordered increasing, $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. All eigenvalues of (1.1) are real and positive. The following lemma indicates that

$$
\lambda_1 > \frac{2^{2s}\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}.
$$

**Lemma 3.1.** *$\forall u \in H_*^{s/2}(\mathbb{B}^d)\backslash\{0\}$, it holds that*

$$\|(-\Delta)^{s/2}u\|_{\mathbb{R}^d} > 2^s\sqrt{\frac{\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}}\|u\|_{\varpi^{-s}} > 2^s\sqrt{\frac{\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}}\|u\|.$$

*Proof.* We first show that for $n, k \in \mathbb{N}_0$,

$$\frac{\Gamma(k+s+1)}{\Gamma(k+1)} = \frac{(k+s)(k+s-1)\cdots(s+1)}{k!}\Gamma(s+1) > \Gamma(s+1),$$

$$\frac{\Gamma(n+k+s+d/2)}{\Gamma(n+k+d/2)} = \frac{(n+k+d/2+s-1)(n+k+d/2+s-2)\cdots(d/2+s)\Gamma(d/2+s)}{(n+k+d/2-1)(n+k+d/2-2)\cdots d/2\Gamma(d/2)}$$
$$> \frac{\Gamma(d/2+s)}{\Gamma(d/2)}.$$

Then, for any

$$u(\boldsymbol{x}) = \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\widehat{u}_{k,\ell}^{s,n}\varpi^s P_{k,\ell}^{s,n}(\boldsymbol{x}) \in H_*^{s/2}(\mathbb{B}^d),$$

it is derived from (2.2), (2.20), Theorem 2.4 together with the Parseval's identity that

$$\begin{aligned}\|(-\Delta)^{s/2}u\|_{\mathbb{R}^d}^2 &= \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\frac{2^{2s}\Gamma(k+s+1)\Gamma(n+k+d/2+s)}{\Gamma(k+1)\Gamma(k+n+d/2)}|\widehat{u}_{k,\ell}^{s,n}|^2\|P_{k,\ell}^{s,n}\|_{\varpi^s}^2\\ &> \frac{2^{2s}\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}|\widehat{u}_{k,\ell}^{s,n}|^2\|P_{k,\ell}^{s,n}\|_{\varpi^s}^2\\ &= \frac{2^{2s}\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}\|u\|_{\varpi^{-s}}^2 > \frac{2^{2s}\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)}\|u\|^2,\end{aligned}$$

which ends the proof. □

Define

$$W_{N,K} = \text{span}\bigg\{\sqrt{\frac{2k+s+n+d/2}{2^{2s-1}}}\frac{\Gamma(k+n+d/2)Q_{k,\ell}^{-s,n}(\boldsymbol{x})}{\Gamma(k+n+s+d/2)} =: \widetilde{Q}_{k,\ell}^{-s,n}(\boldsymbol{x}) :$$
$$1 \le \ell \le a_n^d,\ 0 \le k \le K,\ 0 \le n \le N\bigg\}, \tag{3.2}$$

the Galerkin spectral approximation to (1.1) amounts to find a nontrivial eigenpair $(\lambda_{N,K}, u_{N,K}) \in \mathbb{R} \times W_{N,K}$ such that

$$a(u_{N,K}, v) = \lambda_{N,K}b(u_{N,K}, v), \quad \forall v \in W_{N,K}. \tag{3.3}$$

Let us expand $u_{N,K}$ as follows:

$$u_{N,K}(\boldsymbol{x}) = \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{K}\widehat{u}_{k,\ell}^{n}\widetilde{Q}_{k,\ell}^{-s,n}(\boldsymbol{x}),$$

and denote

$$\widehat{u} = \Big(\widehat{u}_1^0, \widehat{u}_2^0, \cdots, \widehat{u}_{a_0^d}^0, \widehat{u}_1^1, \widehat{u}_2^1, \cdots, \widehat{u}_{a_1^d}^1, \cdots, \widehat{u}_1^N, \widehat{u}_2^N, \cdots, \widehat{u}_{a_N^d}^N\Big)^T, \quad \widehat{u}_\ell^n = \big(\widehat{u}_{0,\ell}^n, \widehat{u}_{1,\ell}^n, \cdots, \widehat{u}_{K,\ell}^n\big),$$

$$S=\mathrm{diag}\big\{(S^n_\ell)_{0\le n\le N,1\le \ell\le a^d_n}\big\},\qquad S^n_\ell=\big[a\big(\widetilde{Q}^{-s,n}_{k,\ell},\widetilde{Q}^{-s,m}_{j,l}\big)\big]_{0\le k,j\le K},$$
$$M=\mathrm{diag}\big\{(M^n_\ell)_{0\le n\le N,1\le \ell\le a^d_n}\big\},\quad M^n_\ell=\big[b\big(\widetilde{Q}^{-s,n}_{k,\ell},\widetilde{Q}^{-s,m}_{j,l}\big)\big]_{0\le k,j\le K}.$$

Then the algebraic eigen system associated with (3.3) can be written as

$$S\widehat{u}=\lambda_{N,K}M\widehat{u}.$$

The following lemma indicates that the stiffness matrix $S$ is diagonal. Thanks to the orthogonality of spherical harmonics, $M$ is a block diagonal matrix.

**Lemma 3.2.** *For $k,j,n,m\in\mathbb{N}_0$ and $1\le\ell\le a^d_n, 1\le\iota\le a^d_m$, it holds that*

$$\big((-\Delta)^{s/2}\widetilde{Q}^{-s,n}_{k,\ell},(-\Delta)^{s/2}\widetilde{Q}^{-s,m}_{j,\iota}\big)_{\mathbb{R}^d}:=\delta_{m,n}\delta_{\ell,\iota}\delta_{k,j},\tag{3.4}$$
$$\big(\widetilde{Q}^{-s,n}_{k,\ell},\widetilde{Q}^{-t,m}_{j,\iota}\big):=m^{s,t,n}_{k,j}\delta_{m,n}\delta_{\ell,\iota},\tag{3.5}$$

*where*

$$m^{s,t,n}_{k,j}=\frac{(-1)^{k+j}\sqrt{(2k+s+n+d/2)(2j+t+n+d/2)}\,\Gamma(n+k+j+d/2)\Gamma(s+t+1)}{2^{s+t}\Gamma(j+t-k+1)\Gamma(k+s-j+1)\Gamma(s+t+n+k+j+1+d/2)}.$$

*Proof.* In view of (2.26), one readily gets (3.4). To prove (3.5), we first resort to the following connection identity of two Jacobi polynomials with different indexes of the weight function [5, Theorem 7.1.3]:

$$\begin{aligned}&P^{(\gamma,\beta)}_n(\boldsymbol{x})=\sum_{m=0}^{n}c^{\gamma,\alpha}_{n,m}P^{(\alpha,\beta)}_m(\boldsymbol{x}),\\&c^{\gamma,\alpha}_{n,m}=\frac{(\beta+1)_n(\gamma-\alpha)_{n-m}(\alpha+\beta+1)_m(\alpha+\beta+2m+1)(\beta+\gamma+n+1)_m}{(\alpha+\beta+2)_n(n-m)!(\beta+1)_m(\alpha+\beta+1)(\alpha+\beta+n+2)_m}.\end{aligned}\tag{3.6}$$

We now concentrate on the main body of the proof. By Rodrigues' formula and integration by parts, one can obtain

$$\begin{aligned}&\big(Q^{-s,n}_{k,\ell},Q^{-t,m}_{j,\iota}\big)\\&=\int_{\mathbb{S}^{d-1}}Y^n_\ell(\hat{\boldsymbol{x}})Y^m_\iota(\hat{\boldsymbol{x}})\mathrm{d}\sigma(\hat{\boldsymbol{x}})\\&\quad\times\int_0^1P^{(s,n+d/2-1)}_k(2r^2-1)P^{(t,m+d/2-1)}_j(2r^2-1)(1-r^2)^{s+t}r^{n+m+d-1}\mathrm{d}r\\&=\Big(\frac12\Big)^{s+t+n+d/2+1}\delta_{m,n}\delta_{\ell,\iota}\int_{-1}^1P^{(s,n+d/2-1)}_k(\rho)P^{(t,n+d/2-1)}_j(\rho)(1-\rho)^{s+t}(1+\rho)^{n+d/2-1}\mathrm{d}\rho\\&=\Big(\frac12\Big)^{s+t+n+d/2+1}\delta_{m,n}\delta_{\ell,\iota}\frac{(-1)^k}{2^kk!}\\&\quad\times\int_{-1}^1\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\big[(1-\rho)^{s+k}(1+\rho)^{k+n+d/2-1}\big]\big[P^{(t,n+d/2-1)}_j(\rho)(1-\rho)^t\big]\mathrm{d}\rho\\&=\frac{1}{2^{s+t+n+d/2+1+k}k!}\delta_{m,n}\delta_{\ell,\iota}\int_{-1}^1(1-\rho)^{s+k}(1+\rho)^{k+n+d/2-1}\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\big[P^{(t,n+d/2-1)}_j(\rho)(1-\rho)^t\big]\mathrm{d}\rho,\end{aligned}$$

where the second equality sign is derived from variable substitution $\rho=2r^2-1$.

To proceed, we recall the property of the hypergeometric function derived from [5, p. 123, Exercise 43(b)]

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\big[\zeta^{c-1}{}_2F_1(a,b;c;\zeta)\big]=(c-1)\zeta^{c-2}{}_2F_1(a,b;c-1;\zeta),$$

which is equivalent to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\bigg[\Big(\frac{1-\zeta}{2}\Big)^{c-1}{}_2F_1\Big(a,b;c;\frac{1-\zeta}{2}\Big)\bigg]=-\frac{c-1}{2}\Big(\frac{1-\zeta}{2}\Big)^{c-2}{}_2F_1\Big(a,b;c-1;\frac{1-\zeta}{2}\Big). \tag{3.7}$$

Using (3.7) repeatedly together with (2.5) yields that

$$\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\big[P_j^{(t,n+d/2-1)}(\rho)(1-\rho)^t\big]=(-1)^k(j+t-k+1)_k(1-\rho)^{t-k}P_j^{(t-k,n+d/2-1+k)}(\rho).$$

As a result, one obtains that

$$\begin{aligned}&\int_{-1}^{1}(1-\rho)^{s+k}(1+\rho)^{k+n+d/2-1}\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\big[P_j^{(t,n+d/2-1)}(\rho)(1-\rho)^t\big]\mathrm{d}\rho\\ &=(-1)^k(j+t-k+1)_k\int_{-1}^{1}(1-\rho)^{s+t}(1+\rho)^{k+n+d/2-1}P_j^{(t-k,n+d/2-1+k)}(\rho)\mathrm{d}\rho,\end{aligned}$$

where $P_j^{(t-k,n+d/2-1+k)}$ is the generalized Jacobi polynomial as discussed in [37, §4.22] whenever $t-k\le -1$ and/or $n+d/2-1+k\le -1$. Then, we use (3.6) to expand $P_j^{(t-k,n+d/2-1+k)}$ as $\sum_{m=0}^{j}c_{j,m}^{t-k,s+t}P_j^{(s+t,n+d/2-1+k)}$. Owing to the orthogonality of Jacobi polynomials (2.6), it holds that

$$\begin{aligned}&\big(Q_{k,\ell}^{-s,n},Q_{j,\iota}^{-t,m}\big)\\ &=\frac{(-1)^k}{2^{s+t+n+d/2+1+k}k!}\frac{\Gamma(j+t+1)}{\Gamma(j+t-k+1)}\delta_{m,n}\delta_{\ell,\iota}\\ &\quad\times\int_{-1}^{1}(1-\rho)^{s+t}(1+\rho)^{k+n+d/2-1}c_{j,0}^{t-k,s+t}P_0^{(s+t,n+d/2-1+k)}\mathrm{d}\rho\\ &=\frac{(-1)^{k+j}}{2^{s+t+n+d/2+1+k}k!}\frac{\Gamma(j+t+1)}{\Gamma(j+t-k+1)}\frac{(n+d/2+k)_j(s+k-j+1)_j}{(s+t+n+k+d/2+1)_j j!}h_0^{s+t,n+d/2-1+k}\delta_{m,n}\delta_{\ell,l}\\ &=\frac{(-1)^{k+j}\Gamma(j+t+1)\Gamma(n+d/2+k+j)\Gamma(s+k+1)\Gamma(s+t+1)}{2k!j!\Gamma(t+j-k+1)\Gamma(s+k-j+1)\Gamma(s+t+n+k+d/2+1+j)}\delta_{m,n}\delta_{\ell,l}.\end{aligned}$$

Thus, (3.5) is an immediate consequence of the above equation. □

### 3.2. Estimate of the spectral condition number

The goal of this subsection is to give estimates on the smallest and greatest numerical eigenvalues, and thus on the condition number of the mass matrix $M$.

Let $\{\lambda_{N,K}^i,\psi_{N,K}^i\}_{i=1}^{(N+1)^2}$ be eigensolutions of (3.3) such that $\lambda_{N,K}^1\le\lambda_{N,K}^2\le\cdots\le\lambda_{N,K}^{(N+1)^2}$, here we take $K=N$ in the finite dimensional space (3.2). The following lemma indicates that each numerical eigenvalue satisfies $\lambda_{N,N}^i\lesssim N^{4s}$.

**Lemma 3.3.** *For any $u_N\in W_{N,N}$, it holds that*

$$\|(-\Delta)^{s/2}u_N\|_{\mathbb{R}^d}^2\lesssim N^{4s}\|u_N\|^2. \tag{3.8}$$

*Proof.* For

$$u_N = \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{N} \widehat{u}_{k,\ell}^{s,n} Q_{k,\ell}^{-s,n}(\boldsymbol{x}) \in W_{N,N},$$

it follows from (2.2), (2.20) and (2.24):

$$\begin{aligned}\|(-\Delta)^{s/2}u_N\|_{\mathbb{R}^d}^2 &= \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{N} |\widehat{u}_{k,\ell}^{s,n}|^2 \sigma_{k,n}^{s,0}\|P_{k,\ell}^{s,n}\|_{\varpi^s}^2 \\ &\lesssim \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{N} k^s(k+n)^s |\widehat{u}_{k,\ell}^{s,n}|^2 \|P_{k,\ell}^{s,n}\|_{\varpi^s}^2 \\ &\lesssim N^{2s}\sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{N} |\widehat{u}_{k,\ell}^{s,n}|^2 \|P_{k,\ell}^{s,n}\|_{\varpi^s}^2, \end{aligned} \tag{3.9}$$

where we have used the limit for asymptotic approximations [35, Eq. (1.66)]

$$\lim_{n\to+\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)n^\alpha} = 1. \tag{3.10}$$

On the other hand, we resort to the following inequality which is derived from [8, Corollary 6.2] extended to high dimension that for any $\phi_N \in \mathbb{P}_N(\mathbb{B}^d)$,

$$\|\phi_N\|_{\varpi^s} \lesssim N^s \|\phi_N\|_{\varpi^{2s}}. \tag{3.11}$$

The detailed proof of (3.11) is postponed to Appendix B. Since $\varpi^{-s}u_N \in \mathbb{P}_N(\mathbb{B}^d)$, one obtains

$$\|u_N\|^2 = \|\varpi^{-s}u_N\|_{\varpi^{2s}}^2 \gtrsim N^{-2s}\|\varpi^{-s}u_N\|_{\varpi^s}^2 = N^{-2s}\sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{N} |\widehat{u}_{k,\ell}^{s,n}|^2 \|P_{k,\ell}^{s,n}\|_{\varpi^s}^2.$$

Combining this with (3.9) leads to the desired result. □

**Theorem 3.1.** *As $N$ tends to infinity, it holds that*

$$\lambda_{N,N}^1 = \mathcal{O}(1), \quad \lambda_{N,N}^{(N+1)^2} = \mathcal{O}(N^{4s}). \tag{3.12}$$

*Moreover, the spectral condition number of the mass matrix $M$ satisfies*

$$\chi_N(M) = \frac{\lambda_{N,N}^{(N+1)^2}}{\lambda_{N,N}^1} = \mathcal{O}(N^{4s}). \tag{3.13}$$

*Proof.* By means of the min-max principle and Lemma 3.1, we get that

$$\frac{2^{2s}\Gamma(s+1)\Gamma(d/2+s)}{\Gamma(d/2)} < \lambda_1 = \min_{u\in H_*^{s/2}(\mathbb{B}^d)} \frac{\|(-\Delta)^{s/2}u\|_{\mathbb{R}^d}^2}{\|u\|^2} \le \min_{u\in W_N} \frac{\|(-\Delta)^{s/2}u\|_{\mathbb{R}^d}^2}{\|u\|^2} = \lambda_{N,N}^1,$$

$$\begin{aligned}\lambda_{N,N}^1 &= \min_{u\in W_N} \frac{\|(-\Delta)^{s/2}u\|_{\mathbb{R}^d}^2}{\|u\|^2} \le \frac{\big\|(-\Delta)^{s/2}\widetilde{Q}_{0,\ell}^{-s,n}\big\|_{\mathbb{R}^d}^2}{\big\|\widetilde{Q}_{0,\ell}^{-s,n}\big\|^2} = \frac{1}{m_{0,0}} \\ &= \frac{2^{2s}\Gamma(s+1)^2\Gamma(2s+n+d/2+1)}{(s+n+d/2)\Gamma(n+d/2)\Gamma(2s+1)} = \frac{\sqrt{\pi}(2s+n+d/2)\Gamma(2s+n+d/2)\Gamma(s+1)}{(s+n+d/2)\Gamma(n+d/2)\Gamma(s+1/2)},\end{aligned}$$

where the identity $\Gamma(z)\Gamma(z+1/2)=2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ is used for the last equality sign. This leads to the conclusion $\lambda^1_{N,N}=\mathcal{O}(1)$.

In the sequel, it follows from the equality (3.8) that

$$\lambda_{N,N}^{(N+1)^2}=\max_{u\in W_N}\frac{\|(-\Delta)^{s/2}u\|^2_{\mathbb{R}^d}}{\|u\|^2}\lesssim N^{4s}.$$

To prove $\lambda_{N,N}^{(N+1)^2}=\mathcal{O}(N^{4s})$, it suffices to verify that for

$$\phi_N(\boldsymbol{x})=\frac{\Gamma(2s+n+d/2+N+1)}{\Gamma(n+d/2+N)}(1-\|\boldsymbol{x}\|^2)^s P_N^{(2s+1,n+d/2-1)}(2\|\boldsymbol{x}\|^2-1)Y_\ell^n(\boldsymbol{x})\in W_{N,N},$$
$$\|(-\Delta)^{s/2}\phi_N\|^2_{\mathbb{R}^d}=\mathcal{O}(N^{8s+2}),\quad \|\phi_N\|^2=\mathcal{O}(N^{4s+2}),\quad N\ \to\ +\infty.$$

In fact, thanks to (3.6), we obtain that

$$\phi_N(\boldsymbol{x})=(1-\|\boldsymbol{x}\|^2)^s\sum_{m=0}^N\frac{(2s+n+d/2+2m)\Gamma(2s+n+d/2+m)}{\Gamma(n+d/2+m)}\times P_m^{(2s,n+d/2-1)}(2\|\boldsymbol{x}\|^2-1)Y_\ell^n(\boldsymbol{x}).$$

By the orthogonality relation of (2.15) and (2.6) together with (3.10)

$$\begin{aligned}\|\phi_N\|^2&=\sum_{m=0}^N\frac{(2s+n+d/2+2m)^2\Gamma(2s+n+d/2+m)^2}{\Gamma(n+d/2+m)^2}\\&\quad\times\int_0^1 P_m^{(2s,n+d/2-1)}(2r^2-1)P_m^{(2s,n+d/2-1)}(2r^2-1)(1-r^2)^{2s}r^{2n+d-1}dr\\&=\left(\frac{1}{2}\right)^{2s+n+d/2+1}\sum_{m=0}^N\frac{(2s+n+d/2+2m)^2\Gamma(2s+n+d/2+m)^2}{\Gamma(n+d/2+m)^2}\\&\quad\times\int_{-1}^1 P_m^{(2s,n+d/2-1)}(\rho)P_m^{(2s,n+d/2-1)}(\rho)(1-\rho)^{2s}(1+\rho)^{n+d/2-1}d\rho\\&=\sum_{m=0}^N\frac{(2s+n+d/2+2m)\Gamma(2s+n+d/2+m)\Gamma(2s+m+1)}{2\Gamma(n+d/2+m)\Gamma(m+1)}=\mathcal{O}(N^{4s+2}),\end{aligned}$$

where the second equality sign is derived from variable substitution $\rho=2r^2-1$.

Meanwhile, by the connection relation (3.10) once again

$$\phi_N(\boldsymbol{x})=\sum_{m=0}^N\frac{\Gamma(s+1+N-m)\Gamma(s+n+d/2+m)\Gamma(n+d/2+2s+1+N+m)(s+n+d/2+2m)}{\Gamma(s+1)\Gamma(N-m+1)\Gamma(n+d/2+m)\Gamma(s+n+d/2+m+1+N)}\times P_m^{(s,n+d/2-1)}(2\|\boldsymbol{x}\|^2-1)Y_\ell^n(\boldsymbol{x})(1-\|\boldsymbol{x}\|^2)^s,$$

which together with (2.4) and (2.26) gives that

$$\|(-\Delta)^{s/2}\phi_N\|^2_{\mathbb{R}^d}=\frac{2^{2s-1}}{\Gamma(s+1)^2}\sum_{m=0}^N\frac{\Gamma(s+1+N-m)^2\Gamma(s+n+d/2+m)^2\Gamma(n+d/2+2s+1+N+m)^2}{\Gamma(N-m+1)^2\Gamma(n+d/2+m)^2\Gamma(n+d/2+s+1+N+m)^2}\times\frac{\Gamma(m+s+1)^2(s+n+2m+d/2)}{\Gamma(m+1)^2}=\mathcal{O}(N^{8s+2}).$$

This completes the proof that $\lambda_{N,N}^{(N+1)^2}=\mathcal{O}(N^{4s})$. And (3.13) is an immediate consequence of (3.12). □

## 4. Regularity Theory and Error Estimates

### 4.1. Regularity theory

We first introduce a weighted Sobolev space for our regularity analysis and error estimates, then present our regularity results in weighted Sobolev spaces and their proofs.

Define the weighted space $\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)$ for any $\mu,\nu>0$,

$$\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d):=\big\{u\in L^2_{\varpi^s}(\mathbb{B}^d):|u|_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}<\infty\big\},$$

furnished with the semi-norm

$$|u|^2_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}=\big\|\nabla_0^{\mu}u\big\|^2_{\varpi^s}+\|u\|^2_{\nu,\varpi^s}, \tag{4.1}$$

where

$$\nabla_0^{\mu}=\begin{cases}(-\Delta_0)^{\mu/2}, & \mu \quad \text{is even},\\ (-\Delta_0)^{\mu-1/2}\nabla_0, & \mu \quad \text{is odd},\end{cases}$$

and

$$\|u\|_{\nu,\varpi^s}=\begin{cases}\|u\|_{\varpi^s}, & \nu=0,\\ \big(\|\nabla u\|^2_{\varpi^{s+1}}+\|\nabla_0 u\|^2_{\varpi^s}\big)^{1/2}, & \nu=1,\\ \big\|\big(-\Delta+\nabla\cdot\boldsymbol{x}(2\alpha+\boldsymbol{x}\cdot\nabla)-2\alpha d\big)^l u\big\|_{t,\varpi^s}, & \nu=2l+t\geq 2, \quad t\in\{0,1\}.\end{cases}$$

The semi-norm $|u|_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}$ admits an equivalent norm

$$|u|^2_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}=\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\big[n^{2\mu}+(n+k)^{2\nu}\big]\big|\widetilde{u}_{k,\ell}^{s,n}\big|^2\chi_{k,n}^s,\quad u=\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\widetilde{u}_{k,\ell}^{s,n}P_{k,\ell}^{s,n}. \tag{4.2}$$

The norm in this space is defined for any $\mu,\nu>0$ as $\|\cdot\|_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}$, that is

$$\|u\|^2_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}=\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\big[1+n^{2\mu}+(n+k)^{2\nu}\big]\big|\widehat{u}_{k,\ell}^{s,n}\big|^2\chi_{k,n}^s. \tag{4.3}$$

#### 4.1.1. Regularity of solution to the source problem

In order to obtain the regularity estimate for the eigenfunction of the eigenvalue problem (1.1), the regularity of solution to the following source problem of the fractional Laplacian in weighted Sobolev spaces is firstly considered

$$\begin{cases}(-\Delta)^s u+c\,u=f, & \boldsymbol{x}\in\mathbb{B}^d,\\ u(\boldsymbol{x})=0, & \boldsymbol{x}\in\mathbb{R}^d\setminus\mathbb{B}^d,\end{cases} \tag{4.4}$$

where $f$ is a smooth and given function, $c\geq 0$ is a constant. A combination of (2.2), (2.3), Lemma 2.1 as well as the Lax-Milgram theorem leads to the well-posedness of the problem (4.4) shown in the following theorem.

**Theorem 4.1.** *For the problem* (4.4) *with* $c\geq 0$ *and* $f\in H_*^{-s}(\mathbb{B}^d)$, *there exists a unique solution* $u\in H_*^s(\mathbb{B}^d)$ *such that* $\|u\|_{s,*}\leq\|f\|_{s,\mathbb{R}^d}$, *where* $H_*^{-s}(\mathbb{B}^d)$ *is the dual space of* $H_*^s(\mathbb{B}^d)$ *with respect to the inner product* $(u,v)_{\omega,\Omega}$.

The full regularity for the solution $(1-\|\boldsymbol{x}\|^2)^{-s}u$ instead of $u$ to the problem (4.4) is obtained as stated in Theorem 4.2. For the proof, please see Appendix C.

**Theorem 4.2.** *For the problem* (4.4) *with* $c = 0$*, if* $f \in \mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d) \bigcap H_*^{-s}(\mathbb{B}^d)$ *with* $\mu, \nu \geq 0$*, then*

$$(1 - \|\boldsymbol{x}\|^2)^{-s} u \in \mathcal{B}_s^{\mu+s,\nu+2s}(\mathbb{B}^d).$$

However, the regularity result in above theorem does not hold for the fractional Laplace equation (4.4) with $c > 0$. The following lemma plays an essential role in the analysis of regularity for the Eq. (4.4) when $c \neq 0$.

**Lemma 4.1.** *If* $v \in \mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)$ *with* $\mu, \nu \geq 0$*, then* $(1-\|\boldsymbol{x}\|^2)^s v \in \mathcal{B}_s^{\mu,\min(\nu,3s+1-\varepsilon)}(\mathbb{B}^d)$ *with* $\varepsilon > 0$*.*

The proof of this lemma is given in Appendix C.

**Theorem 4.3.** *For the problem* (4.4) *with* $c > 0$*, and* $f \in \mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d) \bigcap H_*^{-s}(\mathbb{B}^d)$ *with* $\mu, \nu \geq 0$*, we have for any arbitrary small* $\varepsilon > 0$ *that*

$$(1 - \|\boldsymbol{x}\|^2)^{-s} u \in \mathcal{B}_s^{\mu+s,\min(\nu,3s+1-\varepsilon)+2s}(\mathbb{B}^d).$$

*Proof.* Denote $\tilde{u} = (1 - \|\boldsymbol{x}\|^2)^{-s}u$. Since $f \in H_*^s(\mathbb{B}^d)$, it follows $u \in L^2_{\varpi^s}(\mathbb{B}^d)$ from Theorems 4.1 and 2.4. Then it yields that

$$\tilde{f} = f - cu \in \mathcal{B}_s^{0,0}(\mathbb{B}^d).$$

Now we use the bootstrapping technique to lift the regularity of solution $\tilde{u}$. Due to the conclusion of Theorem 4.2 and $(-\Delta)^s u = \tilde{f}$, it yields

$$\tilde{u} \in \mathcal{B}_s^{\min(\mu,0)+s,\min(\nu,0)+2s}(\mathbb{B}^d).$$

Thence by Lemma 4.1, we obtain

$$u \in \mathcal{B}_s^{\min(\mu,0)+s,\min(\nu,0)+2s}(\mathbb{B}^d).$$

Repeating the above procedure if $\mu, \nu > 0$, we obtain

$$\tilde{f} = f - cu \in \mathcal{B}_s^{\min(\mu,s),\min(\nu,2s)}(\mathbb{B}^d),$$

and then by Theorem 4.2 and $(-\Delta)^s u = \tilde{f}$, it yields that

$$\tilde{u} \in \mathcal{B}_s^{\min(\mu,s)+s,\min(\nu,2s)+2s}(\mathbb{B}^d).$$

If we further have $\mu \geq s$ and $\nu \geq 2s$, we repeat the above procedure and obtain

$$\tilde{u} \in \mathcal{B}_s^{\min(\mu,2s)+s,\min(\nu,4s)+2s}(\mathbb{B}^d).$$

If $\nu \geq 4s$, by Lemma 4.1, it yields that

$$\tilde{u} \in \mathcal{B}_s^{\min(\mu,2s)+s,\min(\nu,3s+1-\varepsilon)+2s}(\mathbb{B}^d).$$

If $\mu$ is large enough, we can repeat the above procedure $k$ times and obtain

$$\tilde{u} \in \mathcal{B}_s^{\min(\mu,ks)+s,\min(\nu,3s+1-\varepsilon)+2s}(\mathbb{B}^d)$$

until $k \geq \mu/s$. Thus by Lemma 4.1, we get the desired conclusion. □

#### 4.1.2. Regularity of eigenfunctions for the fractional eigenvalue problem

In this subsection, it is to obtain the regularity estimate for the eigenfunction of the eigenvalue problem (1.1).

**Theorem 4.4.** *Assume $u$ is an eigenfunction of the fractional eigenvalue problem* (1.1)*, we have for any arbitrary small $\varepsilon > 0$ that*

$$(1-\|\boldsymbol{x}\|^2)^{-s}u \in \mathcal{B}_s^{\infty,5s+1-\varepsilon}(\mathbb{B}^d). \tag{4.5}$$

*Proof.* For the problem (4.4) with $c > 0$ and $f = 0$, using the result of Theorem 4.3, it leads to the desired conclusion (4.5). □

### 4.2. Error estimate

This subsection studies the orthogonal polynomial approximation on the unit ball. Then, an optimal error estimate for the spectral-Galerkin method in weighted Sobolev space is given at last.

Introduce the orthogonal projection $\Pi^s_{N,K} : H^s_*(\mathbb{B}^d) \to W_{N,K}$ such that

$$\big((-\Delta)^s(\Pi^s_{N,K}u - u), v\big)_{\mathbb{R}^d} = 0, \quad v \in W_{N,K}.$$

Owing to the orthogonality relation (2.20), $\forall u \in H^s_*(\mathbb{B}^d)$ can be represented as

$$u(\boldsymbol{x}) = \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty} \widehat{u}^{s,n}_{k,\ell}(1-\|\boldsymbol{x}\|^2)^s P^{s,n}_{k,\ell}(\boldsymbol{x}), \quad \widehat{u}^{s,n}_{k,\ell} = \frac{\big(u, (1-\|\boldsymbol{x}\|^2)^s P^{s,n}_{k,\ell}\big)_{\varpi^{-s}}}{\chi^s_{k,n}}, \tag{4.6}$$

and $\Pi^s_{N,K}u$ is then a truncated series of $u$,

$$\Pi^s_{N,K}u(\boldsymbol{x}) = \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{K} \widehat{u}^{s,n}_{k,\ell}(1-\|\boldsymbol{x}\|^2)^s P^{s,n}_{k,\ell}(\boldsymbol{x}). \tag{4.7}$$

**Lemma 4.2.** *For $(1-\|\boldsymbol{x}\|^2)^{-s}u \in \mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)$ with $\mu, \nu \geq 0$, we have the following estimate:*

$$\big\|u - \Pi^s_{N,K}u\big\|_{\varpi^{-s}} \lesssim (N^{-\gamma_1} + K^{-\gamma_2})|\varpi^{-s}u|_{\mathcal{B}^{\gamma_1,\gamma_2}_s(\mathbb{B}^d)}, \tag{4.8}$$

*where $\gamma_1 = \mu + s$ and $\gamma_2 = 2s + \min(3s+1-\varepsilon, \nu)$ with arbitrary small $\varepsilon > 0$.*

*Proof.* It follows from (4.6), (4.7) and Parseval's theorem that

$$\begin{aligned}
\big\|\Pi^s_{N,K}u - u\big\|^2_{\varpi^{-s}} &= \left(\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty} - \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{K}\right)\widehat{u}^{s,n}_{k,\ell}(1-\|\boldsymbol{x}\|^2)^s\big\|P^{s,n}_{k,\ell}\big\|^2_{\varpi^{-s}} \\
&= \left(\sum_{n=N+1}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty} + \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=K+1}^{\infty} - \sum_{n=N+1}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=K+1}^{\infty}\right)\big|\widehat{u}^{s,n}_{k,\ell}\big|^2\chi^{s,n}_k \\
&\leq \left(\sum_{n=N+1}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty} + \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=K+1}^{\infty}\right)\big|\widehat{u}^{s,n}_{k,\ell}\big|^2\chi^{s,n}_k.
\end{aligned}$$

Using the facts that

$$\begin{aligned} N^{2\mu} &\le n^{2\mu} + (n+k)^{2\nu}, \quad n \ge N+1, \quad k \ge 0, \\ K^{2\nu} &\le n^{2\mu} + (n+k)^{2\nu}, \quad n \ge 0, \qquad\quad k \ge K+1, \end{aligned} \tag{4.9}$$

we have

$$\big\|\Pi^s_{N,K} u - u\big\|^2_{\varpi^{-s}} \le (N^{-2\mu} + K^{-2\nu}) \sum_{n=0}^{\infty} \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\infty} \big[n^{2\mu} + (n+k)^{2\nu}\big] \big|\widehat{u}^{s,n}_{k,\ell}\big|^2 \chi^{s,n}_k,$$

which implies by the definition of the semi-norm (4.2) that

$$\big\|u - \Pi^s_{N,K} u\big\|_{\varpi^{-s}} \lesssim (N^{-\mu} + K^{-\nu}) \big|(1 - \|\boldsymbol{x}\|^2)^{-s} u\big|_{\mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)}. \tag{4.10}$$

Above result together with the regularity estimate in Theorem 4.3 leads to (4.8). □

**Lemma 4.3.** *For $(1 - \|\boldsymbol{x}\|^2)^{-s} u \in \mathcal{B}^{\mu+s,\nu+s}_s(\mathbb{B}^d)$ with $\mu, \nu \ge 0$, we have the following estimate:*

$$\big|u - \Pi^s_{N,K} u\big|_{s,*} \lesssim (N^{s-\gamma_1} + K^{s-\gamma_2}) |\varpi^{-s} u|_{\mathcal{B}^{\gamma_1,\gamma_2}_s}, \tag{4.11}$$

*where $\gamma_1 = \mu + s$ and $\gamma_2 = 2s + \min(3s + 1 - \varepsilon, \nu)$ with arbitrary small $\varepsilon > 0$.*

*Proof.* It follows from (4.6), (4.7), (2.2), (2.3), (2.22), (2.24) that

$$\begin{aligned} &\big|u - \Pi^s_{N,K} u\big|^2_{s,*} \\ &= \big\|(-\Delta)^{s/2}\big(u - \Pi^s_{N,K} u\big)\big\|^2_{\mathbb{R}^d} \\ &= \big((-\Delta)^s\big(u - \Pi^s_{N,K} u\big), u - \Pi^s_{N,K} u\big)_{\mathbb{R}^d} \\ &\le \left( \sum_{n=N+1}^{\infty} \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\infty} + \sum_{n=0}^{\infty} \sum_{\ell=1}^{a_n^d} \sum_{k=K+1}^{\infty} \right) \frac{2^{2s}\Gamma(k+s+1)\Gamma(k+n+s+d/2)}{\Gamma(k+1)\Gamma(k+n+d/2)} \big|\widehat{u}^{s,n}_{k,\ell}\big|^2 \chi^s_{k,n}. \end{aligned}$$

Using the facts (4.9) and the estimate (C.2), we have

$$\begin{aligned} &|u - \Pi^s_{N,K} u|^2_{s,*} \\ &\le (N^{-2\mu} + K^{-2\nu}) \sum_{n=0}^{\infty} \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\infty} \frac{2^{2s}\Gamma(k+s+1)\Gamma(k+n+s+d/2)}{\Gamma(k+1)\Gamma(k+n+d/2)} \big[n^{2\mu} + (n+k)^{2\nu}\big] \big|\widehat{u}^{s,n}_{k,\ell}\big|^2 \chi^s_{k,n} \\ &\lesssim (N^{-2\mu} + K^{-2\nu}) \sum_{n=0}^{\infty} \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\infty} \big[n^{2\mu+2s} + (n+k)^{2\nu+2s}\big] \big|\widehat{u}^{s,n}_{k,\ell}\big|^2 \chi^s_{k,n}, \end{aligned}$$

which implies by the definition of the semi-norm (4.2) that

$$\big|u - \Pi^s_{N,K} u\big|_{s,*} \lesssim (N^{-\mu} + K^{-\nu}) \big|(1 - \|\boldsymbol{x}\|^2)^{-s} u\big|_{\mathcal{B}^{\mu+s,\nu+s}_s}.$$

Above conclusion together with the regularity estimate in Theorem 4.3 leads to (4.11). □

**Remark 4.1.** Thanks to Lemmas 4.2 and 4.3, the convergence order of the spectral-Galerkin method for source problems of the fractional Laplace equation (4.4) with $c > 0$ can be derived, which is $5s + 1$ in $L^2_{\varpi^{-s}}$-norm and $4s + 1$ in $H^s_*$-norm.

Recall that $a(\cdot,\cdot)$ is symmetric, continuous and coercive on $H_*^{s/2}(\mathbb{B}^d)\times H_*^{s/2}(\mathbb{B}^d)$, $b(\cdot,\cdot)$ is continuous on $L^2(\mathbb{B}^d)\times L^2(\mathbb{B}^d)$, and $H_*^{s/2}(\mathbb{B}^d)$ is compactly imbedded in $L^2(\mathbb{B}^d)$. Thus, based on the approximation theory of Babuška and Osborn on the Ritz method for self-adjoint and positive-definite eigenvalue problems [6, pp. 697-700], we now arrive at the following main theorem.

**Theorem 4.5.** *Let $\{\lambda_{N,K}^i\}$ be the eigenvalues of* (3.3) *ordered non-decreasingly with respect to $i$, repeated according to their multiplicities. Further let $\lambda_k$ be an eigenvalue of* (1.1) *with the geometric multiplicity $q$ and assume that $\lambda_k=\lambda_{k+1}=\cdots=\lambda_{k+q-1}$. It holds that*

$$0\le \lambda_{N,K}^j-\lambda_k\lesssim K^{-8s-2+\varepsilon},\quad j=k,k+1,\dots,k+q-1,$$

*where*

$$E(\lambda_k):=\Big\{\psi \text{ is an eigenfunction corresponding to } \lambda_k \text{ with } \|(-\Delta)^{s/2}\psi\|_{\mathbb{R}^d}=1\Big\}.$$

*Let $\psi_{N,K}^j$ be an eigenfuction corresponding to $\lambda_{N,K}^j$ for $j=k,k+1,\dots,k+q-1$, then there holds*

$$\inf_{u\in E(\lambda_k)}\big\|u-\psi_{N,K}^j\big\|_{s,*}\lesssim K^{-4s-1+\varepsilon}.$$

*Let $\psi_k$ be an eigenfuction corresponding to $\lambda_k$, there exist a function*

$$v_N\in \operatorname{span}\big\{\psi_{N,K}^k,\cdots,\psi_{N,K}^{k+q-1}\big\}$$

*such that*

$$\|\psi_k-v_N\|_{s,*}\lesssim K^{-4s-1+\varepsilon}.$$

## 5. Numerical Tests

In this section, we present several numerical examples to illustrate the accuracy and efficiency of our spectral-Galerkin method and validate the theoretical results related. We first present the numerical results on the condition number of the mass matrix $M$, then show numerical approximation results to verify the expected convergence orders of numerical eigenvalues.

### 5.1. Condition number

In this subsection, we present in Fig. 5.1 the condition number versus $N$ with various fractional order $s$ from 0.3 to 1.0 in logarithm-logarithm scale. The result shows that, for each $s$, the condition number $\chi_N(M)$ grows algebraically with respect to $N$. In order to investigate the growth tendency of the condition number numerically, we draw the condition number together with the function $N^{4s}$ in logarithm-logarithm scale. The straight lines indicate that $\chi_N(M)=\mathcal{O}(N^{4s})$, which is predicted by Theorem 3.1.

### 5.2. Convergence order

At the beginning of this subsection, our results are compared with those available in the literature. In [20], the authors provide sharp estimates on the eigenvalues of the fractional Laplacian in the unit ball of any dimension; meanwhile, the finite element approximation for the fractional Laplacian eigenvalue problem is studied in [10]. Table 5.1 reports the first eigenvalue

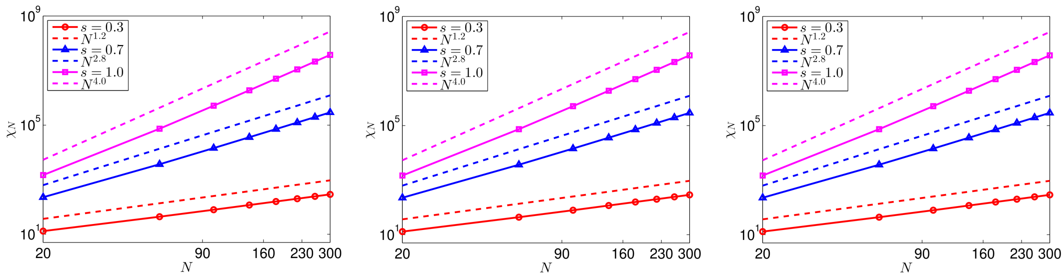
Table 5.1: The first eigenvalue in the unit ball in $\mathbb{R}^2$. Estimate from [20], upper bound obtained by FEM in [10] with a meshsize $h \sim 0.02$, results by our spectral-Galerkin method with $N = 6$.

| s | Ref. [20] | Ref. [10] | Present |
|---|---|---|---|
| 0.005 | 1.00475 | 1.00480 | 1.00475 |
| 0.05 | 1.05095 | 1.05145 | 1.05095 |
| 0.25 | 1.34373 | 1.34626 | 1.34373 |
| 0.5 | 2.00612 | 2.01060 | 2.00612 |
| 0.75 | 3.27594 | 3.28043 | 3.27594 |

by our spectral-Galerkin method with $N = 6$ for various $s$ and the numerical approximation to $\lambda_1$ computed by the methods in [20] and [10]. We can observe that these outcomes are in good agreement.

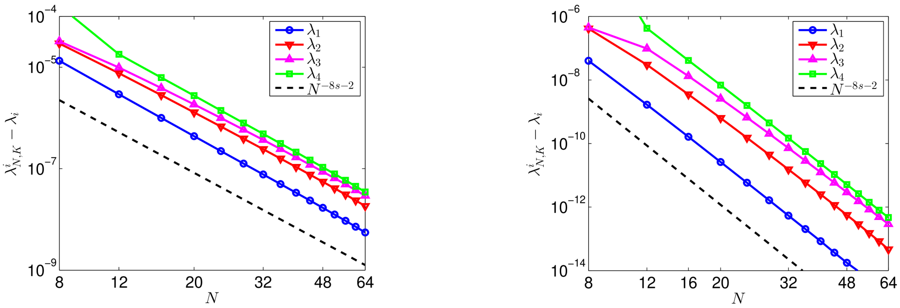
On the other hand, in logarithm-logarithm scale, the approximation errors of the 4 smallest eigenvalues with different fractional orders are presented in Figs. 5.2-5.4. Error plots reveal that the computational eigenvalues converge at a rate of $\mathcal{O}(N^{-8s-2})$, which is consistent with the result in Theorem 4.5.

Finally, we present some figures of the first two eigenfunctions in two dimensions with different fractional order $s = 0.6, 0.8, 1.0$ and $K = N = 10$ in Figs. 5.5-5.7, respectively. Fig. 5.8 visualize of $\psi^i_{N,K}$ with $s = 0.8$ and different $n, k, \ell$ in three dimensions.



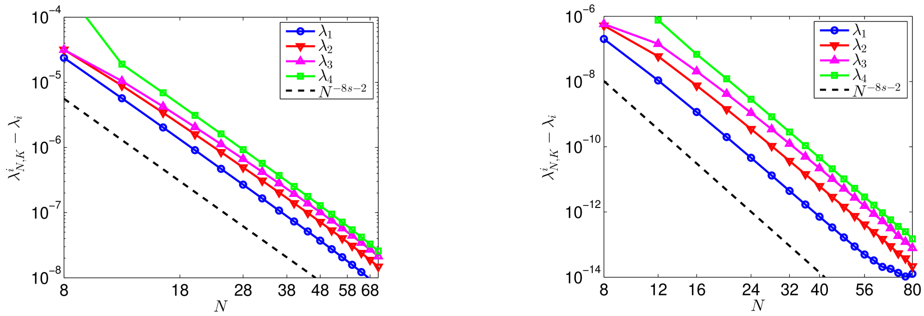

(a) $d = 1$ (b) $d = 2$ (c) $d = 3$

Fig. 5.1. The condition number and polynomial function $N^{4s}$ versus $N$ with different fractional orders $s = 0.3, 0.7, 1.0$ in logarithm-logarithm scale.



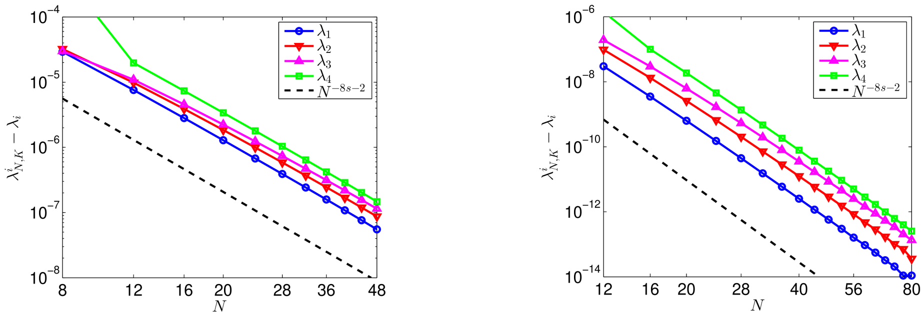

(a) $s = 0.2$ (b) $s = 0.8$

Fig. 5.2. Approximation errors $\lambda^i_{N,K} - \lambda_i(\circ : \lambda_1, \nabla : \lambda_2, \triangle : \lambda_3, \square : \lambda_4)$ versus the reference function $N^{-8s-2}$ with different fractional orders in logarithm-logarithm scale in one dimension.



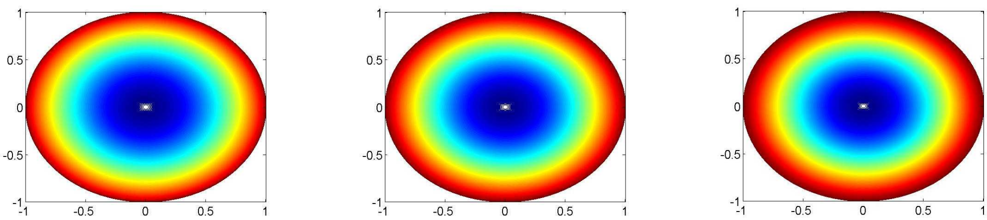

(a) $s = 0.2$ (b) $s = 0.8$

Fig. 5.3. Approximation errors $\lambda^i_{N,K} - \lambda_i(\circ : \lambda_1, \nabla : \lambda_2, \triangle : \lambda_3, \square : \lambda_4)$ versus the reference function $N^{-8s-2}$ with different fractional orders in logarithm-logarithm scale in two dimensions.



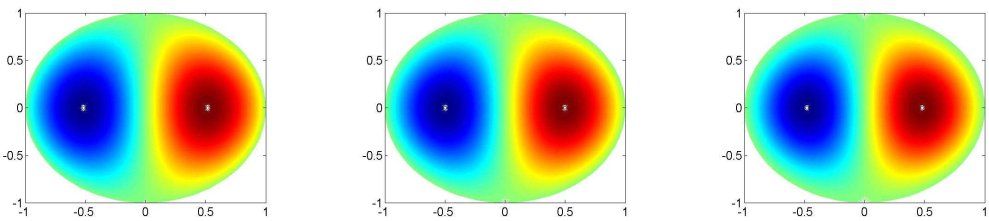

(a) $s = 0.2$ (b) $s = 0.8$

Fig. 5.4. Approximation errors $\lambda^i_{N,K} - \lambda_i(\circ : \lambda_1, \nabla : \lambda_2, \triangle : \lambda_3, \square : \lambda_4)$ versus the reference function $N^{-8s-2}$ with different fractional orders in logarithm-logarithm scale in three dimensions.



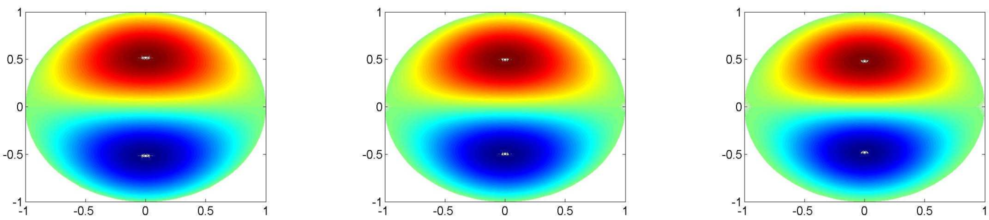
(a) $s = 0.6$ (b) $s = 0.8$ (c) $s = 1.0$

Fig. 5.5. The figures of $\psi^1_{N,K}$ for different fractional orders with $N = K = 10$ in two dimensions.



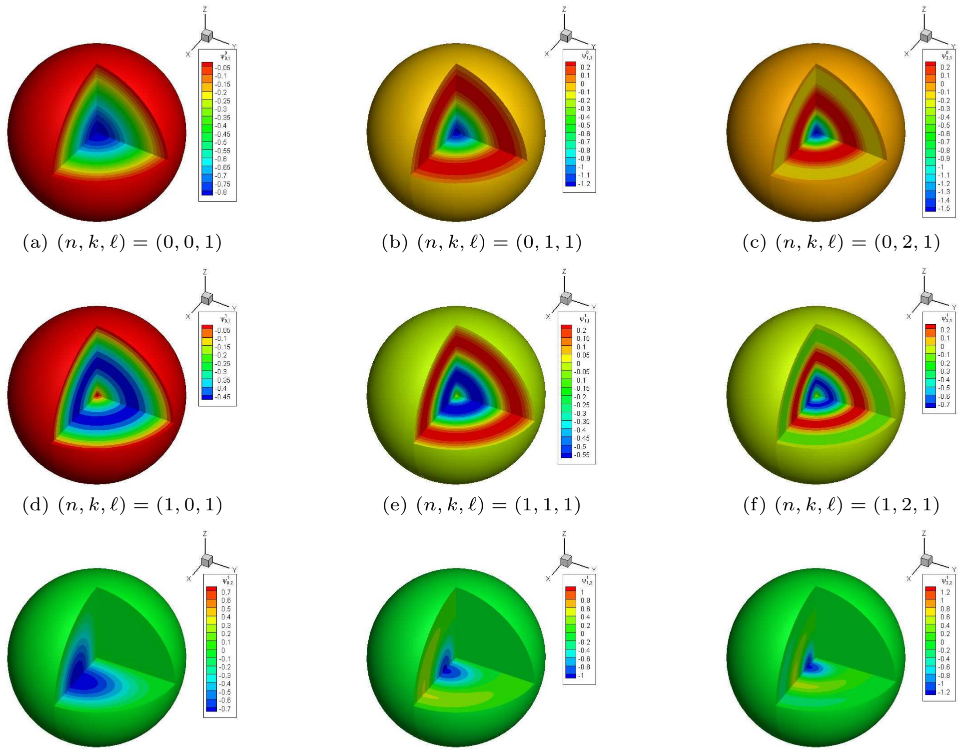
(a) $s = 0.6$ (b) $s = 0.8$ (c) $s = 1.0$

Fig. 5.6. The figures of $\psi^2_{N,K}$ corresponding to $Y^1_1 = r\cos\theta$ for different fractional orders with $N = K = 10$ in two dimensions.




(a) $s = 0.6$ (b) $s = 0.8$ (c) $s = 1.0$

Fig. 5.7. The figures of $\psi^2_{N,K}$ corresponding to $Y^1_2 = r\sin\theta$ for different fractional orders with $N = K = 10$ in two dimensions.



(a) $(n,k,\ell) = (0,0,1)$ (b) $(n,k,\ell) = (0,1,1)$ (c) $(n,k,\ell) = (0,2,1)$

(d) $(n,k,\ell) = (1,0,1)$ (e) $(n,k,\ell) = (1,1,1)$ (f) $(n,k,\ell) = (1,2,1)$

(g) $(n,k,\ell) = (1,0,2)$ (h) $(n,k,\ell) = (1,1,2)$ (i) $(n,k,\ell) = (1,2,2)$

Fig. 5.8. The figures of $\psi^i_{N,K}$ corresponding to $k$ and $Y^n_\ell$ with $s = 0.8$ in three dimensions.

# Appendix A. The Proofs in Section 2

## A.1. The proof of Lemma 2.5

It is obvious that (2.21) holds for $\nu = 0$. When $\nu = 1$, one can obtain from (2.16), (2.17) and (2.14) that

$$\begin{aligned}\frac{\Delta P^{\alpha,n}_{k,\ell}(\boldsymbol{x})}{Y^n_\ell(\boldsymbol{x})} &= \frac{1}{r^n}\left[\partial_r^2 + \frac{d-1}{r}\partial_r - \frac{n(n+d-2)}{r^2}\right]\left[r^n P_k^{(\alpha,n+d/2-1)}(2r^2-1)\right] \\ &= \left[\partial_r^2 + \frac{2n+d-1}{r}\partial_r\right] P_k^{(\alpha,n+d/2-1)}(2r^2-1),\end{aligned}$$

where the last equality sign is derived from a direct reduction. Making a change of variable $\rho = 2r^2 - 1$, one deduces by (2.9) that

$$\begin{aligned}
&\left[\partial_r^2 + \frac{2n+d-1}{r}\partial_r\right] P_k^{(\alpha,n+d/2-1)}(2r^2-1)\\
&= 8\left[(\rho+1)\partial_\rho^2 + \left(n+\frac{d}{2}\right)\partial_\rho\right] P_k^{(\alpha,n+d/2-1)}(\rho)\\
&= 2\left(k+\alpha+n+\frac{d}{2}\right)\left[\left(k+\alpha+n+\frac{d}{2}+1\right)(\rho+1)P_{k-2}^{(\alpha+2,n+d/2+1)}(\rho)\right.\\
&\qquad\left. + 2\left(n+\frac{d}{2}\right)P_{k-1}^{(\alpha+1,n+d/2)}(\rho)\right].
\end{aligned}$$

Note that a combination of (2.8) and (2.11), (2.10) leads to the following relations:

$$\left(k+\frac{\alpha+\beta}{2}+1\right)(1+z)P_k^{(\beta,\alpha+1)}(z) = (k+\alpha+1)P_k^{(\beta,\alpha)}(z) + (k+1)P_{k+1}^{(\beta,\alpha)}(z), \tag{A.1}$$

$$(2k+\alpha+\beta+1)P_k^{(\beta,\alpha)}(z) = (k+\alpha+\beta+1)P_k^{(\beta,\alpha+1)}(z) + (k+\beta)P_{k-1}^{(\beta,\alpha+1)}(z). \tag{A.2}$$

Thus, one can get from (2.10), (A.1) and (A.2) that

$$\begin{aligned}
\frac{\Delta P_{k,\ell}^{\alpha,n}(\boldsymbol{x})}{Y_\ell^n(\boldsymbol{x})} &= \frac{4(k+\alpha+n+d/2)_2}{2k+\alpha+n+d/2}\left[\left(k+n+\frac{d}{2}-1\right)P_{k-2}^{(\alpha+2,n+d/2)}(\rho) + (k-1)P_{k-1}^{(\alpha+2,n+d/2)}(\rho)\right]\\
&\quad + \frac{4(n+d/2)(k+\alpha+n+d/2)}{2k+\alpha+n+d/2}\left[\left(k+\alpha+n+\frac{d}{2}+1\right)P_{k-1}^{(\alpha+2,n+d/2)}(\rho)\right.\\
&\qquad\left. - \left(k-1+n+\frac{d}{2}\right)P_{k-2}^{(\alpha+2,n+d/2)}(\rho)\right]\\
&= \frac{4(k+\alpha+n+d/2)(k+n+d/2-1)}{2k+\alpha+n+d/2}\left[\left(k+\alpha+n+\frac{d}{2}+1\right)P_{k-1}^{(\alpha+2,n+d/2)}(\rho)\right.\\
&\qquad\left. + (k+\alpha+1)P_{k-2}^{(\alpha+2,n+d/2)}(\rho)\right]\\
&= 4\left(k+n+\frac{d}{2}-1\right)\left(k+n+\alpha+\frac{d}{2}\right)P_{k-1}^{(\alpha+2,n+d/2-1)}(\rho),
\end{aligned}$$

which proves (2.21) in the case that $\nu = 1$. For the integer $\nu > 1$, we repeat the above process and deduces that

$$\begin{aligned}
(-\Delta)^\nu P_{k,\ell}^{\alpha,n}(\boldsymbol{x}) &= (-4)^{\nu-1}\left(k+n+\alpha+\frac{d}{2}\right)_{\nu-1}\left(k+n+\frac{d}{2}-\nu+1\right)_{\nu-1}\Delta P_{k-\nu+1,\ell}^{\alpha+2(\nu-1),n}(\boldsymbol{x})\\
&= (-4)^\nu\left(k+n+\alpha+\frac{d}{2}\right)_\nu\left(k+n+\frac{d}{2}-\nu\right)_\nu P_{k-\nu,\ell}^{\alpha+2\nu,n}(\boldsymbol{x}).
\end{aligned}$$

This ends the proof. □

### A.2. The proof of Lemma 2.6

We first note the fact that

$$\int_{\mathbb{B}^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_0^1 r^{d-1}\mathrm{d}r\int_{\mathbb{S}^{d-1}} f(r\hat{\boldsymbol{x}})\mathrm{d}\sigma(\hat{\boldsymbol{x}}).$$

Then, by the definition (2.22), a technical reduction and Lemma 2.4, one can obtain

$$\begin{aligned}
&\int_{\mathbb{B}^d}(1-\|\boldsymbol{x}\|^2)^\alpha P_{k,\ell}^{\alpha,n}(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x}\\
&=\int_0^1(1-r^2)^\alpha P_k^{(\alpha,n+d/2-1)}(2r^2-1)r^{n+d-1}\mathrm{d}r\int_{\mathbb{S}^{d-1}}\mathrm{e}^{-\mathrm{i}\rho r\langle\hat{\boldsymbol{\xi}},\hat{\boldsymbol{x}}\rangle}Y_\ell^n(\hat{\boldsymbol{x}})\mathrm{d}\sigma(\hat{\boldsymbol{x}})\\
&=\int_0^1(1-r^2)^\alpha P_k^{(\alpha,n+d/2-1)}(2r^2-1)r^{n+d-1}\frac{(2\pi)^{d/2}(-\mathrm{i})^n}{(\rho r)^{d/2-1}}J_{n+d/2-1}(r\rho)\mathrm{d}r\,Y_\ell^n(\hat{\boldsymbol{\xi}})\\
&=\frac{(2\pi)^{d/2}(-\mathrm{i})^n}{\rho^{d/2-1}}\bigg[\int_0^1(1-r^2)^\alpha P_k^{(\alpha,n+d/2-1)}(2r^2-1)r^{n+d/2}J_{n+d/2-1}(r\rho)\mathrm{d}r\bigg]Y_\ell^n(\hat{\boldsymbol{\xi}}).
\end{aligned}\tag{A.3}$$

Using the expression of the Bessel function of the first kind

$$J_\nu(z)=\sum_{m=0}^\infty\frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\Big(\frac{z}{2}\Big)^{2m+\nu},$$

the variable transformation $z=2r^2-1$ and (2.12), one further derives

$$\begin{aligned}
&\int_0^1(1-r^2)^\alpha P_k^{(\alpha,n+d/2-1)}(2r^2-1)r^{n+d/2}J_{n+d/2-1}(r\rho)\mathrm{d}r\\
&=\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+n+d/2)}\int_0^1\Big(\frac{r\rho}{2}\Big)^{2m+n+d/2-1}(1-r^2)^\alpha P_k^{(\alpha,n+(d-2)/2)}(2r^2-1)r^{n+d/2}\mathrm{d}r\\
&=\sum_{m=0}^\infty\frac{(-1)^m2^{-\alpha-m-n-d/2-1}}{m!\Gamma(m+n+d/2)}\Big(\frac{\rho}{2}\Big)^{2m+n+d/2-1}\int_{-1}^1P_k^{(\alpha,n+(d-2)/2)}(z)(1-z)^\alpha(1+z)^{m+n+d/2-1}\mathrm{d}z\\
&=\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+n+d/2)}\Big(\frac{\rho}{2}\Big)^{2m+n+d/2-1}\frac{(m-k+1)_k}{2\,k!}\frac{\Gamma(k+\alpha+1)\Gamma(m+n+d/2)}{\Gamma(k+\alpha+m+n+d/2+1)}\\
&=\frac{\Gamma(k+\alpha+1)}{2(-1)^kk!}\Big(\frac{\rho}{2}\Big)^{-\alpha-1}\sum_{m=0}^\infty\frac{(-1)^{m-k}}{(m-k)!\Gamma(k+\alpha+m+n+d/2+1)}\Big(\frac{\rho}{2}\Big)^{2(m-k)+(n+d/2+2k+\alpha)}\\
&=\frac{(-1)^k2^\alpha\Gamma(k+\alpha+1)}{k!}\rho^{-\alpha-1}J_{n+2k+d/2+\alpha}(\rho).
\end{aligned}$$

This together with (A.3) leads to the desired conclusion (2.23) immediately. □

### A.3. The proof of Theorem 2.4

Owing to (2.21), it suffices to prove Theorem 2.4 with the integer $\nu\in[s/2-1/2,s+d/2)$. To this end, we start with (2.23) in view of the definition of the fractional Laplacian (1.3)

$$\|\boldsymbol{\xi}\|^{2s-2\nu}\big[\mathscr{F}Q_{k,\ell}^{-s,n}\big](\boldsymbol{\xi})=\frac{(-\mathrm{i})^{n+2k}(2\pi)^{d/2}2^s\Gamma(k+n+s+d/2)}{\|\boldsymbol{\xi}\|^{d/2-s+2\nu}\Gamma(k+n+d/2)}J_{n+2k+d/2+s}(\|\boldsymbol{\xi}\|)\,Y_\ell^n(\hat{\boldsymbol{\xi}}),$$

and get from (2.18) that

$$\begin{aligned}
&\mathscr{F}^{-1}\big[\|\boldsymbol{\xi}\|^{2s-2\nu}\mathscr{F}Q_{k,\ell}^{-s,n}\big]\\
&=\int_{\mathbb{R}^d}\frac{2^s\Gamma(k+n+s+d/2)}{(2\pi)^{d/2}\mathrm{i}^{n+2k}\Gamma(k+n+d/2)}\frac{J_{n+2k+d/2+s}(\|\boldsymbol{\xi}\|)}{\|\boldsymbol{\xi}\|^{d/2-s+2\nu}}Y_\ell^n(\hat{\boldsymbol{\xi}})\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{\xi}\\
&=\int_0^\infty\frac{(-\mathrm{i})^{n+2k}2^s\Gamma(k+n+s+d/2)}{(2\pi)^{d/2}\rho^{1-d/2-s+2\nu}\Gamma(k+n+d/2)}J_{n+2k+d/2+s}(\rho)\mathrm{d}\rho\int_{\mathbb{S}^{d-1}}Y_\ell^n(\hat{\boldsymbol{\xi}})\mathrm{e}^{\mathrm{i}\rho r\langle\hat{\boldsymbol{\xi}},\hat{\boldsymbol{x}}\rangle}\mathrm{d}\sigma(\hat{\boldsymbol{\xi}})
\end{aligned}$$

$$= \int_0^\infty \frac{(-\mathrm{i})^{n+2k} 2^s \Gamma(k+n+s+d/2)}{(2\pi)^{d/2}\rho^{1-d/2-s+2\nu}\Gamma(k+n+d/2)} J_{n+2k+d/2+s}(\rho) \frac{(2\pi)^{d/2}\mathrm{i}^n}{(\rho r)^{d/2-1}} J_{n+d/2-1}(\rho r)\mathrm{d}\rho Y_\ell^n(\hat{\boldsymbol{x}})$$
$$= \frac{2^s\Gamma(k+n+s+d/2)}{(-1)^k\Gamma(k+n+d/2)r^{d/2-1}} \int_0^\infty \rho^{s-2\nu} J_{n+2k+d/2+s}(\rho) J_{n+d/2-1}(\rho r)\mathrm{d}\rho Y_\ell^n(\hat{\boldsymbol{x}}), \tag{A.4}$$

where $\boldsymbol{\xi} = \rho\hat{\boldsymbol{\xi}}$ with $\rho = \|\boldsymbol{\xi}\|, \hat{\boldsymbol{\xi}} \in \mathbb{S}^{d-1}$.

Further, we derive from [21, Eq. (8.11.9)] that when $\mu+\nu+1 > \lambda > -1$

$$\int_0^{+\infty} J_\mu(x)J_\nu(xy)x^{-\lambda}y^{1/2}\mathrm{d}x$$
$$= \frac{\Gamma\big((\mu+\nu-\lambda+1)/2\big)y^{\nu+1/2}}{2^\lambda\Gamma(\nu+1)\Gamma\big((\lambda+\mu-\nu+1)/2\big)}$$
$$\times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; y^2\right), \quad 0<y<1. \tag{A.5}$$

For $0 \le r \le 1$ and $\nu \in (s-1/2, s+d/2)$, it yields by (A.5) and (2.5) that

$$\mathscr{F}^{-1}\big[\|\boldsymbol{\xi}\|^{2s-2\nu}\mathscr{F}Q_{k,\ell}^{-s,n}(\boldsymbol{x})\big]$$
$$= \frac{2^{2s-2\nu}\Gamma(k+n+s+d/2)\Gamma(n+k+d/2+s-\nu)}{(-1)^k\Gamma(k+n+d/2)\Gamma(n+d/2)\Gamma(k+\nu+1)} Y_\ell^n(\boldsymbol{x})$$
$$\times {}_2F_1\Big(-k-\nu, n+k+\frac{d}{2}+s-\nu; n+\frac{d}{2}; r^2\Big).$$

If $\nu$ is integer, then

$$\mathscr{F}^{-1}\big[\|\boldsymbol{\xi}\|^{2s-2\nu}\mathscr{F}Q_{k,\ell}^{-s,n}\big]$$
$$= \frac{\Gamma(n+k+s+d/2)\Gamma(n+k+d/2+s-\nu)}{(-1)^k 2^{2\nu-2s}\Gamma(n+k+d/2)\Gamma(n+k+\nu+d/2)} Y_\ell^n(\boldsymbol{x}) P_{k+\nu,\ell}^{(n+d/2,s-2\nu)}(1-2r^2)$$
$$= \frac{\Gamma(n+k+d/2+s)\Gamma(n+k+d/2+s-\nu)}{(-1)^\nu 2^{2\nu-2s}\Gamma(n+k+d/2)\Gamma(n+k+\nu+d/2)} P_{k+\nu,\ell}^{s-2\nu,n}(\boldsymbol{x}).$$

The proof is complete. □

## Appendix B. The Proof of Eq. (3.11)

We prove (3.11) in two steps. The first step asserts that any polynomial $\varphi_N$ in $\mathbb{P}_N(\mathbb{B}^d)$ satisfies

$$\|\varphi_N\|_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} \lesssim N\|\varphi_N\|_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}, \quad \alpha > -1. \tag{B.1}$$

Indeed, for

$$\varphi_N(\boldsymbol{x}) = \sum_{n=0}^N \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \widehat{u}_{k,\ell}^{\alpha+1,n} P_{k,\ell}^{\alpha+1,n}(\boldsymbol{x}),$$

we have

$$\|\varphi_N\|_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)} = \left(\sum_{n=0}^N \sum_{\ell=1}^{a_n^d} \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \big|\widehat{u}_{k,\ell}^{\alpha+1,n}\big|^2 \big\|P_{k,\ell}^{\alpha+1,n}\big\|^2_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}\right)^{1/2}.$$

Meanwhile, it follows from the triangle inequality and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\|\varphi_N\|_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} &\le \sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\lfloor\frac{N-n}{2}\rfloor} |\widehat{u}_{k,\ell}^{\alpha+1,n}|\|P_{k,\ell}^{\alpha+1,n}\|_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} \\
&\le \left(\sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\lfloor\frac{N-n}{2}\rfloor} |\widehat{u}_{k,\ell}^{\alpha+1,n}|^2\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}\right)^{1/2} \\
&\quad \times \left(\sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\lfloor\frac{N-n}{2}\rfloor} \frac{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)}}{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}}\right)^{1/2} \\
&\le \|\varphi_N\|_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)} \left(\sum_{n=0}^{N}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\lfloor\frac{N-n}{2}\rfloor} \frac{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)}}{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}}\right)^{1/2}.
\end{aligned}$$

To evaluate this inequality further, we derive from (3.6) that

$$\begin{aligned}
P_k^{(\alpha+1,n+d/2-1)}(r) &= \frac{(n+d/2)_k}{(\alpha+n+d/2+1)_k} \\
&\quad \times \sum_{m=0}^{k} \frac{(1)_{k-m}(\alpha+n+d/2)_m(\alpha+n+d/2+2m)}{(k-m)!(n+d/2)_m(\alpha+n+d/2)} P_k^{(\alpha,n+d/2-1)}(r) \\
&= \frac{\Gamma(n+d/2+k)}{\Gamma(\alpha+n+d/2+k+1)} \\
&\quad \times \sum_{m=0}^{k} \frac{\Gamma(\alpha+n+d/2+m)(\alpha+n+d/2+2m)}{\Gamma(n+d/2+m)} P_m^{(\alpha,n+d/2-1)}(r).
\end{aligned}$$

Then one can get from (2.20) that

$$\begin{aligned}
\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} &= \frac{\Gamma(n+d/2+k)^2}{\Gamma(\alpha+n+d/2+k+1)^2} \\
&\quad \times \sum_{m=0}^{k} \frac{\Gamma(\alpha+n+d/2+m)^2(\alpha+n+d/2+2m)^2}{\Gamma(n+d/2+m)^2} \|P_{m,\ell}^{\alpha,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} \\
&= \frac{\Gamma(n+d/2+k)^2}{(\alpha+n+d/2+k)^2\Gamma(\alpha+n+d/2+k)^2} \\
&\quad \times \sum_{m=0}^{k} \frac{\Gamma(\alpha+n+d/2+m)\Gamma(m+\alpha+1)(\alpha+n+d/2+2m)}{2\Gamma(n+d/2+m)\Gamma(m+1)},
\end{aligned}$$

which indicates that $\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} = \mathcal{O}(1)$. Recalling that $\|P_{k,\ell}^{\alpha,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)}$ behaves like $(n+2k)^{-1}$, it yields that

$$\frac{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^\alpha}(\mathbb{B}^d)}}{\|P_{k,\ell}^{\alpha+1,n}\|^2_{L^2_{\varpi^{\alpha+1}}(\mathbb{B}^d)}} \lesssim n+2k,$$

which completes the proof of (B.1).

Next we obtain the following result via an interpolation argument that any polynomial $\varphi_N$ in $\mathbb{P}_N(\mathbb{B}^d)$ satisfies

$$\|\varphi_N\|_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} \lesssim N^{\beta-\alpha}\|\varphi_N\|_{L^2_{\varpi^\beta}(\mathbb{B}^d)}, \quad \alpha > -1, \quad \beta > 0, \quad \alpha \le \beta. \tag{B.2}$$

In the case that $\beta-\alpha\le 1$, we consider the identity operator which is continuous from the space $\mathbb{P}_N(\mathbb{B}^d)$ with the norm $L^2_\beta(\mathbb{B}^d)$ into the space $\mathbb{P}_N(\mathbb{B}^d)$ with the norm $L^2_\alpha(\mathbb{B}^d)$. One can obtain by the interpolation argument that

$$\|\varphi_N\|_{L^2_{\varpi^{\beta-\lambda}}(\mathbb{B}^d)} \lesssim N^\lambda \|\varphi_N\|_{L^2_{\varpi^\beta}(\mathbb{B}^d)}, \quad 0\le\lambda\le 1,$$

which is based on the fact

$$\begin{aligned}
&\|\varphi_N\|_{L^2_{\varpi^\beta}(\mathbb{B}^d)} \lesssim \|\varphi_N\|_{L^2_{\varpi^\beta}(\mathbb{B}^d)},\\
&\|\varphi_N\|_{L^2_{\varpi^{\beta-1}}(\mathbb{B}^d)} \lesssim N\|\varphi_N\|_{L^2_{\varpi^\beta}(\mathbb{B}^d)}.
\end{aligned}$$

Taking $\alpha=\beta-\lambda$ gives the result (B.2). In the case that $\beta-\alpha\ge 1$, one can obtain by iterating inequality (B.1)

$$\|\varphi_N\|_{L^2_{\varpi^\alpha}(\mathbb{B}^d)} \lesssim N^k \|\varphi_N\|_{L^2_{\varpi^{\alpha+k}}(\mathbb{B}^d)}.$$

This ends the proof of (B.2). Thus (3.11) is an immediate consequence of (B.1) combined with (B.2). The proof is complete. □

## Appendix C. The Proofs in Section 4

### C.1. The proof of Theorem 4.2

For

$$f(\boldsymbol{x}) = \sum_{n=0}^\infty\sum_{\ell=1}^{a_n^d}\sum_{k=0}^\infty \widehat{f}^{s,n}_{k,\ell} P^{s,n}_{k,\ell}(\boldsymbol{x}) \in \mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d),$$

it follows from the norm (4.3) that

$$\|f\|^2_{\mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)} = \sum_{n=0}^\infty\sum_{\ell=1}^{a_n^d}\sum_{k=0}^\infty \big[1+n^{2\mu}+(n+k)^{2\nu}\big]\big|\widehat{f}^{s,n}_{k,\ell}\big|^2\chi^{s,n}_k<\infty.$$

For $f\in H^{-s}_*(\mathbb{B}^d)$, it is derived $u\in H^s_*(\mathbb{B}^d)$ from Theorem 4.1. And it is readily to derive $u\in L^2_{\varpi^{-s}}(\mathbb{B}^d)$ due to (2.4). Then it follows that $(1-\|\boldsymbol{x}\|^2)^{-s}u\in L^2_{\varpi^s}(\mathbb{B}^d)$. Thus, using the definition (2.22), it yields that

$$u(\boldsymbol{x}) = \sum_{n=0}^\infty\sum_{\ell=1}^{a_n^d}\sum_{k=0}^\infty \frac{1}{\varrho^s_{k,n}}\widehat{u}^{s,n}_{k,\ell} Q^{-s,n}_{k,\ell}(\boldsymbol{x}).$$

One gets from (2.22), (2.24) with $\nu=0$, the equation $(-\Delta)^s u=f$ that

$$\widehat{u}^{s,n}_{k,\ell} = \frac{\varrho^s_{k,n}}{\sigma^{s,0}_{k,n}}\widehat{f}^{s,n}_{k,\ell} = \frac{2^{-2s}\Gamma(k+1)\Gamma(k+n+d/2)}{\Gamma(k+s+1)\Gamma(k+n+s+d/2)}\widehat{f}^{s,n}_{k,\ell}.$$

Then it follows from the norm (4.3) that

$$\begin{aligned}
&\|(1-\|\boldsymbol{x}\|^2)^{-s}u\|^2_{\mathcal{B}^{\mu+s,\nu+2s}_s(\mathbb{B}^d)}\\
=&\sum_{n=0}^\infty\sum_{\ell=1}^{a_n^d}\sum_{k=0}^\infty \big[1+n^{2\mu+2s}+(n+k)^{2\nu+4s}\big]\big|\widehat{u}^{s,n}_{k,\ell}\big|^2\chi^{s,n}_k
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty} \frac{2^{-4s}\Gamma^2(k+1)\Gamma^2(k+n+d/2)}{\Gamma^2(k+s+1)\Gamma^2(k+n+s+d/2)} \times \big[1+n^{2\mu+2s}+(n+k)^{2\nu+4s}\big]\big|\widehat{f}_{k,\ell}^{s,n}\big|^2\chi_k^{s,n}. \tag{C.1}$$

Based on the asymptotic formula (3.10), we have the asymptotic estimate

$$\frac{\Gamma(k+1)\Gamma(k+n+d/2)}{\Gamma(k+s+1)\Gamma(k+n+s+d/2)} \asymp k^{-s}(k+n)^{-s}. \tag{C.2}$$

Substituting (C.2) into (C.1), we get

$$\begin{aligned}&\big\|(1-\|\boldsymbol{x}\|^2)^{-s}u\big\|^2_{\mathcal{B}_s^{\mu+s,\nu+2s}(\mathbb{B}^d)}\\ &\le C\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\big[1+n^{2\mu}+(n+k)^{2\nu}\big]\big|\widehat{f}_{k,\ell}^{s,n}\big|^2\chi_k^{s,n}<+\infty.\end{aligned} \tag{C.3}$$

This completes the proof. □

### C.2. The proof of Lemma 4.1

$$v(\boldsymbol{x}) = \sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d}\sum_{k=0}^{\infty}\widehat{v}_{k,\ell}^{s,n}P_{k,\ell}^{s,n}(\boldsymbol{x}) \in \mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)$$

can be rewritten as

$$v=\sum_{n=0}^{\infty}\sum_{\ell=1}^{a_n^d} v_\ell^n(r)r^nY_\ell^n(\hat{\boldsymbol{x}}), \quad v_\ell^n(r)=\sum_{k=0}^{\infty}\widehat{v}_{k,\ell}^{s,n}P_{k,\ell}^{(s,n+d/2-1)}(2r^2-1), \quad \boldsymbol{x}=r\hat{\boldsymbol{x}}.$$

Define an equivalent semi-norm of $|\cdot|_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)}$ analogous to the two-dimensional case in [27]

$$|v|^2_{\mathcal{B}_s^{\mu,\nu}(\mathbb{B}^d)} = \sum_{n=0}^{\infty}(n+k)^{2\mu}\|\widetilde{v}_n\|^2_{L^2_{s,n+d/2-1}(\Lambda)} + \sum_{n=0}^{\infty}|\widetilde{v}_n|^2_{B^{\nu}_{s,n+d/2-1}(\Lambda)}, \tag{C.4}$$

where

$$\widetilde{v}_n(z) = v_\ell^n\frac{\sqrt{(1+z)/2}}{2^{(n+d/2-1)/2}}.$$

Here $L^2_{\gamma,\beta}(\Lambda)$ with the associated norm $\|\cdot\|_{L^2_{\gamma,\beta}(\Lambda)}(\gamma,\beta\in\mathbb{R})$ in one dimension is denoted by the weighted space of all functions defined on the unit interval $\Lambda=(-1,1)$

$$\|u\|^2_{L^2_{\gamma,\beta}(\Lambda)} = \int_{-1}^{1}u^2(z)(1-z)^{\gamma}(1+z)^{\beta}\mathrm{d}z<\infty.$$

The non-uniformly Jacobi-weighted Sobolev space $B^J_{\gamma,\beta}(\Lambda)$, when $J$ is a nonnegative integer, is defined by

$$B^J_{\gamma,\beta}(\Lambda) := \big\{u \,|\, \partial_r^j u \in L^2_{\gamma+j,\beta+j}(\Lambda),\ j=0,1,\ldots,J\big\},$$

which is equipped with the following norm:

$$\|u\|_{B^J_{\gamma,\beta}(\Lambda)} = \left(\sum_{j=0}^{J}|u|^2_{B^j_{\gamma,\beta}(\Lambda)}\right)^{1/2}, \quad |u|_{B^j_{\gamma,\beta}(\Lambda)} = \|\partial_r^j u\|_{L^2_{\gamma+j,\beta+j}(\Lambda)}.$$

When $J = s$ is not an integer, the space can be defined via classic interpolation method, e.g. K-method [3]. To prove the lemma, we need the following conclusion [28].

If $v(z) \in B^{\nu}_{s,m}(\Lambda)$, then

$$\big(1 - r^2(z)\big)^s v(z) \in B^{\min(\nu, 3s+1-\varepsilon)}_{s,m}(\Lambda), \tag{C.5}$$

where $r(z) = \sqrt{(1+z)/2}$ and $\varepsilon > 0$ arbitrarily small.

Since $v \in \mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)$ with $\mu, \nu \geq 0$, the semi-norm (C.4) of $v$ can be bounded

$$|v|^2_{\mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)} = \sum_{n=0}^{\infty} (n+k)^{2\mu} \|\widetilde{v}_n\|^2_{L^2_{s,n+d/2-1}(\Lambda)} + \sum_{n=0}^{\infty} |\widetilde{v}_n|^2_{B^{\nu}_{s,n+d/2-1}(\Lambda)} < \infty, \tag{C.6}$$

where

$$\widetilde{v}_n(z) = v^n_\ell \frac{\sqrt{(1+z)/2}}{2^{(n+d/2-1)/2}}.$$

It suffices to show semi-norm of product $(1 - r^2(z))^s v$ can also be bounded. Applying the result (C.5), we have

$$\big|(1-r^2)^s \widetilde{v}_n\big|^2_{B^{\min(\nu,3s+1-\varepsilon)}_{s,n+d/2-1}(\Lambda)} \lesssim |\widetilde{v}_n|^2_{B^{\nu}_{s,n+d/2-1}(\Lambda)} < \infty. \tag{C.7}$$

Then it is obtained

$$\big|(1-r^2)^s v\big|^2_{\mathcal{B}^{\mu,\min(\nu,3s+1-\varepsilon)}_s(\mathbb{B}^d)} \leq |v|^2_{\mathcal{B}^{\mu,\nu}_s(\mathbb{B}^d)},$$

which completes the proof. □

**Acknowledgments.** The first author was supported by the National Natural Science Foundation of China (Grant No. 12101325) and by the NUPTSF (Grant No. NY220162). The second author was supported by the National Natural Science Foundation of China (Grant Nos. 12131005, 11971016). The third author was supported by the National Natural Science Foundation of China (Grant No. 12131005). The fifth author was supported by the National Natural Science Foundation of China (Grant Nos. 12131005, U2230402).

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