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TIME MULTIPOINT NONLOCAL PROBLEM FOR A STOCHASTIC SCHRÖDINGER EQUATION*

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Abstract

A time multipoint nonlocal problem for a Schrödinger equation driven by a cylindrical Q-Wiener process is presented. The initial value depends on a finite number of future values. Existence and uniqueness of a solution formulated as a mild solution is obtained. A single-step implicit Euler-Maruyama difference scheme, a Rothe-Maryuama scheme, is suggested as a numerical solution. Convergence rate for the solution of the difference scheme is established. The theoretical statements for the solution of this difference scheme is supported by a numerical example.

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Key words: Time nonlocal problem, Mild solution, Cylindrical Wiener process, Time discretization, Abstract time-dependent stochastic Schrödinger equation, Euler-Maruyama method.

1. Introduction

Typically, phenomena evolving in time in various fields such as natural sciences, engineering, and finance are described in terms of differential equations. In models for which uncertainty is needed to build into the model, inherent randomness is a natural additional ingredient.

The most well-known differential equations subjected to randomness are stochastic ordinary differential equations of which among the most profound example model stock prices, see e.g. [28]. Partial differential equations with uncertainty can also be handled as stochastic partial differential equations, see e.g. [26] where applications to environmental pollution models and bond market models appear. In the above examples the initial value is typically independent of the time-evolving random noise. For backward stochastic differential equations, suitable for stochastic control and option pricing, the final value is a random variable adapted to the filtration at the final time point where the solution is nevertheless non-anticipating, see e.g. [23, Section 3]. In [22, Chapter 3.3] a stochastic two point boundary problem is considered. It is a finite-dimensional linear Stratonovich stochastic differential equation where the initial value depends linearly on the final value and is therefore anticipating, existence and uniqueness

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of a solution is shown. For a shooting method as a numerical solution to a stochastic two-point boundary value problem, see [6].

Existence and uniqueness of solution to evolution equations with nonlocal bondary conditions, formulated as abstract nonlocal Cauchy problems, are typically shown by fixed point arguments, see e.g. [2,3,11,16]; in [3] an application is given by a diffusion of a small amount of gas in a transparent tube which allows measurements at different time points and not just at time zero. In [8], a deterministic Schrödinger equation is studied where the initial value is a linear combination of future values, i.e. there are non-local boundary conditions in time. Existence, uniqueness, as well as stability of a numerical approximation is obtained. Also in [17,18], infinite dimensional time-nonlocal problems for deterministic Schrödinger type equations is considered. In [33] various applications of deterministic time-nonlocal dynamical systems are reviewed, also including Schrödinger equations. Time-nonlocal dynamical systems allow couplings of the initial conditions with the system in a nonlocal manner, rather than at a single point [33]. Note that time-nonlocal problems are generalisations of for instance periodic conditions, see e.g. [32]. The most applications refereed to are in quantum mechanics where for instance it can be seen as a way to mitigate the influence of initial conditions, or include the ability to impose initial and final boundary conditions on the evolution of a quantum system. For nonlocal in time problems applied to radionuclides propagation in Stokes fluid and problems of predicting the state of a medium see e.g. references in [9].

A stochastically dispersed Schrödinger equation with a linear diffusive term is known as the Belavkin equation; for a rigorous treatment of such equations see e.g. [25]. For analysis of stochastic Schrödinger equations with cubic nonlinear drift subjected to a multiplicative finite-dimensional Wiener process and standard initial value condition, see for instance [1, 10, 15, 30]; in [30] and its references, applications appear in optical fiber communication; in [15] stability of finite element approximation in space combined with various time discretization schemes such as explicit and implicit Euler and Crank-Nicolson schemes is obtained where the noise is of Stratonovich type and the equation is formulated in a variational weak form; for more references of numerics of stochastic nonlinear Schrödinger equations with given initial value see for instance the references in [15]. In [5], an infinite-dimensional Q-Wiener process is allowed where for the case of linear stochastic Schrödinger equation, temporal discretization convergence of an exponential integrator scheme is obtained of order one for additive noise and $1/2$ for multiplicative noise. In [12–14] strong and weak convergence rates of several numerical schemes for stochastic nonlinear Schrödinger equations with non-monotone coefficients and multiplicative noise with given initial value are derived. In [31], numerics for deterministic nonlocal-in-time Schrödinger equations is considered.

In this paper, a stochastic Schrödinger equation with a time-dependent Gaussian excitation and time non-local initial condition is considered, which, to our knowledge, is novel in combining temporal discretization of a stochastic Schrödinger equation subjected to a cylindrical Q-Wiener process with time multi-point initial condition. Since the initial value is a linear combination of future values, the solution is not adapted to the given filtration. Here the drift is linear and the dispersion is non-anticipating. That makes it possible to formulate a solution within the framework of Itô-integrals in infinite dimensions, here in a mild form, which can be compared to the finite-dimensional linear two-point boundary stochastic differential equation [22] where a mild form is not needed.

The involved operator in [7] is self-adjoint positive definite while in [8] and in this paper the operator is only assumed to be self-adjoint. In [7], writing the equation in a mild form, the equation is driven by a one-dimensional Wiener process. In this paper the solution is also in a mild form driven by a noise of the type $f(t)dW$ where W is a cylindrical Q-Wiener process and $f(t)$ fulfills a Hilbert-Schmidt boundedness making stochastic integrals $\int_0^t f(s)dW(s)$ well-defined. Existence and uniqueness is shown together with stability and convergence of a Rothe-Maruyama difference scheme which is an implicit scheme of Euler-Maruyama type, here adapted to the time multi-point boundary condition.

In Section 2, the stochastic Schrödinger problem is formulated. In Section 3, existence and uniqueness result are obtained. In Section 4, the Rothe-Maruyama time multipoint boundary together with convergence of the scheme is shown. In Section 5, the numerical scheme is applied to a non-trivial example, including a spatial discretization scheme, supporting the convergence. The paper ends with conclusions in Section 6.

2. Formulation of The Stochastic Shrödinger Equation

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]} \subset \mathcal{F}$ and U and H separable Hilbert spaces. Let $(\cdot\,,\cdot)$ denote the inner product and $\|.\|$ denote the norm of H. For an $\mathcal{F}$-measurable random variable X with values in H let $\|X\|_{L^2(\Omega,H)} = (E[\|X\|^2])^{1/2}$.

Following [19, 27] we introduce a cylindrical Q-Wiener process which is said to be in U as follows: Consider a linear self-adjoint, positive semidefinite operator $Q : U \mapsto U$, not necessarily with a finite trace. Then $U_0 := Q^{1/2}(U)$ can be Hilbert-Schmidt embedded into a Hilbert space U_1 by a linear map $\mathfrak{J} : U_0 \mapsto U_1$, for which

$$\sum_{i=1}^{\infty} \|\mathfrak{J}e_i\|_{U_1}^2 < \infty,$$

where $\{e_i\}_{i=1}^{\infty}$ denotes an orthonormal basis of U_0 and $\|\cdot\|_{U_1}$ is the norm in U_1. We let throughout $W = \{W(t)\}_{t\in[0,T]}$ be a cylindrical Q-Wiener process in U adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. More explicitly, it means that the cylindrical Q-Wiener process can be represented in the form

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t)\mathfrak{J}e_i, \quad \forall\, t \in [0, T],$$

where $\{\beta_i\}_{i=1}^{\infty}$ is a sequence of standard one-dimensional Wiener processes and the series is convergent in U_1 almost surely (which in fact means that $W(t)$ takes values in U_1 and not necessarily in U). If Q has finite trace we can select $\mathfrak{J} = I$ and $U_1 = U$, [19, 27].

Let, as in [19, 27], $L(U_0, H)$ denote the set of linear maps $U_0 \mapsto H$ and $\mathcal{N}_W$ be the set of predictable processes $\Psi : \Omega\times[0, T]\mapsto L(U_0, H)$ such that the probability $P(\int_0^T \|\Psi(s)\|_{L_2^0}^2 ds<\infty)$ is one, where

$$\|\Psi(s)\|_{L_2^0}^2 := \sum_{i=1}^{\infty} \|\Psi(s)e_i\|^2$$

the squared Hilbert-Schmidt norm of $\Psi(s) = \Psi(\cdot, s)$. For stochastic processes φ in $\mathcal{N}_W$, the H-valued Itô-integral

$$\int_0^T \Psi(s)dW(s) := \int_0^T \Psi(s) \circ \mathfrak{J}^{-1}dW(s)$$

with respect to the cylindrical Wiener process W is well-defined.

Denote by $L(H) = L(H, H)$ the set of linear continuous maps $G : H \mapsto H$ with norm

$$\|G\|_{L(H)} = \sup\{\|G(x)\| : \|x\| = 1\}.$$

In this article, the time multipoint nonlocal problem for the stochastic Schrödinger equation

$$\begin{cases} idu(t) + Au(t)dt = f(t)dW(t), \quad 0 < t < T, \\ u(0) = \sum_{m=1}^{p} \alpha_m u(\lambda_m) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big), \\ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq T \end{cases} \tag{2.1}$$

is considered, where $\varphi : \mathbb{R}^p \mapsto H$ is a map on H such that $\|\varphi(W(\lambda_1), \cdots, W(\lambda_p))\|_{L^2(\Omega,H)} < \infty$, $i^2 = -1, \alpha_1, \cdots, \alpha_p$ are constants, and $f \in \mathcal{N}_W$. A is a linear not necessarily bounded self-adjoint operator on H, domain of A is dense in H and p is a positive integer.

3. Time Nonlocal Problem

Since A is a self-adjoint operator on the Hilbert space H, by Stone's theorem together with [4, Example 4.12], the operators

$$e^{iAt} := \sum_{k=0}^{\infty} \frac{(iAt)^k}{k!}$$

form a strongly continuous semigroup where e^{iAt} is unitary and the adjoint of e^{iAt} is

$$(e^{iAt})^* = (e^{iAt})^{-1} = e^{-iAt}.$$

The sum $\sum_{k=0}^{\infty}((iAt)^k/k!)$ is uniformly convergent if A is bounded and strongly convergent if A is unbounded. The operator iA is the infinitesimal generator of a strongly continuous semigroup $S(t) := e^{iAt}$. The domain of $A, D(A)$, is dense in H and for any $f \in D(A)$,

$$-i\frac{d}{dt}S(t)f = AS(t)f = S(t)Af,$$

see [4, Example 4.12]. Using the spectral representation of self-adjoint operator (see [4, p. 153]), we have

$$\|e^{iAt}\|_{L(H)} = \sup_{\lambda\in\sigma(A)} |e^{i\lambda t}| \leq 1,$$

where $\sigma(A)$ is the spectrum of the a self-adjoint operator A. See also [24] for a suitable reference.

Assume throughout that

$$\sup_{t\in[0,T]} \int_0^t E\left[\left\|e^{iA(t-s)}f(s)\right\|^2_{L_2^0}\right] ds < \infty. \tag{3.1}$$

Note that (3.1) implies that $\int_0^t e^{iA(t-s)}f(s)dW(s)$ is a well-defined element in H, cf. [27]. Furthermore, by the Itô-isomorphism [27],

$$E\left(\left\|\int_0^t e^{iA(t-s)}f(s)dW(s)\right\|^2\right) = \int_0^t E\left(\left\|e^{iA(t-s)}f(s)\right\|^2_{L_2^0}\right) ds.$$

Definition 3.1 (Mild Solution). *A solution to* (2.1) *is an H-valued process $\{u(t)\}_{t\in[0,T]}$ such that*

$$\begin{cases} u(t) = e^{iAt}u(0) - i\int_0^t e^{iA(t-s)}f(s)dW(s), \quad t \in [0,T], \\ u(0) = \sum_{m=1}^{p} \alpha_m u(\lambda_m) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big) \end{cases} \tag{3.2}$$

is satisfied.

Theorem 3.1. *Assume* $\sum_{m=1}^{p} |\alpha_m| < 1$. *Then there exists a unique solution* $u(t)$ *of the problem* (2.1) *and the following stability inequality is satisfied:*

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(\Omega,H)} \leq \left(1 - \sum_{k=1}^{m} |\alpha_m|\right)^{-1} \times \left(\sup_{t\in[0,T]} \left(\int_0^t E\left(\left\|e^{-iA(t-s)} f(s)\right\|^2_{L_2^0}\right) ds \right)^{\frac{1}{2}} + \left\|\varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\right\|_{L^2(\Omega,H)} \right). \tag{3.3}$$

Note that $u(0) \notin \mathcal{F}_0$ which means the proof of Theorem 3.1 is somewhat different from standard proofs of existence and uniqueness of solutions of infinite dimensional stochastic differential equations.

Proof. Firstly, existence of a mild solution will be shown. Since

$$\left\| \left(I - \sum_{m=1}^{p} \alpha_m e^{iA\lambda_m}\right) - I \right\|_{L(H)} \leq \sum_{m=1}^{p} |\alpha_m| \, \|e^{iA\lambda_m}\|_{L(H)} \leq \sum_{m=1}^{p} |\alpha_m| < 1,$$

the operator $I - \sum_{m=1}^{p} \alpha_m e^{iA\lambda_m} : H \mapsto H$ is bijective and has an inverse

$$J = \left(I - \sum_{m=1}^{p} \alpha_m e^{iA\lambda_m}\right)^{-1} = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} \alpha_m e^{iA\lambda_m} \right)^k$$

with

$$\|J\|_{L(H)} \leq \sum_{k=1}^{\infty} \left\| \sum_{m=1}^{p} \alpha_m e^{iA\lambda_m} \right\|^k_{L(H)} \leq \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} |\alpha_m| \, \|e^{iA\lambda_m}\|_{L(H)} \right)^k \leq \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} |\alpha_m| \right)^k = \left(1 - \sum_{m=1}^{p} |\alpha_m|\right)^{-1}.$$

Since

$$-\sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s) dW_s + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big) \in H,$$

there exists a unique element $\bar{u}_0$ in H such that

$$\bar{u}_0 = J\left(-\sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s) dW_s + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big) \right), \tag{3.4}$$

i.e.

$$\left(I - \sum_{m=1}^{p} \alpha_m e^{iA\lambda_m}\right) \bar{u}_0 = -\sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s) dW(s) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big).$$

Consequently,

$$\bar{u}_0 = \sum_{m=1}^{p} \alpha_m \left[e^{iA\lambda_m} \bar{u}_0 - \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s) dW_s \right] + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big).$$

For this unique $\bar{u}_0$, we define $u : \Omega \times [0,T] \mapsto H$ by

$$u(t) = e^{iAt}\bar{u}_0 - i\int_0^t e^{iA(t-s)} f(s)dW(s). \tag{3.5}$$

In particular, $u(0) = \bar{u}_0$. Hence,

$$u(t) = e^{iAt}u(0) - i\int_0^t e^{iA(t-s)} f(s)dW(s), \tag{3.6}$$

and

$$u(0) = \sum_{m=1}^{p} \alpha_m \left[e^{iA\lambda_m}u(0) - \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s)dW(s)\right] + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big) \tag{3.7}$$

can be written as

$$u(0) = \sum_{m=1}^{p} \alpha_m u(\lambda_m) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big).$$

It means that (2.1) has a mild solution.

Now assume that u is a solution of (3.2). The estimate (3.3) will be established of which uniqueness follows. By applying the Itô-isomorphism to the second line of (3.2) in which we have put the first line of (3.2), i.e. (3.7),

$$\begin{aligned}
\|u(0)\|_{L^2(\Omega,H)} &\le \sum_{m=1}^{p} |\alpha_m| \left[\|e^{iA\lambda_m}u(0)\|_{L^2(\Omega,H)} + \left\| \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s)dW(s)\right\|_{L^2(\Omega,H)} \right] \\
&\quad + \big\|\varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \\
&\le \sum_{m=1}^{p} |\alpha_m| \left[\|u(0)\|_{L^2(\Omega,H)} + \left(\int_0^{\lambda_m} E\left[\big\|e^{iA(\lambda_m - s)} f(s)\big\|^2_{L_2^0} \right] ds \right)^{\frac{1}{2}} \right] \\
&\quad + \big\|\varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\big\|_{L^2(\Omega,H)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u(0)\|_{L^2(\Omega,H)} \le \Bigg(\sum_{m=1}^{p} |\alpha_m| \bigg(\int_0^{\lambda_m} E\big\|e^{iA(\lambda_m - s)} f(s)\big\|^2_{L_2^0} ds \bigg)^{\frac{1}{2}} \\
+ \big\|\varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \Bigg) \left(1 - \sum_{m=1}^{p} |\alpha_m| \right)^{-1}.
\end{aligned} \tag{3.8}$$

Similarly, for $u(t)$ given by (3.6),

$$\begin{aligned}
\sup_{t\in[0,T]} \|u(t)\|_{L^2(\Omega,H)} &\le \sup_{t\in[0,T]} \left(\|e^{iAt}\|^2_{L(H)} E\,\|u(0)\|^2 \right)^{\frac{1}{2}} \\
&\quad + \sup_{t\in[0,T]} \left\| \int_0^t e^{iA(t-s)} f(s)dW(s) \right\|_{L^2(\Omega,H)} \\
&\le \|u(0)\|_{L^2(\Omega,H)} + \sup_{t\in[0,T]} \left(\int_0^t E\left[\big\|e^{iA(t-s)} f(s)\big\|^2_{L_2^0} \right] ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

Using (3.8) and utilizing

$$\sum_{m=1}^{p}|\alpha_m|\left(1-\sum_{m=1}^{p}|\alpha_m|\right)^{-1}+1=\left(1-\sum_{m=1}^{p}|\alpha_m|\right)^{-1},$$

we write

$$\begin{aligned}
&\sup_{t\in[0,T]}\|u(t)\|_{L^2(\Omega,H)}\\
&\leq\left(\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{\lambda_m}E\big\|e^{iA(t-s)}f(s)\big\|_{L_2^0}^2ds\right)^{\frac{1}{2}}+\|\varphi(W_{\lambda_1},\cdots,W_{\lambda_p})\|_{L^2(\Omega,H)}\right)\left(1-\sum_{m=1}^{p}|\alpha_m|\right)^{-1}\\
&\quad+\max_{1\leq m\leq p}\left(\int_0^{\lambda_m}E\big\|e^{iA(\lambda_m-s)}f(s)\big\|_{L_2^0}^2ds\right)^{\frac{1}{2}}\\
&\leq\left(\left(\max_{1\leq m\leq p}\int_0^{\lambda_m}E\big\|e^{iA(\lambda_m-s)}f(s)\big\|_{L_2^0}^2ds\right)^{\frac{1}{2}}+\big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)}\right)\left(1-\sum_{m=1}^{p}|\alpha_m|\right)^{-1},
\end{aligned}$$

i.e. (3.3). Uniqueness of a mild solution follows directly from (3.3). □

Observe that for the proof of Theorem 3.1, $\sum_{m=1}^{p}|\alpha_m|<1$ is a critical condition. However, in [21, 32], a more general condition of the type

$$\sum_{m=1}^{p}|\alpha_m|e^{d\lambda_m}<1$$

is allowed for a deterministic time non-local Schrödinger equation for a certain constant $d>0$.

4. The Rothe-Maruyama Time Multipoint Boundary Scheme

To find an approximate solution for the time multipoint nonlocal boundary value problem for stochastic Schrödinger equation (2.1), on the time interval $[0,T]$ we consider the uniform grid space

$$[0,T]_\tau=\{t_k=k\tau,\ k=0,1,\ldots,N,\ N\tau=T\} \tag{4.1}$$

with step size $\tau>0$ and N an arbitrary but fixed positive integer.

Let us associate the time multipoint non-local problem for the stochastic Schrödinger equation (2.1) with the corresponding first order implicit Rothe-Maruyama difference scheme

$$\begin{cases} i(u_k-u_{k-1})+\tau Au_k=f(t_{k-1})(W_k-W_{k-1}), & 1\leq k\leq N,\\ u_0=\displaystyle\sum_{m=1}^{p}\alpha_m u_{l_m}+\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big), \end{cases} \tag{4.2}$$

$N\tau=T, t_k=k\tau, W_k=W(t_k)$. Here $l_m=\lfloor\lambda_m/\tau\rfloor$, the integer part of λ_m/τ for $1\leq m\leq p$. Note that (4.2) is equivalent to

$$\begin{cases} u_k=R^k u_0-i\displaystyle\sum_{j=1}^{k}R^{k-j+1}f(t_{j-1})\big(W(t_j)-W(t_{j-1})\big),\\ u_0=\displaystyle\sum_{m=1}^{p}\alpha_m u_{l_m}+\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big). \end{cases} \tag{4.3}$$

For deterministic differential equations the implicit Euler difference scheme is known under the name Rothe scheme, [7,8]. The Euler type finite difference scheme, adapted for stochastic differential equations is known under the name Euler-Maruyama scheme, [20]. The Euler-Maruyama scheme considered here is also implicit and is therefore named as the Rothe-Maruyama scheme as in [29], where $u(0)$ was given which is different from here where $u(0)$ depends on a finite number of future values. Since A is a self adjoint operator, all eigenvalues of A are real numbers. Therefore, for $\tau > 0$ the complex number $i\tau$ cannot be an eigenvalue of A. Hence $I - i\tau A$ is invertible. Furthermore, for its inverse $R = (I - i\tau A)^{-1}$, with $\sigma(A)$ its resolvent set,

$$\|R\| = \sup_{\mu\in\sigma(A)} |1 - i\tau\mu|^{-1} \le \sup_{\mu\in\mathbb{R}} |1 - i\tau\mu|^{-1} = |1 + \tau^2\mu^2|^{-\frac{1}{2}} \le 1. \tag{4.4}$$

Theorem 4.1. *Assume* $\sum_{m=1}^{p} |\alpha_m| < 1$ *and*

$$\max_{1\le m\le p} \sum_{j=1}^{l_m} E\left\|R^{l_m-j+1} f(t_{j-1})\right\|_{L_2^0}^2 \tau < \infty.$$

Then there exists a unique solution of (4.2).

Proof. Similar to the proof of Theorem 3.1, we first show existence of a solution. The assumption

$$\sum_{k=1}^{N} E\|f(t_{k-1})\|_{L_2^0}^2 \tau < \infty$$

implies that the expression

$$\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} f(t_{k-1}) dW(s) = \sum_{k=1}^{N} f(t_{k-1})\big(W(t_k) - W(t_{k-1})\big)$$

is well-defined in H. Since, by (4.4),

$$\left\| \left(I - \sum_{m=1}^{p} \alpha_m R^{l_m} \right) - I \right\|_{L(H)} \le \sum_{m=1}^{p} |\alpha_m| \, \|R\|_{L(H)}^{l_m} \le \sum_{m=1}^{p} |\alpha_m| < 1,$$

$I - \sum_{m=1}^{p} \alpha_m R^{l_m}$ has an inverse

$$D = \left(I - \sum_{m=1}^{p} \alpha_m R^{l_m} \right)^{-1} = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} \alpha_m R^{l_m} \right)^{k}$$

with

$$\|D\|_{L(H)} \le \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} |\alpha_m| \, \|R\|_{L(H)}^{l_m} \right)^{k} \le \sum_{k=1}^{\infty} \left(\sum_{m=1}^{p} |\alpha_m| \right)^{k} = \left(1 - \sum_{m=1}^{p} |\alpha_m| \right)^{-1}.$$

Hence, there exists a unique element u_0 in H such that

$$u_0 = D\left(-i \sum_{m=1}^{p} \alpha_m \sum_{j=1}^{l_m} R^{l_m-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big) \right), \tag{4.5}$$

i.e.

$$\left(I - \sum_{m=1}^{p} \alpha_m R^{l_m}\right) u_0 = -i \sum_{m=1}^{p} \alpha_m \sum_{j=1}^{l_m} R^{l_m-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big) + \varphi(W_{\lambda_1}, \cdots, W_{\lambda_p}),$$

so that

$$u_0 = \sum_{m=1}^{p} \alpha_m \left[R^{l_m} u_0 - i \sum_{j=1}^{l_m} R^{l_m-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big)\right] + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big). \tag{4.6}$$

For this u_0 we define u_k as

$$u_k = R^k u_0 - i \sum_{j=1}^{k} R^{k-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big). \tag{4.7}$$

Hence, (4.6) can be written as

$$u_0 = \sum_{m=1}^{p} \alpha_m u_{l_m} + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big). \tag{4.8}$$

From (4.7) we get for $1 \le k \le p$,

$$\begin{aligned} u_k &= R\left[R^{k-1} u_0 - i \sum_{j=1}^{k} R^{k-1-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big)\right] \\ &= R\left[R^{k-1} u_0 - i \sum_{j=1}^{k-1} R^{k-1-j+1} f(t_{j-1})\big(W(t_j) - W(t_{j-1})\big) - i f(t_{k-1})\big(W(t_k) - W(t_{k-1})\big)\right] \\ &= R\big[u_{k-1} - i f(t_{k-1})\big(W(t_k) - W(t_{k-1})\big)\big], \end{aligned}$$

i.e.

$$(I - i\tau A) u_k = u_{k-1} - i f(t_{k-1})\big(W(t_k) - W(t_{k-1})\big).$$

We have shown that $\{u_k\}_{k=1}^N$ satisfies (4.2). The uniqueness of (4.3) follows since u_0 is uniquely defined as well as $\{u_k\}_{k=1}^N$. $\square$

Theorem 4.2. *Assume that $\sum_{m=1}^{p} |\alpha_m| < 1$. Then the unique solution of the difference scheme (4.2) obeys the following stability inequality:*

$$\max_{0\le k\le N} \|u_k\|_{L^2(\Omega,H)} \le \left(1 - \sum_{i=1}^{p} |\alpha_m|\right)^{-1} \left(\max_{1\le k\le N} \left(\sum_{j=1}^{k} E\left[\big\|R^{k-j+1} f(t_{j-1})\big\|_{L_2^0}^2\right] \tau\right)^{\frac{1}{2}} + \big\|\varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\big\|_{L^2(\Omega,H)}\right). \tag{4.9}$$

Proof. By (4.6) and the Itô-isomorphism,

$$
\begin{aligned}
\|u_0\|_{L^2(\Omega,H)} \le \sum_{m=1}^{p} |\alpha_m| \Bigg[\big\|R^{l_m} u_0\big\|_{L^2(\Omega,H)} + \Bigg\| \sum_{j=1}^{l_m} R^{l_m-j+1} f(t_{j-1})\big(W(t_j)-W(t_{j-1})\big) \Bigg\|_{L^2(\Omega,H)} \Bigg] \\
+ \big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \\
\le \sum_{m=1}^{p} |\alpha_m| \Bigg[\|u_0\|_{L^2(\Omega,H)} + \max_{0\le m\le p} \Bigg(\sum_{j=1}^{l_m} E\big\|R^{l_m-j+1} f(t_{j-1})\big\|^2_{L_2^0}\tau \Bigg)^{\frac{1}{2}} \Bigg] \\
+ \big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(E\|u_0\|^2)^{\frac{1}{2}} \le \Bigg(\sum_{m=1}^{p} |\alpha_m| \Bigg(\max_{0\le m\le p} \sum_{j=1}^{l_m} E\big\|R^{l_m-j+1} f(t_{j-1})\big\|^2_{L_2^0}\tau \Bigg)^{\frac{1}{2}} \\
+ \big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \Bigg) \Bigg(1 - \sum_{m=1}^{p} |\alpha_m| \Bigg)^{-1}. \qquad (4.10)
\end{aligned}
$$

Similarly, for u_k given by (4.7),

$$
\begin{aligned}
\max_{0\le k\le N} \|u_k\|_{L^2(\Omega,H)} \le \max_{0\le k\le N} \big\|R^k u_0\big\|_{L^2(\Omega,H)} \\
+ \max_{0\le k\le N} \Bigg\| \sum_{j=1}^{k} R^{k-j+1} f(t_{j-1})\big(W(t_j)-W(t_{j-1})\big) \Bigg\|_{L^2(\Omega,H)} \\
\le \|u_0\|_{L^2(\Omega,H)} + \max_{0\le k\le N} \Bigg(\sum_{j=1}^{k} E\big\|R^{k-j+1} f(t_{j-1})\big\|_{L_2^0}\tau \Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Using the boundedness (4.10) of u_0 and

$$
\sum_{m=1}^{p} |\alpha_m| \Bigg(1 - \sum_{m=1}^{p} |\alpha_m| \Bigg)^{-1} + 1 = \Bigg(1 - \sum_{m=1}^{p} |\alpha_m| \Bigg)^{-1},
$$

we obtain

$$
\begin{aligned}
&\max_{0\le k\le N} \|u_k\|_{L^2(\Omega,H)} \\
&\le \Bigg(\sum_{m=1}^{p} |\alpha_m| \Bigg(\sum_{j=1}^{l_m} E\big\|R^{l_m-j+1} f(t_{j-1})\big\|^2_{L_2^0}\tau \Bigg)^{\frac{1}{2}} + \big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \Bigg) \\
&\quad \times \Bigg(1 - \sum_{m=1}^{p} |\alpha_m| \Bigg)^{-1} + \max_{1\le k\le N} \Bigg(\sum_{j=1}^{k} E\big\|R^{k-j+1} f(t_{j-1})\big\|_{L_2^0}\tau \Bigg)^{\frac{1}{2}} \\
&= \Bigg(1 - \sum_{m=1}^{p} |\alpha_m| \Bigg)^{-1} \Bigg(\Bigg(\max_{1\le k\le N} \sum_{j=1}^{k} E\big\|R^{k-j+1} f(t_{j-1})\big\|^2_{L_2^0}\tau \Bigg)^{\frac{1}{2}} + \big\|\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \Bigg).
\end{aligned}
$$

The proof is complete. □

4.1. Error analysis

Now we will show that the Rothe-Maruyama difference scheme approximation (4.2) for the time nonlocal problem for the stochastic Schrödinger equation (2.1) has a strong convergence of order 1/2. For this, we need some related estimates which will be stated in the following lemma with the convention $0^0 = 1$.

Lemma 4.1. *Let A be a self-adjoint operator. Then the following estimates hold:*

$$\|A^\alpha R^k\|_{L(H)} \le \alpha^{\frac{\alpha}{2}}(\sqrt{k}\tau)^{-\alpha}, \qquad k \ge 1, \quad 0 \le \alpha < k, \tag{4.11}$$

$$\|A^k R^k\|_{L(H)} \le \tau^{-k}, \qquad k \ge 1, \tag{4.12}$$

$$\big\|A^{-\beta}\big(R^k - e^{ik\tau A}\big)\big\|_{L(H)} \le \beta^{-1}(2-\beta)^{\frac{2-\beta}{2}}(\sqrt{k}\tau)^\beta, \quad \beta \ne 0, \quad 1-k < \beta \le 2, \tag{4.13}$$

in particular this holds for $k \ge 1$ and $0 < \beta \le 2$. Moreover, for any $\nu \in [0,1]$,

$$\big\|A^{-\nu}(e^{itA} - I)\big\|_{L(H)} \le 2^{1+\nu}t^\nu. \tag{4.14}$$

Proof. To prove (4.11) and (4.13), we modify the proof in [29], where the case of $0 \le \alpha \le 1$ and $1 \le \beta \le 2$ was shown.

Proof of (4.11). We have

$$\big\|(\tau A)^\alpha R^k\big\|_{L(H)} \le \sup_{\lambda\in\mathbb{R}} \frac{(\tau\lambda)^\alpha}{(1+\tau^2\lambda^2)^{k/2}} = \sup_{\mu\in\mathbb{R}} \frac{\mu^\alpha}{(1+\mu^2)^{k/2}}.$$

With $f(\mu) = \mu^\alpha(1+\mu^2)^{-k/2}$, for $0 \le \alpha < k$,

$$f'(\mu) = -(k-\alpha)\mu^{\alpha-1}\left(\mu^2 - \frac{\alpha}{k-\alpha}\right)(1+\mu^2)^{-1-\frac{k}{2}},$$

which gives maxima of f for $\mu^* = (\alpha/(k-\alpha))^{1/2}$ with

$$f(\mu^*) = \frac{\big(\sqrt{\alpha/(k-\alpha)}\big)^\alpha}{\big(1+\alpha/(k-\alpha)\big)^{k/2}} = \left(\frac{k-\alpha}{k}\right)^{\frac{k-\alpha}{2}}\left(\frac{\alpha}{k}\right)^{\frac{\alpha}{2}} \le \left(\frac{\alpha}{k}\right)^{\frac{\alpha}{2}} = \alpha^{\frac{\alpha}{2}}\big(\sqrt{k}\big)^{-\alpha}$$

implying (4.11). For $\alpha = k, f'(\mu) \ge 0$ so

$$f(\mu) \le \lim_{\mu\to\infty} \mu^k(1+\mu^2)^{\frac{k}{2}} = 1,$$

which means that (4.12) also holds.

Proof of (4.13). With $R^k(s) = (1 - i\tau sA)^{-k}$,

$$\begin{aligned}
\big\|A^{-\beta}\big(R^k - e^{ik\tau A}\big)\big\|_{L(H)} &= \left\|A^{-\beta}\int_0^1 \frac{d}{ds}\big(R^k(s)e^{ik\tau(1-s)A}\big)ds\right\|_{L(H)} \\
&= \left\|A^{-\beta}\int_0^1 ik\tau AR^{k+1}(s)e^{ik\tau(1-s)A}i\tau sA ds\right\|_{L(H)} \\
&= k\tau^2\left\|\int_0^1 A^{2-\beta}sR^{k+1}(s)e^{ik\tau(1-s)A}ds\right\|_{L(H)} \\
&\le k\tau^2\int_0^1 s\big\|A^{2-\beta}R^{k+1}(s)\big\|_{L(H)}ds.
\end{aligned}$$

For $0 \le 2-\beta < k+1$, i.e. for $1-k < \beta \le 2$, in particular $0 < \beta \le 2$, we get by (4.11),

$$\begin{aligned}\big\|A^{-\beta}\big(R^k - e^{ik\tau A}\big)\big\|_{L(H)} &\le k\tau^2 \int_0^1 s\big\|A^{2-\beta}R^{k+1}(s)\big\|_{L(H)} ds \\ &\le (2-\beta)^{\frac{2-\beta}{2}} k\tau^2 \int_0^1 \frac{s}{(\sqrt{k+1}s\tau)^{2-\beta}} ds \\ &= (2-\beta)^{\frac{2-\beta}{2}} \frac{k}{(k+1)^{1-\beta/2}} \tau^\beta \int_0^1 s^{\beta-1} ds \\ &= (2-\beta)^{\frac{2-\beta}{2}} k^{\frac{\beta}{2}} \left(\frac{k}{k+1}\right)^{1-\frac{\beta}{2}} \tau^\beta \int_0^1 s^{\beta-1} ds \\ &\le (2-\beta)^{\frac{2-\beta}{2}} \beta^{-1} k^{\frac{\beta}{2}} \tau^\beta.\end{aligned}$$

The proof of (4.14) is a minor modification of [19, Lemma B.9(ii)].

$$\begin{aligned}\big\|(tA)^{-\nu}(e^{itA} - I)\big\|_{L(H)} &\le \sup_{\lambda\in\mathbb{R}\setminus\{0\}} (t\lambda)^{-\nu}|e^{it\lambda} - 1| \\ &= \sup_{\mu\in\mathbb{R}\setminus\{0\}} \mu^{-\nu}|e^{it\mu} - 1| \\ &= 2 \sup_{\mu\in\mathbb{R}\setminus\{0\}} \mu^{-\nu}|\sin(\mu/2)| \\ &= 2\cdot 2^\gamma \sup_{\gamma\in\mathbb{R}\setminus\{0\}} \gamma^{-\nu}|\sin\gamma| = 2^{1+\nu},\end{aligned}$$

from which (4.14) follows. □

Theorem 4.3. *Assume that*

$$\big\|A\varphi\big(W(t_1),\cdots,W(t_p)\big)\big\|_{L^2(\Omega,H)} \le M_1, \tag{4.15}$$

$$\sup_{\tau\in[0,T]} \max_{1\le k\le N} \sum_{j=1}^k E\Big[\big\|AR^{k-j}f(t_{j-1})\big\|^2_{L_2^0}\Big]\tau < \infty. \tag{4.16}$$

Assume furthermore for $0 \le t_1 < t_2 \le T$,

$$\int_{t_1}^{t_2} E\Big[\big\|e^{iA(t_2-s)}f(s)\big\|^2_{L_2^0}\Big] + E\Big[\big\|Ae^{iA(t_2-s)}f(s)\big\|^2_{L_2^0}\Big] ds \le M_2|t_2-t_1|, \tag{4.17}$$

$$\int_{t_1}^{t_2} \big\|e^{i(t_2-s)A}f(s) - e^{i(t_2-t_1)A}f(t_1)\big\|^2_{L^2(\Omega,L_2^0)} ds \le L_1(t_2-t_1)^2. \tag{4.18}$$

Then the Rothe-Maruyama difference scheme (4.2) *for the stochastic Schrödinger equation* (2.1) *has a strong convergence of order* $1/2$ *at the initial point* t_0*. That is, at the initial point the convergence estimate*

$$\|u(t_0) - u_0\|_{L^2(\Omega,H)} \le M\tau^{\frac{1}{2}} \tag{4.19}$$

holds. Here the positive constants L_1, M_1, M_2, M *do not depend on* τ.

The temporal convergence order of $1/2$ is in line with that of semilinear stochastic differential equations with Lipschitz assumptions on the drift and dispersion and a standard initial condition, see e.g. [19, Theorem 3.14]. Recall however that temporal discretization convergence of an exponential integrator scheme for a stochastic Schrödinger equation with Cauchy condition is obtained in [5] of order one for additive noise and 1/2 for multiplicative noise.

Clearly, (4.16) holds if

$$\sup_{\tau\in[0,T]}\sum_{j=1}^{N}E\Big[\|Af(t_{j-1})\|^2_{L_2^0}\Big]\tau<\infty. \tag{4.20}$$

Similarly (4.17) holds if

$$\sup_{t\in[0,T]}\Big(E\Big[\|f(t)\|^2_{L_2^0}\Big]+E\Big[\|Af(t)\|^2_{L_2^0}\Big]\Big)<\infty. \tag{4.21}$$

It is also clear that (4.18) holds under the Hölder-continuity of $e^{-isA}f(s)$ with exponent 1/2 in the sense

$$\big\|e^{-isA}f(s)-e^{-itA}f(t)\big\|_{L^2(\Omega,L_2^0)}\le L_2|t-s|^{\frac12} \tag{4.22}$$

for a constant L_2. It is furthermore obvious that (4.22) holds if

$$\sup_{t\in[0,T]}\Big(E\Big[\|f(t)\|^2_{L_2^0}\Big]\Big)<\infty,$$

$$\|e^{itA}-I\|_{L(H)}\le Ct^{\frac12},\quad \|f(t)-f(s)\|_{L_2^0}\le L_3|t-s|^{\frac12} \tag{4.23}$$

for constants C and L_3 since

$$\big\|e^{-isA}f(s)-e^{-itA}f(t)\big\|_{L_2^0}\le\big\|e^{-itA}\big(e^{i(t-s)A}-I\big)f(s)\big\|_{L_2^0}+\big\|e^{-itA}\big(f(s)-f(t)\big)\big\|_{L_2^0}.$$

Proof. Using the expressions (3.4) and (4.5) we have

$$\begin{aligned}u(0)-u_0&=(J-D)\varphi(W_{\lambda_1},\cdots,W_{\lambda_p})-Ji\sum_{m=1}^{p}\alpha_m\int_0^{\lambda_m}e^{iA(\lambda_m-s)}f(s)dW_s\\&\quad+Di\sum_{m=1}^{p}\alpha_m\sum_{j=1}^{l_m}R^{l_m-j+1}f(t_{j-1})(W_{t_j}-W_{t_{j-1}})\\&=T_1+\cdots+T_7,\end{aligned}$$

where

$$\begin{aligned}T_1&=(J-D)\varphi\big(W(t_1),\cdots,W(t_p)\big)\\&=DJ\sum_{m=1}^{p}\alpha_m\big(e^{iA\lambda_m}-R^{l_m}\big)\varphi\big(W(t_1),\cdots,W(t_p)\big),\end{aligned} \tag{4.24}$$

$$T_2=-iDJ\sum_{n=1}^{p}\alpha_n\big(e^{iA\lambda_n}-R^{l_n}\big)\sum_{m=1}^{p}\alpha_m\int_0^{\lambda_m}e^{i(\lambda_m-s)A}f(s)dW(s), \tag{4.25}$$

$$T_3=-iD\sum_{m=1}^{p}\alpha_m\int_{l_m\tau}^{\lambda_m}e^{i(\lambda_m-s)A}f(s)dW(s), \tag{4.26}$$

$$T_4=-iD\sum_{m=1}^{p}\alpha_m\int_0^{l_m\tau}\big[e^{i(\lambda_m-s)A}-e^{i(l_m\tau-s)A}\big]f(s)dW(s) \tag{4.27}$$

$$T_5 = -iD \sum_{m=1}^{p} \alpha_m \Bigg(\sum_{j=1}^{l_m} \int_{(j-1)\tau}^{j\tau} e^{i(l_m\tau - s)A} f(s) dW_s - \sum_{j=1}^{l_m} \int_{(j-1)\tau}^{j\tau} R^{l_m - j} e^{i(j\tau - s)A} f(s) dW(s) \Bigg), \tag{4.28}$$

$$T_6 = -iD \sum_{m=1}^{p} \alpha_m \Bigg(\sum_{j=1}^{l_m} \int_{(j-1)\tau}^{j\tau} R^{l_m - j} e^{i(j\tau - s)A} f(s) dW(s) - \sum_{j=1}^{l_m} \int_{(j-1)\tau}^{j\tau} R^{l_m - j} e^{i\tau A} f(t_{j-1}) dW(s) \Bigg), \tag{4.29}$$

$$T_7 = -iD \sum_{m=1}^{p} \alpha_m \sum_{j=1}^{l_m} R^{l_m - j} \big(e^{i\tau A} - R \big) f(t_{j-1}) \Delta W_{l_j}. \tag{4.30}$$

Using (4.14) with $\nu = 1$ and (4.13) with $\beta = 1$,

$$\begin{aligned}
\|T_1\|_{L^2(\Omega,H)} &= \Bigg\| DJ \sum_{m=1}^{p} \alpha_m \big(e^{iA\lambda_m} - R^{l_m} \big) \varphi\big(W(t_1), \cdots, W(t_p) \big) \Bigg\|_{L^2(\Omega,H)} \\
&\le \|D\| \, \|J\| \sum_{m=1}^{p} |\alpha_m| \big\| A^{-1} e^{iAl_m\tau} \big(e^{iA(\lambda_m - l_m\tau)} - I \big) A \varphi\big(W(t_1), \cdots, W(t_p) \big) \big\|_{L^2(\Omega,H)} \\
&+ \|D\| \, \|J\| \sum_{m=1}^{p} |\alpha_m| \big\| A^{-1} \big(e^{iAl_m\tau} - R^{l_m} \big) A \varphi\big(W(t_1), \cdots, W(t_p) \big) \big\|_{L^2(\Omega,H)} \\
&\le \|D\| \, \|J\| \sum_{m=1}^{p} |\alpha_m| \big(|\lambda_m - l_m\tau| M_1 + \sqrt{l_m} \tau M_1 \big) \\
&\le \|D\| \, \|J\| \sum_{m=1}^{p} |\alpha_m| \Big(\tau M_1 + \sqrt{T} \tau^{\frac{1}{2}} M_1 \Big) \le M \tau^{\frac{1}{2}}.
\end{aligned}$$

We use the same approach to T_1 – viz. (4.14) with $\nu = 1$ and (4.13) with $\beta = 1$, the inequalities $\lambda_n - l_n\tau \le \tau, l_n\tau \le T$ and (4.17) to get an estimate for T_2 as follows:

$$\begin{aligned}
\|T_2\|_{L^2(\Omega,H)} &= \Bigg\| DJ \sum_{n=1}^{p} \alpha_n \big(e^{iA\lambda_n} - R^{l_n} \big) \sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} e^{i(\lambda_m - s)A} f(s) dW(s) \Bigg\|_{L^2(\Omega,H)} \\
&\le \|D\| \, \|J\| \sum_{n=1}^{p} |\alpha_n| \Bigg\| e^{iA\lambda_n} A^{-1} \big(e^{iA(\lambda_n - l_n\tau)} - I \big) \\
&\qquad \times \sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} A e^{i(\lambda_m - s)A} f(s) dW(s) \Bigg\|_{L^2(\Omega,H)} \\
&+ \|D\| \, \|J\| \sum_{n=1}^{p} |\alpha_n| \Bigg\| e^{iA\lambda_n} A^{-1} \big(e^{iAl_n\tau} - R^{l_n} \big) \\
&\qquad \times \sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} A e^{i(\lambda_m - s)A} f(s) dW(s) \Bigg\|_{L^2(\Omega,H)} \\
&\le \|D\| \, \|J\| \sum_{n=1}^{p} |\alpha_n| (\lambda_n - l_n\tau) \Bigg\| \sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} A e^{i(\lambda_m - s)A} f(s) dW(s) \Bigg\|_{L^2(\Omega,H)}
\end{aligned}$$

$$
\begin{aligned}
&+\|D\|\,\|J\|\sum_{n=1}^{p}|\alpha_n|\sqrt{l_n}\tau\left\|\sum_{m=1}^{p}\alpha_m\int_0^{\lambda_m}Ae^{i(\lambda_m-s)A}f(s)dW(s)\right\|_{L^2(\Omega,H)}\\
&\le\|D\|\,\|J\|\sum_{n=1}^{p}|\alpha_n|\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{T}E\left[\left\|Ae^{i(\lambda_m-s)A}f(s)\right\|_{L_2^0}^2\right]ds\right)^{\frac{1}{2}}\tau\\
&+\|D\|\,\|J\|\sum_{n=1}^{p}|\alpha_n|\sqrt{T}\sqrt{\tau}\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{\lambda_m}E\left[\left\|Ae^{i(\lambda_m-s)A}f(s)\right\|_{L_2^0}^2\right]ds\right)^{\frac{1}{2}}\\
&\le M\tau^{\frac{1}{2}}.
\end{aligned}
$$

By (4.17) and using $|\lambda_m-l_m\tau|\le\tau$,

$$
\begin{aligned}
\|T_3\|_{L^2(\Omega,H)}&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\left\|\int_{l_m\tau}^{\lambda_m}e^{iA(\lambda_m-s)}f(s)dW(s)\right\|_{L^2(\Omega,H)}\\
&=\|D\|\sum_{m=1}^{p}|\alpha_m|\left(\int_{l_m\tau}^{\lambda_m}E\left(\left\|e^{iA(\lambda_m-s)}f(s)\right\|_{L_2^0}^2\right)ds\right)^{\frac{1}{2}}\le M\tau^{\frac{1}{2}}.
\end{aligned}
$$

By (4.14) with $\nu=1,|\lambda_m-l_m\tau|\le\tau$ and by (4.17)

$$
\begin{aligned}
\|T_4\|_{L^2(\Omega,H)}&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\left\|\int_0^{l_m\tau}\left[e^{i(\lambda_m-s)A}-e^{i(l_m\tau-s)A}\right]f(s)dW(s)\right\|_{L^2(\Omega,H)}\\
&=\|D\|\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{l_m\tau}\left\|A^{-1}\left[e^{i(\lambda_m-l_m\tau)A}-I\right]Ae^{i(l_m\tau-s)A}f(s)\right\|_{L_2^0}^2ds\right)^{\frac{1}{2}}\\
&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{l_m\tau}(\lambda_m-l_m\tau)^2\left\|Ae^{i(l_m\tau-s)A}f(s)\right\|_{L_2^0}^2ds\right)^{\frac{1}{2}}\\
&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\left(\int_0^{l_m\tau}\left\|Ae^{i(l_m\tau-s)A}f(s)\right\|_{L_2^0}^2ds\right)^{\frac{1}{2}}\tau\le M\tau.
\end{aligned}
$$

By (4.13) with $\beta=1,(l_m-j)\tau\le T$ for $j=1,\dots,l_m$, and (4.17),

$$
\begin{aligned}
\|T_5\|_{L^2(\Omega,H)}&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\Bigg(\sum_{j=1}^{l_m}\int_{(j-1)\tau}^{j\tau}\left\|A^{-1}(e^{i(l_m-j)\tau A}-R^{l_m-j})\right\|_{L(H)}^2\\
&\qquad\times E\left[\left\|Ae^{i(j\tau-s)A}f(s)\right\|_{L_2^0}^2\right]ds\Bigg)^{\frac{1}{2}}\\
&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\left(\sum_{j=1}^{l_m}\int_{(j-1)\tau}^{j\tau}\left(\sqrt{l_m-j}\tau\right)^2E\left[\left\|Ae^{i(j\tau-s)A}f(s)\right\|_{L_2^0}^2\right]ds\right)^{\frac{1}{2}}\\
&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\sqrt{T}\left(\sum_{j=1}^{l_m}\int_{(j-1)\tau}^{j\tau}E\left[\left\|Ae^{i(j\tau-s)A}f(s)\right\|_{L_2^0}^2\right]ds\right)^{\frac{1}{2}}\sqrt{\tau}\\
&\le\|D\|\sum_{m=1}^{p}|\alpha_m|\sqrt{T}\left(\sum_{j=1}^{l_m}\tau\right)^{\frac{1}{2}}\sqrt{\tau}\le M\tau^{\frac{1}{2}}.
\end{aligned}
$$

By (4.18) and that $l_m\tau \le T$,

$$\begin{aligned}\|T_6\|_{L^2(\Omega,H)} &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left\| \sum_{j=1}^{l_m} \int_{t_{j-1}}^{t_j} R^{l_m-j}\big(e^{i(t_j-s)A} f(s) - e^{i\tau A} f(t_{j-1})\big) dW(s) \right\|_{L^2(\Omega,H)} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left(\sum_{j=1}^{l_m} \int_{t_{j-1}}^{t_j} E\left[\left\| e^{i(t_j-s)A} f(s) - e^{i(t_j-t_{j-1})A} f(t_{j-1}) \right\|_{L_2^0}^2 \right] ds \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left(\sum_{j=1}^{l_m} (t_j - t_{j-1})^2 \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sqrt{T}\sqrt{\tau} \le M\tau^{\frac{1}{2}}.\end{aligned}$$

By the use of (4.13) with $\beta = 1$ and (4.16),

$$\begin{aligned}\|T_7\|_{L^2(\Omega,H)} &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left\| \sum_{j=1}^{l_m} R^{l_m-j}(e^{i\tau A} - R) f(t_{j-1}) \Delta W_{l_j} \right\|_{L^2(\Omega,H)} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left(\sum_{j=1}^{l_m} E\left[\left\| A^{-1}(e^{i\tau A} - R) \right\|_{L(H)}^2 \left\| A R^{l_m-j} f(t_{j-1}) \right\|_{L_2^0}^2 \right] \tau \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \left(\sum_{j=1}^{l_m} \left(\sqrt{1}\tau\right)^2 E\left[\left\| A R^{l_m-j} f(t_{j-1}) \right\|_{L_2^0}^2 \right] \tau \right)^{\frac{1}{2}} \\ &= \|D\| \sum_{m=1}^{p} |\alpha_m| \left(\sum_{j=1}^{l_m} E\left[\left\| A R^{l_m-j} f(t_{j-1}) \right\|_{L_2^0}^2 \right] \tau \right)^{\frac{1}{2}} \tau \le M\tau.\end{aligned}$$

The proof is complete. □

Remark 4.1. (ι) An alternative way to show convergence of T_4, here with convergence rate 1/2, is

$$\begin{aligned}\|T_4\|_{L^2(\Omega,H)} &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{m} \left\| \int_{l_{j-1}\tau}^{l_j\tau} \left[e^{i(\lambda_m-s)A} - e^{i(l_m\tau-s)A} \right] f(s) dW(s) \right\|_{L^2(\Omega,H)} \\ &= \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{m} \left(\int_{l_{j-1}\tau}^{l_j\tau} E\left(\left\| A^{-1}\left[e^{i(\lambda_m-l_m\tau)A} - I \right] A e^{i(l_m\tau-s)A} f(s) \right\|_{L_2^0}^2 \right) ds \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{m} \left(\int_{l_{j-1}\tau}^{l_j\tau} (\lambda_m - l_m\tau)^2 E\left(\left\| A e^{i(l_m\tau-s)A} f(s) \right\|_{L_2^0}^2 \right) ds \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{m} \left(\tau^2 \int_{l_{j-1}\tau}^{l_j\tau} E\left(\left\| A e^{i(l_m\tau-s)A} f(s) \right\|_{L_2^0}^2 \right) ds \right)^{\frac{1}{2}} \\ &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{m} (\tau^3)^{\frac{1}{2}} \le M\tau^{\frac{1}{2}}.\end{aligned}$$

($\iota\iota$) An alternative way to show convergence of T_7, here with convergence rate 1/2, is

$$\begin{aligned}
\|T_7\|_{L^2(\Omega,H)} &\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{l_m} \big\|A^{-1}(e^{i\tau A}-R)\big\|_{L^2(\Omega,H)} \big\|AR^{l_m-j} f(t_{j-1})\Delta W_{l_j}\big\|_{L^2(\Omega,H)} \\
&\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{l_m} \Big(E\big\|A^{-1}(e^{i\tau A}-R)\big\|^2_{L(H)}\Big)^{\frac{1}{2}} \Big(E\|AR^{l_m-j} f(t_{j-1}\|^2_{L^0_2}\tau\Big)^{\frac{1}{2}} \\
&\le \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{l_m} \Big(\big((\sqrt{1}\tau)^1\big)^2\Big)^{\frac{1}{2}} \Big(E\Big[\big\|AR^{l_m-j} f(t_{j-1})\big\|^2_{L^0_2}\Big]\tau\Big)^{\frac{1}{2}} \\
&= \|D\| \sum_{m=1}^{p} |\alpha_m| \sum_{j=1}^{l_m} \tau^{\frac{3}{2}} \Big(E\Big[\big\|AR^{l_m-j} f(t_{j-1})\big\|^2_{L^0_2}\Big]\Big)^{\frac{1}{2}} \le M\tau^{\frac{1}{2}}.
\end{aligned}$$

Here we may assume that

$$\max_{1\le j\le N} \Big(E\Big[\big\|AR^{l_m-j} f(t_{j-1})\big\|^2_{L^0_2}\Big]\Big)^{\frac{1}{2}} < \infty$$

or assume the weaker condition

$$\max_{1\le l_m\le N} \sum_{j=1}^{l_m} \tau \Big(E\Big[\big\|AR^{l_m-j} f(t_{j-1})\big\|^2_{L^0_2}\Big]\Big)^{\frac{1}{2}} < \infty. \tag{4.31}$$

Theorem 4.4. *Assume that the assumptions of the previous theorem hold. Then the Rothe-Maruyama difference scheme* (4.2) *for time nonlocal problem for the stochastic Schrödinger equation* (2.1) *has a mean-square convergence of order* 1/2*. That is, for* $0 \le k \le N$*,*

$$\|u(t_k) - u_k\|_{L^2(\Omega,H)} \le M\tau^{\frac{1}{2}}$$

holds for $\tau \le 1$*. Here the positive constant* M *does not depend on* τ*.*

Proof. For $k = 0$, the result follows from the previous theorem. For $1 \le k \le N$, we can write

$$u(t_k) - u_k = T_{8k} + T_{9k} + T_{10k} + T_{11k} + T_{12k},$$

where

$$\begin{aligned}
T_{8k} &= (e^{ik\tau A} - R^k)J\left[-\sum_{m=1}^{p} \alpha_m \int_0^{\lambda_m} e^{iA(\lambda_m - s)} i f(s)dW(s) + \varphi\big(W(\lambda_1), \cdots, W(\lambda_p)\big)\right], \\
T_{9k} &= R^k\big(u(0) - u_0\big), \\
T_{10k} &= -i\sum_{j=1}^{k} \big[e^{i(k-j)\tau A} - R^{k-j}\big] \int_{t_{j-1}}^{t_j} e^{i(t_j - s)A} f(s)dW(s), \\
T_{11k} &= -i\sum_{j=1}^{k} R^{k-j}\left[\int_{t_{j-1}}^{t_j} e^{i(t_j-s)A} f(s)dW(s) - e^{i\tau A} f(t_{j-1})\Delta W_j\right], \\
T_{12k} &= -i\sum_{j=1}^{k} [e^{i\tau A} - R]R^{k-j} f(t_{j-1})\Delta W_j.
\end{aligned}$$

We will estimate these five terms separately. By (4.15), (4.17), (4.13) with $\beta = 1$ and (4.14) with $\nu = 1$, and $k\tau \le T$,

$$\begin{aligned}\|T_{8k}\|_{L^2(\Omega,H)} &\le \|J\|\big\|A^{-1}(e^{ik\tau A} - R^k)\big\|_{L(H)}\Bigg[\big\|A\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \\ &\qquad + \sum_{m=1}^{p}|\alpha_m|\Big\|\int_0^{\lambda_m} e^{iA(\lambda_m-s)}iAf(s)dW(s)\Big\|_{L^2(\Omega,H)}\Bigg] \\ &\le \|J\|\big\|A^{-1}(e^{ik\tau A} - R^k)\big\|_{L(H)}\Bigg[\big\|A\varphi\big(W(\lambda_1),\cdots,W(\lambda_p)\big)\big\|_{L^2(\Omega,H)} \\ &\qquad + \sum_{m=1}^{p}|\alpha_m|\int_0^{\lambda_m}\big\|e^{iA(\lambda_m-s)}Af(s)_{L^2(\Omega,H)}\big\|^2 ds\Bigg] \\ &\le \|J\|\sqrt{k}\tau(M_1 + M_2\lambda_p) \le M\tau^{\frac{1}{2}}.\end{aligned}$$

Estimate for T_{9k} follows from the previous theorem. Now let us obtain an estimate for T_{10k}. Using the triangle inequality, inequality (4.13) with $\beta = 1$, Itô isometry and estimate (4.17), we have

$$\begin{aligned}E\|T_{10k}\|_{L^2(\Omega,H)} &\le \left(\sum_{j=1}^{k}\int_{t_j-1}^{t_j}\big\|A^{-1}\big(e^{i(k-j)\tau A} - R^{k-j}\big)\big\|_{L(H)}^2\big\|Ae^{i(t_j-s)A}f(s)\big\|_{L_2^0}^2 ds\right)^{\frac{1}{2}} \\ &\le \left(\sum_{j=1}^{k}\int_{t_j-1}^{t_j}\big(\sqrt{k-j}\tau\big)^2\big\|Ae^{-isA}f(s)\big\|_{L_2^0}^2 ds\right)^{\frac{1}{2}} \\ &\le \left(\sum_{j=1}^{k}\left(\int_{t_j-1}^{t_j}(k-j)\tau\right)\big\|Ae^{-isA}f(s)\big\|_{L_2^0}^2 ds\right)^{\frac{1}{2}}\sqrt{\tau} \\ &\le \sqrt{T}\left(\sum_{j=1}^{k}\int_{t_j-1}^{t_j}\big\|Ae^{-isA}f(s)\big\|_{L_2^0}^2 ds\right)^{\frac{1}{2}}\sqrt{\tau} \\ &\le \sqrt{T}\left(\int_0^T\big\|Ae^{-isA}f(s)\big\|_{L_2^0}^2 ds\right)^{\frac{1}{2}}\sqrt{\tau} \le M\tau^{\frac{1}{2}}.\end{aligned}$$

By (4.18),

$$\begin{aligned}\|T_{11k}\|_{L^2(\Omega,H)} &= \left\|\sum_{j=1}^{k}R^{k-j}\int_{t_j-1}^{t_j}\left[e^{i(t_j-s)A}f(s)dW_s - e^{i\tau A}f(t_{j-1})\Delta W_j\right]\right\|_{L^2(\Omega,H)} \\ &\le \left(\sum_{j=1}^{k}\int_{t_j-1}^{t_j}E\left(\big\|e^{i(t_j-s)A}f(s) - e^{i(t_j-t_{j-1})A}f(t_{j-1})\big\|_{L_2^0}^2\right)ds\right)^{\frac{1}{2}} \\ &\le \left(\sum_{j=1}^{k}L_1(t_k - t_{j-1})^2 ds\right)^{\frac{1}{2}} \le M\tau^{\frac{1}{2}}.\end{aligned}$$

By (4.14) with $\nu = 1$ and (4.16),

$$\|T_{12k}\|_{L^2(\Omega,H)} = \left\|\sum_{j=1}^{k}[e^{i\tau A} - R]R^{k-j}f(t_{j-1})\Delta W_j\right\|_{L^2(\Omega,H)}$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{k} \left\| [e^{i\tau A} - R] R^{k-j} f(t_{j-1}) \right\|^2_{L^2(\Omega,H)} \tau \right)^{\frac{1}{2}} \\
&\le \left(\sum_{j=1}^{k} \left\| A^{-1} [e^{i\tau A} - R] \right\|^2_{L(H)} E \left[\left\| A R^{k-j} f(t_{k-1}) \right\|^2_{L_2^0} \right] \tau \right)^{\frac{1}{2}} \\
&\le \left(\sum_{j=1}^{k} \tau^2 E \left[\left\| A R^{k-j} f(t_{k-1}) \right\|^2_{L_2^0} \right] \tau \right)^{\frac{1}{2}} \\
&\le \left(\sum_{j=1}^{k} E \left[\left\| A R^{k-j} f(t_{k-1}) \right\|^2_{L_2^0} \right] \tau \right)^{\frac{1}{2}} \tau \le M\tau.
\end{aligned}
$$

Or in an alternative way using the assumptions of Remark 4.1($\iota\iota$), namely (4.31), we can obtain

$$
\begin{aligned}
\|T_{12k}\|_{L^2(\Omega,H)} &\le \sum_{j=1}^{k} \left\| [e^{i\tau A} - R] R^{k-j} f(t_{j-1}) \Delta W_j \right\|_{L^2(\Omega,H)} \\
&\le \sum_{j=1}^{k} \left(\left\| A^{-1} [e^{i\tau A} - R] \right\|^2_{L(H)} E \left[\left\| A R^{k-j} f(t_{k-1}) \right\|^2_{L_2^0} \right] \tau \right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{k} \left(\tau^2 E \left[\left\| A R^{k-j} f(t_{k-1}) \right\|^2_{L_2^0} \right] \tau \right)^{\frac{1}{2}} \le M\tau^{\frac{1}{2}}.
\end{aligned}
$$

Hence, the result follows from the estimates of $T_{8k}, T_{9k}, T_{10k}, T_{11k}$, and T_{12k}. □

5. Numerical Verification

In this section, numerical experiment of the time nonlocal problem

$$
\begin{cases}
\quad idu(t,x) + \left(u_{xx}(t,x) + \left(\pi^2 - \dfrac{i}{2} + 1 \right) u(t,x) \right) dt & \\
= ie^{it} \sin(\pi x) e^{W(t)} dW(t), & 0 < t, x < 1, \\
\quad u(0,x) = \dfrac{1}{3} u\left(\dfrac{1}{3}, x \right) + \varphi\left(x, W\left(\dfrac{1}{3} \right) \right), & 0 < x < 1, \\
\quad u(t,0) = u(t,1) = 0, & 0 < t < 1, \\
\quad \varphi\left(x, W\left(\dfrac{1}{3} \right) \right) = \sin(\pi x) \left(1 - \dfrac{1}{3} e^{\frac{i}{3}} e^{W(\frac{1}{3})} \right) &
\end{cases}
\tag{5.1}
$$

for the Schrödinger equation by using first order Rothe-Maruyama difference scheme is investigated. The Eq. (5.1) can be formulated in the mild form (3.2). Here $W = \{W(t)\}_{t\ge 0}$ is a standard one-dimensional Wiener process with values in $U = U_0 = \mathbb{R}$; for fixed t, $f(t) : U \ni u \mapsto f(t,\cdot)u \in H = L^2([0,1])$ with

$$
f(t,x) = ie^{it} \sin(\pi x) \exp\big(W(t)\big),
$$

for which

$$
\|f(t)\|^2_{L_0^2} = \|f(t)\cdot 1\|^2_{L^2(0,1)} = \int_0^1 \sin^2(\pi x) dx e^{2W(t)} = \frac{1}{2} e^{2W(t)},
$$

hence, by the use of $E[e^{\alpha W(t)}] = \exp(\alpha^2 t/2)$,

$$\int_0^t E\left[\|e^{i(t-s)A}f(s)\|^2_{L_2^0}\right] ds \le \int_0^t E\left[\|f(s)\|^2_{L_2^0}\right] ds \le \int_0^t E\big[e^{2W(s)}\big] ds = \int_0^t e^{2^2\frac{s}{2}} ds,$$

and (3.1) is valid. Furthermore, $\varphi : \mathbb{R} \ni u \mapsto \varphi(\cdot, u) \in H = L^2([0,1])$ with

$$\varphi(x,u) = \sin(\pi x)\left(1 - \frac{1}{3}e^{\frac{i}{3}}\exp(u)\right),$$

for which

$$\begin{aligned} E\left\|\varphi\left(W\left(\frac{1}{3}\right)\right)\right\|^2_H &= E\left\|\varphi\left(W\left(\frac{1}{3}\right)\right)\right\|^2_{L^2([0,1])} \\ &= \int_0^1 \sin^2(\pi x) dx E\left[\left|1 - \frac{1}{3}e^{\frac{i}{3}}\exp\left(W\left(\frac{1}{3}\right)\right)\right|^2\right] < \infty. \end{aligned}$$

Besides, A is defined by

$$A\phi(x) = \left(\pi^2 - \frac{i}{2} + 1\right)\phi(x) + \phi_{xx}(x), \quad \phi \in C_0^\infty\big((0,1)\big),$$

where $C_0^\infty((0,1))$ is the set of infinitely differentiable real-valued functions on $(0,1)$ with compact support. For $\phi \in C_0^\infty((0,1))$, define

$$\|\phi\|_{1,2} = \left(\int_0^1 |\phi(y)|^p + |\phi_x(y)|^2 dy\right)^{\frac{1}{2}}.$$

The domain $D(A)$ of A is the completion of $C_0^\infty((0,1))$ in $H = L^2((0,1))$ with respect to $\|\cdot\|_{1,2}$. By applying the Itô formula to $t \mapsto e^{it}e^{W(t)}$, it is easy to see that

$$u(t,x) = e^{it}\sin(\pi x)e^{W(t)} \tag{5.2}$$

is the unique mild solution to (5.1). Concerning the convergence conditions (4.15)-(4.18), we recall that for (4.16) it is sufficient to show (4.20), for (4.17) it is sufficient to validate (4.21), and for (4.18) it is sufficient to show (4.22). We have for (4.15),

$$\begin{aligned} \left\|A\varphi\Big(W\left(\frac{1}{3}\right)\Big)\right\|^2_{L^2(\Omega,H)} &= E\left[\left\|A\varphi\Big(W\left(\frac{1}{3}\right)\Big)\right\|^2_H\right] = E\left[\left\|A\varphi\Big(W\left(\frac{1}{3}\right)\Big)\right\|^2_{L^2([0,1])}\right] \\ &= E\left[\int_0^1 \left|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right)\varphi\left(x, W\left(\frac{1}{3}\right)\right)\right|^2 dx\right] \\ &= E\Bigg[\int_0^1 \Bigg|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right)[\sin(\pi x)] \\ &\qquad \left(1 - \frac{1}{3}e^{\frac{i}{3}}\exp\left(W\left(\frac{1}{3}\right)\right)\right)\Bigg|^2 dx\Bigg] < \infty, \end{aligned}$$

and for (4.21),

$$\|Af(t)\|^2_{L^2(\Omega, L_2^0)} = E\left[\|Af(t)\|^2_{L^2(0,1)}\right]$$

$$
\begin{aligned}
&= E\left[\int_0^1 \left|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right) f(t,x)\right|^2 dx\right] \\
&= E\left[\int_0^1 \left|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right) ie^{it}\sin(\pi x)\exp\big(W(t)\big)\right|^2 dx\right] \\
&= E\left[\int_0^1 \left|\left(-\frac{i}{2} + 1\right) ie^{it}\sin(\pi x)\exp\big(W(t)\big)\right|^2 dx\right] \\
&= \frac{5}{4}\int_0^1 \sin^2(\pi x)\exp(2t)dx = \frac{5}{8}\exp(2t).
\end{aligned}
$$

For (4.20),

$$
\begin{aligned}
\sum_{j=1}^{N} E\left[\|Af(t_{j-1})\|^2_{L_2^0}\right]\tau &= \sum_{j=1}^{N} E\left[\int_0^1 \left|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right) f(t_{j-1},x)\right|^2 dx\right]\tau \\
&\le \sum_{j=1}^{N} E\left[\int_0^1 \left|\left(\pi^2 - \frac{i}{2} + 1 + \frac{\partial^2}{\partial x^2}\right) ie^{it_{j-1}}\sin(\pi x)\exp\big(W(t_{j-1})\big)\right|^2 dx\right]\tau \\
&\le \sum_{j=1}^{N} E\left[\int_0^1 \left|\left(-\frac{i}{2} + 1\right) ie^{it_{j-1}}\sin(\pi x)\exp\big(W(t_{j-1})\big)\right|^2 dx\right]\tau \\
&\le \sum_{j=1}^{N} \frac{5}{4}\int_0^1 \sin^2(\pi x)\exp\big(2(t_{j-1})\big)dx\tau \\
&\le \frac{5}{8}\sum_{j=1}^{N} \exp\big(2(t_{j-1})\big)\tau \le \frac{5}{8}T\exp(2T).
\end{aligned}
$$

Furthermore, where we recall $f(t,x) = ie^{it}\sin(\pi x)\exp(W(t))$, for $0 \le s \le t \le 1$,

$$
\begin{aligned}
&\left\|e^{-iAs}f(s) - e^{-iAt}f(t)\right\|^2_{L^2(\Omega, L_2^0)} \\
&= E\left[\left\|e^{-iAs}f(s) - e^{-iAt}f(t)\right\|^2_{L_2(0,1)}\right] \\
&= E\left[\left\|e^{-(\frac{1}{2}+i)s}f(s) - e^{-(\frac{1}{2}+i)t}f(t)\right\|^2_{L_2(0,1)}\right] \\
&= \|\sin(\pi\cdot)\|^2_{L^2(0,1)}E\left[\left|e^{-(\frac{1}{2}+i)s}e^{is}\exp\big(W(s)\big) - e^{-(\frac{1}{2}+i)t}e^{it}\exp\big(W(t)\big)\right|^2\right] \\
&\le \frac{1}{2}E\left[\left|\exp\left(-\frac{s}{2} + W(s)\right) - \exp\left(-\frac{t}{2} + W(t)\right)\right|^2\right] \\
&= \frac{1}{2}E\left[\left|\exp\left(-\frac{s}{2} + W(s)\right)(1 - \exp\left(-\frac{t-s}{2} + \big(W(t) - W(s)\big)\right)\right|^2\right] \\
&= \frac{1}{2}E\left[\left|\exp\left(-\frac{s}{2} + W(s)\right)\right|^2\right]E\left[\left|1 - \exp\left(-\frac{t-s}{2} + \big(W(t) - W(s)\big)\right)\right|^2\right] \\
&= \frac{1}{2}E\big[\exp\big(-s + 2W(s)\big)\big]E\Big[1 - 2\exp\left(-\frac{t-s}{2} + \big(W(t) - W(s)\big)\right) \\
&\qquad + \exp\big(-(t-s) + 2\big(W(t) - W(s)\big)\big)\Big] \\
&= \frac{1}{2}\left(\exp\left(-s + 2^2\frac{s}{2}\right)\right)\left(1 - 2\exp\left(-\frac{t-s}{2} + \frac{t-s}{2}\right) + \exp\left(-(t-s) + 2^2\frac{t-s}{2}\right)\right)
\end{aligned}
$$

$$= \frac{1}{2} e^{s} \int_{0}^{t-s} e^{\theta} d\theta \leq \frac{1}{2} e^{s} \int_{0}^{t-s} e^{t-s} d\theta \leq e^{1}(t-s),$$

hence (4.22). For the approximate solution of problem (5.1), the set $[0,1]_\tau \times [0,1]_h$ of a family of grid points depending on the small parameters τ and h,

$$[0,1]_\tau \times [0,1]_h = \{(t_k, x_n) \,:\, t_k = k\tau,\ k = 1,\ldots,N-1,\ N\tau = 1,\\ x_n = nh,\ n = 1,\ldots,M-1,\ Mh = 1\}$$

is defined. For problem (5.1), the Rothe-Maruyama scheme including a spatial discretization can be formulated as follows:

$$\begin{cases} i\big(u_n^k - u_n^{k-1}\big) + \left(\dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \left(\pi^2 - \dfrac{i}{2} + 1\right) u_n^k\right)\tau \\ = f(t_{k-1}, x_n)\Delta W_{k-1}, \qquad k = 1,\ldots,N = \dfrac{1}{\tau}, \quad n = 1,\ldots,M-1, \\ u_n^0 = \dfrac{1}{3} u_n^{[\frac{N}{3}]} + \varphi\left(x_n, \dfrac{1}{3}W\right), \quad n=1,\ldots,M-1, \quad u_0^k = 0, \quad u_M^k = 0, \quad k = 0,\ldots,N, \end{cases} \tag{5.3}$$

where $\Delta W_{k-1} = W(t_k) - W(t_{k-1})$ and $[N/3]$ is the integer part of $N/3$. With $a = \tau/h^2$, $b = -i$, $c = i - 2\tau/h^2 + \pi^2\tau$, $d = \tau/h^2$, we have for $k = 1,\ldots,N$,

$$a u_{n+1}^k + b u_n^{k-1} + c u_n^k + d u_{n-1}^k = f(t_{k-1}, x_n)\Delta W_{k-1}, \quad n = 1,\ldots,M-1.$$

By also taking into consideration the non-local initial condition we can write

$$AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad n = 1,\ldots,M-1, \tag{5.4}$$

where

$$U_n = \begin{bmatrix} u_n^0 \\ u_n^1 \\ \cdots \\ u_n^{N-1} \\ u_n^N \end{bmatrix}, \quad n = 1,\ldots,M-1, \quad U_0 = U_M = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix}_{(N+1)\times 1}, \tag{5.5}$$

and A, B, C and D are $(N+1)\times(N+1)$ matrices with

$$A(i,i+1) = a, \quad B(i,i) = b, \quad B(i,i+1) = c, \quad C(i,i+1) = d, \quad D(i,i) = 1$$

with all other entries zero, and

$$\varphi_n = \begin{bmatrix} \varphi_n^1 \\ \cdots \\ \varphi_n^N \\ \varphi_n^0 \end{bmatrix} = \begin{bmatrix} f(t_0, x_n)\Delta W_0 \\ \cdots \\ f(t_{N-1}, x_n)\Delta W_{N-1} \\ \varphi(x_n, W) \end{bmatrix}, \quad n = 1,\ldots,M-1.$$

To solve (5.4) together with (5.5) we apply the modified Gauss elimination method for the difference equation with respect to n with matrix coefficients as in [8]. It means that we seek a solution of the matrix in the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1,\ldots,2,1,0, \tag{5.6}$$

where α_j $(j = 1, \ldots, M)$ are $(N+1) \times (N+1)$ square matrices and β_j $(j = 1, \ldots, M)$ are $(N+1) \times 1$ columns. By plugging (5.6) into (5.4) we get

$$AU_{n+1} + B(\alpha_{n+1}U_{n+1} + \beta_{n+1}) + C\big(\alpha_n(\alpha_{n+1}U_{n+1}) + \beta_n\big) = D\varphi_n,$$

i.e.

$$(A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1})U_{n+1} + B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n = D\varphi_n,$$

which for each n has a solution iteratively given by

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \quad \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, 2, 3, \ldots, M-1.$$

Note that for obtaining $\alpha_{n+1}, \beta_{n+1}, n = 1, \ldots, M-1$, first we need to find α_1 and β_1. Since $U_0 = \alpha_1 U_1 + \beta_1$, where U_0 is a zero vector we let, as in [8], α_1 be the zero matrix and β_1 also be the zero column vector. We thus obtain α_n and β_n forward and U_n backward by n.

Numerically verifying Theorem 4.4, supported by Figs. 5.1-5.3, the numerical solution of the difference equation (5.3) is compared with the analytical solution of the differential equation

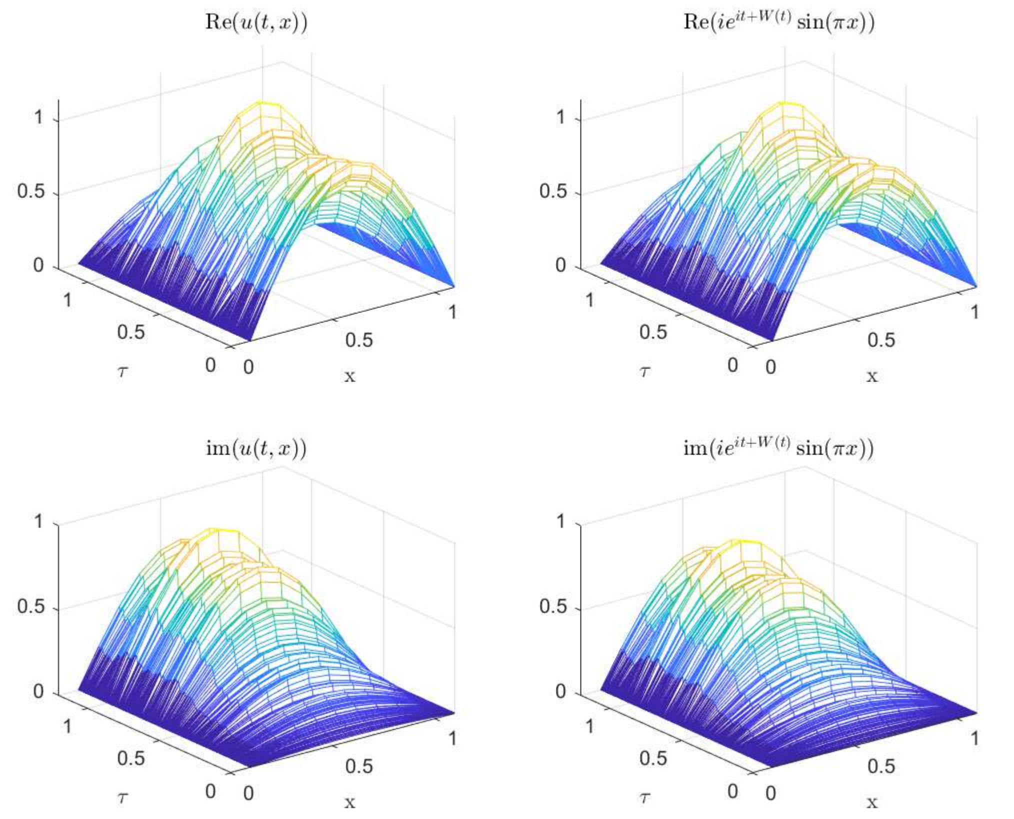

Fig. 5.1. Visualization of one simulation of the approximative Rothe-Maruyama solution (5.1) and the exact solution (5.2). Real and imaginary part of computed values are on the left, and that of exact values are on the right. Here $M = 100$ and $N = 400$. Note the slight deviation of the approximative solution and the exact solution for the imaginary part of the solution at the initial time point $t = 0$.

(5.1) with the L^2 error $\|u(t_N) - u_N\|_{L^2(\Omega,H)}$ at the final time point approximated by

$$E_M^N = \left(\frac{1}{N_{sim}} \sum_{j=1}^{N_{sim}} \sum_{n=1}^{M-1} \left[u(t_N, x_n) - u_n^N \right]^2 h \right)^{\frac{1}{2}}, \tag{5.7}$$

where $u(t_N, x_n)$ is the exact solution at the final time point $t_N = 1$ given by (5.2) and u_n^N is the numerical solution (5.1) at $(t_N, x_n) = (1, x_n)$.

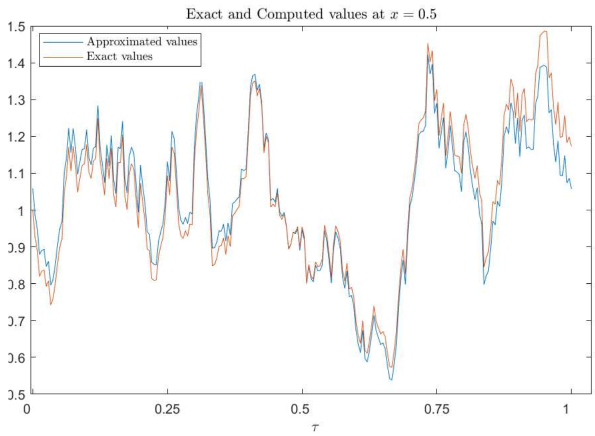

Fig. 5.2. A simulated time-evolutionary trajectory of the real part of the approximative (5.1) and the exact solution (5.2) also here with $M = 100$ and $N = 400, N = 100$ at $x = 0.5$. Note the typical Wiener process type trajectory in time which is not apparent from Fig. 5.1.

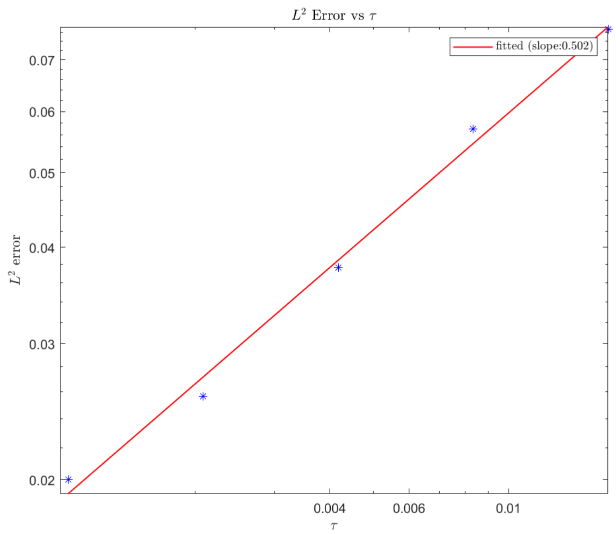

Fig. 5.3. Loglog plot of the L^2-error estimate (5.7) of the numerical scheme for large M ($M = 1000$) and $N = 25, 50, 100, 200, 400$ based on 1000 number of simulations. The solid line is the fitted regression line with slope 0.502. The empirical convergence rate as the rate of the regression line slope seems to be in line with Theorem 4.4 stating the convergence 1/2.

6. Conclusions

In this paper, existence and uniqueness of a solution to a time multipoint Schrödinger equation with values in a Hilbert space driven by a cylindrical Wiener process is obtained. Mean square convergence rate of order 1/2 of an implicit Euler-Maruyama scheme, which should be the main contribution of the paper due to more intricate calculations, is also achieved. Numerical experiments for a non-trivial example corroborates the convergence rate. Convergence rate of temporal-space discretization will be investigated in future studies.

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References

[1] M.A.E. Abdelrahman, W.W. Mohammed, M. Alesemi, and S. Albosaily, The effect of multiplicative noise on the exact solutions of nonlinear Schrödinger equation, *AIMS Math.*, **6**:3 (2021), 2970–2980.

[2] S. Aizicovici and Y. Gao, Functional differential equations with nonlocal initial conditions, *Journal of Applied Mathematics and Stochastic Analysis*, **10**:2 (1997), 145–156.

[3] S. Aizicovici and H. Lee, Nonlinear nonlocal Cauchy problems in Banach spaces, *Appl. Math. Lett.*, **18** (2005), 401–407.

[4] W.O. Amrein, *Hilbert Space Methods in Quantum Mechanics*, EPFL Press, 2009.

[5] R. Anton and D. Cohen, Exponential integrators for stochastic Schrödinger equations driven by Itô noise, *J. Comput. Math.*, **36**:2 (2005), 276–309.

[6] A. Arciniega, Shooting methods for numerical solution of nonlinear stochastic boundary-value problems, *Stoch. Anal. Appl.*, **25** (2007) 187–200.

[7] A. Ashyralyev and M.E. San, An approximation of semigroups method for stochastic parabolic equations, *Abstr. Appl. Anal.*, **2012** (2012), 684248.

[8] A. Ashyralyev and A. Sirma, Nonlocal boundary value problems for the Schrödinger equation, *Comput. Math. with Appl.*, **55** (2008), 392–407.

[9] G. Avalishvili, M. Avalishvili, and B. Miara, Nonclassical problems with nonlocal initial conditions for second-order evolution equations, *Asymptot. Anal.*, **76** (2012), 171–192.

[10] V. Barbu, M. Röckner, and D. Zhang, Stochastic nonlinear Schrödinger equations with linear multiplicative noise: Rescaling approach, *J. Nonlinear Sci.*, **24** (2014), 383–409.

[11] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.*, **40**:1 (1991), 11–19.

[12] J. Cui and J. Hong, Analysis of a splitting scheme for damped stochastic nonlinear Schrödinger equation with multiplicative noise, *SIAM J. Numer. Anal.*, **56**:4 (2018), 2045–2069.

[13] J. Cui, J. Hong, and Z. Liu, Strong convergence rate of finite difference approximations for stochastic cubic Schrödinger equations, *J. Differential Equations*, **263**:7 (2017), 3687–3713.

[14] J. Cui, J. Hong, Z. Liu, and W. Zhou, Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations, *J. Differential Equations*, **266**:9 (2019), 5625–5663.

[15] X. Feng and S. Ma, Stable numerical methods for a stochastic nonlinear Schrödinger equation with linear multiplicative noise, *Discrete Contin. Dyn. Syst. Ser. S*, **15**:4, (2022), 687–711.

[16] E. Gatsori, S.K. Ntouyas, and Y.G. Sficas, On a nonlocal Cauchy problem for differential inclusions, *Abstr. Appl. Anal.*, **2004**:5 (2004), 425–434.

[17] D.G. Gordeziani and G.A. Avalishvili, Time-nonlocal problems for Schrödinger type equations: I. Problems in abstract spaces, *Differ. Equ.*, **41**:5 (2005), 703–711.

[18] D.G. Gordeziani and G.A. Avalishvili, Time-nonlocal problems for Schrödinger type equations: II. Results for specific problems, *Differ. Equ.*, **41**:6 (2005), 852–859.

[19] R. Kruse, *Strong and Weak Approximation of Semilinear Stochastic Evolution Equation*, in: *Lectures Notes in Mathematics*, Vol. 2093, Springer, 2014.

[20] G.J. Lord, C.E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, 2014.

[21] S.K. Ntouyas and P.C. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, *J. Math. Anal. Appl.*, **210** (1997), 679–687.

[22] D. Nualart, *Malliavin Calculus and Related Topics*, Springer, 2006.

[23] E. Pardoux, *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, Springer, 2014.

[24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, in: *Applied Mathematical Sciences*, Vol. 44, Springer, 1983.

[25] C. Pellegrini, Existence, uniqueness and approximation of a stochastic Schrödinger equation: The diffusive case, *Ann. Probab.*, **36**:6 (2008), 2332–2353.

[26] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, Cambridge University Press, 2007.

[27] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, in: *Lecture Notes in Mathematics*, Springer, 2007.

[28] S.E. Shreve, *Stochastic Calculus for Finance II*, Springer, 2004.

[29] A. Sirma, Rothe-Maruyama difference-scheme for the stochastic Schrödinger equation, *Int. J. Appl. Math.*, **34**:2 (2021), 249–258.

[30] F. Sun, Optical wave patterns of nonlinear Schrödinger equation with anti-cubic nonlinearity in optical fiber, *Results Phys.*, **31** (2021), 104889.

[31] D. Sytnyk, Parallel numerical method for nonlocal-in-time Schrödinger equation, *J. Coupled Syst. Multiscale Dyn.*, **5**:2 (2017), 204–211.

[32] D. Sytnyk and R. Melnik, Linear nonlocal problem for the abstract time-dependent non-homogeneous Schrödinger equation, *arXiv:1609.08670*, 2018.

[33] D. Sytnyk and R. Melnik, Mathematical models with nonlocal initial conditions: An exemplification from quantum mechanics, *Math. Comput. Appl.*, **26** (2021), 73.