Finite Element Scheme with H2N2 Interpolation for Multi-Term Time-Fractional Mixed Sub-Diffusion and Diffusion-Wave Equation

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Abstract. In this paper, two numerical schemes for the multi-term fractional mixed diffusion and diffusion-wave equation (of order \(\alpha\), with \(0 < \alpha < 2\)) are developed to solve the initial value problem. Firstly, we study a direct numerical scheme that uses quadratic Charles Hermite and Newton (H2N2) interpolation polynomials approximations in the temporal direction and finite element discretization in the spatial direction. We prove the stability of the direct numerical scheme by the energy method and obtain a priori error estimate of the scheme with an accuracy of order \(3 - \alpha\). In order to improve computational efficiency, a new fast numerical scheme based on H2N2 interpolation and an efficient sum-of-exponentials approximation for the kernels is proposed. Numerical examples confirm the error estimation results and the validity of the fast scheme.

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Key words: The multi-term fractional mixed diffusion and diffusion-wave equation, finite element method, H2N2 interpolation, fast algorithm, stability and convergence.

1 Introduction

Inspired by the nonlocality of fractional derivatives, the study of fractional partial differential equations (FPDEs) has attracted much attention in recent years, see [3,10,23,24]. To our knowledge, although many experts and scholars have made many meaningful contributions in their theoretical analysis, the exact solutions of most fractional equations are still difficult to obtain [8, 11]. Even if an exact solution is obtained, it is often too

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complexity to apply in real life. Due to the difficulties in obtaining accurate solutions for FPDEs, there is an urgent need to study effective numerical methods for FPDEs. In recent decades, the numerical methods for FPDEs has been well applied, such as the finite element methods [12,29], finite difference methods [3,10,26,27] and spectral methods [5,17] et al..

With the rapid development of science and technology, FPDEs have been widely studied in rheology, wave propagation, fluid flows, etc. as seen in [6, 13, 16, 21]. However, compared to classical diffusion equations or wave equations, multi-term time fractional mixed sub-diffusion and diffusion wave equations may be more consistent and accurate in modeling some processes of disturbance. Zhang et al. [32] established a compact ADI scheme for the two-dimensional time fractional diffusion-wave equation. Jin et al. [13] derived a fully discrete scheme for the multi-term time fractional diffusion equation with bounded convex polyhedral domain conditions. Using the L2-1σ formula, Gao et al. [10] constructed a temporal second order difference scheme for the time multi-term and distributed-order fractional sub-diffusion equations. In [6], the authors adopted a high order difference scheme for the multi-term time fractional partial differential equations applied the energy method and the Galerkin spectral method. By an effective asymptotic expansion of the error equation of the shifted Grünwald–Letnikov formula, Zeng et al. [30] investigated a new modified weighted shifted Grünwald–Letnikov formula for multi-term fractional ordinary and partial differential equations. Sun [20] et al. proved two temporal second-order difference schemes for the multi-term time fractional diffusion-wave equation based on the order reduction technique. Feng et al. [9] presented a mixes L scheme to approximate the novel two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on convex domains. A fully discrete approximate scheme for the 2D multi-term time-fractional mixed diffusion and diffusion-wave equations were established by using linear triangle finite element method in space and classical L1 time-stepping method combined with Crank-Nicolson scheme in time [31]. Jiang et al. [11] proposed the multi-term space-time Caputo-Riesz fractional advective-diffusion equation with Dirichlet inhomogeneous boundary conditions. Bhrawy and Zaky [2] considered an efficient operational formulation of spectral tau method for multi-term time-space fractional differential equation under Dirichlet boundary conditions. Based on a mixed difference scheme in time and an unstructured mesh finite element method in space, Fan et al. [7] studied a fully discrete scheme for two-dimensional multi-term time-space fractional diffusion-wave equations, which defined on an irregular convex domain with the time fractional orders belonging to the whole interval (0,2).

However, most of the above results only considered the L1 approximation in the time direction, or the numerical methods were used in the relevant equations with high computational costs and large storage problems. To overcome these difficulties, several techniques were employed to reduce the cost of calculating and storing derivation methods. Ke et al. [14] attempted to deal with the block lower triangular Toeplitz-like with tridiagonal blocks system which arises from the time-fractional partial differential equa-
tion. They proved a fast direct method for solving the linear system which was much faster than the classical block forward substitution method. Shen et al. [18] obtained the fast H2N2 scheme for the time-fractional diffusion-wave equation by the kernel function in the fractional operators was approximated by the sum-of-exponentials. Lyu [15] et al. applied a linearized finite difference method to solve the nonlinear time-fractional wave equation involving multi-term time fractional derivatives based on the fast L2-1 formula.

In the paper, we investigate the following multi-term time fractional mixed sub-diffusion and diffusion wave equation

$$\partial_t u(x,t) + \sum_{j=1}^{M_1} p_j \partial^\alpha_j u(x,t) + \sum_{j=1}^{M_2} q_j \partial^\beta_j u(x,t) - \Delta u(x,t) + gu(x,t) = f(x,t)$$  \hspace{1cm} (1.1)

for \((x,t) \in \Omega \times (0,T]\), with the initial conditions and boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (1.2a)$$

$$u(x,t) = 0, \quad \Omega \subset \mathbb{R}^d, \quad u_t(x,t) = 0, \quad \Omega \subset \mathbb{R}^d, \quad t \in (0,T], \quad (1.2b)$$

where \(M_1, M_2\) are positive integers, \(p_i, q_j, g\) are nonnegative constants, \(\Omega \subset \mathbb{R}^d, (d = 1,2)\), and \(\partial \Omega\) is the boundary of \(\Omega\), \(x = (x^{(1)}, x^{(2)}) \in \Omega\). \(\Delta\) is Laplace operator, and \(u_0(x), u_1(x)\) are given sufficiently smooth functions. Meanwhile, \(\partial^\alpha_j u(x,t), \partial^\beta_j u(x,t)\) denote the Caputo fractional derivative by

$$\partial^\alpha_j u(x,t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha_i}} ds, \quad (1.3a)$$

$$\partial^\beta_j u(x,t) = \frac{1}{\Gamma(2-\beta_j)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{1}{(t-s)^{\beta_j-1}} ds, \quad (1.3b)$$

with \(0 < \alpha_0 < \cdots < \alpha_i < \cdots < \alpha_{M_1} < 1, (i = 1,2,\cdots,M_1)\) and \(1 < \beta_0 < \cdots < \beta_j < \cdots < \beta_{M_2} < 2 (j = 1,2,\cdots,M_2)\). For convenience, we define \(\alpha_{M_1} = \alpha, \beta_{M_2} = \beta\).

**Lemma 1.1.** Assume that \(\{\{(\lambda_i, v_i) : i = 1,2,\cdots\}\}\) be the eigenvalues and eigenfunctions for the Sturm–Liouville boundary value problem [19,22]

$$L v_i := -\Delta v_i = \lambda_i v_i \quad \text{on} \quad \Omega, \quad v_i = 0 \quad \text{on} \quad \partial \Omega,$$

where \(\|v_i\|_{L^2(\Omega)} = 1\) for any \(i\) is required to normalize the eigenfunction. Let \(u(\cdot,t) \in D(L^3)\) for each \(t \in [0,T]\), then

$$u_1(\cdot,t), u_H(\cdot,t) \in D(L^3) \quad \text{for each} \quad t \in [0,T],$$

with

$$\|u(\cdot,t)\|_{L^3(\Omega)} + \|u_1(\cdot,t)\|_{L^3(\Omega)} + \|u_H(\cdot,t)\|_{L^3(\Omega)} \leq C.$$
In this paper, we use $C$ with or without subscript, to represent positive constants independent of mesh parameters and it may takes different values at different places. The unique solution $u$ of (1.1)-(1.2b) exists with a constant $C$, for all $(x,t) \in \Omega \times (0,T]$, such that
\[ |u^{(l)}(t)| \leq C(1+t^{\beta-l}) \quad \text{for } l = 0,1,2,3. \]

The structure of this paper is presented as follows. In Section 2, we do some preliminary work and construct a direct numerical scheme based on H2N2 interpolation for (1.1)-(1.2b). Section 3, the stability of the finite element direct numerical scheme is studied. The proof of the error analysis for the direct numerical scheme is analyzed in Section 4. To improve computational efficiency, we propose a new fast numerical scheme in Section 5. In Section 6, three examples have verified the effectiveness of the error estimation results for the direct numerical scheme and the fast numerical scheme based on H2N2 interpolation. The conclusion is provided at the end of the article.

2 Construction of fully discrete scheme

In this section, we will propose a fully discrete approximation scheme of (1.1)-(1.2b). To begin with, we introduce triangular FEMs for the spatial discretization and summarize some mathematical notations, which will be useful later. For brevity, we denote $\|\cdot\|_1 = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$, and the semi-norm $|u|_1$, $H^1$ norm $\|u\|_1$ as follow
\[ |u|_1 = \left\| \frac{\partial u}{\partial x} \right\|, \quad \|u\|_1 = \left( \|u\|^2 + |u|_1^2 \right)^{\frac{1}{2}}. \]

We define the time-step length $\tau = \frac{T}{N}$, $t_n = n\tau$, $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n+t_{n+1})$, $n = 1,2,\cdots,N$. And $\Omega_\tau = \{t_k | 0 \leq k \leq N\}$ for $u = \{u^n | 0 \leq n \leq N\}$ defined on $\Omega_\tau$, introduced the following notations
\[ u^{n-\frac{1}{2}} = \frac{u^n + u^{n-1}}{2}, \quad \delta_t u^{n-\frac{1}{2}} = \frac{u^n - u^{n-1}}{\tau}, \quad 1 \leq n \leq N. \]

For any function $u$ defined on the interval $[0,t_1]$, we consider the $H_{2,0}(t)$ its Hermite quadratic interpolation polynomial
\[ H_{2,0}(t) = u_0 + u_0'(t-t_0) + \frac{1}{\tau}(\delta_t u_0^\frac{1}{2} - u_0^\frac{1}{2})(t-t_0)^2. \] (2.1)

For any function $u$ defined on the interval $[t_{k-1},t_{k+1}]$, $(1 \leq k \leq N-1)$, the Newton quadratic interpolation polynomial $N_{2,k}(t)$ as follows
\[ N_{2,k}(t) = u(t_{k-1}) + \delta_t u^{k-\frac{1}{2}}(t-t_{k-\frac{1}{2}}) + \frac{1}{2}\delta_t^2 u^k(t-t_k)(t-t_{k-1}). \] (2.2)
Using the H2N2 method for the Caputo-fractional derivatives (1.3b) on the time grid points of the form \(\{t_\frac{1}{2}, t_\frac{3}{2}, \cdots, t_{N-\frac{1}{2}}\}\) [18], we have

\[
\sum_{j=1}^{M_2} q_j \partial_j^{\alpha_i} u(t_{n-\frac{1}{2}}) \\
\approx \sum_{j=1}^{M_2} \frac{q_j}{\Gamma(2-\beta_j)} \left[ \int_{t_0}^{t_\frac{1}{2}} H_2^\alpha(t)(t_{n-\frac{1}{2}}-t)^{1-\beta_j} dt + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_2^\alpha(t)(t_{n-\frac{1}{2}}-t)^{1-\beta_j} dt \right] \\
= \sum_{j=1}^{M_2} \frac{q_j}{\Gamma(2-\beta_j)} \left[ b_0^{(n,\beta_j)} \delta_i u^n - \sum_{k=1}^{n-1} (b_k^{(n,\beta_j)} - b_{n-k}^{(n,\beta_j)}) \delta_i u^{k-\frac{1}{2}} - b_{n-k}^{(n,\beta_j)} u_i^0 \right] \\
:= \sum_{j=1}^{M_2} q_j D_j^{\alpha_i} u(t_{n-\frac{1}{2}}) + E_{\beta_j}^{n-\frac{1}{2}}. \quad (2.3)
\]

Here

\[
b_{n-1}^{(n,\beta_j)} = \frac{2}{\tau} \int_{t_0}^{t_\frac{1}{2}} (t_{n-\frac{1}{2}}-t)^{1-\beta_j} dt = \frac{2^{1-\beta_j}}{2-\beta_j} \left[ \left( n-1 \right)^{2-\beta_j} - (n-1)^{2-\beta_j} \right], \quad (2.4a)
\]

\[
b_{n-1-k}^{(n,\beta_j)} = \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} (t_{n-\frac{1}{2}}-t)^{1-\beta_j} dt = \frac{\tau^{1-\beta_j}}{2-\beta_j} \left[ (n-k)^{2-\beta_j} - (n-1-k)^{2-\beta_j} \right], \quad (2.4b)
\]

where \(1 \leq k \leq n-1\). Similarly, we can obtain the discrete scheme of (1.3a) through the Hermite first order interpolation polynomial and the Newton first order interpolation polynomial, we can obtain

\[
\sum_{i=1}^{M_1} p_i \partial_i^{\alpha_i} u(t_{n-\frac{1}{2}}) \\
\approx \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(1-\alpha_i)} \left[ \int_{t_0}^{t_\frac{1}{2}} H_2^\alpha(t)(t_{n-\frac{1}{2}}-t)^{-\alpha_i} dt + \sum_{k=1}^{n-1} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_2^\alpha(t)(t_{n-\frac{1}{2}}-t)^{-\alpha_i} dt \right] \\
= \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(1-\alpha_i)} \left\{ \left( t_{n-\frac{1}{2}} - t_0 \right)^{-\alpha_i} u_i^0 \left( t_{n-\frac{1}{2}} - t_1 \right)^{-\alpha_i} \delta_i u^\frac{1}{2} \\
- \left( t_{n-\frac{1}{2}} - t_{k-\frac{1}{2}} \right)^{-\alpha_i} \delta_i u_{k-\frac{1}{2}}^\frac{1}{2} + \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_2^\alpha(t)(t_{n-\frac{1}{2}}-t)^{-\alpha_i} dt \right\} \\
= \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(1-\alpha_i)} \left[ a_{\alpha_i-1}^{(n,\alpha_i)} \delta_i u^{n-\frac{1}{2}} + \sum_{k=1}^{n-1} \left( a_{\alpha_i-1}^{(n,\alpha_i)} - a_{\alpha_i}^{(n,\alpha_i)} \right) \delta_i u_{k-\frac{1}{2}}^\frac{1}{2} + A_0^{(n,\alpha_i)} - a_{\alpha_i}^{(n,\alpha_i)} u_i^0 \right]
\]
where

\[ A_0^{(n, \alpha_i)} = (t_{n-\frac{1}{2}} - t_0)^{1-\alpha_i} = t_{n-\frac{1}{2}}^{1-\alpha_i}, \] (2.6a)

\[ a_0^{(n, \alpha_i)} = \frac{2}{\tau} \int_{t_0}^{t_{n-\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\alpha_i} dt = \frac{2\tau^{1-\alpha_i}}{2-\alpha_i} \left[ \left( \frac{n}{2} - \frac{1}{2} \right)^{2-\alpha_i} - \left( \frac{n-1}{2} \right)^{2-\alpha_i} \right], \] (2.6b)

\[ a_k^{(n, \alpha_i)} = \frac{1}{\tau} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} (t_{n-\frac{1}{2}} - t)^{1-\alpha_i} dt = \frac{\tau^{1-\alpha_i}}{2-\alpha_i} \left[ (n-k)^{2-\alpha_i} - (n-1-k)^{2-\alpha_i} \right], \] (2.6c)

with \( 1 \leq k \leq n - 1. \)

**Lemma 2.1** ([18]). Assume that \( u(t) \in C^3[t_0, t_n] \), it holds

\[
\left| \sum_{i=1}^{M_1} p_i \partial_i^0 u(t_{n-\frac{1}{2}}) - \sum_{i=1}^{M_1} p_i D_1^{(n, \alpha_i)} u(t_{n-\frac{1}{2}}) \right| \leq C_0 \max_{t_0 \leq t \leq t_{n-\frac{1}{2}}} |u'''(t)| \tau^{3-\alpha_i},
\]

\[
\left| \sum_{j=1}^{M_2} q_j \partial_j^0 u(t_{n-\frac{1}{2}}) - \sum_{j=1}^{M_2} q_j D_1^{(n, \beta_j)} u(t_{n-\frac{1}{2}}) \right| \leq C_1 \max_{t_0 \leq t \leq t_{n-\frac{1}{2}}} |u''(t)| \tau^{3-\beta_j},
\]

For any \( v \in H_0^1(\Omega) \), based on (2.3), (2.5) and Lemma 2.1, we can find \( u^n \in H_0^1(\Omega) \), it yields that

\[
(\delta_i u^{n-\frac{1}{2}}, v) + \left( \sum_{i=1}^{M_1} p_i D_1^{(n, \alpha_i)} u^{n-\frac{1}{2}}, v \right) + \left( \sum_{j=1}^{M_2} q_j D_1^{(n, \beta_j)} u^{n-\frac{1}{2}}, v \right) = -B(u^n, v) + (f^{n-\frac{1}{2}}, v) + (E_1^{n-\frac{1}{2}}, v),
\] (2.7)

where

\[
|E_1^{n-\frac{1}{2}}| = |E_{\alpha_i}^{n-\frac{1}{2}} + E_{\beta_j}^{n-\frac{1}{2}}| \leq C \min \{ 3-\alpha_i, 3-\beta_j \}.
\]

The bilinear operator \( B(u, v) \) is defined as

\[ B(u, v) := (\nabla u, \nabla v) + (gu, v), \]

and it is easy to know that the bilinear form is symmetrical, continuous and coercive, such that

\[ B(u, v) \leq C \| u \|_1 \| v \|_1, \quad B(u, u) \geq C \| u \|_1^2. \] (2.8)

Let \( T_h \) be a partition of a domain \( \Omega \) composed of closed and simply connected polygons in \( \mathbb{R} \) that satisfy a set of shape regularity assumptions. Then there exists \( c_h \) be the edges in \( T_h \). Denote by \( h_k \) the diameter for every element \( k \in T_h \) and \( h = \max_{k \in T_h} h_k \) the
mesh size for $T_h$. Therefore $V_h$ be the Galerkin finite element space associated with $T_h$ defined as follows

$$V_h = \{v_h|v_h \in C(\Omega) \cap H_0^1(\Omega); v_h|_e \in \mathcal{P}_r(e), \forall e \in T_h \},$$

where $\mathcal{P}_r$ is the space of polynomials of total degree $r$ or less. For any $v_h^n \in V_h$, select $u_h^n \in V_h, (n = 1, 2, \cdots, N)$, the direct numerical scheme for the equation (1.1)-(1.2b) can be written as follows:

$$(\delta_t u_h^{n-\frac{1}{2}},v_h) + \sum_{i=1}^{M_1} p_i(D_t^{\alpha_i} u_h^{n-\frac{1}{2}},v_h) + \sum_{j=1}^{M_2} q_j(D_t^{\beta_j} u_h^{n-\frac{1}{2}},v_h) = -B(u_h^{n-\frac{1}{2}},v_h) + (f^{n-\frac{1}{2}},v_h), \quad (2.9a)$$

$$u_h^0 = u_0(x), \quad u_h^{0i} = u_1(x), \quad x \in \Omega. \tag{2.9b}$$

3 Stability analysis for the direct numerical scheme

In this section, we will discuss the unconditional stability for (2.9).

**Lemma 3.1** ([18]). For any $n \geq 2$, according to $b^{(n,\beta_i)}_k$ defined by (2.4a)-(2.4b) and $a^{(n,\alpha_i)}_k$ as defined by (2.6a)-(2.6c), we gain

$$0 < a^{(n,\alpha_i)}_{n-1} < a^{(n,\alpha_i)}_{n-2} < \cdots < a^{(n,\alpha_i)}_0 < A^{(n,\alpha_i)}_0, \quad (3.1a)$$

$$0 < t_n^{-\beta_i} < b^{(n,\beta_i)}_{n-1} < b^{(n,\beta_i)}_{n-2} < \cdots < b^{(n,\beta_i)}_0 = \frac{t_n^{-\beta_i}}{2-\beta_i}. \quad (3.1b)$$

**Proof.** According to the definition of (2.4a) and using the Binomial theorem, then we can calculate

$$b^{(n,\beta_i)}_{n-1} = \frac{2t_n^{-\beta_i}}{2-\beta_i} \left( -\frac{1}{2} \right)^{2-\beta_i} \left[ 1 - \left( 1 - \frac{1}{2n-1} \right)^{2-\beta_i} \right]$$

$$= \frac{2t_n^{-\beta_i}}{2-\beta_i} \left( -\frac{1}{2} \right)^{2-\beta_i} \left[ 2 - \beta_i \right] \left( \frac{2-\beta_i}{2n-1} \right)^2 \left( -\frac{1}{2n-1} \right)^2$$

$$- \frac{3!}{3!} \frac{(2-\beta_i)(1-\beta_i)(-\beta_i)}{2n-1} \left( -\frac{1}{2n-1} \right)^3 - \cdots$$

$$> \frac{2t_n^{-\beta_i}}{2-\beta_i} \left( -\frac{1}{2} \right)^{2-\beta_i} \left( -\frac{1}{2n-1} \right)^2 = t_n^{-\beta_i}. \tag{3.2}$$

This completes the proof. \qed
Lemma 3.2 ([25]). We denote vector quantity $S_n = [S_1, S_2, \ldots, S_N]^T$ for a constant $P$, which satisfies the following property

$$
\sum_{n=1}^{N} b^{(n, \beta_j)}_n S_n = - \sum_{k=1}^{n-1} \left( b^{(n, \beta_j)}_{n-k} - b^{(n, \beta_j)}_{n-1} P \right) S_n
$$

\[ \geq \frac{1}{2} \sum_{n=1}^{N} b^{(n, \beta_j)}_{n-k} S_n^2 - \frac{1}{2} \sum_{n=1}^{N} b^{(n, \beta_j)}_{n-1} p^2. \]  

(3.3)

Theorem 3.1. The direct numerical scheme (2.9) is unconditionally stable, that is

$$
\left\| \xi^N_h \right\|^2 = \left\| \xi^0_h \right\|^2 + \sum_{j=1}^{M_2} \frac{4q_j}{\Gamma(3 - \beta_j)} \left[ a^{(n, \alpha_i)}_{n-1, \alpha_i} \delta t \xi^{n-\frac{1}{2}}_h + \sum_{k=1}^{n-1} \left( a^{(n, \alpha_i)}_{k-1, \alpha_i} - a^{(n, \alpha_i)}_{k, \alpha_i} \right) \delta t \xi^{k-\frac{1}{2}}_h \right] \left( \delta t \xi^{n-\frac{1}{2}}_h, \check{v}_h \right) + \left( A^{(n, \beta_j)}_0 - a^{(n, \beta_j)}_0 \right) \left( \delta t \xi^{n-\frac{1}{2}}_h, \check{v}_h \right) + \sum_{j=1}^{M_2} \frac{q_j}{\Gamma(2 - \beta_j)} \left[ b^{(n, \beta_j)}_0 \left( \delta t \xi^{n-\frac{1}{2}}_h, \check{v}_h \right) - \sum_{k=1}^{n-1} \left( b^{(n, \beta_j)}_{n-k} - b^{(n, \beta_j)}_{n-1} \right) \left( \delta t \xi^{k-\frac{1}{2}}_h, \check{v}_h \right) \right] \left( \delta t \xi^{n-\frac{1}{2}}_h, \check{v}_h \right) = 0. \]

(3.5)

Taking $\check{v}_h = \xi^{n-\frac{1}{2}}_h$ and noticing the (3.1), it holds that

$$
\left( \delta t \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) - \sum_{j=1}^{M_1} \frac{p j}{\Gamma(2 - \alpha_i)} \left( \delta t \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) + \sum_{j=1}^{M_2} \frac{q j}{\Gamma(2 - \beta_j)} \left[ b^{(n, \beta_j)}_0 \left( \delta t \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) \right] - \sum_{k=1}^{n-1} \left( b^{(n, \beta_j)}_{n-k} - b^{(n, \beta_j)}_{n-1} \right) \left( \delta t \xi^{k-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) - b^{(n, \beta_j)}_{n-1} \left( \delta t \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) \leq 0. \]

(3.6)

Using the Cauchy-Schwarz inequality, which can be expressed by

$$
\left( \delta t \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) = \frac{\xi^{n-1}_h - \xi^{n-1}_h}{\tau}, \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) \geq \frac{\xi^{n-1}_h - \xi^{n-1}_h}{2 \tau} \cdot \xi^{n-\frac{1}{2}}_h, \xi^{n-\frac{1}{2}}_h \right) \geq \frac{\xi^{n-1}_h - \xi^{n-1}_h}{2 \tau} \leq 0. \]

(3.7)
Combining with (3.9), (3.10) and Young inequality, we deduce

\[
\sum_{n=1}^{M_2} \sum_{j=1}^{2\tau q_j} \frac{2\tau q_j}{\Gamma(2-\beta_j)} \left[ b_0^{(n,\beta_j)} \left( \delta_i \epsilon_h^{n-\frac{1}{2}} \epsilon_h^{n-\frac{1}{2}} \right) \right. \\
\left. - \sum_{k=1}^{n-1} \left( b_{n-1}^{(n,\beta_j)} - b_{n-k}^{(n,\beta_j)} \right) \cdot \left( \delta_i \epsilon_h^{k-\frac{1}{2}} \epsilon_h^{n-\frac{1}{2}} \right) - b_{n-1}^{(n,\beta_j)} \left( \epsilon_{ht}^{n}, \epsilon_h^{n-\frac{1}{2}} \right) \right]
\]

\[
\leq \sum_{j=1}^{M_2} \frac{\tau q_j}{\Gamma(2-\beta_j)} \sum_{n=1}^{N} b_{n-1}^{(n,\beta_j)} \| \delta_i \epsilon_h^{n-\frac{1}{2}} \|^2 - \sum_{j=1}^{M_2} \frac{\tau q_j}{\Gamma(2-\beta_j)} \sum_{n=1}^{N} b_{n-1}^{(n,\beta_j)} \| \epsilon_{ht}^{n} \|^2.
\] (3.8)

Next we claim that for the (3.1), we have the estimate

\[
\frac{\tau}{\Gamma(2-\beta_j)} \sum_{n=1}^{N} b_{n-1}^{(n,\beta_j)} = 2^{2-\beta_j} \frac{\tau}{\Gamma(3-\beta_j)} \sum_{n=1}^{N} \left[ \left( n - \frac{1}{2} \right) 2^{-\beta_j} - (n - 1)^2 \beta_j \right]
\leq 2^{2-\beta_j} \frac{\tau}{\Gamma(3-\beta_j)} \sum_{n=1}^{N} \left[ n^2 \beta_j - (n - 1)^2 \beta_j \right]
\leq \frac{2}{\Gamma(3-\beta_j)} \left( \frac{T}{N} \right)^{2-\beta_j} N^{2\beta_j} \leq \frac{2 T^{2-\beta_j}}{\Gamma(3-\beta_j)}.
\] (3.9)

In view of expression (3.1) and (3.9), thereby obtaining

\[
\frac{2\tau}{\Gamma(2-\alpha_i)} \sum_{n=1}^{N} a_0^{(n,\alpha_i)} = 4 \frac{\tau^{2-\alpha_i}}{\Gamma(3-\alpha_i)} \sum_{n=1}^{N} \left[ \left( n - \frac{1}{2} \right) 2^{-\alpha_i} - (n - 1)^2 \alpha_i \right]
\leq \frac{4 T^{2-\alpha_i}}{\Gamma(3-\alpha_i)}.
\] (3.10)

Combining with (3.9), (3.10) and Young inequality, we deduce

\[
2 \tau \sum_{n=1}^{N} \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(2-\alpha_i)} a_0^{(n,\alpha_i)} \left( \epsilon_{ht}^{n}, \epsilon_h^{n-\frac{1}{2}} \right)
\leq 2 \tau \sum_{n=1}^{N} \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(2-\alpha_i)} a_0^{(n,\alpha_i)} \left\{ \sum_{j=1}^{M_2} \frac{\tau q_j}{\Gamma(2-\beta_j)} \sum_{n=1}^{N} b_{n-1}^{(n,\beta_j)} \right\}
\left[ \sum_{n=1}^{N} \frac{2\tau q_j}{\Gamma(2-\beta_j)} \sum_{i=1}^{M_1} a_0^{(n,\alpha_i)} \| \epsilon_h^{n-\frac{1}{2}} \|^2 \right]
\leq \frac{M_2}{\Gamma(3-\beta_j)} \left\| \epsilon_h^{n-\frac{1}{2}} \right\|^2 + \frac{M_2}{\Gamma(3-\beta_j)} \frac{\tau q_j}{\Gamma(2-\beta_j)} \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(3-\alpha_i)} \| \epsilon_{ht}^{n} \|^2.
\] (3.11)
Finally, inserting (3.8)-(3.11) into (3.7) and recalling (3.1), we conclude that
\[ \left\| e_h^N \right\|^2 - \left\| e_h^0 \right\|^2 - \sum_{j=1}^{M_2} \frac{4 \eta_j T^{2-\beta_j}}{\Gamma(3-\beta_j)} \left\| e_{ht}^0 \right\|^2 - \sum_{j=1}^{M_2} \frac{\Gamma(3-\beta_j)}{2 \eta_j T^{2-\beta_j}} \sum_{i=1}^{M_1} p_i T^{2-\alpha_i} \left\| e_{ht}^0 \right\|^2 \leq 0. \] (3.12)

The proof is completed. \( \square \)

4 Error analysis for the direct numerical scheme

In this section, in order to study the convergence of the direct numerical scheme, some approximation operators and their approximation properties are introduced, which will be used in the later.

Lemma 4.1 ([1, 4]). Defining Ritz-projection operator \( R_h : H_0^1(\Omega) \to V_h \),
\[ (\nabla (z - R_h z), \nabla z_h) = 0, \quad z_h \in V_h, \] (4.1)
with the following estimate inequality for any \( z \in H_0^1(\Omega) \cap H^{r+1}(\Omega) \)
\[ \left\| z - R_h z \right\| + h \left\| z - R_h z \right\|_1 \leq C h^{r+1} \left\| z \right\|_{r+1}. \] (4.2)

Define the polynomial norm in the formula
\[ \left\| z \right\|_1 = \left( \sum_{0 \leq a \leq t} \int_\Omega \left\| D^a z \right\|^2 dx \right)^{\frac{1}{2}} \]
and \( D^a z \) as fractional derivatives of order \( a \).

Theorem 4.1. We assume that \( u^N \) and \( u_N^h \) are the solutions of (2.7) and (2.9), respectively. There exists solution \( u \in L^\infty(0,T;H_0^1(\Omega) \cap H^{r+1}(\Omega)) \), \( D_t^{(n,a)} u, D_t^{(n,a)} u \in L^2(0,T;H_0^1(\Omega) \cap H^{r+1}(\Omega)) \), where \( r \) is a non-negative integer, then there is a normal number \( C \), making the following error estimates valid
\[ \left\| u^N - u_N^h \right\|_1 \leq C T^{2(3-\beta)} + C h^{2r} \left( \left\| u \right\|_{r+1}^2 + \left\| u_0 \right\|_{r+1}^2 + \left\| u_1 \right\|_{r+1}^2 + \left\| \delta_t u^{n-\frac{1}{2}} \right\|_{r+1}^2 \right) \]
\[ + \max_{1 \leq n \leq N} \left\| D_t^{(n,a)} u^{n-\frac{1}{2}} \right\|_{r+1}^2 + \max_{1 \leq n \leq N} \left\| D_t^{(n,a)} u^{n-\frac{1}{2}} \right\|_{r+1}^2. \] (4.3)

Proof. Let us prove (4.3) by the definition of (4.1). For this, for any \( v_h \in V_h \), and denote \( e^u = u^{n} - R_h u^m + R_h u^m - u^m = \xi^n + \eta^n \), \( (n = 1,2,\ldots,N) \), then a straightforward calculation shows
\[ B(\xi^{n-\frac{1}{2}},v_n) = B(u^{n-\frac{1}{2}} - R_h u^{n-\frac{1}{2}},v_n) \]
\[ = (\nabla (u^{n-\frac{1}{2}} - R_h u^{n-\frac{1}{2}}, \nabla v_n)) + g(\xi^{n-\frac{1}{2}},v_n) \]
\[ = g(\xi^{n-\frac{1}{2}},v_n). \]
Substituting of the function (2.9) into Eq. (2.7), taking \( v_0 = \eta^{n-\frac{1}{2}} \) and summing \( n \) from 1 to \( N \) lead to the equation

\[
\begin{align*}
\sum_{n=1}^{N} (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) + \sum_{i=1}^{M_1} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} \left[ a^{(n,\alpha_i)}_n \right. & \left. (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) 
\right] \\
+ \sum_{k=1}^{n-1} (d_{k-1}^{(n,\alpha_i)} - d_k^{(n,\alpha_i)}) \cdot (\delta_i \eta^{k-\frac{1}{2}}, \delta_i \eta^{k-\frac{1}{2}}) - a_0^{(n,\alpha_i)} (\delta_i \eta^{n-\frac{1}{2}}) \\
+ \sum_{j=1}^{M_2} \sum_{n=1}^{N} \frac{q_j}{\Gamma(2-\beta_j)} \left[ b_0^{(n,\beta_j)} (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) 
\right] \\
- \sum_{k=1}^{n-1} (b_{n-1-k}^{(n,\beta_j)} - b_{k}^{(n,\beta_j)}) \cdot (\delta_i \eta^{k-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) - b_{n-1}^{(n,\beta_j)} (\delta_i \eta^{n-\frac{1}{2}}) + \sum_{n=1}^{N} B(\eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) \\
= \sum_{n=1}^{N} (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) - \sum_{i=1}^{M_1} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} \left[ a^{(n,\alpha_i)}_n \right. & \left. (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) 
\right] \\
+ \sum_{i=1}^{M_2} \sum_{n=1}^{N} \frac{q_j}{\Gamma(2-\beta_j)} \left[ b_0^{(n,\beta_j)} (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) 
\right] \\
- \sum_{k=1}^{n-1} (b_{n-1-k}^{(n,\beta_j)} - b_{k}^{(n,\beta_j)}) \cdot (\delta_i \eta^{k-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) - \sum_{k=1}^{n-1} (b_{n-1-k}^{(n,\beta_j)} - b_{k}^{(n,\beta_j)}) \cdot (\delta_i \eta^{k-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \quad (4.4)
\end{align*}
\]

Note that

\[
\sum_{n=1}^{N} (\delta_i \eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) \leq \sum_{n=1}^{N} \| \delta_i \eta^{n-\frac{1}{2}} \|^2. \quad (4.5)
\]

Notice also that \( \| \cdot \| \leq \| \cdot \|_1 \). Applying formula (2.6a), we find for

\[
\begin{align*}
\sum_{n=1}^{N} B(\eta^{n-\frac{1}{2}}, \delta_i \eta^{n-\frac{1}{2}}) = \sum_{n=1}^{N} \frac{1}{2\tau} \left[ B(\eta^{n}, \eta^{n}) - B(\eta^{n-1}, \eta^{n-1}) \right] \\
= \frac{1}{2\tau} \left[ B(\eta^{N}, \eta^{N}) - B(\eta^{0}, \eta^{0}) \right]. \quad (4.6)
\end{align*}
\]

Owing then to the condition of \( \epsilon^0 = 0 \) and noticing the (4.2), we deduce

\[
B(\eta^{0}, \eta^{0}) \leq \| \eta^{0} \|_1^2 \leq \left( \| u(t_0 - u_0) \|_1^2 + h^{2r} \| u_0 \|_{r+1}^2 \right) \\
= h^{2r} \| u_0 \|_{r+1}^2. \quad (4.7)
\]
According to $∥·∥ ≤ ∥·∥$ Lemma 4.1, is equivalent to the following equation

The first term on the right-hand side of Eq. (4.4), together with the Young inequality and Lemma 4.1, is equivalent to the following equation
By the compatibility condition (2.6a) and (4.10). Furthermore

\[ I_2 = \sum_{i=1}^{M_i} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} A_0^{(n,\alpha_i)} (\eta_i^0, \delta_i \eta_i^{n-\frac{1}{2}}) \]

\[ \leq \sum_{i=1}^{M_i} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} t^{1-\alpha_i} n^{-\frac{1}{2}} \left( \frac{1}{4} \| \eta_i^0 \|_2^2 + \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2 \right) \]

\[ \leq N T^{1-\alpha_i} C h^2 \| u_1 \|_{r+1}^2 + N \sum_{i=1}^{M_i} \frac{p_i T^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2. \quad (4.13) \]

The estimate of \( I_3 \) follows from the Young inequality and (4.11)

\[ I_3 = \sum_{n=1}^{N} g \left( \xi^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}} \right) \leq \sum_{n=1}^{N} g \left( \frac{1}{2} \| \xi^{n-\frac{1}{2}} \|_2^2 + \frac{1}{2} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2 \right) \]

\[ \leq N C h^2 \| u \|_{r+1}^2 + \frac{1}{2} \sum_{n=1}^{N} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2. \quad (4.14) \]

Now notice from (3.2) that

\[ I_4 = \sum_{n=1}^{N} \left( E_{1,n}, \delta_i \eta_i^{n-\frac{1}{2}} \right) \leq \sum_{n=1}^{N} \left[ \left( N - \frac{1}{2} \right) \tau \right] \beta^{-1} \| E_{1,n} \|_2^2 + \sum_{n=1}^{N} \frac{b_{n,\beta}}{N-n} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2 \]

\[ \leq N C \tau^{2 \min(3-a,3-\beta)} + \sum_{n=1}^{N} \frac{b_{n,\beta}}{4} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2. \quad (4.15) \]

We employ (4.2) and (4.13) to estimate

\[ I_5 = \sum_{i=1}^{M_i} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} \left[ d_{n,\alpha_i}^{(n,\alpha_i)} (\delta_i \eta_i^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}}) + \sum_{k=1}^{n-1} (a_k^{(n,\alpha_i)} - a_k^{(n,\alpha_i)}) (\delta_i \eta_i^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}}) (A_0^{(n,\alpha_i)} - A_0^{(n,\alpha_i)} (\xi_i^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}})) \right] \]

\[ = \sum_{n=1}^{N} \left( D_t^{(n,\alpha_i)} \xi^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}} \right) + \sum_{i=1}^{M_i} \sum_{n=1}^{N} \frac{p_i}{\Gamma(2-\alpha_i)} A_0^{(n,\alpha_i)} (\xi_i^{n-\frac{1}{2}}, \delta_i \eta_i^{n-\frac{1}{2}}) \]

\[ \leq N C \left( \tau^{2(3-a)} + h^2 \max_{1 \leq n \leq N} \| D_t^{(n,\alpha_i)} u_1 \|_{r+1}^2 \right) + \sum_{i=1}^{M_i} \sum_{n=1}^{N} \frac{p_i (n,\alpha_i)}{2 \Gamma(2-\alpha_i)} \| \delta_i \eta_i^{n-\frac{1}{2}} \|_2^2. \]
The proof is completed.

and similarly, we compute that

\[
I_6 = \sum_{j=1}^{M_1} \sum_{n=1}^{N} \frac{q_j}{\Gamma(2 - \beta_j)} \left[ b^{(n,\beta_j)}_0 (\delta \xi^{n-\frac{1}{2}}, \delta \eta^{n-\frac{1}{2}}) - \sum_{k=1}^{n-1} (b^{(n,\beta_j)}_{n-1-k} - b^{(n,\beta_j)}_{n-k}) (\delta \xi^{k-\frac{1}{2}}, \delta \eta^{n-\frac{1}{2}}) \right]
\]

\[
\leq \sum_{j=1}^{M_1} \frac{q_j}{\Gamma(2 - \beta_j)} \sum_{n=1}^{N} \left( D^{(n,\beta_j)}_t (\xi^{n-\frac{1}{2}}, \delta \eta^{n-\frac{1}{2}}) \right) \leq NC \left( \tau^{2(3-\beta)} + h^{2r} \max_{1 \leq n \leq N} \| D^{(n,\beta_j)}_t u^{n-\frac{1}{2}} \|_{r+1}^2 \right).
\]

Let us insert (4.5)-(4.17) into (4.4) and thereafter compute

\[
\frac{1}{2\tau} \left[ B(\eta^N, \eta^N) - B(\eta^0, \eta^0) \right] \leq NC \left( \tau^{2\min(3-\alpha, 3-\beta)} + h^{2(r+1)} \| u \|_{r+1}^2 + h^{2(r+1)} \| \delta u^{n-\frac{1}{2}} \|_{r+1}^2 + \frac{h^{2r}}{2} \max_{1 \leq n \leq N} \| D_t^{(n,\alpha)} u^{n-\frac{1}{2}} \|_{r+1}^2 \right).
\]

Hence we rewrite above and obtain the relation

\[
\| \eta^N \|_1 \leq C \tau^{2(3-\beta)} + Ch^{2r} \left( \| u \|_{r+1}^2 + \| u_0 \|_{r+1}^2 + \| u_1 \|_{r+1}^2 + \| \delta u^{n-\frac{1}{2}} \|_{r+1}^2 \right) + \max_{1 \leq n \leq N} \| D_t^{(n,\alpha)} u^{n-\frac{1}{2}} \|_{r+1}^2.
\]

Note: When \( r = 1 \), \( V_h \) is a linear finite space, then the Theorem 4.1 satisfies

\[
\| \eta^N \|_1^2 \leq C \tau^{2(3-\beta)} + Ch^2 \left( \| u \|_{1}^2 + \| u_0 \|_{1}^2 + \| u_1 \|_{1}^2 + \| \delta u^{n-\frac{1}{2}} \|_{2}^2 \right) + \max_{1 \leq n \leq N} \| D_t^{(n,\alpha)} u^{n-\frac{1}{2}} \|_{2}^2.
\]

The proof is completed.

5 Construction of the fast numerical scheme

In this section, we propose a fast numerical scheme based on H\( 2N2 \) interpolation and kernel for efficient exponents and approximations.
Lemma 5.1 ([18]). Let denote tolerance error \( \epsilon \), cut-off time restriction \( \delta \) and final time \( T \). Then there is a natural number \( N_{\text{exp}} \) and positive numbers \( s_i \) and \( \omega_i \), such that

\[
\left| t^{-\beta} - \sum_{l=1}^{N_{\text{exp}}} \omega_l e^{-s_l t} \right| \leq \epsilon, \quad t \in [\delta,T], \quad l = 1,2,\cdots, N_{\text{exp}},
\]

where

\[
N_{\text{exp}} = \mathcal{O}\left( (\log e^{-1}) \left( \log \log e^{-1} + \log(T\delta^{-1}) \right) + (\log \delta^{-1}) \left( \log \log e^{-1} + \log \delta^{-1} \right) \right).
\]

Now our applying the derivation of (2.1) and Lemma 5.1 to construct fast algorithm for approximating (1.3a), choose \( \delta = \frac{x}{2} \), \( \delta_i u^{-\frac{1}{2}} = u_i^0 \) yields

\[
\sum_{i=1}^{M_1} p_i \partial_t^\alpha u(t_{n-\frac{1}{2}}) \approx \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(1-\beta_i)} \left[ \int_{t_0}^{t_{n-\frac{1}{2}}} H_{2,0}''(t) \sum_{l=1}^{N_{\text{exp}}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} dt + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N_{2,k}'(t) \sum_{l=1}^{N_{\text{exp}}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} dt \right] \\
+ \int_{t_{n-\frac{1}{2}}}^{t_{n-1}} N_{2,n-1}'(t)(t_{n-\frac{1}{2}}-t)^{-\beta_i} dt \]

\[
= \sum_{i=1}^{M_1} \frac{p_i}{\Gamma(1-\beta_i)} \sum_{l=1}^{N_{\text{exp}}} \omega_l F_i^n + \frac{1-\alpha_i}{\Gamma(2-\alpha_i)} (\delta_i u_i^n - 2 - \delta_i u_i^n - \frac{3}{2}) \\
:= \sum_{i=1}^{M_1} p_i D_i^n u(t_{n-\frac{1}{2}}), \quad 1 \leq n \leq N.
\]

(5.2)

Following from the construction of (2.1), (2.2) and (2.3). Arranging, we deduce

\[
\sum_{j=1}^{M_2} q_j \partial_t^\beta u(t_{n-\frac{1}{2}}) \approx \sum_{j=1}^{M_2} \frac{q_j}{\Gamma(2-\beta_j)} \left[ \int_{t_0}^{t_{n-\frac{1}{2}}} H_{2,0}''(t) \sum_{l=1}^{N_{\text{exp}}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} dt + \sum_{l=1}^{N_{\text{exp}}} \omega_l e^{-s_l(t_{n-\frac{1}{2}}-t)} dt \right] \\
+ \int_{t_{n-\frac{1}{2}}}^{t_{n-1}} N_{2,n-1}'(t)(t_{n-\frac{1}{2}}-t)^{-\beta_i} dt \\
= \sum_{j=1}^{M_2} \frac{q_j}{\Gamma(2-\beta_j)} \sum_{l=1}^{N_{\text{exp}}} \omega_l F_j^n + \frac{1-\alpha_j}{\Gamma(2-\alpha_j)} (\delta_i u_i^n - 2 - \delta_i u_i^n - \frac{3}{2}) \\
:= \sum_{j=1}^{M_2} q_j D_j^n u(t_{n-\frac{1}{2}}), \quad 1 \leq n \leq N.
\]

(5.3)
where
\[
F^n_{11} = \int_{t_0}^{t_1} H'_{2,0}(t) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N'_{2,k}(t) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt,
\]
\[
F^n_{12} = \int_{t_0}^{t_1} H''_{2,0}(t) e^{-s_2(t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-2} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N''_{2,k}(t) e^{-s_2(t_{n-\frac{1}{2}} - t)} dt.
\]

We invoke the recursive algorithm to compute the integral for \( F^n_{11}, F^n_{12} \) to discover
\[
F^n_{11} = e^{-s_1} \left[ \int_{t_0}^{t_1} H'_{2,0}(t) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-3} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N'_{2,k}(t) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt \right] \\
+ \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} N'_{2,n-2}(t) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt \\
= e^{-s_1} F^n_{11} - 1 + B^n_{1,1} \delta(t_{u_{n-\frac{1}{2}}} - t) + B^n_{1,l_1} (\delta(t_{u_{n-\frac{1}{2}}} - t) - \delta(t_{u_{n-\frac{3}{2}}} - t)), \quad n = 2, 3, \ldots,
\]
\[
F^n_{12} = e^{-s_2} \left[ \int_{t_0}^{t_1} H'_{2,0}(t) e^{-s_2(t_{n-\frac{1}{2}} - t)} dt + \sum_{k=1}^{n-3} \int_{t_{k-\frac{1}{2}}}^{t_{k+\frac{1}{2}}} N'_{2,k}(t) e^{-s_2(t_{n-\frac{1}{2}} - t)} dt \right] \\
+ \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} N''_{2,n-2}(t) e^{-s_2(t_{n-\frac{1}{2}} - t)} dt \\
= e^{-s_2} F^n_{12} - 1 + B^n_{2,1} (\delta(t_{u_{n-\frac{1}{2}}} - t) - \delta(t_{u_{n-\frac{3}{2}}} - t)), \quad n = 2, 3, \ldots,
\]
where
\[
\delta(t_{u_{n-\frac{1}{2}}} - t) = \begin{cases} 0, & F^n_{11} = 0, \quad F^n_{12} = 0, \quad B^n_{1,1} = \int_{t_0}^{t_1} e^{-s_1(t_{n-\frac{1}{2}} - t)} dt, \\
B^n_{1,1} = \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} e^{-s_1(t_{n-\frac{1}{2}} - t)} dt, \quad n \geq 3, \quad 1 \leq l_1 \leq N_{1,\exp}, \quad B^n_{2,1} = \frac{2}{\tau} \int_{t_0}^{t_1} (t_{l_1}) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt, \\
B^n_{2,1} = \frac{1}{2\tau} \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} (2t_{l_n-2} - t_{l_n-3}) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt, \quad n \geq 3, \quad 1 \leq l_1 \leq N_{1,\exp}, \\
B^n_{2,1} = \frac{1}{2\tau} \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} (t_{l_n-2} - t_{l_n}) e^{-s_1(t_{n-\frac{1}{2}} - t)} dt, \quad B^n_{2,2} = \frac{1}{\tau} \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{3}{2}}} e^{-s_2(t_{n-\frac{1}{2}} - t)} dt, \quad B^n_{2,2} = \frac{1}{\tau} \int_{t_{n-\frac{3}{2}}}^{t_{n-\frac{1}{2}}} e^{-s_2(t_{n-\frac{1}{2}} - t)} dt, \quad n \geq 3, \quad 1 \leq l_2 \leq N_{2,\exp}.
\]

In the first term on the right side of (5.2), we can calculate it by using (2.1)
\[
\sum_{i=1}^{M_1} p_i \left( \sum_{l_1=1}^{N_{1,\exp}} \frac{\omega_{l_1} u_{l_1}^0}{\Gamma(1-\alpha_i) e^{-s_1(t_{n-\frac{1}{2}} - l_1)}} \int_{l_1}^{t_1} \frac{H''_{2,0}(t)}{\Gamma(1-\alpha_i)} \left( \sum_{l_2=1}^{N_{2,\exp}} \frac{\omega_{l_2} u_{l_2}^0}{s_{l_2}} e^{-s_2(t_{n-\frac{1}{2}} - l_2)} dt \right) dt \right)
\]
\[\begin{align*}
&\sum_{i=1}^{N_{\text{exp}}} \omega_i e^{-s_i t} \frac{\partial^n}{\partial \tau^n} \delta_i u_{t=\frac{1}{2}}^n - \frac{N''_{2,i} (t)}{\Gamma(1-\alpha_i)} \sum_{k=1}^{n-2} \frac{\omega_{k,i} e^{-s_k (t_{n-1} - \frac{1}{2} - t)}}{s_{k,i}} dt \\
&+ \frac{\tau^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \delta_i u_{t=\frac{1}{2}}^{n-\frac{3}{2}} + \frac{N''_{2,i} (t)}{\Gamma(2-\alpha_i)} \int_{t_{n-\frac{1}{2}}}^{t_{n-\frac{1}{2}} + (t_{n-1} - t)^{1-\alpha_i}} dt \\
&= \sum_{i=1}^{M_1} p_i \left\{ \tilde{A}_0^{(n, i)} u^0_t \left( \frac{\partial^n}{\partial \tau^n} \delta_i u_{t=0}^n - \frac{\omega_0 e^{-s_0 t}}{s_0} \right) + \sum_{k=1}^{n-2} \frac{\omega_k e^{-s_k t}}{s_k} \left( \delta_i u_{t=\frac{1}{2}}^{k+\frac{1}{2}} - \delta_i u_{t=\frac{1}{2}}^{\frac{k}{2}} \right) + \frac{\tau^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \delta_i u_{t=\frac{1}{2}}^{n-\frac{3}{2}} \right\},
\end{align*}\]

where
\[\begin{align*}
\tilde{A}_0^{(n, i)} &= -\sum_{l=1}^{N_{\text{exp}}} \frac{\omega_l e^{-s_l t}}{s_l} \\
\tilde{a}_k^{(n, i)} &= \begin{cases} \\
- \frac{2}{\tau} \int_{t_0}^{t_{k+1}} \sum_{l=1}^{N_{\text{exp}}} \frac{\omega_l e^{-s_l (t_{n-1} - t)}}{s_l} dt, & k = 0, \\
- \frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+1}} \sum_{l=1}^{N_{\text{exp}}} \frac{\omega_l e^{-t_l (t_{n-1} - t)}}{s_l} dt, & k = 1, \ldots, n-2, \\
\tilde{a}_{n-1}^{(n, i)}, & k = n-1.
\end{cases}
\end{align*}\]

Combining the definition of (2.2) and integrating it by parts, then
\[\begin{align*}
\sum_{j=1}^{M_2} F_j \tilde{B}_j u(t_{n-\frac{1}{2}}) &= \frac{2}{\Gamma(2-\beta_j)} \int_{t_0}^{t_{n-\frac{1}{2}}} \frac{\partial^n}{\partial \tau^n} \delta_i u_{t=\frac{1}{2}}^n \frac{1}{\Gamma(1-\beta_j)} \sum_{l=1}^{N_{\text{exp}}} \omega_{l,i} e^{-s_{l,i} \left( t_{n-\frac{1}{2}} - t \right)} + \frac{\tilde{b}_{n-1}^{(n, \beta_j)}}{\Gamma(2-\beta_j)} \left[ \delta_i u_{t=\frac{1}{2}}^{n-\frac{3}{2}} - \delta_i u_{t=\frac{1}{2}}^{\frac{n}{2}} \right] \\
&+ \frac{1}{\Gamma(2-\beta_j)} \sum_{k=1}^{n-1} \frac{\tau^{1-\beta_j}}{\Gamma(2-\beta_j)} \int_{t_{k-\frac{1}{2}}}^{t_{k+1}} \frac{\partial^n}{\partial \tau^n} \delta_i u_{t=\frac{1}{2}}^{k+\frac{1}{2}} - \delta_i u_{t=\frac{1}{2}}^{\frac{k}{2}} \sum_{l=1}^{N_{\text{exp}}} \omega_{l,i} e^{-s_{l,i} \left( t_{n-1} - t \right)} dt \\
&= \frac{1}{\Gamma(2-\beta_j)} \left[ \sum_{k=1}^{n-1} \tilde{b}_{n-1}^{(n, \beta_j)} \left( \delta_i u_{t=\frac{1}{2}}^{k+\frac{1}{2}} - \delta_i u_{t=\frac{1}{2}}^{\frac{k}{2}} \right) + \tilde{b}_{n-1}^{(n, \beta_j)} \left( \delta_i u_{t=\frac{1}{2}}^{n-\frac{3}{2}} - \delta_i u_{t=\frac{1}{2}}^{\frac{n}{2}} \right) \right],
\end{align*}\]

with
\[\begin{align*}
\tilde{b}_{n-1}^{(n, \beta_j)} &= \begin{cases} \\
\frac{2}{\tau} \int_{t_0}^{t_{n-\frac{1}{2}}} \sum_{l=1}^{N_{\text{exp}}} \omega_{l,i} e^{-s_{l,i} \left( t_{n-\frac{1}{2}} - t \right)} dt, & k = 0, \\
\frac{1}{\tau} \int_{t_{k-\frac{1}{2}}}^{t_{k+1}} \sum_{l=1}^{N_{\text{exp}}} \omega_{l,i} e^{-s_{l,i} \left( t_{n-1} - t \right)} dt, & k = 1, \ldots, n-2, \\
b_{n-1}^{(n, \beta_j)}, & k = n-1.
\end{cases}
\end{align*}\]
In the practical numerical computation, $N_{\exp}$ is usually much smaller than $N$ in [28]. It shows that the fast algorithm (5.1) effectively reduces the computation costs compared to the direct method. By the definition of $a_0^{(n,a_i)}$, $b_0^{(n,b_i)}$, and $\tilde{a}_0^{(n,a_i)}$, $\tilde{b}_0^{(n,b_i)}$, it easy to know that

$$
\begin{align*}
  a_0^{(n,a_i)} &= \tilde{a}_0^{(n,a_i)}, \\
  b_0^{(n,b_i)} &= \tilde{b}_0^{(n,b_i)}, \\
  |a_k^{(n,a_i)} - \tilde{a}_k^{(n,a_i)}| &\leq \epsilon, \quad 0 \leq k \leq n-2, \\
  |b_k^{(n,b_i)} - \tilde{b}_k^{(n,b_i)}| &\leq \epsilon, \quad 0 \leq k \leq n-2.
\end{align*}
$$

**Lemma 5.2** ([18]). Assume that $u(t) \in C^3[0,T]$ for any $1 \leq n \leq N$, it holds

$$
\begin{align*}
  \left| \sum_{i=1}^{M_1} p_i^t F^a_i u(t_{n-\frac{1}{2}}) - \sum_{j=1}^{M_2} q_j^t F^b_j u(t_{n-\frac{1}{2}}) \right| &\leq C_0 \tau^{3-a_i} + \frac{e_1 t_{n-\frac{1}{2}}}{(1-\alpha_1)} \left\{ \max_{0 \leq t \leq t_n} |u''(t)| + \max_{0 \leq t \leq t_n} |u''(t)| \right\}, \\
  \left| \sum_{i=1}^{M_1} p_i^t F^a_i u(t_{n-\frac{1}{2}}) - \sum_{j=1}^{M_2} q_j^t F^b_j u(t_{n-\frac{1}{2}}) \right| &\leq C_1 \tau^{3-b_j} + \frac{e_1 t_{n-\frac{1}{2}}}{(2-\beta_j)} \max_{0 \leq t \leq t_n} |u''(t)|.
\end{align*}
$$

We can find $u^n \in H^1_0(\Omega)$, for any $v \in H^1_0(\Omega)$, then

$$
\begin{align*}
  (\delta_t u^{n-\frac{1}{2}}, v) + \sum_{i=1}^{M_1} p_i^t (F^a_i u^{n-\frac{1}{2}}, v) + \sum_{j=1}^{M_2} q_j^t (F^b_j u^{n-\frac{1}{2}}, v) \\
  = -B(u^{n-\frac{1}{2}}, v) + (f^{n-\frac{1}{2}}, v) + (E_{2,n}, v),
\end{align*}
$$

where

$$
|E_{2,n}| \leq C(\tau^{\min(3-a_i,3-b_j)} + h^2 + \epsilon).
$$

We can seek $u^n_h \in V_h$, $(n = 1, 2, \cdots, N)$, for any $v^n_h \in V_h$, fast numerical scheme for the (1.1)-(1.2b) is written as follows:

$$
\begin{align*}
  (\delta_t u^{n-\frac{1}{2}}, v_h) + \sum_{i=1}^{M_1} p_i^t (F^a_i u^{n-\frac{1}{2}}, v_h) + \sum_{j=1}^{M_2} q_j^t (F^b_j u^{n-\frac{1}{2}}, v_h) + B(u^{n-\frac{1}{2}}, v_h) &= (f^{n-\frac{1}{2}}, v_h), \\
  u^0_h &= u_0(x), \quad u^0_{h0} = u_1(x), \quad x \in \Omega.
\end{align*}
$$

**6 Numerical experiments**

In this section, we illustrate the effectiveness of the numerical scheme by two examples and verify our theoretical results. In order to calculate the error, next we give the definition of the $L_2$ norm errors between the exact solutions and the numerical solutions

$$
E_{\beta,n}(h, \tau) = \max_{1 \leq n \leq N} \|e^n\|,
$$
the temporal convergence order $Rate_τ$ and the spatial convergence order $Rate_h$ can be expressed as

$$Rate_τ = \log_2 \left( \frac{E_2(h,2τ)}{E_2(h,τ)} \right), \quad Rate_h = \log_2 \left( \frac{E_2(2h,τ)}{E_2(h,τ)} \right).$$

**Example 6.1.** Without loss of generality, we add a force term $f(x,t)$ on the right hand side of Eq. (2.9), we consider the following equation

$$\partial_t u(x,t) + \partial_x^α u(x,t) + \partial_t^β u(x,t) = \partial_x^2 u(x,t) + f(x,t), \quad (x,t) \in \Omega \times (0,T], \quad (6.1)$$

with the initial conditions and boundary conditions as follows

$$u(x,0) = 0, \quad u_t(x,t) = 3t^2 \sin(πx), \quad (x,t) \in \Omega \times (0,T],$$

$$u(0,t) = u(2,t) = 0, \quad t \in (0,T],$$

where $\Omega = [0,1], \quad T = 1$, the exact solution $u(x,t) = t^3 \sin(πx)$, Then, we can calculate the source item of the equation is

$$f(x,t) = \left[ \frac{6t^{3-α}}{Γ(4-α)} + \frac{6t^{3-β}}{Γ(4-β)} + π^2 t^3 + 3t^2 \right] \sin(πx).$$

**Example 6.2.** In this example, we solve a multi-term fractional mixed diffusion and diffusion-wave equation with the initial value is not 0

$$\begin{align*}
\partial_t u(x,t) + \partial_x^α u(x,t) + \partial_t^β u(x,t) & = \partial_x^2 u(x,t) + f(x,t), 
0 \leq x \leq 2\pi, \quad 0 < t \leq 1, 
\quad (6.2a) \\
u(x,0) = \sin x, \quad u_t(x,t) & = (3t^2 + 6t) \sin(πx), 
0 \leq x \leq 2\pi, \quad 0 < t \leq 1, 
\quad (6.2b) \\
u(0,t) = 0, \quad u(2π,t) & = 0, 
0 < t \leq 1, 
\quad (6.2c)
\end{align*}$$

the exact solution

$$u(x,t) = (t^3 + 3t^2 + 1) \sin(πx).$$

Thus, we can get the source item

$$f(x,t) = \left[ \frac{6t^{3-α}}{Γ(4-α)} + \frac{6t^{2-α}}{Γ(3-α)} + \frac{6t^{3-β}}{Γ(4-β)} + \frac{6t^{2-β}}{Γ(3-β)} + π^2 (t^3 + 3t^2 + 1) + 3t^2 + 6t \right] \sin(πx).$$

We define the tolerance error $ε = 10^{-12}$ and the cut-off time $δ = 10^{-12}$ in the fast numerical scheme (FS). In addition, to check the efficiency of two schemes, we compared it with the direct numerical scheme (DS). In Table 1 and Table 3, we can directly observe the time convergence order is $Rate_τ = 3-β$, which is consistent with the theory.

In Table 2 and Table 4, the spatial convergence rate of the proposed algorithm was measured through Example 6.1 and Example 6.2. The step sizes in time remain small enough and fixed $τ = 1/1000$, by taking different steps in space, Table 2 and Table 4 show the second-order convergence of the two schemes in space. With the large $N$, we can observe from that the fast numerical scheme takes less CPU time (in seconds) than the direct algorithm with the same accuracy.
### Table 1: Numerical convergence orders in temporal direction and elapsed CPU time with different \((\alpha, \beta)\).

| \((\alpha, \beta)\) |  \(h\)  | Direct Scheme | | Fast Scheme | |
|---------------------|---------|---------------|----------------|---------------|
| \((0.3, 1.3)\)      | 1/160   | 3.987e-4      | 0.60           | 3.987e-4      | 0.60          |
|                     | 1/320   | 1.2183e-4     | 3.18           | 1.2183e-4     | 3.18          |
|                     | 1/640   | 3.7303e-5     | 7.26           | 3.7303e-5     | 7.26          |
|                     | 1/1280  | 1.1408e-5     | 19.85          | 1.1408e-5     | 19.85         |
|                     | 1/2560  | 3.4974e-6     | 91.64          | 3.4980e-6     | 91.64         |
| \((0.3, 1.5)\)      | 1/160   | 1.0111e-3     | 0.53           | 1.0111e-3     | 0.53          |
|                     | 1/320   | 3.5214e-4     | 2.58           | 3.5214e-4     | 2.58          |
|                     | 1/640   | 1.2360e-4     | 6.77           | 1.2360e-4     | 6.77          |
|                     | 1/1280  | 4.4116e-5     | 14.18          | 4.4116e-5     | 14.18         |
|                     | 1/2560  | 1.5558e-5     | 37.50          | 1.5558e-5     | 37.50         |
| \((0.3, 1.7)\)      | 1/160   | 2.3895e-3     | 0.23           | 2.3895e-3     | 0.23          |
|                     | 1/320   | 1.0110e-3     | 1.40           | 1.0110e-3     | 1.40          |
|                     | 1/640   | 4.1571e-4     | 6.08           | 4.1571e-4     | 6.08          |
|                     | 1/1280  | 1.6905e-4     | 12.48          | 1.6905e-4     | 12.48         |
|                     | 1/2560  | 6.8392e-5     | 27.80          | 6.8392e-5     | 27.80         |
| \((0.9, 1.3)\)      | 1/160   | 3.7803e-4     | 1.13           | 3.7803e-4     | 1.13          |
|                     | 1/320   | 1.1555e-3     | 3.59           | 1.1555e-3     | 3.59          |
|                     | 1/640   | 4.3591e-4     | 8.59           | 4.3591e-4     | 8.59          |
|                     | 1/1280  | 1.0826e-4     | 22.34          | 1.0826e-4     | 22.34         |
|                     | 1/2560  | 3.3194e-5     | 27.80          | 3.3194e-5     | 27.80         |
| \((0.9, 1.5)\)      | 1/160   | 9.6433e-4     | 1.58           | 9.6433e-4     | 1.58          |
|                     | 1/320   | 3.3590e-4     | 3.67           | 3.3590e-4     | 3.67          |
|                     | 1/640   | 1.1792e-4     | 7.57           | 1.1792e-4     | 7.57          |
|                     | 1/1280  | 4.2093e-4     | 16.44          | 4.2093e-4     | 16.44         |
|                     | 1/2560  | 1.4846e-5     | 43.22          | 1.4846e-5     | 43.22         |
| \((0.9, 1.7)\)      | 1/160   | 2.3008e-3     | 0.53           | 2.3008e-3     | 0.53          |
|                     | 1/320   | 9.7340e-4     | 1.36           | 9.7340e-4     | 1.36          |
|                     | 1/640   | 4.0027e-4     | 7.01           | 4.0027e-4     | 7.01          |
|                     | 1/1280  | 1.6277e-4     | 14.88          | 1.6277e-4     | 14.88         |
|                     | 1/2560  | 2.9781e-5     | 32.20          | 2.9781e-5     | 32.20         |

**Example 6.3.** We will present an example for solving the two-dimensional multi-term time-fractional mixed diffusion and diffusion wave equation

\[
\begin{align*}
\partial_t u(x,y,t) + \partial_x^\alpha u(x,y,t) + \partial_y^\beta u(x,y,t) &= \Delta u(x,y,t) + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,1), \\
u(x,y,0) &= 0, \quad u_t(x,y,0) = 0, \quad (x,y) \in (0,1) \times (0,1), \\
u(0,y,t) &= 0, \quad u(1,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,1,t) = 0,
\end{align*}
\]
Table 2: Numerical convergence orders in special direction and elapsed CPU time with $N=1000$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$h$</th>
<th>$\alpha=0.3$</th>
<th>$\alpha=0.9$</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>CPU(s)</td>
<td>CPU(s)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_2(h,\tau)$</td>
<td>$Rate_h$</td>
</tr>
<tr>
<td>1.3</td>
<td>1/16</td>
<td>8.6500e-3</td>
<td>6.66</td>
</tr>
<tr>
<td></td>
<td>1/32</td>
<td>2.1497e-3</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>1/64</td>
<td>5.3663e-4</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>1/128</td>
<td>1.3408e-4</td>
<td>2.00</td>
</tr>
<tr>
<td>1.5</td>
<td>1/16</td>
<td>8.3022e-3</td>
<td>6.17</td>
</tr>
<tr>
<td></td>
<td>1/32</td>
<td>2.0596e-3</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>1/64</td>
<td>5.1197e-4</td>
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<td>1/128</td>
<td>1.2591e-4</td>
<td>2.02</td>
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<tr>
<td>1.7</td>
<td>1/16</td>
<td>7.8885e-3</td>
<td>7.33</td>
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<tr>
<td></td>
<td>1/32</td>
<td>1.9432e-3</td>
<td>7.73</td>
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<td>1/64</td>
<td>4.7155e-4</td>
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<tr>
<td></td>
<td>1/128</td>
<td>1.0589e-4</td>
<td>32.48</td>
</tr>
</tbody>
</table>

where $\Omega=(0,1)^2$, $t \in (0,1]$, $0 < \alpha < 1$, $1 < \beta < 2$, and the exact solution is

$$u = 100t^{2+\beta}\sin(\pi x)\sin(\pi y),$$

the source item

$$f(x,y,t) = 100\left[\frac{\Gamma(3+\beta)\Gamma(2+\beta-\alpha)}{\Gamma(3+\beta-\alpha)} + \frac{\Gamma(3+\beta)\Gamma(2)}{\Gamma(3)} t^{2+\beta} + 2\pi^2 t^{2+\beta} + (2+\beta) t^{1+\beta}\right] \sin(\pi x)\sin(\pi y).$$

Let $h_x = h_y = \lceil \tau^{(3-\beta)/2} \rceil$, $\tau = 1/1000$. Then we test the numerical accuracy and CPU time by using the direct numerical scheme (2.9) and the fast numerical scheme (5.4), as shown in Table 5. From it, the convergence orders are of the $3-\beta$ order in time, which is same as the Theorem 4.1. In particular, if $N$ grows, the computational complexity of FS (5.4) is more pronounced than that of DS (2.9).

Regarding the accuracy test of spatial orientation, we fixed the tolerance error $\epsilon = 10^{-12}$, the cut-off time $\delta = 10^{-12}$, and took different spatial steps in the DS and FS. Besides illustrates the convergence order of both schemes are second order, which is consistent with the theoretical results. Table 6 also shows the computational cost of the DS is largely higher than the FS with the large $N$.

### 7 Conclusions

In this paper, we propose the direct numerical scheme and the fast numerical scheme based on H2N2 interpolation for multi-term time fractional mixed sub-diffusion and diffusion wave equations with order $\alpha \in (0,1)$, $\beta \in (1,2)$. The convergence order of the direct
Table 3: Numerical convergence orders in temporal direction and elapsed CPU time with different $(\alpha, \beta)$.

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$h$</th>
<th>Direct Scheme</th>
<th></th>
<th>Fast Scheme</th>
<th></th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td>$E_2(h, \tau)$</td>
<td>Rate</td>
<td>CPU(s)</td>
<td>$E_2(h, \tau)$</td>
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<td>2.3517e-3 *</td>
<td>1.99</td>
<td>2.3517e-3 *</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td>1/320</td>
<td>7.2308e-4 1.7015 4.48</td>
<td>7.2308e-4 1.7015</td>
<td>3.78</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/640</td>
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<td>2.2249e-4 1.7004</td>
<td>7.80</td>
<td></td>
</tr>
<tr>
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<td>0.63</td>
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<td>1/160</td>
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<td>1.03</td>
<td>2.2691e-3 *</td>
<td>0.92</td>
</tr>
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<td>1/320</td>
<td>6.9762e-4 1.7016 4.29</td>
<td>6.9762e-4 1.7016</td>
<td>3.52</td>
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</tr>
<tr>
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<td>1/640</td>
<td>2.1465e-4 1.7005 26.12</td>
<td>2.1465e-4 1.7005</td>
<td>7.47</td>
<td></td>
</tr>
<tr>
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<td>6.5904e-5 1.7035 345.95</td>
<td>6.5904e-5 1.7035</td>
<td>21.12</td>
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<td>2.0269e-5 1.7011 14326.34</td>
<td>2.0269e-5 1.7011</td>
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<td>(0.9,1.5)</td>
<td>1/160</td>
<td>6.0795e-3 *</td>
<td>0.73</td>
<td>6.0795e-3 *</td>
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<td>2.1258e-3 1.5160 3.08</td>
<td>2.1258e-3 1.5160</td>
<td>2.90</td>
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<tr>
<td></td>
<td>1/640</td>
<td>7.4813e-4 1.5066 13.69</td>
<td>7.4813e-4 1.5066</td>
<td>6.07</td>
<td></td>
</tr>
<tr>
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<td>1/1280</td>
<td>2.6736e-4 1.4845 83.45</td>
<td>2.6736e-4 1.4845</td>
<td>14.20</td>
<td></td>
</tr>
<tr>
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<td>1/2560</td>
<td>9.4400e-5 1.5019 781.31</td>
<td>9.4400e-5 1.5019</td>
<td>40.91</td>
<td></td>
</tr>
<tr>
<td>(0.9,1.7)</td>
<td>1/160</td>
<td>1.5176e-2 *</td>
<td>0.67</td>
<td>1.5176e-2 *</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>1/320</td>
<td>6.4236e-3 1.2403 1.51</td>
<td>6.4236e-3 1.2403</td>
<td>1.47</td>
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<tr>
<td></td>
<td>1/640</td>
<td>2.6432e-3 1.2811 7.73</td>
<td>2.6432e-3 1.2811</td>
<td>6.53</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/1280</td>
<td>1.0756e-3 1.2972 43.53</td>
<td>1.0756e-3 1.2972</td>
<td>14.11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/2560</td>
<td>4.3541e-4 1.3047 233.79</td>
<td>4.3541e-4 1.3047</td>
<td>28.99</td>
<td></td>
</tr>
</tbody>
</table>

The numerical scheme is $3 - \beta$, which is only dependent on the $\beta$ in the time direction. Furthermore, numerical examples illustrated that the numerical results agree with the theoretical conclusions, while showing that when $N$ is large, the fast numerical scheme can significantly reduce the computational quantity cost without loss of accuracy.
Table 4: Numerical convergence orders in special direction and elapsed CPU time with \( N = 1000 \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( h )</th>
<th>( a = 0.3 )</th>
<th>CPU(s)</th>
<th>( a = 0.9 )</th>
<th>CPU(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E_2(h, \tau) )</td>
<td>Rate</td>
<td>DS</td>
<td>FS</td>
<td>( E_2(h, \tau) )</td>
</tr>
<tr>
<td>1.3</td>
<td>1/16</td>
<td>5.1142e-2</td>
<td>*</td>
<td>6.98</td>
<td>6.13</td>
</tr>
<tr>
<td>1/32</td>
<td>1.2752e-2</td>
<td>2.00</td>
<td>7.29</td>
<td>6.62</td>
<td>1.2305e-2</td>
</tr>
<tr>
<td>1/64</td>
<td>3.1846e-3</td>
<td>2.00</td>
<td>12.38</td>
<td>7.24</td>
<td>3.0743e-3</td>
</tr>
<tr>
<td>1/128</td>
<td>7.9688e-4</td>
<td>2.00</td>
<td>178.09</td>
<td>10.48</td>
<td>1.9259e-4</td>
</tr>
<tr>
<td>1/256</td>
<td>1.9963e-4</td>
<td>2.00</td>
<td>178.09</td>
<td>10.48</td>
<td>1.9259e-4</td>
</tr>
<tr>
<td>1.5</td>
<td>1/16</td>
<td>5.0109e-2</td>
<td>*</td>
<td>6.14</td>
<td>5.98</td>
</tr>
<tr>
<td>1/32</td>
<td>1.2487e-2</td>
<td>2.00</td>
<td>7.80</td>
<td>6.23</td>
<td>1.2053e-2</td>
</tr>
<tr>
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<td>3.1175e-3</td>
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<td>12.89</td>
<td>7.20</td>
<td>3.0087e-3</td>
</tr>
<tr>
<td>1/128</td>
<td>7.7735e-4</td>
<td>2.00</td>
<td>178.09</td>
<td>10.48</td>
<td>7.5004e-4</td>
</tr>
<tr>
<td>1/256</td>
<td>7.7735e-4</td>
<td>2.00</td>
<td>178.09</td>
<td>10.48</td>
<td>1.8555e-4</td>
</tr>
<tr>
<td>1.7</td>
<td>1/16</td>
<td>4.8823e-2</td>
<td>*</td>
<td>6.35</td>
<td>5.93</td>
</tr>
<tr>
<td>1/32</td>
<td>1.2146e-2</td>
<td>2.01</td>
<td>6.89</td>
<td>6.62</td>
<td>1.1739e-2</td>
</tr>
<tr>
<td>1/64</td>
<td>3.0181e-3</td>
<td>2.01</td>
<td>12.53</td>
<td>7.40</td>
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<td>46.00</td>
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<td>7.1322e-4</td>
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<tr>
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<td>1.6955e-4</td>
<td>2.12</td>
<td>172.39</td>
<td>10.52</td>
<td>1.6327e-4</td>
</tr>
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</table>

Table 5: Comparison of the temporal convergence order and elapsed CPU time of DS and FS for Example 5.2 with different \((\alpha, \beta)\), \( h = \lceil N^{3/2} \rceil^{-1} \), \( \epsilon = 10^{-12} \).

<table>
<thead>
<tr>
<th>((\alpha, \beta))</th>
<th>( h )</th>
<th>Direct Scheme</th>
<th>CPU(s)</th>
<th>Rate</th>
<th>Fast Scheme</th>
<th>CPU(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0.3, 1.3) )</td>
<td>1/80</td>
<td>2.5005e-2</td>
<td>*</td>
<td>6.12</td>
<td>2.5003e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>1/160</td>
<td>7.4908e-3</td>
<td>1.7390</td>
<td>58.74</td>
<td>7.4905e-3</td>
<td>1.7383</td>
</tr>
<tr>
<td></td>
<td>1/320</td>
<td>2.2313e-3</td>
<td>1.7472</td>
<td>730.75</td>
<td>2.2313e-3</td>
<td>1.7472</td>
</tr>
<tr>
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<td>1/640</td>
<td>6.6899e-4</td>
<td>1.7378</td>
<td>10562.27</td>
<td>6.6879e-4</td>
<td>1.7383</td>
</tr>
<tr>
<td>( (0.3, 1.5) )</td>
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<td>4.8088e-2</td>
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<td>2.88</td>
<td>4.8105e-2</td>
<td>*</td>
</tr>
<tr>
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<td>1.6756e-2</td>
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<td>1.6760e-2</td>
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<tr>
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<td>5.7370e-3</td>
<td>1.5463</td>
<td>137.70</td>
<td>5.7380e-3</td>
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<tr>
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<td>1.9899e-3</td>
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<td>( (0.3, 1.7) )</td>
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<td>*</td>
<td>1.73</td>
<td>8.7437e-2</td>
<td>*</td>
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<tr>
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<td>1.5042e-2</td>
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<td>34.04</td>
<td>1.5047e-2</td>
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<td>253.04</td>
<td>6.1765e-3</td>
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<tr>
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<td>2.4081e-2</td>
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<td>4.6415e-2</td>
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<td>1.9256e-3</td>
<td>1.5286</td>
<td>1483.86</td>
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<td>2.13</td>
<td>8.5651e-2</td>
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<td>1.2813</td>
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Table 6: Comparison of the temporal convergence order and elapsed CPU time of DS and FS for Example 5.2 with different \((\alpha, \beta)\), \(N = 1000\), \(\epsilon = 10^{-12}\).

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<th>(h)</th>
<th>(\alpha = 0.3)</th>
<th>(\alpha = 0.9)</th>
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<td>(Rate_h)</td>
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</tr>
<tr>
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<tr>
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<td>1/16</td>
<td>1.4088e-1</td>
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<td>1/32</td>
<td>3.5073e-2</td>
<td>2.01</td>
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<tr>
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<td>1/64</td>
<td>8.6734e-3</td>
<td>2.02</td>
</tr>
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<td>1.7</td>
<td>1/16</td>
<td>1.3304e-1</td>
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<td>2.13</td>
</tr>
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Acknowledgements

The authors are grateful to the anonymous referees for their constructive comments. This work is supported by the State Key Program of the National Natural Science Foundation of China (No. 11931003), and by the Grants (Nos. 41974133 and 12271233).

References


