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A HIGH ORDER SCHEME FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH THE CAPUTO-HADAMARD DERIVATIVE*

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Abstract

In this paper, we consider numerical solutions of the fractional diffusion equation with the α order time fractional derivative defined in the Caputo-Hadamard sense. A high order time-stepping scheme is constructed, analyzed, and numerically validated. The contribution of the paper is twofold: 1) regularity of the solution to the underlying equation is investigated, 2) a rigorous stability and convergence analysis for the proposed scheme is performed, which shows that the proposed scheme is $3+\alpha$ order accurate. Several numerical examples are provided to verify the theoretical statement.

Mathematics subject classification: 34A08, 65L05, 65L20.
Key words: Caputo-Hadamard derivative, Fractional differential equations, High order scheme, Stability and convergence analysis.

1. Introduction

Fractional calculus has been paid much attention in recent decades, due to, on one side, its well-recognized applicability in science and engineering, and on the other side, its attractive complementary of the integer order calculus in pure mathematics, see, e.g. [7, 19, 23, 24] and the references cited therein.

Up to now, there exist several kinds of fractional integrals and derivatives, like Riemann-Liouville, Caputo, Riesz, C-Fabrizio and Hadamard integrals and derivatives. The first three have been widely studied in the past decades. Actually, the Hadamard derivative which was proposed early in 1892 [15] is also very worthy of in-depth study, since it has been extensively used in mechanics and engineering, e.g. both planar and three-dimensional elasticities, or the fracture analysis [2] and the Lomnitz logarithmic creep law of special substances, e.g. igneous rock [11, 22]. Moreover, ultraslow diffusion appears in various applications [6, 9]. For instance, vacancy-mediated tracer flow and particle movements in certain strongly heterogeneous media may demonstrate ultraslow diffusive phenomena [4, 5, 28]. Mathematically, the mean square

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displacement of the particles in ultraslow diffusion grows logarithmically in time [16, 25, 26]. Therefore, the Hadamard fractional operators, whose kernels are defined in terms of logarithmic functions, serve as a natural choice for modeling ultraslow diffusion processes and thus attract wide attentions.

For partial differential equations involving Hadamard derivative, although the research is relatively sparse, several studies have been carried out, and we see increasing interest in this topic from both scientific and engineering communities. We mention, among others, the work [18] to develop fractional integration and differentiation in the Hadamard setting. The existence almost everywhere was established for the considered Hadamard-type fractional derivative, the semigroup and reciprocal properties for the Hadamard-type fractional derivative and integration operators were proved. The stability and logarithmic decay of the solution of Hadamard-type fractional differential equation was discussed in [20]. A logarithmic transformation reducing the Caputo-Hadamard (CH) fractional problems to their Caputo analogues was presented in [29]. The well-posedness and regularity of CH fractional stochastic differential equations were studied in [28]. Numerically, Gohar *et al.* [12, 13] and Li *et al.* [21] derived several finite difference schemes to approximate the CH fractional derivative. Very recently, Fan *et al.* [10] derived some new numerical formulas, called as L1-2 formula, L2-1σ formula and H2N2 formula, for discretization of the CH fractional derivative. A second-order scheme with nonuniform time meshes for CH fractional sub-diffusion equations with initial singularity is investigated in [27]. The predictor-corrector numerical method for solving CH fractional differential equations with the graded meshes was considered in [14]. However, to the best of our knowledge, the convergence order of the existing schemes is no more than three.

The aim of this work is to propose and analyze an efficient time stepping scheme having the convergence order more than three for the CH fractional differential equations. The proposed scheme is based on the so-called block-by-block approach, which is a common method for the integral equations, and has been successfully applied to construct high order scheme for the Caputo fractional differential equations in [3]. Although the used idea for the scheme construction is the same as [3], the convergence analysis is a completely different skill from the method used in [3]. The rest of this paper is organized as follows. In Section 2, we present some regularity properties of the solution for the considered problem. In Section 3, we describe the detailed construction of the high order scheme for the Hadamard FDEs under consideration. Then in Section 4, we derive an estimate for the local errors through a series of lemmas. The stability and convergence analysis is given in Section 5. Finally, several numerical examples are provided in Section 6 to support the theoretical statement. Some concluding remarks are given in the final section.

2. Problem and Regularity Properties

We are interested in the following CH fractional equation with $0 < \alpha < 1$:

$$\begin{aligned} &{}^{CH}D^{\alpha}_{a,t}u(t) = f\big(t, u(t)\big), \quad 0 < a < t,\\ &u(a) = u_a, \end{aligned} \tag{2.1}$$

where $f(t, u)$ is a nonlinear function with respect to u, and the initial value u_a is given. The notation ${}^{CH}D^{\alpha}_{a,t}$ is the CH fractional derivative of order α defined by [2, 17],

$$ {}^{CH}D^{\alpha}_{a}v(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{-\alpha} \delta v(s) \frac{\mathrm{d}s}{s}, $$

where, for simplifying the notation, we use the notation $\delta v := s\,\mathrm{d}v/\mathrm{d}s$, which is called the δ-derivative of v. It is known [1] that the problem (2.1) is equivalent to the Volterra integral equation as follows:

$$u(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s}. \tag{2.2}$$

Throughout the paper we assume that the function $f(t,u)$ is continuous and satisfies the Lipschitz condition with respect to the second variable u on a suitable set G, i.e. there exists a Lipschitz constant $L > 0$ such that, for all (t, u_1) and $(t, u_2) \in G$, we have

$$|f(t,u_1) - f(t,u_2)| \le L\,|u_1 - u_2|\,. \tag{2.3}$$

Following [13], there exists a $T > a$, such that the problem (2.1) admits a unique solution on the interval $(a, T]$.

In order to give some regularity properties of the solutions to (2.1), we introduce the following logarithmic transformation:

$$\bar{t} := \log \frac{t}{a}, \tag{2.4}$$

which is a one-to-one mapping from $[a,T]$ to $[0,\bar{T}]$ with $\bar{T} = \log(T/a)$. For any function $g(t)$ on $[a,T]$, one could define $\bar{g}(\bar{t})$ on $[0,\bar{T}]$ by

$$\bar{g}(\bar{t}\,) := g\big(ae^{\bar{t}}\big) = g(t).$$

It can be directly checked, see also [29], that

$$\frac{\mathrm{d}^n}{\mathrm{d}\bar{t}^n}\bar{g}(\bar{t}\,) = \delta^n g(t), \quad n \in \mathbb{N}. \tag{2.5}$$

Using the logarithmic transformation (2.4) and the transformation property (2.5), we see that the Caputo-Hadamard fractional operator is linked to the Caputo fractional operator as follows:

$${}^{CH}D^{\alpha}_{a,t}\; g(t) = {}^{C}D^{\alpha}_{0,\bar{t}}\; \bar{g}(\bar{t}\,),$$

where the Caputo fractional derivative ${}^{C}D^{\alpha}_{0,s}v$ is defined by

$${}^{C}D^{\alpha}_{0,s}v(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-z)^{-\alpha} v'(z) dz.$$

Consequently, the initial value problem (2.1) is equivalent to the initial value problem with Caputo derivative

$$\begin{aligned} &{}^{C}D^{\alpha}_{0,\bar{t}}\bar{u}(\bar{t}\,) = \bar{f}\big(\bar{t}, \bar{u}(\bar{t}\,)\big), \quad \bar{t} > 0, \\ &\bar{u}(0) = u_a. \end{aligned} \tag{2.6}$$

It has been well known that the solution to (2.6) can be expressed as

$$\bar{u}(\bar{t}) = u_a + \frac{1}{\Gamma(\alpha)} \int_0^{\bar{t}} (\bar{t} - s)^{\alpha-1} \bar{f}\big(s, \bar{u}(s)\big) \mathrm{d}s.$$

Then, in virtue of Diethelm's regularity results [7,8], we have the following proposition.

Proposition 2.1. *(1) Assume that* $\delta^3 f \in C(G)$. *Then, there exists a function* $\psi(t)$ *with* $\delta^2\psi(t) \in C[a,T]$ *such that the solution* $u(t)$ *of the initial value problem* (2.1) *can be expressed in the form*

$$u(t) = \psi(t) + \sum_{l=1}^{\lceil 2/\alpha\rceil -1} \lambda_l \left(\log\frac{t}{a}\right)^{\alpha l} + \sum_{l=1}^{\lceil 1/\alpha\rceil -1} \mu_l \left(\log\frac{t}{a}\right)^{1+\alpha l},$$

where $\lambda_l \in \mathbb{R}, l = 1,\dots,\lceil 2/\alpha\rceil - 1, \mu_l \in \mathbb{R}, l = 1,\dots,\lceil 1/\alpha\rceil - 1$.

(2) If $\delta^k f \in C(G), k \in \mathbb{N}$, *then* $u(t) \in C[a,T]$ *and* $\delta^k u(t) \in C(a,T]$. *Moreover, for* $\nu = 1, 2, \dots, k$, *it holds*

$$\delta^\nu u(t) = \mathcal{O}\left(\left(\log\frac{t}{a}\right)^{\alpha-\nu}\right) \quad as \quad t \to a.$$

Proof. These two results are respectively direct consequences of the regularity results established in [8, Theorem 2.1] and [7, Theorem 6.27] for the transformed solution $\bar{u}(\bar{t}\,)$ and then inversely transforming back to $u(t)$. □

We see from Proposition 2.1 that a smooth function f on the right-hand side of the differential equation will necessarily lead to a non-smooth behaviour of the solution due to the singularity at the starting point a. However high order numerical schemes usually require regular enough f and u. In the present work we will assume $\delta^4 f(\cdot, u(\cdot)) \in C[a,T]$, i.e. the fourth order δ-differentiability of the function f and the solution u simultaneously. To see how this assumption is justifiable, we consider a class of the function f for which the required regularity becomes true.

Proposition 2.2. *If* $\delta^k f(\cdot, u(\cdot)) \in C[a,T], k \in \mathbb{N}$, *then* $\delta^k u(t) \in C(a,T]$. *Furthermore,* $\delta^k u(t) \in C[a,T]$ *if and only if* $f(\cdot, u(\cdot))$ *has a k-fold zero at the starting point a.*

Before we come to the proof of Proposition 2.2, we first prove an useful property of Hadamard integral operator. That is, if $g(t) \in C[a,T]$, then ${}_H I_t^\alpha g(t) \in C[a,T]$, where ${}_H I_t^\alpha$ is the Hadamard fractional integral of order α defined by

$${}_H I_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s}.$$

In fact, it has been showed in [29] that

$${}_H I_t^\alpha g(t) = I_{\bar{t}}^\alpha \bar{g}(\bar{t}\,),$$

where

$$I_{\bar{t}}^\alpha \bar{g}(\bar{t}\,) = \frac{1}{\Gamma(\alpha)} \int_0^{\bar{t}} (\bar{t} - s)^{\alpha-1} \bar{g}(s)\mathrm{d}s.$$

Note that $\bar{g}(\bar{t}\,) \in C[0, \bar{T}]$ if $g(t) \in C[a,T]$. It follows from [7, Theorem 2.5] that $I_{\bar{t}}^\alpha \bar{g}(\bar{t}\,) \in C[0,\bar{T}]$, which leads to ${}_H I_t^\alpha g(t) \in C[a,T]$.

Proof of Proposition 2.2. In view of (2.2), the solution u of (2.1) satisfies

$$u(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} z(s)\frac{\mathrm{d}s}{s}, \tag{2.7}$$

where $z(s) := f(s, u(s))$. A straightforward calculation leads

$$
\begin{aligned}
\delta u(t) = t\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{t}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} z(s)\frac{\mathrm{d}s}{s} \\
&= \frac{t}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\log\frac{t}{a}} \tau^{\alpha-1} z(te^{-\tau})\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} z(a) + \frac{t}{\Gamma(\alpha)}\int_0^{\log\frac{t}{a}} \tau^{\alpha-1}\frac{\mathrm{d}z(te^{-\tau})}{\mathrm{d}t}\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} z(a) + \frac{t}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\mathrm{d}z}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}s}{s},
\end{aligned}
\tag{2.8}
$$

where we have changed the variable of integration by letting $\tau = \log(t/s)$ (i.e. $s = te^{-\tau}$). It is apparent from $s = te^{-\tau}$ that $t\,\mathrm{d}s/\mathrm{d}t = s$, thus we obtain

$$
\begin{aligned}
\delta u(t) &= \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} z(a) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\delta z(s)\frac{\mathrm{d}s}{s} \\
&= \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} z(a) + {}_H I_t^\alpha \delta z.
\end{aligned}
\tag{2.9}
$$

Following the same idea, differentiate the above equation $k-1$ times in succession leads to

$$
\begin{aligned}
\delta^k u(t) = {}_H I_t^\alpha \delta^k z + \frac{1}{\Gamma(\alpha)}\Bigg[&\left(\log\frac{t}{a}\right)^{\alpha-1}\delta^{k-1}z(a) \\
&+ \sum_{l=2}^{k}(\alpha-1)\cdots(\alpha-l+1)\left(\log\frac{t}{a}\right)^{\alpha-l}\delta^{k-l}z(a)\Bigg],
\end{aligned}
\tag{2.10}
$$

where $\delta^0 z(a) = z(a)$ and $k \geq 2$. Under the assumption on f, the function $\delta^k z$ is continuous. Consequently, ${}_H I_t^\alpha \delta^k z \in C[a, T]$. For $\alpha < 1$, the right-hand side of (2.10), and therefore also the left-hand side, i.e. the function $\delta^k u$, is continuous on the half-open interval $(a, T]$. Furthermore, $\delta^k u$ is continuous on the closed interval $[a, T]$ if and only if $\delta^l z(a) = 0, l = 0, 1, \ldots, k-1$. $\square$

3. Numerical Scheme

The proposed scheme will be constructed based on the equivalent equation (2.2). For a given positive integer N, we divide the interval $[a, T]$ into $2N$ equal sub-intervals with size $\Delta t = (T-a)/(2N)$, and denote $t_j = a+j\Delta t, \tau_i = \log t_i - \log t_{i-1}, i = 1, \ldots, 2N, j = 0, 1, \ldots, 2N$. The numerical solution of (2.2) at the point t_j is denoted by u_j. Set $u_0 = u_a, f_j = f(t_j, u_j)$.

We first determine the approximations to $u(t)$ at t_1 and t_2. Using the quadratic interpolation [10], $f(t, u(t))$ can be approximated in the interval $[t_0, t_2]$ by means of three point $(t_0, f_0), (t_1, f_1), (t_2, f_2)$ as

$$
f\big(t, u(t)\big) \approx \sum_{i=0}^{2}\phi_{0,i}(t)f_i, \quad t \in [t_0, t_2], \tag{3.1}
$$

where $\phi_{0,i}(t), i = 0, 1, 2$, are quadratic logarithmic interpolations, defined by

$$
\phi_{0,0}(t) = \frac{\log(t/t_1)\log(t/t_2)}{\tau_1(\tau_1+\tau_2)}, \tag{3.2a}
$$

$$\phi_{0,1}(t)=\frac{\log(t/t_0)\log(t/t_2)}{-\tau_1\tau_2}, \tag{3.2b}$$

$$\phi_{0,2}(t)=\frac{\log(t/t_0)\log(t/t_1)}{\tau_2(\tau_1+\tau_2)}. \tag{3.2c}$$

Substituting (3.1) into (2.2), and integrating, we obtain

$$u(t_2)\approx u_0+\sum_{i=0}^{2}c_2^{0,i}f_i, \tag{3.3}$$

where

$$c_2^{0,i}=\frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\phi_{0,i}(s)\frac{\mathrm{d}s}{s},\quad i=0,1,2,$$

which can be exactly computed. Note that (3.3) requires the values of f (or indirectly, the values of u) at t_1 and t_2. To determine u_1, we approximate $f(t,u(t))$ on $[t_0,t_1]$ as

$$f\big(t,u(t)\big)\approx\varphi_{0,0}(t)f_0+\varphi_{0,1}(t)f_{\frac{1}{2}}+\varphi_{0,2}(t)f_1,\quad \forall\, t\in[t_0,t_1], \tag{3.4}$$

where $t_{1/2}=t_0+1/2\Delta t, f_{1/2}=f(t_{1/2},u(t_{1/2}))$, and $\varphi_{0,i}(t), i=0,1,2$, are another set of quadratic logarithmic interpolations, defined by

$$\begin{aligned}\varphi_{0,0}(t)&=\frac{\log\left(t/t_{\frac{1}{2}}\right)\log(t/t_1)}{\log\left(t_0/t_{\frac{1}{2}}\right)\log(t_0/t_1)},\\ \varphi_{0,1}(t)&=\frac{\log(t/t_0)\log(t/t_1)}{\log\left(t_{\frac{1}{2}}/t_0\right)\log\left(t_{\frac{1}{2}}/t_1\right)},\\ \varphi_{0,2}(t)&=\frac{\log(t/t_0)\log\left(t/t_{\frac{1}{2}}\right)}{\log(t_1/t_0)\log\left(t_1/t_{\frac{1}{2}}\right)}.\end{aligned} \tag{3.5}$$

Substituting (3.4) into (2.2) yields

$$u(t_1)\approx u_0+d_1^{0,0}f_0+d_1^{0,1}f_{1/2}+d_1^{0,2}f_1, \tag{3.6}$$

where

$$d_1^{0,i}=\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}\varphi_{0,i}(s)\frac{\mathrm{d}s}{s},\quad i=0,1,2.$$

The value of $f_{1/2}$ is determined according to (3.1), which leads to

$$f_{\frac{1}{2}}\approx\sum_{i=0}^{2}\omega_i f_i \tag{3.7}$$

with $\omega_i=\phi_{0,i}\left(t_{1/2}\right), i=0,1,2$. Substituting (3.7) into (3.6), we obtain

$$u(t_1)\approx u_0+d_1^{0,0}f_0+d_1^{0,1}(\omega_0f_0+\omega_1f_1+\omega_2f_2)+d_1^{0,2}f_1:=u_0+\sum_{i=0}^{2}c_1^{0,i}f_i, \tag{3.8}$$

where

$$c_1^{0,0}=d_1^{0,0}+\omega_0d_1^{0,1},\quad c_1^{0,1}=\omega_1d_1^{0,1}+d_1^{0,2},\quad c_1^{0,2}=\omega_2d_1^{0,1}.$$

This leads to a 2×2 algebraic system for the first two step solutions u_1 and u_2

$$\begin{cases} u_1 = u_0 + \sum_{i=0}^{2} c_1^{0,i} f_i, \\ u_2 = u_0 + \sum_{i=0}^{2} c_2^{0,i} f_i. \end{cases} \tag{3.9}$$

Let us now assume that the approximations u_j are known for $j = 0, 1, \ldots, 2m$, we want to derive approximations to $u(t_{2m+1})$ and $u(t_{2m+2})$. Following the above approach, we have

$$\begin{aligned} u(t_{2m+1}) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{2m+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \Bigg[\int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} \\ &\qquad + \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} \Bigg] \\ &\approx u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \big(\varphi_{0,0}(s) f_0 + \varphi_{0,1}(s) f_{\frac{1}{2}} + \varphi_{0,2}(s) f_1\big) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \big(\varphi_{k,0}(s) f_{2k-1} + \varphi_{k,1}(s) f_{2k} + \varphi_{k,2}(s) f_{2k+1}\big) \frac{\mathrm{d}s}{s}, \end{aligned} \tag{3.10}$$

where $\varphi_{0,i}(t), i = 0, 1, 2$, are defined in (3.5), and $\varphi_{k,i}, k = 1, \ldots, m, i = 0, 1, 2$, are quadratic logarithmic interpolations associated with the points $t_{2k-1}, t_{2k}, t_{2k+1}$

$$\begin{aligned} \varphi_{k,0}(t) &= \frac{\log(t/t_{2k}) \log(t/t_{2k+1})}{\tau_{2k}(\tau_{2k} + \tau_{2k+1})}, \\ \varphi_{k,1}(t) &= \frac{\log(t/t_{2k-1}) \log(t/t_{2k+1})}{-\tau_{2k}\tau_{2k+1}}, \\ \varphi_{k,2}(t) &= \frac{\log(t/t_{2k-1}) \log(t/t_{2k})}{\tau_{2k+1}(\tau_{2k} + \tau_{2k+1})}. \end{aligned} \tag{3.11}$$

Inserting (3.11) into (3.10) gives

$$\begin{aligned} u(t_{2m+1}) \approx u_0 &+ d_{2m+1}^{0,0} f_0 + d_{2m+1}^{0,1} f_{\frac{1}{2}} + d_{2m+1}^{0,2} f_1 \\ &+ \sum_{k=1}^{m} \big[c_{2m+1}^{k,0} f_{2k-1} + c_{2m+1}^{k,1} f_{2k} + c_{2m+1}^{k,2} f_{2k+1}\big], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} d_{2m+1}^{0,i} &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{0,i}(s) \frac{\mathrm{d}s}{s}, \qquad i = 0, 1, 2, \\ c_{2m+1}^{k,i} &= \frac{1}{\Gamma(\alpha)} \int_{t_{2k-1}}^{t_{2k+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k,i}(s) \frac{\mathrm{d}s}{s}, \quad i = 0, 1, 2, \quad k = 1, \ldots, m. \end{aligned}$$

Approximating $f_{1/2}$ in (3.12) by (3.7), we arrive at the following scheme for computing u_{2m+1}:

$$\begin{aligned} u_{2m+1} = u_0 &+ c_{2m+1}^{0,0} f_0 + c_{2m+1}^{0,1} f_1 + c_{2m+1}^{0,2} f_2 \\ &+ \sum_{k=1}^{m} \big[c_{2m+1}^{k,0} f_{2k-1} + c_{2m+1}^{k,1} f_{2k} + c_{2m+1}^{k,2} f_{2k+1}\big], \end{aligned} \tag{3.13}$$

where

$$c_{2m+1}^{0,0} = d_{2m+1}^{0,0} + w_0 d_{2m+1}^{0,1}, \quad c_{2m+1}^{0,1} = w_1 d_{2m+1}^{0,1} + d_{2m+1}^{0,2}, \quad c_{2m+1}^{0,2} = w_2 d_{2m+1}^{0,1}.$$

To compute u_{2m+2}, we use the following approximation:

$$\begin{aligned} u(t_{2m+2}) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{2m+2}} \left(\log \frac{t_{2m+2}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{m} \int_{t_{2k}}^{t_{2k+2}} \left(\log \frac{t_{2m+2}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} \\ &\approx u_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{m} \int_{t_{2k}}^{t_{2k+2}} \left(\log \frac{t_{2m+2}}{s}\right)^{\alpha-1} \big(\phi_{k,0}(s) f_{2k} + \phi_{k,1}(s) f_{2k+1} + \phi_{k,2}(s) f_{2k+2}\big) \frac{\mathrm{d}s}{s}, \end{aligned}$$

where $\phi_{k,i}, k = 1, \ldots, m, i = 0, 1, 2$, are the logarithmic interpolations associated with the points $t_{2k}, t_{2k+1}, t_{2k+2}$

$$\begin{aligned} \phi_{k,0}(t) &= \frac{\log(t/t_{2k+1}) \log(t/t_{2k+2})}{\tau_{2k+1}(\tau_{2k+1} + \tau_{2k+2})}, \\ \phi_{k,1}(t) &= \frac{\log(t/t_{2k}) \log(t/t_{2k+2})}{-\tau_{2k+1}\tau_{2k+2}}, \\ \phi_{k,2}(t) &= \frac{\log(t/t_{2k}) \log(t/t_{2k+1})}{\tau_{2k+2}(\tau_{2k+1} + \tau_{2k+2})}. \end{aligned} \tag{3.14}$$

As a result, we obtain the scheme at the step $2m + 2$

$$u_{2m+2} = u_0 + \sum_{k=0}^{m} \big[c_{2m+2}^{k,0} f_{2k} + c_{2m+2}^{k,1} f_{2k+1} + c_{2m+2}^{k,2} f_{2k+2}\big], \tag{3.15}$$

where

$$c_{2m+2}^{k,i} = \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} \left(\log \frac{t_{2m+2}}{s}\right)^{\alpha-1} \phi_{k,i}(s) \frac{\mathrm{d}s}{s}, \quad i = 0, 1, 2.$$

To summarize, we arrive at the following overall scheme:

$$\begin{cases} u_1 = u_0 + c_1^{0,0} f_0 + c_1^{0,1} f_1 + c_1^{0,2} f_2, \\ u_2 = u_0 + c_2^{0,0} f_0 + c_2^{0,1} f_1 + c_2^{0,2} f_2, \\ u_{2m+1} = u_0 + c_{2m+1}^{0,0} f_0 + c_{2m+1}^{0,1} f_1 + c_{2m+1}^{0,2} f_2 \\ \qquad\qquad + \sum_{k=1}^{m} \big[c_{2m+1}^{k,0} f_{2k-1} + c_{2m+1}^{k,1} f_{2k} + c_{2m+1}^{k,2} f_{2k+1}\big], \\ u_{2m+2} = u_0 + \sum_{k=0}^{m} \big[c_{2m+2}^{k,0} f_{2k} + c_{2m+2}^{k,1} f_{2k+1} + c_{2m+2}^{k,2} f_{2k+2}\big], \\ m = 1, \ldots, N-1. \end{cases} \tag{3.16}$$

4. Estimation of the Truncation Errors

We first present some lemmas which will be used later on. We hereafter denote by C a generic constant which may not be the same at different occurrences, but independent of all discretization parameters.

Lemma 4.1. *It holds*

$$\log \frac{t_1}{t_0} \le \frac{\Delta t}{t_0}, \tag{4.1}$$

and if $\Delta t < (\sqrt{5}-1)a/2$, *then we have*

$$\tau_{j+1} < \tau_j < 2\tau_{j+1}, \quad j = 1, \dots, 2N-1, \tag{4.2}$$

where $\tau_j = \log t_j - \log t_{j-1}$.

Proof. The inequality (4.1) is trivial. For (4.2), it suffices to show that $\tau_j < 2\tau_{j+1}$. A direct calculation gives

$$\tau_j - 2\tau_{j+1} = \log \frac{t_j^3}{t_{j-1}t_{j+1}^2},$$

and

$$t_j^3 - t_{j-1}t_{j+1}^2 = t_j^3 - (t_j - \Delta t)(t_j + \Delta t)^2 = \Delta t \left(\Delta t^2 + t_j \Delta t - t_j^2\right).$$

If $\Delta t < (\sqrt{5}-1)a/2$, then $\Delta t < (\sqrt{5}-1)t_j/2$. Consequently $t_j^3 - t_{j-1}t_{j+1}^2 < 0$, and

$$\log \frac{t_j^3}{t_{j-1}t_{j+1}^2} < 0.$$

This gives $\tau_j < 2\tau_{j+1}$. □

Remark 4.1. Similar to (4.2), if $\Delta t < (\sqrt{5}-1)a/2$, it holds

$$\log \frac{t_1}{t_{\frac{1}{2}}} < \log \frac{t_{\frac{1}{2}}}{t_0} < 2 \log \frac{t_1}{t_{\frac{1}{2}}}. \tag{4.3}$$

Throughout the paper, we will always assume $\Delta t < (\sqrt{5}-1)a/2$.

Lemma 4.2. *For* $i \neq j$ *and* $p \neq q$, *we have*

$$\log \frac{t_j}{t_i} = \left(\frac{j-i}{p-q} + \varepsilon(\Delta t) \right) \log \frac{t_p}{t_q}, \tag{4.4}$$

where $\lim_{\Delta t \to 0} \varepsilon(\Delta t) = 0$.

Proof. A routine computation gives rise to the following formula:

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\log(t_j/t_i)}{\log(t_p/t_q)} &= \lim_{\Delta t \to 0} \frac{\log\left(1 + (j-i)\Delta t/t_i\right)}{\log\left(1 + (p-q)\Delta t/t_q\right)} = \lim_{\Delta t \to 0} \frac{(j-i)t_q}{(p-q)t_i} \\ &= \lim_{\Delta t \to 0} \frac{(j-i)\left(t_i + (q-i)\Delta t\right)}{(p-q)t_i} = \frac{j-i}{p-q}. \end{aligned}$$

This completes the proof. □

Lemma 4.3.

$$\int_{t_j}^{t_k} \left(\log \frac{b}{s} \right)^{\alpha-1} \frac{\mathrm{d}s}{s} \le \left(\log \frac{b}{t_k} \right)^{\alpha-1} \log \frac{t_k}{t_j},$$

where $k > j, b$ *is a positive constant.*

Proof. By virtue of the mean value theorem of integrals, there is $\tau^* \in [\log t_j, \log t_k]$ such that

$$\int_{t_j}^{t_k} \left(\log \frac{b}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} = \int_{\log t_j}^{\log t_k} (\log b - \tau)^{\alpha-1} \mathrm{d}\tau = (\log b - \tau^*)^{\alpha-1} \log \frac{t_k}{t_j}.$$

The statement then follows from the monotonically increasing nature of $(\log b - s)^{\alpha-1}$ with respect to s. □

Lemma 4.4 ([28]). *For $\alpha > 0, \beta > 0, b > a > 0$, there holds*

$$\int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\beta-1} \frac{\mathrm{d}s}{s} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\log \frac{b}{a}\right)^{\alpha+\beta-1}.$$

We recall the modified Gronwall inequality, which is crucial to the proof of the stability and error analysis of our scheme.

Lemma 4.5 (Discrete Gronwall Inequality, [13, Lemma 4.3]). *Let $0 < \alpha < 1$, N be a positive integer, $a = t_0 < t_1 < \cdots < t_{2N} = T$ and*

$$b_{j,n} = \left(\log \frac{t_n}{a} - \log \frac{t_j}{a}\right)^{\alpha-1} \log \frac{t_{j+1}}{t_j}, \quad j = 0, 1, \ldots, n-1, \quad n = 1, 2, \ldots, 2N.$$

Suppose η_0 is a positive constant, the positive sequence $\{e_n\}$ satisfies

$$\begin{aligned} &e_0 \le \eta_0, \\ &e_n \le M \sum_{j=0}^{n-1} b_{j,n} e_j + \eta_0 \end{aligned}$$

with M being a positive constant independent of n. Then

$$e_n \le C\eta_0, \quad n = 1, 2, \ldots, 2N,$$

where C is a positive constant independent of n.

We are now in a position to derive an estimate for the truncation errors of the scheme (3.16). We define the local errors separately for the odd steps and even steps as follows:

$$R_{2m+1}(\Delta t) := u(t_{2m+1}) - \hat{u}_{2m+1}, \tag{4.5}$$

$$R_{2m+2}(\Delta t) := u(t_{2m+2}) - \hat{u}_{2m+2}, \tag{4.6}$$

where $\hat{u}_{2m+1}$ and $\hat{u}_{2m+2}$ are the approximations to $u(t_{2m+1})$ and $u(t_{2m+2})$, respectively evaluated by using the schemes (3.13) and (3.15) with the exact previous solutions, i.e.

$$\begin{aligned} \hat{u}_{2m+1} = u_0 &+ c_{2m+1}^{0,0} f\big(t_0, u(t_0)\big) + c_{2m+1}^{0,1} f\big(t_1, u(t_1)\big) + c_{2m+1}^{0,2} f\big(t_2, u(t_2)\big) \\ &+ \sum_{k=1}^{m} \left[c_{2m+1}^{k,0} f\big(t_{2k-1}, u(t_{2k-1})\big) + c_{2m+1}^{k,1} f\big(t_{2k}, u(t_{2k})\big) + c_{2m+1}^{k,2} f\big(t_{2k+1}, u(t_{2k+1})\big)\right], \end{aligned}$$

$$\hat{u}_{2m+2} = u_0 + \sum_{k=0}^{m} \left[c_{2m+2}^{k,0} f\big(t_{2k}, u(t_{2k})\big) + c_{2m+2}^{k,1} f\big(t_{2k+1}, u(t_{2k+1})\big) + c_{2m+2}^{k,2} f\big(t_{2k+2}, u(t_{2k+2})\big)\right].$$

Proposition 4.1. *If* $\delta^4 f(\cdot, u(\cdot)) \in C[a,T]$, *then it holds*

$$|R_{2m+1}(\Delta t)| \le C\Delta t^{3+\alpha}, \quad |R_{2m+2}(\Delta t)| \le C\Delta t^{3+\alpha}.$$

Proof. We only derive the estimate for $R_{2m+1}(\Delta t)$. The even step error $R_{2m+2}(\Delta t)$ can be estimated similarly. Using (3.10) and (3.12) in the definition (4.5), we have

$$\begin{aligned}
&R_{2m+1}(\Delta t)\\
&= u(t_{2m+1}) - \Big\{ u_0 + \big(d_{2m+1}^{0,0} + w_0 d_{2m+1}^{0,1}\big) f\big(t_0, u(t_0)\big) + \big(w_1 d_{2m+1}^{0,1} + d_{2m+1}^{0,2}\big) f\big(t_1, u(t_1)\big)\\
&\qquad + w_2 d_{2m+1}^{0,1} f\big(t_2, u(t_2)\big) + \sum_{k=1}^{m} \Big[c_{2m+1}^{k,0} f\big(t_{2k-1}, u(t_{2k-1})\big) + c_{2m+1}^{k,1} f\big(t_{2k}, u(t_{2k})\big)\\
&\qquad + c_{2m+1}^{k,2} f\big(t_{2k+1}, u(t_{2k+1})\big)\Big]\Big\}\\
&= u(t_{2m+1}) - \Big\{ u_0 + f\big(t_0, u(t_0)\big) d_{2m+1}^{0,0} + \big[w_0 f\big(t_0, u(t_0)\big) + w_1 f\big(t_1, u(t_1)\big) + w_2 f\big(t_2, u(t_2)\big)\big] d_{2m+1}^{0,1}\\
&\qquad + f\big(t_1, u(t_1)\big) d_{2m+1}^{0,2} + \sum_{k=1}^{m} \Big[c_{2m+1}^{k,0} f\big(t_{2k-1}, u(t_{2k-1})\big)\\
&\qquad + c_{2m+1}^{k,1} f\big(t_{2k}, u(t_{2k})\big) + c_{2m+1}^{k,2} f\big(t_{2k+1}, u(t_{2k+1})\big)\Big]\Big\}\\
&= u_0 + \frac{1}{\Gamma(\alpha)} \left[\int_{t_0}^{t_1} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} f\big(s, u(s)\big) \frac{\mathrm{d}s}{s} + \sum_{k=1}^{m} \int_{t_{2k-1}}^{t_{2k+1}} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} f\big(s, u(s)\big)\frac{\mathrm{d}s}{s}\right]\\
&\quad - \Big\{ u_0 + \frac{1}{\Gamma(\alpha)} \Big[f\big(t_0, u(t_0)\big) \int_{t_0}^{t_1} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{0,0}(s)\frac{\mathrm{d}s}{s}\\
&\qquad + \big(w_0 f\big(t_0, u(t_0)\big) + w_1 f\big(t_1, u(t_1)\big) + w_2 f\big(t_2, u(t_2)\big)\big)\\
&\qquad \times \int_{t_0}^{t_1} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{0,1}(s)\frac{\mathrm{d}s}{s} + f\big(t_1, u(t_1)\big) \int_{t_0}^{t_1} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{0,2}(s)\frac{\mathrm{d}s}{s}\Big]\\
&\qquad + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \Big[f\big(t_{2k-1}, u(t_{2k-1})\big) \int_{t_{2k-1}}^{t_{2k+1}} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k,0}(s)\frac{\mathrm{d}s}{s}\\
&\qquad + f\big(t_{2k}, u(t_{2k})\big) \int_{t_{2k-1}}^{t_{2k+1}} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k,1}(s)\frac{\mathrm{d}s}{s}\\
&\qquad + f\big(t_{2k+1}, u(t_{2k+1})\big) \int_{t_{2k-1}}^{t_{2k+1}} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k,2}(s)\frac{\mathrm{d}s}{s}\Big]\Big\}\\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\\
&\quad \times \Big\{ f\big(s, u(s)\big) - \Big[f\big(t_0, u(t_0)\big)\varphi_{0,0}(s) + f\big(t_{\frac{1}{2}}, u(t_{\frac{1}{2}})\big)\varphi_{0,1}(s) + f\big(t_1, u(t_1)\big)\varphi_{0,2}(s)\Big]\\
&\qquad + \Big[f\big(t_{\frac{1}{2}}, u(t_{\frac{1}{2}})\big) - \big(w_0 f\big(t_0, u(t_0)\big) + w_1 f\big(t_1, u(t_1)\big) + w_2 f\big(t_2, u(t_2)\big)\big)\Big]\varphi_{0,1}(s)\Big\}\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\\
&\times\Big\{f\big(s,u(s)\big)-\Big[\varphi_{k,0}(s)f\big(t_{2k-1},u(t_{2k-1})\big)+\varphi_{k,1}(s)f\big(t_{2k},u(t_{2k})\big)\\
&\qquad+\varphi_{k,2}(s)f\big(t_{2k+1},u(t_{2k+1})\big)\Big]\Big\}\frac{\mathrm{d}s}{s}\\
=&\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\big(r_0(s)+\tilde r_0(s)\varphi_{0,1}(s)\big)\frac{\mathrm{d}s}{s}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}r_{2k-1}(s)\frac{\mathrm{d}s}{s},
\end{aligned}
$$

where

$$
\begin{aligned}
r_0(s)&=f\big(s,u(s)\big)-f\big(t_0,u(t_0)\big)\varphi_{0,0}(s)-f\big(t_{\frac12},u(t_{\frac12})\big)\varphi_{0,1}(s)-f\big(t_1,u(t_1)\big)\varphi_{0,2}(s),\\
\tilde r_0(s)&=f\big(t_{\frac12},u(t_{\frac12})\big)-w_0f\big(t_0,u(t_0)\big)-w_1f\big(t_1,u(t_1)\big)-w_2f\big(t_2,u(t_2)\big),\\
r_{2k-1}(s)&=f\big(s,u(s)\big)-\varphi_{k,0}(s)f\big(t_{2k-1},u(t_{2k-1})\big)-\varphi_{k,1}(s)f\big(t_{2k},u(t_{2k})\big)\\
&\quad-\varphi_{k,2}(s)f\big(t_{2k+1},u(t_{2k+1})\big),\quad k=1,\dots,m.
\end{aligned}
$$

Using the logarithmic interpolation theory employed in [10] for all $s\in[t_0,t_1]$ there exist $\xi_1(s)\in(t_0,t_1),\eta(s)\in(t_0,t_2)$ such that

$$
\begin{aligned}
r_0(s)&=\frac{\delta^3f\big(\xi_1(s),u(\xi_1(s))\big)}{3!}\log\frac{s}{t_0}\log\frac{s}{t_{\frac12}}\log\frac{s}{t_1},\\
\tilde r_0(s)&=\frac{\delta^3f\big(\eta(s),u(\eta(s))\big)}{3!}\log\frac{t_{\frac12}}{t_0}\log\frac{t_{\frac12}}{t_1}\log\frac{t_{\frac12}}{t_2},
\end{aligned}
$$

and for all $s\in[t_{2k-1},t_{2k+1}]$ there exists $\xi_k(s)\in(t_{2k-1},t_{2k+1})$ such that

$$
r_{2k-1}(s)=\frac{\delta^3f\big(\xi_k(s),u(\xi_k(s))\big)}{3!}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}},\quad k=1,\dots,m.
$$

Therefore, we have

$$
\begin{aligned}
&R_{2m+1}(\Delta t) \qquad (4.7)\\
=&\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\frac{\delta^3f\big(\xi_1(s),u(\xi_1(s))\big)}{3!}\log\frac{s}{t_0}\log\frac{s}{t_{\frac12}}\log\frac{s}{t_1}\frac{\mathrm{d}s}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\frac{\delta^3f\big(\eta(s),u(\eta(s))\big)}{3!}\log\frac{t_{\frac12}}{t_0}\log\frac{t_{\frac12}}{t_1}\log\frac{t_{\frac12}}{t_2}\varphi_{0,1}(s)\frac{\mathrm{d}s}{s}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\frac{\delta^3f\big(\xi_k(s),u(\xi_k(s))\big)}{3!}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

It remains to estimate the right-hand side of (4.7) term by term. For the first term, denoted by $R1$, we have

$$
\begin{aligned}
|R1|&\le\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\left|\frac{\delta^3f\big(\xi_1(s),u(\xi_1(s))\big)}{3!}\log\frac{s}{t_0}\log\frac{s}{t_{\frac12}}\log\frac{s}{t_1}\right|\frac{\mathrm{d}s}{s}\\
&\le\frac{M_1}{6\Gamma(\alpha)}\tau_1^3\int_{t_0}^{t_1}\Big(\log\frac{t_{2m+1}}{s}\Big)^{\alpha-1}\frac{\mathrm{d}s}{s},
\end{aligned}
$$

where

$$M_1 = \sup_{t\in[a,T]} \big|\delta^3 f\big(t, u(t)\big)\big|.$$

It follows from Lemmas 4.2 and 4.3 that

$$|R1| \le \frac{M_1}{6\Gamma(\alpha)}\tau_1^4\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1} = \frac{M_1}{6\Gamma(\alpha)}\tau_1^{3+\alpha}\big(2m+\varepsilon(\Delta t)\big)^{\alpha-1} \le \frac{M_1}{6\Gamma(\alpha)}\tau_1^{3+\alpha}, \tag{4.8}$$

where $\lim_{\Delta t\to 0}\varepsilon(\Delta t) = 0$. Similarly, for the second term in (4.7), denoted by $R2$, we have

$$\begin{aligned}
|R2| &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\left|\frac{\delta^3 f\big(\eta(s), u\big(\eta(s)\big)\big)}{3!}\log\frac{t_{\frac{1}{2}}}{t_2}\log\frac{s}{t_0}\log\frac{s}{t_1}\right|\frac{\mathrm{d}s}{s} \\
&\le \frac{M_1}{6\Gamma(\alpha)}\tau_1^2(\tau_1+\tau_2)\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&\le \frac{M_1}{3\Gamma(\alpha)}\tau_1^4\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1} \le \frac{M_1}{3\Gamma(\alpha)}\tau_1^{3+\alpha}.
\end{aligned} \tag{4.9}$$

The third term $R3$ can be bounded by

$$|R3| = \left|\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(\xi_k(s), u\big(\xi_k(s)\big)\big)}{3!}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\right| \tag{4.10}$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m-1}\Bigg\{\left|\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(t_{2k}, u(t_{2k})\big)}{3!}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\right| \\
&\qquad + \Bigg|\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(\xi_k(s), u\big(\xi_k(s)\big)\big) - \delta^3 f\big(t_{2k}, u(t_{2k})\big)}{3!} \\
&\qquad\quad \times\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\Bigg|\Bigg\} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{t_{2m-1}}^{t_{2m+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(\xi_m(s), u\big(\xi_m(s)\big)\big)}{3!}\log\frac{s}{t_{2m-1}}\log\frac{s}{t_{2m}}\log\frac{s}{t_{2m+1}}\frac{\mathrm{d}s}{s}\right|.
\end{aligned}$$

For the first term in the right-hand side of (4.10), denote by $R3_1$, we have

$$\begin{aligned}
R3_1 &= \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m-1}\left|\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(t_{2k}, u(t_{2k})\big)}{3!}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\right| \\
&= \frac{\big|\delta^3 f\big(t_{2k}, u(t_{2k})\big)\big|}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\left|\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\right| \\
&\le \frac{M_1}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\Bigg|\int_{t_{2k-1}}^{t_{2k}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s} \\
&\qquad + \int_{t_{2k}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\Bigg| \\
&= \frac{M_1}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\Bigg|\left(\log\frac{t_{2m+1}}{\tilde{s_k}}\right)^{\alpha-1}\int_{t_{2k-1}}^{t_{2k}}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s} \\
&\qquad + \left(\log\frac{t_{2m+1}}{\bar{s_k}}\right)^{\alpha-1}\int_{t_{2k}}^{t_{2k+1}}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\Bigg|,
\end{aligned} \tag{4.11}$$

where $t_{2k-1} \le \tilde{s_k} \le t_{2k} \le \bar{s_k} \le t_{2k+1}$. By Lemmas 4.4 and 4.2, we have

$$\begin{aligned}
&\int_{t_{2k-1}}^{t_{2k}} \log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\\
&=\int_{t_{2k-1}}^{t_{2k}} \log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\left(\log\frac{s}{t_{2k}}+\log\frac{t_{2k}}{t_{2k+1}}\right)\frac{\mathrm{d}s}{s}\\
&=\int_{t_{2k-1}}^{t_{2k}}\left(\log\frac{t_{2k}}{s}\right)^2\log\frac{s}{t_{2k-1}}\frac{\mathrm{d}s}{s}+\log\frac{t_{2k+1}}{t_{2k}}\int_{t_{2k-1}}^{t_{2k}}\log\frac{t_{2k}}{s}\log\frac{s}{t_{2k-1}}\frac{\mathrm{d}s}{s}\\
&=\frac{1}{12}\tau_{2k}^4+\frac{1}{6}\tau_{2k}^3\tau_{2k+1}=\frac{1}{4}\tau_{2k}^4+\frac{1}{6}\tau_{2k}^4\varepsilon_1(\Delta t),
\end{aligned}\tag{4.12}$$

where $\lim_{\Delta t\to 0}\varepsilon_1(\Delta t)=0$. Similarly, we have

$$\begin{aligned}
&\int_{t_{2k}}^{t_{2k+1}} \log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\\
&=\int_{t_{2k}}^{t_{2k+1}}\left(\log\frac{s}{t_{2k}}+\log\frac{t_{2k}}{t_{2k-1}}\right)\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\\
&=-\int_{t_{2k}}^{t_{2k+1}}\left(\log\frac{s}{t_{2k}}\right)^2\log\frac{t_{2k+1}}{s}\frac{\mathrm{d}s}{s}-\tau_{2k}\int_{t_{2k}}^{t_{2k+1}}\log\frac{t_{2k+1}}{s}\log\frac{s}{t_{2k}}\frac{\mathrm{d}s}{s}\\
&=-\frac{1}{12}\tau_{2k+1}^4-\frac{1}{6}\tau_{2k}\tau_{2k+1}^3=-\frac{1}{4}\tau_{2k}^4-\frac{1}{4}\tau_{2k}^4\varepsilon_2(\Delta t),
\end{aligned}\tag{4.13}$$

where $\lim_{\Delta t\to 0}\varepsilon_2(\Delta t)=0$. Bringing (4.12) and (4.13) into (4.11), we obtain

$$R3_1\le\frac{M_1}{24\Gamma(\alpha)}\sum_{k=1}^{m-1}\left|\tau_{2k}^4\left(\left(\log\frac{t_{2m+1}}{\tilde{s_k}}\right)^{\alpha-1}-\left(\log\frac{t_{2m+1}}{\bar{s_k}}\right)^{\alpha-1}\right)\right|+\tilde{r}_3,\tag{4.14}$$

where

$$\begin{aligned}
\tilde{r}_3&=\frac{M_1}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\left|\tau_{2k}^4\left(\frac{\varepsilon_1(\Delta t)}{6}\left(\log\frac{t_{2m+1}}{\tilde{s_k}}\right)^{\alpha-1}-\frac{\varepsilon_2(\Delta t)}{4}\left(\log\frac{t_{2m+1}}{\bar{s_k}}\right)^{\alpha-1}\right)\right|\\
&\le\frac{M_1}{12\Gamma(\alpha)}\tau_1^4\sum_{k=1}^{m-1}\varepsilon(\Delta t)\left(\log\frac{t_{2m+1}}{t_{2k-1}}\right)^{\alpha-1}\\
&\le\frac{M_1}{12\Gamma(\alpha)}\tau_1^{3+\alpha}\sum_{k=1}^{m-1}\varepsilon(\Delta t)\big(2m-2k+2+\varepsilon_3(\Delta t)\big)^{\alpha-1}\\
&\le\frac{M_1}{12\Gamma(\alpha)}\tau_1^{3+\alpha}\sum_{k=1}^{m-1}\varepsilon(\Delta t)
\end{aligned}$$

with $\varepsilon(\Delta t)=\max_{i=1,2}\{|\varepsilon_i(\Delta t)|\}$, and $\lim_{\Delta t\to 0}\varepsilon_3(\Delta t)=0$. Due to $\lim_{\Delta t\to 0}\varepsilon(\Delta t)=0$, we get $\varepsilon(\Delta t)\le C\Delta t$. It follows that

$$\tilde{r}_3\le\frac{CM_1}{12\Gamma(\alpha)}\tau_1^{3+\alpha}\sum_{k=1}^{m-1}\Delta t\le\frac{CTM_1}{12\Gamma(\alpha)}\tau_1^{3+\alpha}.$$

Applying the mean value theorem to the first term in the right-hand side of (4.14), there exists $\hat{s}\in(\log(t_{2m+1}/\bar{s_k}),\log(t_{2m+1}/\tilde{s_k}))$ such that

$$\left(\log\frac{t_{2m+1}}{\tilde{s_k}}\right)^{\alpha-1}-\left(\log\frac{t_{2m+1}}{\bar{s_k}}\right)^{\alpha-1}=(\alpha-1)\hat{s}^{\alpha-2}\log\frac{\bar{s_k}}{\tilde{s_k}}.$$

This results in

$$\begin{aligned}
R3_1 &\leq \frac{M_1(1-\alpha)}{24\Gamma(\alpha)}\tau_1^4 \sum_{k=1}^{m-1}\left|\hat{s}^{\alpha-2}\log\frac{\bar{s_k}}{\tilde{s_k}}\right| + \tilde{r}_3 \\
&\leq \frac{M_1(1-\alpha)}{24\Gamma(\alpha)}\tau_1^4 \sum_{k=1}^{m-1}\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-2}\frac{\mathrm{d}s}{s} + \tilde{r}_3 \\
&\leq \frac{M_1(1-\alpha)}{24\Gamma(\alpha)}\tau_1^4 \left|\int_{t_1}^{t_{2m-1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-2}\frac{\mathrm{d}s}{s}\right| + \tilde{r}_3 \\
&\leq \frac{M_1}{24\Gamma(\alpha)}\tau_1^4\left(\left|\left(\log\frac{t_{2m+1}}{t_{2m-1}}\right)^{\alpha-1}\right| + \left|\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\right|\right) + \tilde{r}_3 \\
&\leq \frac{M_1}{24\Gamma(\alpha)}\tau_1^{\alpha+3}\left(2^{\alpha-1} + (2m)^{\alpha-1}\right) + \tilde{r}_3 \leq C\tau_1^{\alpha+3}.
\end{aligned}$$

The second term in the right-hand side of (4.10), denote by $R3_2$, can be rewritten as

$$\begin{aligned}
R3_2 = \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\Bigg|\int_{t_{2k-1}}^{t_{2k+1}}&\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(\xi_k(s), u\big(\xi_k(s)\big)\big) - \delta^3 f\big(t_{2k}, u(t_{2k})\big)}{3!} \\
&\times\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\Bigg|.
\end{aligned}$$

By Cauchy mean value theorem, there exists ς_k between ξ_k and t_{2k} such that

$$\frac{\delta^3 f\big(\xi_k, u(\xi_k)\big) - \delta^3 f\big(t_{2k}, u(t_{2k})\big)}{\log\xi_k - \log t_{2k}} = t\frac{\mathrm{d}}{\mathrm{d}t}\delta^3 f\big(t, u(t)\big)\Big|_{t=\varsigma_k} = \delta^4 f\big(\varsigma_k, u(\varsigma_k)\big).$$

Let $M_2 = \sup_{t\in[a,T]}|\delta^4 f(t, u(t))|$, then we have

$$\begin{aligned}
R3_2 &= \frac{1}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\left|\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\delta^4 f\big(\varsigma_k, u(\varsigma_k)\big)\log\frac{\xi_k}{t_{2k}}\log\frac{s}{t_{2k-1}}\log\frac{s}{t_{2k}}\log\frac{s}{t_{2k+1}}\frac{\mathrm{d}s}{s}\right| \\
&\leq \frac{M_2}{6\Gamma(\alpha)}\sum_{k=1}^{m-1}\tau_{2k}^2\left(\tau_{2k}+\tau_{2k+1}\right)^2\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&\leq \frac{2M_2}{3\Gamma(\alpha)}\tau_1^4\int_{t_1}^{t_{2m-1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&= \frac{2M_2}{3\Gamma(\alpha+1)}\tau_1^4\left(\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha} - \left(\log\frac{t_{2m+1}}{t_{2m-1}}\right)^{\alpha}\right) \\
&\leq \frac{2M_2}{3\Gamma(\alpha+1)}\tau_1^4\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha} \leq \frac{2M_2}{3\Gamma(\alpha+1)}\tau_1^4\left(2m\tau_1\right)^{\alpha} \\
&\leq \frac{2M_2}{3\Gamma(\alpha+1)}\tau_1^4\left(\frac{2m\Delta t}{a}\right)^{\alpha} \leq \frac{2M_2T^{\alpha}}{3a^{\alpha}\Gamma(\alpha+1)}\tau_1^4.
\end{aligned}$$

For the third term in the right-hand side of (4.10), denote by $R3_3$, it holds

$$\begin{aligned}
R3_3 &= \frac{1}{\Gamma(\alpha)}\left|\int_{t_{2m-1}}^{t_{2m+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\delta^3 f\big(\xi_m(s), u\big(\xi_m(s)\big)\big)}{3!}\log\frac{s}{t_{2m-1}}\log\frac{s}{t_{2m}}\log\frac{s}{t_{2m+1}}\frac{\mathrm{d}s}{s}\right| \\
&\leq \frac{M_1}{6\Gamma(\alpha)}\tau_{2m}(\tau_{2m}+\tau_{2m+1})^2\left|\int_{t_{2m-1}}^{t_{2m+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\right|
\end{aligned}$$

$$= \frac{M_1}{6\Gamma(\alpha+1)} \left(\tau_{2m} + \tau_{2m+1}\right)^{2+\alpha} \tau_{2m} \le \frac{2^{\alpha+1} M_1}{3\Gamma(\alpha+1)} \tau_1^{3+\alpha}.$$

Thus we have

$$|R3| \le C\big(\tau_1^{3+\alpha} + \tau_1^4\big). \tag{4.15}$$

Combining (4.7)-(4.9), and (4.15) yields

$$|R_{2m+1}(\Delta t)| \le C\big(\tau_1^{3+\alpha} + \tau_1^4\big).$$

Finally, using (4.1), we obtain

$$|R_{2m+1}(\Delta t)| \le C\Delta t^{3+\alpha}. \tag{4.16}$$

The proof is complete. □

5. Stability and Convergence

We turn to perform the stability and convergence analysis for the proposed scheme. We rewrite the scheme (3.16) under an equivalent form as follows:

$$\begin{cases} u_1 = u_0 + c_1^{0,0} f_0 + c_1^{0,1} f_1 + c_1^{0,2} f_2, \\ u_2 = u_0 + c_2^{0,0} f_0 + c_2^{0,1} f_1 + c_2^{0,2} f_2, \\ u_{2m+1} = u_0 + \displaystyle\sum_{j=0}^{2m+1} A_{j,2m+1} f_j, \\ u_{2m+2} = u_0 + \displaystyle\sum_{j=0}^{2m+2} A_{j,2m+2} f_j, \quad m = 1, \ldots, N-1, \end{cases} \tag{5.1}$$

where

$$A_{j,2m+1} = \begin{cases} c_{2m+1}^{0,0}, & j = 0, \\ c_{2m+1}^{0,1} + c_{2m+1}^{1,0}, & j = 1, \\ c_{2m+1}^{0,2} + c_{2m+1}^{1,1}, & j = 2, \\ c_{2m+1}^{k,2} + c_{2m+1}^{k+1,0}, & j = 2k+1, \quad k = 1, \ldots, m-1, \\ c_{2m+1}^{k,1}, & j = 2k, \quad k = 2, \ldots, m, \\ c_{2m+1}^{m,2}, & j = 2m+1, \end{cases} \tag{5.2}$$

and

$$A_{j,2m+2} = \begin{cases} c_{2m+2}^{0,0}, & j = 0, \\ c_{2m+2}^{k-1,2} + c_{2m+2}^{k,0}, & j = 2k, \quad k = 1, 2, \ldots, m, \\ c_{2m+2}^{k,1}, & j = 2k+1, \quad k = 0, 1, \ldots, m, \\ c_{2m+2}^{m,2}, & j = 2m+2. \end{cases} \tag{5.3}$$

Lemma 5.1. *The quadratic logarithmic interpolation functions defined in* (3.2), (3.11), *and* (3.14) *are all bounded.*

Proof. For $t \in [t_0, t_2]$, it holds

$$
\begin{aligned}
&|\phi_{0,0}(t)| \le 1, \\
&|\phi_{0,1}(t)| \le \frac{(\tau_1+\tau_2)^2}{\tau_1\tau_2} \le \left(1+\frac{\tau_2}{\tau_1}\right)\left(1+\frac{\tau_1}{\tau_2}\right) \le 6, \\
&|\phi_{0,2}(t)| \le \frac{\tau_1}{\tau_2} \le 2.
\end{aligned}
$$

Similarly, it can be directly verified that

$$
\begin{aligned}
&|\varphi_{k,0}(t)| \le 1, \quad |\varphi_{k,1}(t)| \le 6, \quad |\varphi_{k,2}(t)| \le 2, \quad \forall\, t \in [t_{2k-1}, t_{2k+1}], \quad k = 1, \ldots, m, \\
&|\phi_{k,0}(t)| \le 1, \quad |\phi_{k,1}(t)| \le 6, \quad |\phi_{k,2}(t)| \le 2, \quad \forall\, t \in [t_{2k}, t_{2k+2}], \quad k = 1, \ldots, m.
\end{aligned}
$$

The lemma is proved. □

Lemma 5.2. *The coefficients $A_{j,2m+1}, j = 0, 1, \ldots, 2m+1$, defined in* (5.2)*, satisfy*

$$
|A_{j,2m+1}| \le \frac{62}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_j}\right)^{\alpha-1} \log \frac{t_{j+1}}{t_j}, \quad j = 0, 1, \ldots, 2m, \tag{5.4}
$$

$$
|A_{2m+1,2m+1}| \le \frac{2^\alpha (2-\alpha)}{a^\alpha \Gamma(3+\alpha)} \Delta t^\alpha. \tag{5.5}
$$

Proof. For $j = 0$, we have

$$
\begin{aligned}
|A_{0,2m+1}| &= \left|c_{2m+1}^{0,0}\right| \le \left|d_{2m+1}^{0,0}\right| + \left|w_0 d_{2m+1}^{0,1}\right| \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} + \left|\phi_{0,0}(t_{\frac{1}{2}})\right| \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} |\varphi_{0,1}(s)| \frac{\mathrm{d}s}{s} \right).
\end{aligned}
$$

The second term in the right-hand side can be bounded by using (3.2), (3.5), and (4.3) as follows:

$$
\begin{aligned}
&\left|\phi_{0,0}(t_{\frac{1}{2}})\right| \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} |\varphi_{0,1}(s)| \frac{\mathrm{d}s}{s} \\
\le\ & \frac{\log(t_1/t_0) \log\left(t_2/t_{\frac{1}{2}}\right)}{\log\left(t_{\frac{1}{2}}/t_0\right) \log(t_2/t_0)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \\
\le\ & \frac{\log(t_1/t_0)}{\log\left(t_{\frac{1}{2}}/t_0\right)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \le 2 \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Furthermore, using Lemma 4.3 and inequalities (4.2), we obtain

$$
\begin{aligned}
|A_{0,2m+1}| &\le \frac{3}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \le \frac{3}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_1}\right)^{\alpha-1} \log \frac{t_1}{t_0} \\
&= \frac{3}{\Gamma(\alpha)} \left(\frac{\log(t_{2m+1}/t_0)}{\log(t_{2m+1}/t_1)}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_0}\right)^{\alpha-1} \log \frac{t_1}{t_0} \\
&= \frac{3}{\Gamma(\alpha)} \left(1 + \frac{\tau_1}{\tau_{2m+1} + \cdots + \tau_2}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_0}\right)^{\alpha-1} \log \frac{t_1}{t_0} \\
&\le \frac{3}{\Gamma(\alpha)} \left(1 + \frac{2\tau_2}{\tau_{2m+1} + \cdots + \tau_2}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_0}\right)^{\alpha-1} \log \frac{t_1}{t_0} \\
&\le \frac{3^{2-\alpha}}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_0}\right)^{\alpha-1} \log \frac{t_1}{t_0}.
\end{aligned} \tag{5.6}
$$

For $j=1$, we have

$$|A_{1,2m+1}| = \left|c_{2m+1}^{0,1} + c_{2m+1}^{1,0}\right| \le \left|w_1 d_{2m+1}^{0,1}\right| + \left|d_{2m+1}^{0,2}\right| + \left|c_{2m+1}^{1,0}\right|. \tag{5.7}$$

The first term in (5.7) is bounded by

$$\begin{aligned}
\left|w_1 d_{2m+1}^{0,1}\right| &= \frac{1}{\Gamma(\alpha)}\left|\frac{\log\left(t_{\frac12}/t_0\right)\log\left(t_{\frac12}/t_2\right)}{\log(t_1/t_0)\log(t_1/t_2)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\log(s/t_0)\log(s/t_1)}{\log\left(t_{\frac12}/t_0\right)\log\left(t_{\frac12}/t_1\right)}\frac{\mathrm{d}s}{s}\right| \\
&\le \frac{2\log\left(t_2/t_{\frac12}\right)}{\Gamma(\alpha)\log\left(t_1/t_{\frac12}\right)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&\le \frac{2\left(\log(t_1/t_0)+\log\left(t_1/t_{\frac12}\right)\right)}{\Gamma(\alpha)\log\left(t_1/t_{\frac12}\right)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&= \frac{2\left(2\log\left(t_1/t_{\frac12}\right)+\log\left(t_{\frac12}/t_0\right)\right)}{\Gamma(\alpha)\log\left(t_1/t_{\frac12}\right)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&\le \frac{8}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}.
\end{aligned}$$

For the second and third terms in (5.7), we have

$$\begin{aligned}
\left|d_{2m+1}^{0,2}\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\varphi_{0,2}(s)\frac{\mathrm{d}s}{s}\right| \le \frac{2}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}, \\
\left|c_{2m+1}^{1,0}\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_3}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\varphi_{1,0}(s)\frac{\mathrm{d}s}{s}\right| \le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_3}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}.
\end{aligned}$$

Thus

$$\begin{aligned}
|A_{1,2m+1}| &\le \frac{10}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_3}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s} \\
&\le \frac{10}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_1}{t_0} + \frac{1}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_3}\right)^{\alpha-1}\log\frac{t_3}{t_1} \\
&\le \frac{10}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1}\frac{\tau_1}{\tau_2} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\frac{\log(t_{2m+1}/t_1)}{\log(t_{2m+1}/t_3)}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1}\frac{\log(t_3/t_1)}{\log(t_2/t_1)} \\
&\le \frac{20}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1} \\
&\quad + \frac{2}{\Gamma(\alpha)}\left(1+\frac{\tau_3+\tau_2}{\tau_{2m+1}+\cdots+\tau_4}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1} \\
&\le \frac{20}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1} \\
&\quad + \frac{2}{\Gamma(\alpha)}\left(1+\frac{6\tau_4}{\tau_{2m+1}+\cdots+\tau_4}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1} \\
&\le \frac{20}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1} + \frac{2\cdot 7^{1-\alpha}}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_1}\right)^{\alpha-1}\log\frac{t_2}{t_1}
\end{aligned}$$

$$\leq \frac{34}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_1}\right)^{\alpha-1} \log \frac{t_2}{t_1}.$$

For $j = 2$, we have

$$|A_{2,2m+1}| = \left|c_{2m+1}^{0,2} + c_{2m+1}^{1,1}\right| \leq |w_2| \left|d_{2m+1}^{0,1}\right| + \left|c_{2m+1}^{1,1}\right|. \tag{5.8}$$

In one side,

$$\begin{aligned}
|w_2| \left|d_{2m+1}^{0,1}\right| &= \left|\frac{\log\left(t_{\frac{1}{2}}/t_0\right)}{\log(t_2/t_0)} \frac{\log\left(t_{\frac{1}{2}}/t_1\right)}{\log(t_2/t_1)}\right| \left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\log(s/t_0)\log(s/t_1)}{\log\left(t_{\frac{1}{2}}/t_0\right)\log\left(t_{\frac{1}{2}}/t_1\right)} \frac{\mathrm{d}s}{s}\right| \\
&\leq \frac{\tau_1^2}{\Gamma(\alpha)(\tau_1+\tau_2)\tau_2} \int_{t_0}^{t_1} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \leq \frac{2}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_1}\right)^{\alpha-1} \log \frac{t_1}{t_0} \\
&= \frac{2}{\Gamma(\alpha)} \left(\frac{\log(t_{2m+1}/t_2)}{\log(t_{2m+1}/t_1)}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2} \frac{\tau_1}{\tau_3} \\
&\leq \frac{8}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2}.
\end{aligned} \tag{5.9}$$

In the other side,

$$\begin{aligned}
\left|c_{2m+1}^{1,1}\right| &= \left|\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_3} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{1,1}(s) \frac{\mathrm{d}s}{s}\right| \\
&\leq \frac{6}{\Gamma(\alpha)} \int_{t_1}^{t_3} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \leq \frac{6}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_3}\right)^{\alpha-1} \log \frac{t_3}{t_1} \\
&= \frac{6}{\Gamma(\alpha)} \left(\frac{\log(t_{2m+1}/t_2)}{\log(t_{2m+1}/t_3)}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2} \frac{\tau_3+\tau_2}{\tau_3} \\
&\leq \frac{18}{\Gamma(\alpha)} \left(1 + \frac{2\tau_4}{\tau_{2m+1} + \cdots + \tau_4}\right)^{1-\alpha} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2} \\
&\leq \frac{18 \cdot 3^{1-\alpha}}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2}.
\end{aligned} \tag{5.10}$$

By substituting (5.9) and (5.10) into (5.8), we get

$$|A_{2,2m+1}| \leq \frac{62}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_2}\right)^{\alpha-1} \log \frac{t_3}{t_2}.$$

Now we derive the estimate for $j = 3, \ldots, 2m$. We distinguish two cases: odd and even j. For $j = 2k+1, k = 1, 2, \ldots, m-1$,

$$\begin{aligned}
|A_{2k+1,2m+1}| &= \left|c_{2m+1}^{k,2} + c_{2m+1}^{k+1,0}\right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_{t_{2k-1}}^{t_{2k+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k,2}(s) \frac{\mathrm{d}s}{s} + \int_{t_{2k+1}}^{t_{2k+3}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \varphi_{k+1,0}(s) \frac{\mathrm{d}s}{s}\right| \\
&\leq \frac{2}{\Gamma(\alpha)} \int_{t_{2k-1}}^{t_{2k+1}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_{2k+1}}^{t_{2k+3}} \left(\log \frac{t_{2m+1}}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s} \\
&\leq \frac{2}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1} \log \frac{t_{2k+1}}{t_{2k-1}} + \frac{1}{\Gamma(\alpha)} \left(\log \frac{t_{2m+1}}{t_{2k+3}}\right)^{\alpha-1} \log \frac{t_{2k+3}}{t_{2k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\Gamma(\alpha)}\Big(\log\frac{t_{2m+1}}{t_{2k+1}}\Big)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}\frac{\tau_{2k+1}+\tau_{2k}}{\tau_{2k+2}}\\
&\quad+\frac{1}{\Gamma(\alpha)}\Big(\frac{\log(t_{2m+1}/t_{2k+1})}{\log(t_{2m+1}/t_{2k+3})}\Big)^{1-\alpha}\Big(\log\frac{t_{2m+1}}{t_{2k+1}}\Big)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}\frac{\tau_{2k+3}+\tau_{2k+2}}{\tau_{2k+2}}\\
&\le \frac{12}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}\\
&\quad+\frac{2}{\Gamma(\alpha)}\left(1+\frac{\tau_{2k+3}+\tau_{2k+2}}{\tau_{2m+1}+\cdots+\tau_{2k+3}}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}\\
&\le \frac{12}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}+\frac{2\cdot 4^{1-\alpha}}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}.\\
&\le \frac{20}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+2}}{t_{2k+1}}.
\end{aligned}
$$

For $j=2k, k=2,\ldots,m$,

$$
\begin{aligned}
|A_{2k,2m+1}| = |c_{2m+1}^{k,1}| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\varphi_{k,1}(s)\frac{\mathrm{d}s}{s}\right|\\
&\le \frac{6}{\Gamma(\alpha)}\int_{t_{2k-1}}^{t_{2k+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\\
&\le \frac{6}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k+1}}\right)^{\alpha-1}\log\frac{t_{2k+1}}{t_{2k-1}}\\
&= \frac{6}{\Gamma(\alpha)}\left(\frac{\log(t_{2m+1}/t_{2k})}{\log(t_{2m+1}/t_{2k+1})}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_{2k}}\right)^{\alpha-1}\log\frac{t_{2k+1}}{t_{2k}}\frac{\tau_{2k+1}+\tau_{2k}}{\tau_{2k+1}}\\
&\le \frac{12}{\Gamma(\alpha)}\left(1+\frac{\tau_{2k+1}}{\tau_{2m+1}+\cdots+\tau_{2k+2}}\right)^{1-\alpha}\left(\log\frac{t_{2m+1}}{t_{2k}}\right)^{\alpha-1}\log\frac{t_{2k+1}}{t_{2k}}\\
&\le \frac{12\cdot 3^{1-\alpha}}{\Gamma(\alpha)}\left(\log\frac{t_{2m+1}}{t_{2k}}\right)^{\alpha-1}\log\frac{t_{2k+1}}{t_{2k}}.
\end{aligned}
\tag{5.11}
$$

This completes the proof of (5.4). It remains to prove (5.5). We proceed as follows:

$$
\begin{aligned}
A_{2m+1,2m+1} = c_{2m+1}^{m,2} &= \frac{1}{\Gamma(\alpha)}\int_{t_{2m-1}}^{t_{2m+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\frac{\log(s/t_{2m-1})\log(s/t_{2m})}{\log(t_{2m+1}/t_{2m-1})\log(t_{2m+1}/t_{2m})}\frac{\mathrm{d}s}{s}\\
&= \left(\Gamma(\alpha)\log\frac{t_{2m+1}}{t_{2m-1}}\log\frac{t_{2m+1}}{t_{2m}}\right)^{-1}\\
&\quad\times\int_{t_{2m-1}}^{t_{2m+1}}\left(\log\frac{t_{2m+1}}{s}\right)^{\alpha-1}\left(\log^2\frac{s}{t_{2m-1}}+\log\frac{s}{t_{2m-1}}\log\frac{t_{2m-1}}{t_{2m}}\right)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

According to Lemma 4.4, we find

$$
\begin{aligned}
A_{2m+1,2m+1} &= \left(\log\frac{t_{2m+1}}{t_{2m-1}}\right)^{\alpha}\left(\frac{2\log(t_{2m+1}/t_{2m-1})}{\Gamma(\alpha+3)\log(t_{2m+1}/t_{2m})}+\frac{\log(t_{2m-1}/t_{2m})}{\Gamma(\alpha+2)\log(t_{2m+1}/t_{2m})}\right)\\
&= \frac{(\tau_{2m}+\tau_{2m+1})^{\alpha}}{\Gamma(3+\alpha)}\left(2-\alpha\frac{\tau_{2m}}{\tau_{2m+1}}\right).
\end{aligned}
$$

Consequently,

$$|A_{2m+1,2m+1}| \le \frac{2^\alpha(2-\alpha)}{\Gamma(3+\alpha)}\tau_1^\alpha \le \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha.$$

This proves (5.5). □

We can likewise prove the following lemma for $A_{j,2m+2}$. The detail of the proof is omitted.

Lemma 5.3. *The coefficients* $A_{j,2m+2}, j = 0, 1, \ldots, 2m+2$, *defined in* (5.3), *satisfy*

$$|A_{j,2m+2}| \le \frac{62}{\Gamma(\alpha)}\left(\log\frac{t_{2m+2}}{t_j}\right)^{\alpha-1}\log\frac{t_{j+1}}{t_j}, \quad j = 0, 1, \ldots, 2m+1,$$

$$|A_{2m+2,2m+2}| \le \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha.$$

Theorem 5.1. *Assume* $\delta^4 f(\cdot, u(\cdot)) \in C[a,T]$. *The scheme* (5.1) *is stable under the condition on the time step size*

$$\frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L < 1.$$

Proof. It is to prove that if the initial condition u_0 is perturbed by $\bar{u}_0$, then the perturbation of the solution u_j, denoted by $\bar{u}_j$, remains bounded. It is readily seen that $\bar{u}_j$ satisfy

$$\bar{u}_{2m+1} = \bar{u}_0 + \sum_{j=0}^{2m+1} A_{j,2m+1}\left[f(t_j, u_j+\bar{u}_j) - f(t_j,u_j)\right],$$

$$\bar{u}_{2m+2} = \bar{u}_0 + \sum_{j=0}^{2m+2} A_{j,2m+2}\left[f(t_j, u_j+\bar{u}_j) - f(t_j,u_j)\right], \quad m = 1, \ldots, N-1.$$

Under the Lipschitz condition on f, we have

$$\begin{aligned}
|\bar{u}_{2m+1}| &= \left|\bar{u}_0 + \sum_{j=0}^{2m+1} A_{j,2m+1}\big(f(t_j,u_j+\bar{u}_j) - f(t_j,u_j)\big)\right| \\
&\le |\bar{u}_0| + \sum_{j=0}^{2m+1}\left|A_{j,2m+1}\big(f(t_j,u_j+\bar{u}_j) - f(t_j,u_j)\big)\right| \\
&\le |\bar{u}_0| + L\sum_{j=0}^{2m+1}|A_{j,2m+1}|\,|\bar{u}_j| \\
&\le |\bar{u}_0| + L\sum_{j=0}^{2m}|A_{j,2m+1}|\,|\bar{u}_j| + \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L\,|\bar{u}_{2m+1}|.
\end{aligned}$$

A simple rearrangement yields

$$\left(1 - \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L\right)|\bar{u}_{2m+1}| \le |\bar{u}_0| + L\sum_{j=0}^{2m}|A_{j,2m+1}|\,|\bar{u}_j|,$$

or

$$|\bar{u}_{2m+1}| \le \tilde{C}\,|\bar{u}_0| + \tilde{C}L\sum_{j=0}^{2m}|A_{j,2m+1}|\,|\bar{u}_j|,$$

where
$$\tilde{C} = \left(1 - \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L\right)^{-1}.$$
Applying Lemma 4.5, we obtain
$$|\bar{u}_{2m+1}| \le C\,|\bar{u}_0|\,.$$
By the same argument, we can prove
$$|\bar{u}_{2m+2}| \le C\,|\bar{u}_0|\,.$$
This completes the proof. □

The convergence of the scheme is proved in the following theorem.

Theorem 5.2. *Let u be the exact solution of (2.1), $\{u_j\}_{j=0}^{2N}$ be the numerical solution of (5.1). Suppose $\delta^4 f(\cdot,u(\cdot)) \in C[a,T], f(t,u)$ satisfies the Lipschitz condition (2.3), and the time step size Δt satisfies*
$$\frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L < 1. \tag{5.12}$$
Then the following error estimate holds:
$$|u(t_j) - u_j| \le C\Delta t^{3+\alpha}, \quad j = 1,2,\ldots,2N. \tag{5.13}$$

Proof. Let $e_j = u(t_j) - u_j, j = 0,1,\ldots,2N$. Then $e_0 = 0$, and $e_j, j \ge 1$, satisfy
$$\begin{cases} e_1 = \displaystyle\sum_{j=0}^{2} c_1^{0,j}\big(f\big(t_j,u(t_j)\big) - f(t_j,u_j)\big) + R_1(\Delta t),\\ e_2 = \displaystyle\sum_{j=0}^{2} c_2^{0,j}\big(f\big(t_j,u(t_j)\big) - f(t_j,u_j)\big) + R_2(\Delta t),\\ e_{2m+1} = \displaystyle\sum_{j=0}^{2m+1} A_{j,2m+1}\big(f\big(t_j,u(t_j)\big) - f(t_j,u_j)\big) + R_{2m+1}(\Delta t),\\ e_{2m+2} = \displaystyle\sum_{j=0}^{2m+2} A_{j,2m+2}\big(f\big(t_j,u(t_j)\big) - f(t_j,u_j)\big) + R_{2m+2}(\Delta t), \quad m = 1,\ldots,N-1, \end{cases}$$
where the coefficients are defined in (5.2) and (5.3). By a direct calculation similar to (5.4) , we know $c_1^{0,j}, c_2^{0,j}, j = 0,1,2$, are bounded. Then it follows from Lemma 5.2 and the Lipschitz condition that
$$\begin{cases} |e_1| \le LC\displaystyle\sum_{j=0}^{2} e_j + |R_1(\Delta t)|\,,\\ |e_2| \le LC\displaystyle\sum_{j=0}^{2} e_j + |R_2(\Delta t)|\,,\\ |e_{2m+1}| \le LC\displaystyle\sum_{j=0}^{2m}\left(\log\frac{t_{2m+1}}{t_j}\right)^{\alpha-1}\log\frac{t_{j+1}}{t_j}\,|e_j| + \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L\,|e_{2m+1}|\\ \qquad + |R_{2m+1}(\Delta t)|\,,\\ |e_{2m+2}| \le LC\displaystyle\sum_{j=0}^{2m+1}\left(\log\frac{t_{2m+2}}{t_j}\right)^{\alpha-1}\log\frac{t_{j+1}}{t_j}\,|e_j| + \frac{2^\alpha(2-\alpha)}{a^\alpha\Gamma(3+\alpha)}\Delta t^\alpha L\,|e_{2m+2}|\\ \qquad + |R_{2m+2}(\Delta t)|\,, \quad m = 1,\ldots,N-1. \end{cases} \tag{5.14}$$

In virtue of the first and second inequalities of (5.14), we have

$$|e_1| \le C(|R_1(\Delta t)| + |R_2(\Delta t)|), \quad |e_2| \le C(|R_1(\Delta t)| + |R_2(\Delta t)|).$$

This, together with Proposition 4.1, leads to (5.13) for $j = 1, 2$.

Next we prove (5.13) for $j > 2$. Under the assumption (5.12), we derive from the last two inequalities of (5.14):

$$\begin{cases} |e_{2m+1}| \le LC \sum_{j=1}^{2m} \left(\log \frac{t_{2m+1}}{t_j}\right)^{\alpha-1} \log \frac{t_{j+1}}{t_j} |e_j| + C\,|R_{2m+1}(\Delta t)|, & m = 1, \ldots, N-1, \\ |e_{2m+2}| \le LC \sum_{j=1}^{2m+1} \left(\log \frac{t_{2m+2}}{t_j}\right)^{\alpha-1} \log \frac{t_{j+1}}{t_j} |e_j| + C\,|R_{2m+2}(\Delta t)|, & m = 1, \ldots, N-1. \end{cases}$$

Then it follows from the Gronwall inequality in Lemma 4.5 that

$$\begin{cases} |e_{2m+1}| \le C\,|R_{2m+1}(\Delta t)|, & m = 1, \ldots, N-1, \\ |e_{2m+2}| \le C\,|R_{2m+2}(\Delta t)|, & m = 1, \ldots, N-1. \end{cases}$$

Finally, using Proposition 4.1 gives

$$|e_j| \le C\Delta t^{3+\alpha}, \quad j = 3, \ldots, 2N.$$

The proof is complete. □

6. Numerical Results

We carry out some numerical experiments to verify the theoretical results obtained in the previous sections. Precisely, our main purpose is to check the convergence property of the numerical solution with respect to the step size.

Example 6.1. Consider the problem (2.1) with $a = 2, u_0 = \log 2$,

$$f(t,u) = \frac{\Gamma(5+\alpha)}{24}\left(\log \frac{t}{2}\right)^4 + \left(\log \frac{t}{2}\right)^{4+\alpha} + \log 2 - u(t).$$

It can be verified that the corresponding exact solution is

$$u(t) = \left(\log \frac{t}{2}\right)^{4+\alpha} + \log 2.$$

All the results reported in this example correspond to the numerical solution captured at $T = 3$. In Table 6.1, we list the maximum errors

$$Err_\infty(\Delta t) = \max_{0 \le i \le 2N} |u(t_i) - u_i|$$

as a function of Δt for $\alpha = 0.3, 0.5$ and 0.7. Also shown are the corresponding decay rates, using the formula as

$$\text{Rate} = \log_2\left(\frac{Err_\infty(\Delta t)}{Err_\infty(\Delta t/2)}\right).$$

From Table 6.1, it is observed that the convergence rate is close to $3+\alpha$. This is in a good agreement with the theoretical prediction.

Table 6.1: Maximum errors and decay rate as functions of Δt for Example 6.1.

Δt	$\alpha = 0.3$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.7$	Rate
1/10	2.7749E-06	–	2.5313E-06	–	1.6310E-06	–
1/20	2.8863E-07	3.2652	2.2719E-07	3.4779	1.2826E-07	3.6687
1/40	2.9980E-08	3.2671	2.0753E-08	3.4526	1.0376E-08	3.6277
1/80	3.0962E-09	3.2755	1.8911E-09	3.4560	8.4198E-10	3.6233
1/160	3.1818E-10	3.2826	1.7130E-10	3.4646	6.7950E-11	3.6312
1/320	3.2561E-11	3.2886	1.5422E-11	3.4734	5.4405E-12	3.6427

Example 6.2. Consider the problem (2.1) with the following right-hand side function:

$$a = 1, \quad u_0 = 0, \quad f(t) = \frac{\Gamma(5+\alpha)}{\Gamma(5)}(\log t)^4 + (\log t)^{8+2\alpha} - u^2.$$

The exact solution is $u(t) = (\log t)^{4+\alpha}$ in this case. It is notable that in the present example, the right-hand side function f is a nonlinear function of u, while the previous example addresses a right-hand side function linearly dependent of u.

We take $T = 2$ and repeat the same calculation as in the Example 6.1 by using the proposed scheme. Table 6.2 shows the maximum errors and decay rates as functions of the time step size for $\alpha = 0.2, 0.4$ and 0.6. Once again the obtained numerical results confirm that the convergence order of the scheme is $3 + \alpha$.

Table 6.2: Maximum errors and decay rate as functions of Δt for Example 6.2.

Δt	$\alpha = 0.2$	Rate	$\alpha = 0.4$	Rate	$\alpha = 0.6$	Rate
1/10	3.5723E-05	–	3.8279E-05	–	2.6428E-05	–
1/20	4.2326E-06	3.0772	4.0699E-06	3.2334	2.5760E-06	3.3589
1/40	4.8136E-07	3.1364	4.1210E-07	3.3040	2.3752E-07	3.4390
1/80	5.3812E-08	3.1611	4.0861E-08	3.3342	2.1105E-08	3.4924
1/160	5.9477E-09	3.1775	3.9857E-09	3.3578	1.8364E-09	3.5226
1/320	6.5316E-10	3.1868	3.8480E-10	3.3727	1.5746E-10	3.5438

Example 6.3. The third example is to test the accuracy of the present numerical scheme, where the analytical solution is unknown. We consider the problem (2.1) with the following right-hand side function:

$$a = 1, \quad u_0 = 0, \quad f(t) = (t-1)^5.$$

To test the accuracy for the present scheme, we chose $T = 2$, and varied the grid size Δt. With this choice, we expect that the numerical solution should be convergent to the exact solution with the error $\mathcal{O}(\Delta t^{3+\alpha})$ based on Theorem 5.2. Since the exact solution is unknown, we calculate the maximum error using

$$Err_\infty(\Delta t) = \max_i \left| u_i(\Delta t) - u_i\left(\frac{\Delta t}{2}\right)\right|.$$

We further estimate the rate of convergence using the formula as

$$\text{Rate} = \log_2\left(\frac{Err_\infty(\Delta t)}{Err_\infty(\Delta t/2)}\right).$$

Results were listed in Table 6.3. One may see that the convergence rate is close to $3+\alpha$ for all cases, which coincides with what we expected.

Table 6.3: Maximum errors and decay rate as functions of Δt for Example 6.3.

Δt	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.7$	Rate
1/10	2.0926E-04	–	1.6923E-04	–	8.3173E-05	–
1/20	2.6080E-05	3.0043	1.7533E-05	3.2708	7.7397E-06	3.4258
1/40	3.0618E-06	3.0905	1.7146E-06	3.3542	6.8139E-07	3.5057
1/80	3.4788E-07	3.1377	1.6186E-07	3.4050	5.7778E-08	3.5599
1/160	3.8808E-08	3.1642	1.4948E-08	3.4367	4.7776E-09	3.5962
1/320	4.2829E-09	3.1797	1.3615E-09	3.4567	3.8854E-10	3.6202

7. Concluding Remarks

In the present work, we have first presented some regularity properties of the solution to the Caputo-Hadamard fractional differential equation. Then an efficient high order scheme was constructed and analyzed for the considered problem. The stability and convergence analysis was carried out to prove that the proposed scheme is stable under a reasonable restriction on the step size. The obtained error estimate shows that the proposed scheme is of order $3+\alpha$. Finally, two numerical examples were provided to confirm the efficiency of the proposed method.

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