Journal of Computational Mathematics
Vol.xx, No.x, 2024, 1–32.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2312-m2023-0185

CRANK-NICOLSON GALERKIN APPROXIMATIONS FOR LOGARITHMIC KLEIN-GORDON EQUATION*

Fang Chen and Meng Li[1)]
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: limeng@zzu.edu.cn
Yanmin Zhao
School of Science and Henan Joint International Research Laboratory of High Performance Computation for Complex Systems, Xuchang University, Xuchang 461000, China
Email: zhaoym@lsec.cc.ac.cn

Abstract

This paper presents three regularized models for the logarithmic Klein-Gordon equation. By using a modified Crank-Nicolson method in time and the Galerkin finite element method (FEM) in space, a fully implicit energy-conservative numerical scheme is constructed for the local energy regularized model that is regarded as the best one among the three regularized models. Then, the cut-off function technique and the time-space error splitting technique are innovatively combined to rigorously analyze the unconditionally optimal and high-accuracy convergence results of the numerical scheme without any coupling condition between the temporal step size and the spatial mesh width. The theoretical framework is uniform for the other two regularized models. Finally, numerical experiments are provided to verify our theoretical results. The analytical techniques in this work are not limited in the FEM, and can be directly extended into other numerical methods. More importantly, this work closes the gap for the unconditional error/stability analysis of the numerical methods for the logarithmic systems in higher dimensional spaces.

Mathematics subject classification: 65N30, 65N06, 65N12.
Key words: Logarithmic Klein-Gordon equation, Finite element method, Cut-off, Error splitting technique, Convergence.

1. Introduction

In this paper, we consider the Klein-Gordon equation with the logarithmic nonlinear term (LogKGE)

$$u_{tt}(\mathbf{x},t)-\Delta u(\mathbf{x},t)+u(\mathbf{x},t)+\lambda u(\mathbf{x},t)f\left(|u(\mathbf{x},t)|^2\right)=0,\quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1a}$$

$$u(\mathbf{x},0)=\phi_0(\mathbf{x}),\quad u_t(\mathbf{x},0)=\phi_1(\mathbf{x}),\qquad \mathbf{x}\in\Omega, \tag{1.1b}$$

$$u(\mathbf{x},t)=0,\qquad (\mathbf{x},t)\in\partial\Omega\times[0,T], \tag{1.1c}$$

where $u(\mathbf{x},t)$ is a real valued scalar field, λ is a parameter measuring the force of the nonlinear interaction, $\phi_0(\mathbf{x})$ and $\phi_1(\mathbf{x})$ are given sufficiently smooth functions, $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ is a bounded convex polygonal or polyhedral domain fixed on a Lipschitz continuous boundary $\partial\Omega$, and

$$f(\rho)=\ln\rho,\quad \rho=|u(\mathbf{x},t)|^2>0. \tag{1.2}$$

* Received September 2, 2023 / Revised version received November 9, 2023 / Accepted December 4, 2023 / Published online April 4, 2024 /
1) Corresponding author

The LogKGE (1.1) admits the law of energy conservation defined by

$$E(t) = \int_\Omega \left(|u_t(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 + |u(\mathbf{x},t)|^2 + \lambda F\left(|u(\mathbf{x},t)|^2\right)\right) d\mathbf{x} \equiv E(0), \quad t \in [0,T], \tag{1.3}$$

where $u(\cdot,t) \in H^1(\mathbb{R}^d), u_t(\cdot,t) \in L^2(\mathbb{R}^d)$ and

$$F(\rho) = \int_0^\rho f(s)ds = \int_0^\rho \ln s ds = \rho \ln \rho - \rho, \quad \rho > 0. \tag{1.4}$$

The logarithmic nonlinearity is widely used in various physical models for different fields of research, such as the logarithmic Schrödinger equation (LogSE) established in quantum mechanics or quantum optics [15, 16], the logarithmic Korteweg-de Vries equation and logarithmic Kadomtsev-Petviashvili equation applied to characterize oceanography and fluid dynamics [22, 45], the Cahn-Hilliard equation with logarithmic potentials [18, 20] studied in material sciences, and so on. Additionally, the LogKGE is regarded as the relativistic version of the LogSE [13], which has been introduced into the quantum field theory by Rosen [37]. This equation has attracted widespread attention due to its fundamental importance in the study of quantum field theory and its connection to various physical phenomena.

In the past decades, many scholars have devoted themselves to studying the well-posedness of the Cauchy problem for LogKGEs. Bartkowski *et al.* [13] proved the existence and uniqueness of weak solutions and classical solutions for one-dimensional LogKGE. Later, Natali *et al.* [35] gave the orbital stability results of periodic standing waves of one-dimensional LogKGE. By employing the auxiliary equation method, Alzaleq *et al.* [2] found new bounded and unbounded exact traveling wave solutions for LogKGE with three different forms. In [46], the author indicated that LogKGE possessed Gaussons: Solitary wave solutions of Gaussian shape. Since the analytical solutions of most nonlinear Klein-Gordon equations are not easy to find, a series of numerical methods have been considered, including finite difference methods (FDMs) [8, 11, 12, 14, 30, 50], FEMs [17, 24, 44], spectral methods [9], exponential wave integrator [8] and operator splitting [10] Fourier pseudospectral methods, and so on. However, due to the blow-up of the logarithmic nonlinear term near the origin, these numerical methods cannot be directly applied to logarithmic equations.

In order to avoid the blow-up, Bao *et al.* [5,6] proposed a regularized FDM and a regularized splitting method for LogSE, and established their error bound. Li *et al.* [27] applied the FDM to solve the numerical solutions of the regularized LogSE in an unbounded domain. Later, for the LogKGE, two energy-conservative regularized FDMs were employed and their error estimates were obtained [48, 49]. It is well known that logarithmic function will only appear numerical blow-up when $\rho \to 0^+$, and this phenomenon will not occur when the value of ρ is large. Therefore, Bao *et al.* [7] recently presented an energy regularized logarithmic Schrödinger equation (ERLogSE) through local energy regularization (LER) technique, that is, a sequence of polynomials approximation to the interaction energy density $F(\rho)$ at near origin. Inspired by above works, Yan *et al.* [47] extended the LER technique to the LogKGE. A conservative Crank-Nicolson FDM and an explicit FDM were raised for the obtained ERLogKGE. Through the above analysis and our knowledge, it is found that there exists no research focusing on the FEM for the LogKGE. However, we must emphasize that the finite element discretization allows us to work in a very low regularity states, which cannot be done by spectral methods or FDMs. Additionally, the FEM exhibits excellent adaptability to complex geometric regions and boundary conditions. In this work, we aim to bridge this gap by developing an energy-conservative FEM for the LogKGE (1.1).

Conservative numerical schemes have been widely studied because they can capture more detailed physical processes by preserving some invariant properties (see [1, 3, 4, 39, 41, 47] and references therein). And the existence of these conservation properties is crucial to ensure the stability and convergence of numerical schemes [19]. In addition, through extensive numerical experiments, Sanz-Serna *et al.* [39] have found that conservative schemes perform better than nonconservative ones, as the latter may be prone to linear blow-up. A classical energy-conservative method for the some nonlinear PDEs is the modified Crank-Nicolson scheme studied by Sanz-Serna [38]. Unfortunately, it is always a full-implicit scheme that presents significant challenges in error analysis, especially for the FEM. In both classical papers [1, 39], the optimal L^2-norm error estimates for modified Crank-Nicolson FEM were analyzed, but they need a time-space ratio constraint. Bao *et al.* [3, 4] established uniform error estimates for the modified Crank-Nicolson FDM, and similar coupling condition was obtained. For the LogKGE (1.1), Yan *et al.* [49] studied the optimal H^1-seminorm error estimates of the Crank-Nicolson FDM in one-dimension space. However, if one wants to generalize the analytical methods to high-dimension space, the time-space ratio constraint must be required.

With the purpose of eliminating the time-space ratio restrictions of the error estimates, a time-space error splitting technique was proposed in [25, 26], which has been widely utilized in error estimates of numerical schemes for a large number of nonlinear models [28, 29, 40, 43]. Indeed, removing the time-space ratio constraint for a numerical scheme can lead to significant improvements in computational efficiency, making it possible to solve larger and more complex problems with greater accuracy and efficiency, and better agreement with experimental data. However, the most applications of the time-space error splitting technique are always limited to the linearized numerical schemes, which may be not enough to analyze the unconditional error estimates of a full-implicit numerical scheme. From [1, 3, 4, 28, 40], we learn that cut-off function technique is an effective method to deal with general nonlinear numerical schemes, which can truncate nonlinear terms into global Lipschitz functions with compact support in d-dimensions (d=1, 2, 3). The cut-off function technique can ensure that once the continuous solutions or the time discrete solutions are bounded, the numerical solutions will not be too far away from them.

In this work, we present an energy conservative numerical scheme for the LogKGE (1.1), which uses the Galerkin FEM for space discretization and the Crank-Nicolson scheme for time discretization. Subsequently, we innovatively combine the time-space error splitting technique with the cut-off function technique to obtain the optimal error estimates of Crank-Nicolson FEM, which not only eliminates any coupling condition between the temporal step size and the spatial mesh width, but also overcomes the difficulties of the fully implicit scheme in error analysis. Moreover, we point that the error analytical method in this work can be naturally applicable to the cases of any dimension spaces, which can be regarded a great improvement compared with the existing references [47, 49]. In addition, in order to achieve the similar H^1-norm error convergence order as in [47, 49] under the condition of low regularity, we devote to the study of methods to improve the accuracy of finite element solutions. At present, one of the methods to improve accuracy is to adopt postprocessing technology, which involves performing some kind of processing on the numerical results obtained using FEMs. The seniors have already done a lot of excellent work in this regard, please refer to [23, 33, 36] and references therein. Lin *et al.* [31, 34] improved the global convergence order by using rectangular grids and combining interpolation postprocessing techniques. Shi *et al.* [29, 40] utilized interpolation postprocessing technique to obtain the global superconvergence results. Motivated by their works, we apply the interpolated postprocessing technique to study the numerical scheme of the ERLogKGE, resulting in significant improvements in the accuracy of finite element solutions in H^1-norm, while keeping the computational complexity within reasonable limits. This contribution is one of the key highlights of this work.

The overall structure of the paper is as follows. In Section 2, three regularized models of LogKGE are introduced. For the ERLogKGE, we propose its Crank-Nicolson FEM and prove the energy conservation of the numerical scheme in Section 3. In Section 4, the main results of this paper are stated, including unconditionally optimal and high accuracy convergence results. Section 5 is devoted to the proof of the convergence results by using the cut-off error splitting technique. In Section 6, we provide numerical experiments to verify the accuracy and validity of the theoretical results. Finally, some conclusions are drawn in Section 7.

2. Several Regularized Models for LogKGE

In order to avoid the blow-up phenomenon of the logarithmic nonlinearity at the origin, three regularized models [7, 47] for LogKGE (1.1) were established, which all rely on a small regularization parameter ε, $0 < \varepsilon \ll 1$. Their specific forms are as follows:

RLogKGE I:

$$u^\varepsilon_{tt}(\mathbf{x},t)-\Delta u^\varepsilon(\mathbf{x},t)+u^\varepsilon(\mathbf{x},t)+\lambda u^\varepsilon(\mathbf{x},t)\widetilde{f^\varepsilon}\left(|u^\varepsilon(\mathbf{x},t)|^2\right)=0, \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{2.1a}$$

$$u^\varepsilon(\mathbf{x},0)=\phi_0(\mathbf{x}), \quad u^\varepsilon_t(\mathbf{x},0)=\phi_1(\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{2.1b}$$

$$u^\varepsilon(\mathbf{x},t)=0, \qquad (\mathbf{x},t)\in\partial\Omega\times[0,T], \tag{2.1c}$$

where

$$\widetilde{f^\varepsilon}\left(|u^\varepsilon(\mathbf{x},t)|^2\right)=2\ln\left(\varepsilon+|u^\varepsilon(\mathbf{x},t)|\right)$$

with

$$\widetilde{F^\varepsilon}(\rho)=\int_0^\rho \widetilde{f^\varepsilon}(s)\,ds=\rho\ln(\varepsilon+\sqrt{\rho})^2+2\varepsilon\sqrt{\rho}-\rho-\varepsilon^2\ln\left(1+\frac{\sqrt{\rho}}{\varepsilon}\right)^2, \quad \rho=|u^\varepsilon(\mathbf{x},t)|^2.$$

RLogKGE II:

$$u^\varepsilon_{tt}(\mathbf{x},t)-\Delta u^\varepsilon(\mathbf{x},t)+u^\varepsilon(\mathbf{x},t)+\lambda u^\varepsilon(\mathbf{x},t)\widehat{f^\varepsilon}\left(|u^\varepsilon(\mathbf{x},t)|^2\right)=0, \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{2.2a}$$

$$u^\varepsilon(\mathbf{x},0)=\phi_0(\mathbf{x}), \quad u^\varepsilon_t(\mathbf{x},0)=\phi_1(\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{2.2b}$$

$$u^\varepsilon(\mathbf{x},t)=0, \qquad (\mathbf{x},t)\in\partial\Omega\times[0,T], \tag{2.2c}$$

where

$$\widehat{f^\varepsilon}\left(|u^\varepsilon(\mathbf{x},t)|^2\right)=\ln\left(\varepsilon^2+|u^\varepsilon(\mathbf{x},t)|^2\right)$$

with

$$\widehat{F^\varepsilon}(\rho)=\int_0^\rho \widehat{f^\varepsilon}(s)\,ds=(\varepsilon^2+\rho)\ln(\varepsilon^2+\rho)-\rho-2\varepsilon^2\ln\varepsilon, \quad \rho=|u^\varepsilon(\mathbf{x},t)|^2.$$

The LER technique is that we regularize the energy density function $F(\rho)$ only locally in the region $\{\rho:\rho<\varepsilon^2\}$ by a sequence of polynomials, and keep it unchanged in $\{\rho:\rho>\varepsilon^2\}$, i.e.

$$F^\varepsilon_n(\rho)=F(\rho)\ \chi_{\{\rho\geq\varepsilon^2\}}+P^\varepsilon_{n+1}(\rho)\chi_{\{\rho<\varepsilon^2\}}, \quad n\geq 2, \tag{2.3}$$

where χ_A is the characteristic function of the set A, and P^ε_{n+1} is a polynomial of degree $n+1$, and the specific expression is as follows (please refer to literature [47] for the detailed derivation process):

$$P^\varepsilon_{n+1}(\rho) = \rho\left(\ln(\varepsilon^2) - 1 - \sum_{k=1}^{n}\frac{1}{k}\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right). \tag{2.4}$$

Differentiating (2.3) with respect to ρ and utilizing (1.4) and (2.4), we get

$$\begin{aligned} f^\varepsilon_n(\rho) &= \left(F^\varepsilon_n\right)'(\rho) = \ln\rho\ \chi_{\{\rho\ge\varepsilon^2\}} + \left(P^\varepsilon_{n+1}\right)'(\rho)\ \chi_{\{\rho<\varepsilon^2\}} \\ &= \ln\rho\chi_{\{\rho\ge\varepsilon^2\}} + \left(\ln(\varepsilon^2) - \frac{n+1}{n}\left(1-\frac{\rho}{\varepsilon^2}\right)^n - \sum_{k=1}^{n-1}\frac{1}{k}\left(1-\frac{\rho}{\varepsilon^2}\right)^k\right)\chi_{\{\rho<\varepsilon^2\}}, \quad \rho\ge 0. \end{aligned} \tag{2.5}$$

Therefore, the ERLogKGE is given as follows:

ERLogKGE:

$$u^\varepsilon_{tt}(\mathbf{x},t) - \Delta u^\varepsilon(\mathbf{x},t) + u^\varepsilon(\mathbf{x},t) + \lambda u^\varepsilon(\mathbf{x},t) f^\varepsilon_n\left(|u^\varepsilon(\mathbf{x},t)|^2\right) = 0, \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{2.6a}$$

$$u^\varepsilon(\mathbf{x},0) = \phi_0(\mathbf{x}), \quad u^\varepsilon_t(\mathbf{x},0) = \phi_1(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{2.6b}$$

$$u^\varepsilon(\mathbf{x},t) = 0, \quad (\mathbf{x},t)\in\partial\Omega\times[0,T], \tag{2.6c}$$

where f^ε_n is defined by (2.5).

Remark 2.1. In ERLogKGE, we demand $F^\varepsilon_n(\rho)\in C^n([0,+\infty))$, and $F^\varepsilon_n(0) = F(0) = 0$ that allows the regularized energy to be well-defined on the whole space. Meanwhile, as the derivative of $F^\varepsilon_n(\rho)$, we observe that $f^\varepsilon_n(\rho)\in C^{n-1}([0,+\infty))$ for $n\ge 2$.

Remark 2.2. Notice that RLogKGE I and RLogKGE II are global regularized models for LogKGE (1.1), involving a direct regularization of $f(\rho)$ in (1.2). In theory, however, logarithmic functions only blow-up when $\rho\to 0^+$, not when ρ is large. Therefore, the ERLogKGE regularizes $F(\rho)$ only locally in the region $\{\rho:\rho<\varepsilon^2\}$, and keep it unchanged in $\{\rho:\rho>\varepsilon^2\}$. Moreover, some numerical experiments have shown that the ERLogKGE is better than other two global regularized models [47].

Remark 2.3. For convenience, we only focus on the numerical method of the ERLogKGE (2.6) with $\lambda = 1$. In what follows, we will build an energy-conservative numerical scheme of the ERLogKGE (2.6), and carry out a series of theoretical analyses on the constructed numerical method, including conservation, and unconditional optimal and high-accuracy convergence. It should be noted that all theories are equally applicable to the other two models, and we only verify them through some numerical examples.

3. Energy-conservative Finite Element Numerical Scheme

Let Ω be a bounded and convex polygon in $\mathbb{R}^2$ (or polyhedron in $\mathbb{R}^3$). We define $\mathcal{T}_h$ as a quasi-uniform partition of Ω, dividing Ω into M elements by triangles or rectangles in $\mathbb{R}^2$ (or tetrahedra or hexahedra in $\mathbb{R}^3$). Define $h = \max_{T_i\in\mathcal{T}_h}\{\operatorname{diam} T_i\}, i = 1,2,\ldots,M$, as the maximum diameter of the element. Let $W_h\subset H^1_0(\Omega)$ be a finite dimensional subspace, which consists of continuous piecewise polynomials of degree r ($r\ge 1$) on $\mathcal{T}_h$. Below we give specific definitions about some finite element spaces in $\mathbb{R}^2$. For Q_1 finite element space, let $\widehat{T}{:=}[-1,1]\times[-1,1]$ be the reference element on $\widehat{x}-\widehat{y}$ plane with the vertices $\widehat{A}_1:=(-1,-1)$, $\widehat{A}_2:=(1,-1)$, $\widehat{A}_3:=(1,1)$ and $\widehat{A}_4:=(-1,1)$. Then, the conforming finite element $\{\widehat{T},\widehat{P},\widehat{\Sigma}\}$ can be defined as

$$\widehat{P}=\operatorname{span}\{1,\widehat{x},\widehat{y},\widehat{xy}\},\quad \widehat{\Sigma}=\big\{\widehat{v}\big(\widehat{A}_i\big),\, i=1,2,3,4\big\}.$$

For P_1 finite element space, let $\widehat{T}$ be an isosceles right triangle on $\lambda_1-\lambda_2$ reference plane with the vertices $\widehat{A}_1:=(1,0)$, $\widehat{A}_2:=(0,1)$ and $\widehat{A}_3:=(0,0)$. Then, the conforming finite element $\{\widehat{T},\widehat{P},\widehat{\Sigma}\}$ can be defined as

$$\widehat{P}=\operatorname{span}\{\lambda_1,\lambda_2,1-\lambda_1-\lambda_2\},\quad \widehat{\Sigma}=\big\{\widehat{v}\big(\widehat{A}_i\big),\, i=1,2,3\big\}.$$

Define the Ritz projection operator R_h: $H_0^1(\Omega)\to W_h$ by [42]

$$\big(\nabla(v-R_hv),\nabla\omega_h\big)=0,\quad \forall\,\omega_h\in W_h, \tag{3.1}$$

which satisfies for any $v\in H^s(\Omega)\cap H_0^1(\Omega)$,

$$\|v-R_hv\|_{L^2}+h\|\nabla(v-R_hv)\|_{L^2}\le C_\Omega h^s\|v\|_{H^s},\quad 1\le s\le r+1, \tag{3.2}$$

where C_Ω is a constant independent of h. For bilinear elements, there exists a constant C_{I_h} independent of h, which satisfies [42]

$$\|I_hv-R_hv\|_{H^1}\le C_{I_h}h^2\|v\|_{H^3},\quad \forall\,v\in H_0^1(\Omega)\cap H^3(\Omega), \tag{3.3}$$

where $I_h: v\in H^1(\Omega)\to I_hv\in W_h$ be the associated interpolation operator. Recall the inverse inequality in the finite element space, where there exists a constant C_{inv} independent of h such that

$$\|\omega_h\|_{L^\infty}\le C_{inv}h^{-\frac{d}{2}}\|\omega_h\|_{L^2},\quad \forall\,\omega_h\in W_h. \tag{3.4}$$

It should be noted that if the mesh partition is quasi-uniform, then (3.2) and (3.4) are always valid in finite element space.

Let $\{t_k|\ t_k=k\tau, 0\le k\le N\}$ be a uniform partition of $[0,T]$ with the time step $\tau=T/N$. For convenience, we let $u^k:=u(\cdot,t_k)$, $k=1,2,\ldots,N$, and define the following operators:

$$\begin{aligned}
&\delta_t^2\omega^k=\frac{\omega^{k+1}-2\omega^k+\omega^{k-1}}{\tau^2},\quad \delta_t^+\omega^k=\frac{\omega^{k+1}-\omega^k}{\tau},\quad \delta_t^-\omega^k=\frac{\omega^k-\omega^{k-1}}{\tau},\\
&\delta_t\omega^k=\frac{\omega^{k+1}-\omega^{k-1}}{2\tau},\qquad \widehat{\omega^k}=\frac{\omega^{k+1}+\omega^{k-1}}{2}.
\end{aligned}$$

Based on the above preparations, the Crank-Nicolson FEM is defined for ERLogKGE (2.6) to seek $U_h^{\varepsilon,k+1}\in W_h$ such that

$$\begin{aligned}
&\big(\delta_t^2U_h^{\varepsilon,k},\omega_h\big)+\big(\nabla\widehat{U_h^{\varepsilon,k}},\nabla\omega_h\big)+\big(\widehat{U_h^{\varepsilon,k}},\omega_h\big)\\
&\quad+\big(G_n^\varepsilon\big(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\big),\omega_h\big)=0,\quad \forall\,\omega_h\in W_h
\end{aligned} \tag{3.5}$$

for $k=1,2,\ldots,N-1$, where the initial and first step finite element solutions are determined by

$$U_h^{\varepsilon,0}=R_h\phi_0,\quad U_h^{\varepsilon,1}=R_h\bigg(\phi_0+\tau\phi_1+\frac{\tau^2}{2}\left(\Delta\phi_0-\phi_0-\phi_0f_n^\varepsilon\left(|\phi_0|^2\right)\right)\bigg). \tag{3.6}$$

Here, $G_n^\varepsilon(z_1,z_2)$ is defined for $z_1,z_2\in\mathbb{R}$ as

$$\begin{aligned} G_n^\varepsilon(z_1,z_2) &:= \int_0^1 f_n^\varepsilon\left(\theta\ |z_1|^2+(1-\theta)\,|z_2|^2\right)d\theta\cdot\frac{z_1+z_2}{2}\\ &= \frac{F_n^\varepsilon\left(|z_1|^2\right)-F_n^\varepsilon\left(|z_2|^2\right)}{|z_1|^2-|z_2|^2}\cdot\frac{z_1+z_2}{2} \end{aligned} \tag{3.7}$$

with F_n^ε is defined by (2.3).

Theorem 3.1. *The fully discrete scheme* (3.5) *satisfies the energy conservation law, i.e.*

$$E^{\varepsilon,N-1}=E^{\varepsilon,N-2}=\cdots=E^{\varepsilon,0},$$

where

$$\begin{aligned} E^{\varepsilon,k} := &\frac{1}{2\tau}\left\|\delta_t^+U_h^{\varepsilon,k}\right\|_{L^2}^2+\frac{1}{4\tau}\left(\left|U_h^{\varepsilon,k+1}\right|_{H^1}^2+\left|U_h^{\varepsilon,k}\right|_{H^1}^2\right)+\frac{1}{4\tau}\left(\left\|U_h^{\varepsilon,k+1}\right\|_{L^2}^2+\left\|U_h^{\varepsilon,k}\right\|_{L^2}^2\right)\\ &+\frac{1}{4\tau}\int_\Omega\left(F_n^\varepsilon\left(\left|U_h^{\varepsilon,k+1}\right|^2\right)+F_n^\varepsilon\left(\left|U_h^{\varepsilon,k}\right|^2\right)\right)d\mathbf{x},\quad 0\le k\le N-1. \end{aligned}$$

Proof. Taking $\omega_h=\delta_tU_h^{\varepsilon,k}$ in (3.5) gives

$$\begin{aligned} &\left(\delta_t^2U_h^{\varepsilon,k},\delta_tU_h^{\varepsilon,k}\right)+\left(\nabla\widehat{U_h^{\varepsilon,k}},\nabla\delta_tU_h^{\varepsilon,k}\right)+\left(\widehat{U_h^{\varepsilon,k}},\delta_tU_h^{\varepsilon,k}\right)\\ &\quad+\left(G_n^\varepsilon\left(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\right),\delta_tU_h^{\varepsilon,k}\right)=0. \end{aligned} \tag{3.8}$$

By using

$$\delta_t^2U_h^{\varepsilon,k}=\frac{1}{\tau}\left(\delta_t^+U_h^{\varepsilon,k}-\delta_t^+U_h^{\varepsilon,k-1}\right),\quad \delta_tU_h^{\varepsilon,k}=\frac{1}{2}\left(\delta_t^+U_h^{\varepsilon,k}+\delta_t^+U_h^{\varepsilon,k-1}\right),$$

we can easily get

$$\begin{aligned} \left(\delta_t^2U_h^{\varepsilon,k},\delta_tU_h^{\varepsilon,k}\right) &= \frac{1}{2\tau}\left(\delta_t^+U_h^{\varepsilon,k}-\delta_t^+U_h^{\varepsilon,k-1},\delta_t^+U_h^{\varepsilon,k}+\delta_t^+U_h^{\varepsilon,k-1}\right)\\ &= \frac{1}{2\tau}\left(\left\|\delta_t^+U_h^{\varepsilon,k}\right\|_{L^2}^2-\left\|\delta_t^+U_h^{\varepsilon,k-1}\right\|_{L^2}^2\right). \end{aligned} \tag{3.9}$$

It is obvious that

$$\begin{aligned} \left(\nabla\widehat{U_h^{\varepsilon,k}},\nabla\delta_tU_h^{\varepsilon,k}\right) &= \frac{1}{4\tau}\left(\nabla U_h^{\varepsilon,k+1}+\nabla U_h^{\varepsilon,k-1},\nabla U_h^{\varepsilon,k+1}-\nabla U_h^{\varepsilon,k-1}\right)\\ &= \frac{1}{4\tau}\left(\left|U_h^{\varepsilon,k+1}\right|_{H^1}^2-\left|U_h^{\varepsilon,k-1}\right|_{H^1}^2\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \left(\widehat{U_h^{\varepsilon,k}},\delta_tU_h^{\varepsilon,k}\right) &= \frac{1}{4\tau}\left(U_h^{\varepsilon,k+1}+U_h^{\varepsilon,k-1},U_h^{\varepsilon,k+1}-U_h^{\varepsilon,k-1}\right)\\ &= \frac{1}{4\tau}\left(\left\|U_h^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|U_h^{\varepsilon,k-1}\right\|_{L^2}^2\right). \end{aligned} \tag{3.11}$$

Meanwhile, according to the definition of G_n^ε in (3.7), we have

$$\begin{aligned} &\left(G_n^\varepsilon\left(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\right),\delta_tU_h^{\varepsilon,k}\right)\\ &=\int_\Omega\frac{F_n^\varepsilon\left(\left|U_h^{\varepsilon,k+1}\right|^2\right)-F_n^\varepsilon\left(\left|U_h^{\varepsilon,k-1}\right|^2\right)}{\left|U_h^{\varepsilon,k+1}\right|^2-\left|U_h^{\varepsilon,k-1}\right|^2}\cdot\frac{U_h^{\varepsilon,k+1}+U_h^{\varepsilon,k-1}}{2}\cdot\delta_tU_h^{\varepsilon,k}d\mathbf{x}\\ &=\frac{1}{4\tau}\int_\Omega\left(F_n^\varepsilon\left(\left|U_h^{\varepsilon,k+1}\right|^2\right)-F_n^\varepsilon\left(\left|U_h^{\varepsilon,k-1}\right|^2\right)\right)d\mathbf{x}. \end{aligned} \tag{3.12}$$

Substituting (3.9)-(3.12) into (3.8) gives

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|\delta_t^{+}U_h^{\varepsilon,k}\right\|_{L^2}^2-\left\|\delta_t^{+}U_h^{\varepsilon,k-1}\right\|_{L^2}^2\right)+\frac{1}{4\tau}\left(\left|U_h^{\varepsilon,k+1}\right|_{H^1}^2-\left|U_h^{\varepsilon,k-1}\right|_{H^1}^2\right)\\&\quad+\frac{1}{4\tau}\left(\left\|U_h^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|U_h^{\varepsilon,k-1}\right\|_{L^2}^2\right)\\&\quad+\frac{1}{4\tau}\int_\Omega\left(F_n^\varepsilon\left(\left|U_h^{\varepsilon,k+1}\right|^2\right)-F_n^\varepsilon\left(\left|U_h^{\varepsilon,k-1}\right|^2\right)\right)d\mathbf{x}=0,\quad 1\le k\le N-1,\end{aligned}$$

which completes the proof. □

Remark 3.1. In Theorem 3.1, the law of energy conservation has been proven. However, the energy here is not a discrete version of the energy (1.3) of the original equation and it corresponds only to the energy of the ERLogKGE. Thankfully, we have learned that the energy of ERLogKGE converges twice to the energy of the original equation depending on the small regularization parameter ε, i.e $\mathcal{O}(\varepsilon^2)$, which has been proven in reference [47].

4. Main Results

Suppose that the solution of system (2.6) exists and satisfies

$$\begin{aligned}&\|\phi_0\|_{H^{r+1}(\Omega)}+\|u^\varepsilon\|_{L^\infty(0,T;H^{r+1}(\Omega))}+\left\|u_t^\varepsilon\right\|_{L^2(0,T;H^2(\Omega))}\\&\quad+\left\|u_{tt}^\varepsilon\right\|_{L^2(0,T;H^{r+1}(\Omega))}+\left\|u_{tttt}^\varepsilon\right\|_{L^2(0,T;H^1(\Omega))}\le C_r.\end{aligned}\tag{4.1}$$

Under the regularity assumption (4.1), we define

$$K_0:=\max_{1\le k\le N}\|u^{\varepsilon,k}\|_{L^\infty(\Omega)}+1,\tag{4.2}$$

where K_0 is a constant independent of τ,h and N.

Next, we will state the first main result of this paper. Under the assumption of regularity (4.1), the unconditional optimal error estimates of the modified Crank-Nicolson finite element scheme under the L^2-norm and H^1-norm are presented.

Theorem 4.1. *Assume that the system* (2.6) *has unique solution u^ε satisfying* (4.1)*. Then the fully discrete scheme* (3.5)-(3.6) *has unique solution $U_h^{\varepsilon,k}$. Moreover, there exist $\tau_1>0,h_1>0$, such that when $\tau\le\tau_1$ and $h\le h_1$, there are*

$$\left\|u^{\varepsilon,k}-U_h^{\varepsilon,k}\right\|_{L^2}\le C(\tau^2+h^{r+1}),\quad \left\|\nabla\big(u^{\varepsilon,k}-U_h^{\varepsilon,k}\big)\right\|_{L^2}\le C(\tau^2+h^r),\quad 0\le k\le N,\tag{4.3}$$

where C is a positive constant that is independent of τ and h but related to u^ε. In addition, it could be different in different places.

Remark 4.1. The optimal error estimates in Theorem 4.1 are established without the restriction of time-space ratio, which is different from the work in [49] that must require $\tau\le\mathcal{O}(|\ln h|^{-1/2})$ in two-dimensions.

In the following theorem, we shall show such a phenomenon that when the rectangle meshes are used for quasi-uniform partition of $\Omega\subset\mathbb{R}^2$ and bilinear finite element is selected for $W_h\subset H_0^1(\Omega)$, the error between finite element interpolation and finite element solution is much smaller than that between analytic solution and finite element solution in H^1-seminorm sense.

Theorem 4.2. *Assuming* (3.3) *and* (4.3) *hold. Then, the superclose results are obtained*

$$\left\|\nabla\left(I_h u^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} \leq C(\tau^2 + h^2), \quad 0 \leq k \leq N, \tag{4.4}$$

where the interpolation operator I_h is the same as in (3.3).

Remark 4.2. By comparing (4.4) with the second inequality of $r = 1$ in (4.3), it can be seen that the convergence order of (4.4) is one order higher in space under H^1-seminorm, which is the phenomenon of superclose.

Under the premise of (4.4), we can further obtain the following high-accuracy convergence results by using interpolated postprocessing technique, which is another main contribution of this paper.

Theorem 4.3. *Under the assumptions of Theorem* 4.2*, we have*

$$\left\|\nabla\left(u^{\varepsilon,k} - I_{2h} U_h^{\varepsilon,k}\right)\right\|_{L^2} \leq C(\tau^2 + h^2), \quad 0 \leq k \leq N, \tag{4.5}$$

where the interpolation postprocessing operator I_{2h} is defined in [32].

Remark 4.3. Take $r = 1$ in (4.3). Comparing (4.3) with (4.5), we find that the accuracy of the processed finite element solutions is improved by one order in space, that is, the global convergence order is developed from $\mathcal{O}(\tau^2 + h)$ to $\mathcal{O}(\tau^2 + h^2)$ under H^1-seminorm. In addition, at the same rate of convergence, the result in Theorem 4.5 allows us to use lower regularity compared to the Crank-Nicolson finite difference scheme [49].

Remark 4.4. At present, there are roughly two ways to improve the accuracy of finite element solutions: one is to encrypt the mesh or increase the number of piecewise polynomials in the finite element space, and the other is to adopt postprocessing technique. Using the first method will greatly increase computation, but the speed of computer technology development always cannot keep up with the demand for FEMs. However, the second approach only needs to add the postprocessing process to the finite element solutions, which improves the accuracy while the computation increases little. Therefore, we adopt interpolated postprocessing technique in this paper to improve the global accuracy of finite element solutions.

The proof of Theorems 4.1-4.3 will be given in Section 5.

5. Error Analysis

In this section, we will consider unconditional optimal error estimates for the fully discrete scheme. Then combine this result with interpolation postprocessing technique to obtain our high-accuracy convergence results.

5.1. Error estimates for the time discrete system

Now, we consider the following time discrete system for the modified Crank-Nicolson finite element scheme (3.5)-(3.6):

$$\delta_t^2 U^{\varepsilon,k} - \Delta\widehat{U^{\varepsilon,k}} + \widehat{U^{\varepsilon,k}} + G_n^{\varepsilon}(U^{\varepsilon,k+1}, U^{\varepsilon,k-1}) = 0, \quad k = 1, 2, \dots, N-1 \tag{5.1}$$

with the initial and boundary conditions

$$U^{\varepsilon,0}(\mathbf{x}) = \phi_0(\mathbf{x}), \quad U^{\varepsilon,1}(\mathbf{x}) = \phi_0(\mathbf{x}) + \tau\phi_1 + \frac{\tau^2}{2}\left(\Delta\phi_0 - \phi_0 - \phi_0 f_n^\varepsilon\left(|\phi_0|^2\right)\right), \quad \mathbf{x}\in\Omega, \tag{5.2}$$

$$U^{\varepsilon,k}(\mathbf{x}) = 0, \quad \mathbf{x}\in\partial\Omega, \quad k = 0,1,\ldots,N. \tag{5.3}$$

According to (5.1), we have

$$\delta_t^2 u^{\varepsilon,k} - \Delta\widehat{u^{\varepsilon,k}} + \widehat{u^{\varepsilon,k}} + G_n^\varepsilon(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) = R^{\varepsilon,k}, \quad k = 1,2\ldots,N-1, \tag{5.4}$$

where

$$\begin{aligned} R^{\varepsilon,k} = &\left(\delta_t^2 u^{\varepsilon,k} - u_{tt}^{\varepsilon,k}\right) - \Delta\left(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\right) + \left(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\right) \\ &+ \left(G_n^\varepsilon(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - u^{\varepsilon,k} f_n^\varepsilon\left(|u^{\varepsilon,k}|^2\right)\right). \end{aligned} \tag{5.5}$$

Denote

$$e_\tau^{\varepsilon,k} = u^{\varepsilon,k} - U^{\varepsilon,k}, \quad 0\le k\le N. \tag{5.6}$$

Since the fully discrete scheme (3.5)-(3.6) is a full-implicit scheme, the boundedness of the numerical solution $U_h^{\varepsilon,k}$ under the L^∞-norm cannot be obtained directly. To solve this problem, we establish an auxiliary problem to handle the nonlinearity G_n^ε using the cut-off function technique. This is the core of our theoretical analysis. Choose a smooth function $\mu(s)\in C^\infty(\mathbb{R})$ such that

$$\mu(s) = \begin{cases} 1, & |s|\in[0,1), \\ e^{1-\frac{1}{1-(1-|s|)^2}}, & |s|\in[1,2), \\ 0, & |s|\in[2,+\infty). \end{cases} \tag{5.7}$$

Define

$$g_{n,A}^\varepsilon(s) = s\mu\left(\frac{s}{K_0^2}\right), \quad f_{n,A}^\varepsilon(s) = f_n^\varepsilon(s)\mu\left(\frac{s}{K_0^2}\right), \tag{5.8}$$

where $s\ge 0, s\in\mathbb{R}$. The function $f_{n,A}^\varepsilon(s)$ will be used as a truncated function of $f_n^\varepsilon(s)$. Moreover, $f_{n,A}^\varepsilon(s)$ and $g_{n,A}^\varepsilon(s)$ are global Lipschitz functions with compact support in d-dimensions ($d = 1,2,3$), and the following properties are valid [3]:

$$\left|g_{n,A}^\varepsilon(s_1) - g_{n,A}^\varepsilon(s_2)\right| \le C_{K_0}\left|s_1 - s_2\right|, \tag{5.9}$$

$$\left|f_{n,A}^\varepsilon(s_1) - f_{n,A}^\varepsilon(s_2)\right| \le C_{K_0}\left|\sqrt{s_1} - \sqrt{s_2}\right|, \quad \forall\, s_1, s_2\ge 0, \tag{5.10}$$

where C_{K_0} is a positive and bounded constant that is related to K_0 and independent of τ and h.

In the following lemma, we will give the error estimates of the time discrete system and the L^∞-norm boundedness of the time discrete solution $U^{\varepsilon,k}$.

Lemma 5.1. *Suppose that u^ε is the solution of the system* (2.6) *satisfying the regularity of* (4.1)*, and the time discrete system* (5.1)-(5.3) *have unique solutions $U^{\varepsilon,k}, 0\le k\le N$. Then there exists $\tau_2 > 0$ such that when $\tau\le\tau_2$*

$$\left\|e_\tau^{\varepsilon,k}\right\|_{L^2} + \left\|\nabla e_\tau^{\varepsilon,k}\right\|_{L^2} \le C_1^*\tau^2, \tag{5.11}$$

$$\|U^{\varepsilon,k}\|_{H^2} \le C_2^*, \quad \|U^{\varepsilon,k}\|_{L^\infty} \le K_0, \tag{5.12}$$

where C_1^* and C_2^* are positive constants independent of τ and h.

Proof. Let us introduce the following auxiliary problem:

$$\delta_t^2 U_A^{\varepsilon,k} - \Delta \widehat{U_A^{\varepsilon,k}} + \widehat{U_A^{\varepsilon,k}} + G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big) = 0, \quad k = 1,2,\ldots,N-1, \tag{5.13}$$

$$U_A^{\varepsilon,0}(\mathbf{x}) = U^{\varepsilon,0}(\mathbf{x}), \quad U_A^{\varepsilon,1}(\mathbf{x}) = U^{\varepsilon,1}(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{5.14}$$

with $U_A^{\varepsilon,k}(\mathbf{x}) = 0$ on $\partial\Omega$ for $k = 0,1,\ldots,N$. Here, $G_{n,A}^{\varepsilon}(z_1,z_2)$ for $z_1, z_2 \in \mathbb{R}$ is

$$\begin{aligned} G_{n,A}^{\varepsilon}(z_1,z_2) &= \int_0^1 f_{n,A}^{\varepsilon}\left(\theta|z_1|^2 + (1-\theta)\,|z_2|^2\right) d\theta \; g_{n,A}^{\varepsilon}\left(\frac{z_1+z_2}{2}\right) \\ &= \frac{F_{n,A}^{\varepsilon}\left(|z_1|^2\right) - F_{n,A}^{\varepsilon}\left(|z_2|^2\right)}{|z_1|^2 - |z_2|^2} \; g_{n,A}^{\varepsilon}\left(\frac{z_1+z_2}{2}\right). \end{aligned} \tag{5.15}$$

From (5.13), we get

$$\delta_t^2 u^{\varepsilon,k} - \Delta \widehat{u^{\varepsilon,k}} + \widehat{u^{\varepsilon,k}} + G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) = R_A^{\varepsilon,k}, \quad 1 \le k \le N-1, \tag{5.16}$$

where

$$\begin{aligned} R_A^{\varepsilon,k} = {} & \big(\delta_t^2 u^{\varepsilon,k} - u_{tt}^{\varepsilon,k}\big) - \Delta\big(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\big) + \big(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\big) \\ & + \big(G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - u^{\varepsilon,k} f_n^{\varepsilon}\big(|u^{\varepsilon,k}|^2\big)\big). \end{aligned} \tag{5.17}$$

From (4.2) and the definition of $G_{n,A}^{\varepsilon}$, we find that $R_A^{\varepsilon,k} = R^{\varepsilon,k}$. Define

$$e_{\tau,A}^{\varepsilon,k} = u^{\varepsilon,k} - U_A^{\varepsilon,k}, \quad 0 \le k \le N.$$

Using Taylor's expansion and the definitions of $U_A^{\varepsilon,0}$ and $U_A^{\varepsilon,1}$, we obtain

$$e_{\tau,A}^{\varepsilon,0} = 0, \quad e_{\tau,A}^{\varepsilon,1} = u^{\varepsilon,1} - U_A^{\varepsilon,1} \le C\tau^2. \tag{5.18}$$

For $k \ge 2$, subtracting (5.16) from (5.13), the error equation is given

$$\delta_t^2 e_{\tau,A}^{\varepsilon,k} - \Delta \widehat{e_{\tau,A}^{\varepsilon,k}} + \widehat{e_{\tau,A}^{\varepsilon,k}} + G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big) = R_A^{\varepsilon,k}. \tag{5.19}$$

Multiplying (5.19) by $\delta_t e_{\tau,A}^{\varepsilon,k}$, and then integrating it over Ω arrives at

$$\begin{aligned} & \frac{1}{2\tau}\left(\big\|\delta_t^- e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|\delta_t^- e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2\right) + \frac{1}{4\tau}\left(\big\|\nabla e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|\nabla e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\right) \\ & \quad + \frac{1}{4\tau}\left(\big\|e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\right) \\ & = \big(R_A^{\varepsilon,k}, \delta_t e_{\tau,A}^{\varepsilon,k}\big) - \Big(G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big), \delta_t e_{\tau,A}^{\varepsilon,k}\Big). \end{aligned} \tag{5.20}$$

Next, we estimate the two terms on the right-hand side of (5.20). Using Taylor's expansion for u^{ε} at $t = t_k$, the following results are derived:

$$\big\|\delta_t^2 u^{\varepsilon,k} - u_{tt}^{\varepsilon,k}\big\|_{L^2}^2 \le \frac{\tau^3}{18}\int_{t_{k-1}}^{t_{k+1}} \big\|u_{tttt}^{\varepsilon}\big\|_{L^2}^2 dt, \tag{5.21}$$

$$\big\|\Delta(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k})\big\|_{L^2}^2 \le \frac{\tau^3}{2}\int_{t_{k-1}}^{t_{k+1}} \big\|u_{tt}^{\varepsilon}\big\|_{H^2}^2 dt, \tag{5.22}$$

$$\big\|\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\big\|_{L^2}^2 \le \frac{\tau^3}{2}\int_{t_{k-1}}^{t_{k+1}} \big\|u_{tt}^{\varepsilon}\big\|_{L^2}^2 dt. \tag{5.23}$$

Reviewing the definitions of the truncation function $f_{n,A}^{\varepsilon}(s), g_{n,A}^{\varepsilon}(s)$ and (4.2), we have

$$g_{n,A}^{\varepsilon}(u^{\varepsilon,k}) = u^{\varepsilon,k}, \quad f_{n,A}^{\varepsilon}(u^{\varepsilon,k}) = f_n^{\varepsilon}(u^{\varepsilon,k}), \quad f_{n,A}^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right) = f_n^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right). \tag{5.24}$$

According to (5.15) and (5.24), we derive

$$\begin{aligned}
&\left\|G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - u^{\varepsilon,k} f_n^{\varepsilon}\big(|u^{\varepsilon,k}|^2\big)\right\|_{L^2}^2 \\
=& \left\| \int_0^1 f_{n,A}^{\varepsilon}\left(\theta|u^{\varepsilon,k+1}|^2 + (1-\theta)|u^{\varepsilon,k-1}|^2\right) d\theta \; g_{n,A}^{\varepsilon}\big(\widehat{u^{\varepsilon,k}}\big) - u^{\varepsilon,k} f_n^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right)\right\|_{L^2}^2 \\
=& \Bigg\| \underbrace{\int_0^1 \Big[f_n^{\varepsilon}\left(\theta|u^{\varepsilon,k+1}|^2 + (1-\theta)|u^{\varepsilon,k-1}|^2\right) - f_n^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right)\Big] d\theta \; \widehat{u^{\varepsilon,k}}}_{=:\mathbf{I}} \\
&+ \underbrace{\int_0^1 f_n^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right) d\theta \; \big(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\big)}_{=:\mathbf{II}} \Bigg\|_{L^2}^2 .
\end{aligned} \tag{5.25}$$

By the differential mean value theorem, Taylor's expansion and $f_n^{\varepsilon} \in C^{n-1}([0,+\infty))$ for $n \geq 2$, we get

$$\begin{aligned}
\mathbf{I} \le& \left| \int_0^1 \Big[f_n^{\varepsilon}\big(\theta|u^{\varepsilon,k+1}|^2 + (1-\theta)|u^{\varepsilon,k-1}|^2\big) - f_n^{\varepsilon}\big(|u^{\varepsilon,k}|^2\big)\Big] d\theta \right| \\
\le& \left| \int_0^1 (f_n^{\varepsilon})'\big(\xi_1(\theta)\big)\Big[\theta\,|u^{\varepsilon,k+1}|^2 + (1-\theta)\,|u^{\varepsilon,k-1}|^2 - |u^{\varepsilon,k}|^2\Big] d\theta \right| \\
\le& \max_{\theta} \left|(f_n^{\varepsilon})'\big(\xi_1(\theta)\big)\right| \left| \int_0^1 \Big[\theta\,|u^{\varepsilon,k+1}|^2 + (1-\theta)|u^{\varepsilon,k-1}|^2 - |u^{\varepsilon,k}|^2\Big] d\theta \right| \\
\le& \max_{\theta} \left|(f_n^{\varepsilon})'\big(\xi_1(\theta)\big)\right| \left| \frac{|u^{\varepsilon,k+1}|^2 + |u^{\varepsilon,k-1}|^2}{2} - |u^{\varepsilon,k}|^2 \right| \\
\le& \max_{\theta} \left|(f_n^{\varepsilon})'\big(\xi_1(\theta)\big)\right| \left| \frac{1}{2}\int_{t_k}^{t_{k+1}} (t_{k+1}-t)\partial_{tt}\big(|u^{\varepsilon}|^2\big)(\mathbf{x},t)dt + \frac{1}{2}\int_{t_k}^{t_{k-1}} (t_{k-1}-t)\partial_{tt}\big(|u^{\varepsilon}|^2\big)(\mathbf{x},t)dt \right| \\
\le& \frac{\tau}{2} \max_{\theta} \left|(f_n^{\varepsilon})'\big(\xi_1(\theta)\big)\right| \left| \int_{t_{k-1}}^{t_{k+1}} \partial_{tt}\big(|u^{\varepsilon}|^2\big)(\mathbf{x},t)dt \right| \le C\tau \left| \int_{t_{k-1}}^{t_{k+1}} \partial_{tt}\big(|u^{\varepsilon}|^2\big)(\mathbf{x},t)dt \right|,
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
\mathbf{II} \le& \left| \int_0^1 f_n^{\varepsilon}\big(|u^{\varepsilon,k}|^2\big) d\theta \big(\widehat{u^{\varepsilon,k}} - u^{\varepsilon,k}\big) \right| \\
\le& \frac{\tau}{2}\left\|f_n^{\varepsilon}\right\|_{L^\infty} \left| \int_{t_{k-1}}^{t_{k+1}} u_{tt}^{\varepsilon}(\mathbf{x},t)dt \right| \le C\tau \left| \int_{t_{k-1}}^{t_{k+1}} u_{tt}^{\varepsilon}(\mathbf{x},t)dt \right|,
\end{aligned} \tag{5.27}$$

where $\xi_1(\theta)$ is a bounded function between $|u^{\varepsilon,k}|^2$ and $\theta\,|u^{\varepsilon,k+1}|^2 + (1-\theta)\,|u^{\varepsilon,k-1}|^2$. Substituting (5.26) and (5.27) into (5.25), we have

$$\begin{aligned}
&\left\|G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - u^{\varepsilon,k} f_n^{\varepsilon}\left(|u^{\varepsilon,k}|^2\right)\right\|_{L^2}^2 \\
\le& C\tau^3 \int_{t_{k-1}}^{t_{k+1}} \Big(\left\|\partial_{tt}\big(|u^{\varepsilon}|^2\big)\right\|_{L^2}^2 + \left\|u_{tt}^{\varepsilon}\right\|_{L^2}^2\Big) dt.
\end{aligned} \tag{5.28}$$

Combining (5.17), (5.21)-(5.23) and (5.28), we obtain the estimate of the first term on the right side of (5.20)

$$\left|\big(R_A^{\varepsilon,k}, \delta_t e_{\tau,A}^{\varepsilon,k}\big)\right| \le \frac{1}{2}\left\|R_A^{\varepsilon,k}\right\|_{L^2}^2 + \frac{1}{2}\left\|\delta_t e_{\tau,A}^{\varepsilon,k}\right\|_{L^2}^2 \le C\tau^3 P^k + \frac{1}{2}\left\|\delta_t e_{\tau,A}^{\varepsilon,k}\right\|_{L^2}^2, \tag{5.29}$$

where

$$P^k = \int_{t_{k-1}}^{t_{k+1}} \Big(\big\| u_{tttt}^{\varepsilon} \big\|_{L^2}^2 + \big\| u_{tt}^{\varepsilon} \big\|_{H^2}^2 + \big\| \partial_{tt} \big(|u^{\varepsilon}|^2 \big) \big\|_{L^2}^2 \Big) dt.$$

Now, for the convenience of writing, we denote

$$\begin{aligned} \eta^{\varepsilon,k}(\theta) &= \theta \ |u^{\varepsilon,k+1}|^2 + (1-\theta) \ |u^{\varepsilon,k-1}|^2, \\ \eta_{\tau,A}^{\varepsilon,k}(\theta) &= \theta \big| U_A^{\varepsilon,k+1} \big|^2 + (1-\theta) \big| U_A^{\varepsilon,k-1} \big|^2. \end{aligned} \tag{5.30}$$

For the second term on the right side of (5.20), we use the Cauchy-Schwarz inequality, Young's inequality, (5.15) and (5.30) to get

$$\begin{aligned} &\Big| \Big(G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big), \delta_t e_{\tau,A}^{\varepsilon,k} \Big) \Big| \\ \le& \frac{1}{2} \big\| G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big) \big\|_{L^2}^2 + \frac{1}{2} \big\| \delta_t e_{\tau,A}^{\varepsilon,k} \big\|_{L^2}^2 \\ \le& \frac{1}{2} \bigg\| \int_0^1 f_{n,A}^{\varepsilon}\big(\eta^{\varepsilon,k}(\theta)\big) d\theta \ g_{n,A}^{\varepsilon}\big(\widehat{u^{\varepsilon,k}}\big) - \int_0^1 f_{n,A}^{\varepsilon}\big(\eta_{\tau,A}^{\varepsilon,k}(\theta)\big) d\theta \ g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big) \bigg\|_{L^2}^2 + \frac{1}{2} \big\| \delta_t e_{\tau,A}^{\varepsilon,k} \big\|_{L^2}^2 \\ \le& \frac{1}{2} \bigg\| \underbrace{\int_0^1 f_{n,A}^{\varepsilon}\big(\eta^{\varepsilon,k}(\theta)\big) d\theta}_{=:\mathbf{III}} \ \underbrace{\Big[g_{n,A}^{\varepsilon}\big(\widehat{u^{\varepsilon,k}}\big) - g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big) \Big]}_{=:\mathbf{IV}} \\ &+ \underbrace{\int_0^1 \Big[f_{n,A}^{\varepsilon}\big(\eta^{\varepsilon,k}(\theta)\big) - f_{n,A}^{\varepsilon}\big(\eta_{\tau,A}^{\varepsilon,k}(\theta)\big) \Big] d\theta}_{=:\mathbf{V}} \ \underbrace{g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big)}_{=:\mathbf{VI}} \bigg\|_{L^2}^2 + \frac{1}{2} \big\| \delta_t e_{\tau,A}^{\varepsilon,k} \big\|_{L^2}^2. \end{aligned} \tag{5.31}$$

According to (5.8), (5.9) and $f_n^{\varepsilon} \in C^{n-1}([0,+\infty))$ for $n \ge 2$, we can easily obtain

$$\mathbf{III} \le \bigg| \int_0^1 f_{n,A}^{\varepsilon} \left(\eta^{\varepsilon,k}(\theta) \right) d\theta \bigg| \le \bigg| \int_0^1 f_n^{\varepsilon} \left(\eta^{\varepsilon,k}(\theta) \right) d\theta \bigg| \le C, \tag{5.32}$$

$$\begin{aligned} \mathbf{IV} &\le \Big| g_{n,A}^{\varepsilon}\big(\widehat{u^{\varepsilon,k}}\big) - g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big) \Big| \le C_{K_0} \bigg| \frac{u^{\varepsilon,k+1} - U_A^{\varepsilon,k+1}}{2} + \frac{u^{\varepsilon,k-1} - U_A^{\varepsilon,k-1}}{2} \bigg| \\ &\le \frac{1}{2} C_{K_0} \big(\big| e_{\tau,A}^{\varepsilon,k+1} \big| + \big| e_{\tau,A}^{\varepsilon,k-1} \big| \big). \end{aligned} \tag{5.33}$$

Similarly, by utilizing the properties of $f_{n,A}^{\varepsilon}$ in (5.10), we have

$$\begin{aligned} \mathbf{V} &\le \bigg| \int_0^1 \Big[f_{n,A}^{\varepsilon}\big(\eta^{\varepsilon,k}(\theta)\big) - f_{n,A}^{\varepsilon}\big(\eta_{\tau,A}^{\varepsilon,k}(\theta)\big) \Big] d\theta \bigg| \\ &\le C_{K_0} \bigg| \int_0^1 \Big(\sqrt{\eta^{\varepsilon,k}(\theta)} - \sqrt{\eta_{\tau,A}^{\varepsilon,k}(\theta)} \Big) d\theta \bigg| \\ &\le C_{K_0} \bigg| \int_0^1 \frac{\theta\big(|u^{\varepsilon,k+1}|^2 - |U_A^{\varepsilon,k+1}|^2\big) + (1-\theta)\big(|u^{\varepsilon,k-1}|^2 - |U_A^{\varepsilon,k-1}|^2\big)}{\sqrt{\eta^{\varepsilon,k}(\theta)} + \sqrt{\eta_{\tau,A}^{\varepsilon,k}(\theta)}} d\theta \bigg| \\ &\le C_{K_0} \bigg| \int_0^1 \frac{\theta\big(|u^{\varepsilon,k+1}| - |U_A^{\varepsilon,k+1}|\big)\big(|u^{\varepsilon,k+1}| + |U_A^{\varepsilon,k+1}|\big)}{\sqrt{\theta}|u^{\varepsilon,k+1}| + \sqrt{\theta}|U_A^{\varepsilon,k+1}|} d\theta \bigg| \\ &\quad + C_{K_0} \bigg| \int_0^1 \frac{(1-\theta)\big(|u^{\varepsilon,k-1}| - |U_A^{\varepsilon,k-1}|\big)\big(|u^{\varepsilon,k-1}| + |U_A^{\varepsilon,k-1}|\big)}{\sqrt{1-\theta}|u^{\varepsilon,k-1}| + \sqrt{1-\theta}|U_A^{\varepsilon,k-1}|} d\theta \bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq C_{K_0}\left(\int_0^1 \sqrt{\theta}\big|e_{\tau,A}^{\varepsilon,k+1}\big|d\theta+\int_0^1 \sqrt{1-\theta}\big|e_{\tau,A}^{\varepsilon,k-1}\big|d\theta\right)\\
&\leq \frac{2}{3}C_{K_0}\big(\big|e_{\tau,A}^{\varepsilon,k+1}\big|+\big|e_{\tau,A}^{\varepsilon,k-1}\big|\big).
\end{aligned}
\tag{5.34}
$$

Next we are going to discuss the boundedness of the term **VI**. From (5.8), we have

$$
\mathbf{VI}\leq \big|g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big)\big|=\left|\widehat{U_A^{\varepsilon,k}}\,\mu\frac{\widehat{U_A^{\varepsilon,k}}}{K_0^2}\right|.
$$

When $|\widehat{U_A^{\varepsilon,k}}|\in[0,K_0^2)$, we get

$$
\mu\frac{\widehat{U_A^{\varepsilon,k}}}{K_0^2}=1,\quad \mathbf{VI}\leq \big|\widehat{U_A^{\varepsilon,k}}\big|\leq K_0^2.
\tag{5.35}
$$

When $|\widehat{U_A^{\varepsilon,k}}|\in[K_0^2,2K_0^2)$, we set

$$
|c|=\left|\frac{\widehat{U_A^{\varepsilon,k}}}{K_0^2}\right|,
$$

which satisfies

$$
|c|\in[1,2),\quad 1-\frac{1}{1-(1-|c|)^2}\in(-\infty,0].
\tag{5.36}
$$

Then, according to the definition of (5.7) and (5.36), we obtain

$$
0<\mu(c)=e^{1-\frac{1}{1-(1-|c|)^2}}\leq 1,\quad \mathbf{VI}\leq \big|\widehat{U_A^{\varepsilon,k}}\big|\,|\mu(c)|\leq 2K_0^2.
\tag{5.37}
$$

When $|\widehat{U_A^{\varepsilon,k}}|\in[2K_0^2,+\infty)$, we get from (5.7) that

$$
\mu\frac{\widehat{U_A^{\varepsilon,k}}}{K_0^2}=0,\quad \mathbf{VI}=0.
\tag{5.38}
$$

Based on (5.35), (5.37) and (5.38), we infer that

$$
\mathbf{VI}\leq 2K_0^2.
\tag{5.39}
$$

Substituting (5.32)-(5.34) and (5.39) into (5.31), we derive the estimate of the second term on the right side of (5.20)

$$
\begin{aligned}
&\Big|\Big(G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1},u^{\varepsilon,k-1})-G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1},U_A^{\varepsilon,k-1}\big),\delta_t e_{\tau,A}^{\varepsilon,k}\Big)\Big|\\
&\leq C\Big(\big\|e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)+\frac{1}{2}\big\|\delta_t e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2.
\end{aligned}
\tag{5.40}
$$

Thus, using (5.20), (5.29) and (5.40), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\Big(\big\|\delta_t^- e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2-\big\|\delta_t^- e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2\Big)+\frac{1}{4\tau}\Big(\big\|\nabla e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2-\big\|\nabla e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)\\
&\quad+\frac{1}{4\tau}\Big(\big\|e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2-\big\|e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)\\
&\leq C\tau^3P^k+C\Big(\big\|e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|e_{\tau,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)+\big\|\delta_t e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2.
\end{aligned}
$$

Summing above the inequality for time step $1, 2, \ldots, m$, $1 \le m \le N-1$, the following result is given:

$$\begin{aligned}&\big\|\delta_t^- e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2}^2 + \big\|\nabla e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2}^2\\ &\le 2\big\|\delta_t^- e_{\tau,A}^{\varepsilon,1}\big\|_{L^2}^2 + C\tau^4\sum_{k=1}^{m} P^k + C\tau\sum_{k=0}^{m+1}\Big(\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2 + \big\|\delta_t^- e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2\Big).\end{aligned} \tag{5.41}$$

Furthermore, we obtain from (4.1) and (5.41) that

$$\begin{aligned}&\big\|\delta_t^- e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2}^2 + \big\|\nabla e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2}^2\\ &\le \frac{2}{1-C\tau}\big\|\delta_t^- e_{\tau,A}^{\varepsilon,1}\big\|_{L^2}^2 + \frac{C\tau^4}{1-C\tau} + \frac{C\tau}{1-C\tau}\sum_{k=0}^{m}\Big(\big\|\nabla e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2 + \big\|\delta_t^- e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^2\Big).\end{aligned} \tag{5.42}$$

By applying the discrete Grönwall inequality to (5.42), there exists positive constants τ_1^* and C_1^* such that when $\tau \le \tau_1^*$,

$$\big\|\delta_t^- e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2} + \big\|\nabla e_{\tau,A}^{\varepsilon,m+1}\big\|_{L^2} \le C_1^*\tau^2, \quad 1 \le m \le N-1. \tag{5.43}$$

Multiplying (5.19) by $\Delta\widehat{e_{\tau,A}^{\varepsilon,k}}$ and integrating the result over Ω, we obtain

$$\begin{aligned}\big\|\Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\big\|_{L^2}^2 &= \big(\delta_t^2 e_{\tau,A}^{\varepsilon,k}, \Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\big) + \big(\widehat{e_{\tau,A}^{\varepsilon,k}}, \Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\big) - \big(R_A^{\varepsilon,k}, \Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\big)\\ &\quad + \Big(G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big), \Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\Big).\end{aligned} \tag{5.44}$$

By utilizing the Cauchy-Schwarz inequality, (5.29), (5.40), and (5.43), we deduce from (5.44) that

$$\begin{aligned}\big\|\Delta\widehat{e_{\tau,A}^{\varepsilon,k}}\big\|_{L^2} &\le \big\|\delta_t^2 e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} + \big\|\widehat{e_{\tau,A}^{\varepsilon,k}}\big\|_{L^2}\\ &\quad + \big\|G_{n,A}^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_{n,A}^{\varepsilon}\big(U_A^{\varepsilon,k+1}, U_A^{\varepsilon,k-1}\big)\big\|_{L^2} + \big\|R_A^{\varepsilon,k}\big\|_{L^2} \le C\tau.\end{aligned} \tag{5.45}$$

According to (5.45) and (5.18), we get

$$\begin{aligned}\big\|\Delta e_{\tau,A}^{\varepsilon,k+1}\big\|_{L^2} + \big\|\Delta e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} &\le 2\sum_{i=1}^{k}\big\|\Delta\widehat{e_{\tau,A}^{\varepsilon,i}}\big\|_{L^2} + \big\|\Delta e_{\tau,A}^{\varepsilon,1}\big\|_{L^2}\\ &\le CT + C\tau^2 \le CT + 1,\end{aligned} \tag{5.46}$$

when $\tau \le \tau_2^* = \sqrt{1/C}$. Furthermore, using (5.18), (5.43) and (5.46), we obtain

$$\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{H^2} \le \big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} + \big\|\nabla e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} + \big\|\Delta e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} \le C, \qquad 0 \le k \le N, \tag{5.47}$$

$$\big\|U_A^{\varepsilon,k}\big\|_{H^2} \le \|u^{\varepsilon,k}\|_{H^2} + \big\|e_{\tau,A}^{\varepsilon,k}\big\|_{H^2} \le \|u^{\varepsilon,k}\|_{H^2} + C \le C_2^*, \quad 0 \le k \le N. \tag{5.48}$$

Based on Gagliardo-Nirenberg inequality, we can easily conclude

$$\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^\infty} \le C\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{H^2}^{\frac{3}{4}}\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^2}^{\frac{1}{4}} + C\big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^2} \le C\tau^{\frac{1}{2}} + C\tau^2 \le \tau^{\frac{1}{4}}, \tag{5.49}$$

when $\tau \le \tau_3^* = C^{-4}$. It further implies that

$$\begin{aligned}\big\|U_A^{\varepsilon,k}\big\|_{L^\infty} &\le \|u^{\varepsilon,k}\|_{L^\infty} + \big\|e_{\tau,A}^{\varepsilon,k}\big\|_{L^\infty} \le \|u^{\varepsilon,k}\|_{L^\infty} + \tau^{\frac{1}{4}}\\ &\le \|u^{\varepsilon,k}\|_{L^\infty} + 1 \le K_0, \quad 0 \le k \le N,\end{aligned} \tag{5.50}$$

when $\tau \le \tau_4^* = 1$. Obviously, by the definitions of $f_{n,A}^{\varepsilon}$ and $g_{n,A}^{\varepsilon}$, and thanks to (5.50), we have the following results:

$$f_{n,A}^{\varepsilon}\big(\eta_{\tau,A}^{\varepsilon,k}(\theta)\big) = f_n^{\varepsilon}\big(\eta_{\tau,A}^{\varepsilon,k}(\theta)\big), \quad g_{n,A}^{\varepsilon}\big(\widehat{U_A^{\varepsilon,k}}\big) = \widehat{U_A^{\varepsilon,k}}. \tag{5.51}$$

From (5.51), we find that the auxiliary problem (5.13)-(5.14) is totally equivalent to the time discrete system (5.1)-(5.3), which implies

$$U^{\varepsilon,k} = U_A^{\varepsilon,k}, \quad k = 0,1,\ldots,N.$$

Taking $\tau_2 = \min\{\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*\}$, we have when $\tau \le \tau_2$,

$$\begin{aligned} &\big\|e_\tau^{\varepsilon,k}\big\|_{L^2} + \big\|\nabla e_\tau^{\varepsilon,k}\big\|_{L^2} \le C_1^*\tau^2, \quad &&\big\|e_\tau^{\varepsilon,k}\big\|_{H^2} \le C,\\ &\|U^{\varepsilon,k}\|_{H^2} \le C_2^*, &&\|U^{\varepsilon,k}\|_{L^\infty} \le K_0, \quad k = 0,1,\ldots,N. \end{aligned}$$

Therefore, the proof is complete. □

Although we have obtained the boundedness of the solutions of the time discrete system in the sense of L^∞-norm, we notice that the convergence order of $\|\Delta e_\tau^{\varepsilon,k}\|_{L^2}$ is only $\mathcal{O}(1)$. Then, in order to obtain error estimates for fully discrete numerical scheme, we need to use the conclusion of Lemma 5.1 to raise the convergence order of $\|\Delta e_\tau^{\varepsilon,k}\|_{L^2}$ to $\mathcal{O}(\tau^2)$.

Lemma 5.2. *Suppose $f_n^{\varepsilon} \in C^2([0,+\infty))$. Under the assumptions of Lemma 5.1, there exists $\tau_3 > 0$, when $\tau \le \tau_3$ the following results hold:*

$$\big\|e_\tau^{\varepsilon,k}\big\|_{H^2} \le C_3^*\tau^2, \quad \big\|\delta_t U^{\varepsilon,k}\big\|_{H^2} \le C_4^*, \quad \big\|\delta_t^2 U^{\varepsilon,k}\big\|_{H^2} \le C_4^*, \quad 0 \le k \le N, \tag{5.52}$$

where C_3^* and C_4^* are positive constants independent of τ and h.

Proof. Subtracting (5.1) from (5.4), we get the error equation

$$\begin{aligned} &\delta_t^2 e_\tau^{\varepsilon,k} - \Delta\widehat{e_\tau^{\varepsilon,k}} + \widehat{e_\tau^{\varepsilon,k}} + G_n^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1})\\ &\quad - G_n^{\varepsilon}(U^{\varepsilon,k+1}, U^{\varepsilon,k-1}) = R^{\varepsilon,k}, \quad 1 \le k \le N-1. \end{aligned} \tag{5.53}$$

By the definitions of $U^{\varepsilon,0}$ and $U^{\varepsilon,1}$, we have

$$e_\tau^{\varepsilon,0} = 0, \quad e_\tau^{\varepsilon,1} = u^{\varepsilon,1} - U^{\varepsilon,1} \le C\tau^2. \tag{5.54}$$

Multiplying (5.53) by $\Delta\delta_t e_\tau^{\varepsilon,k}$, and integrating the result over Ω, we deduce

$$\begin{aligned} &\frac{1}{2\tau}\left(\big\|\nabla\delta_t^- e_\tau^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|\nabla\delta_t^- e_\tau^{\varepsilon,k}\big\|_{L^2}^2\right) + \frac{1}{4\tau}\left(\big\|\Delta e_\tau^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|\Delta e_\tau^{\varepsilon,k-1}\big\|_{L^2}^2\right)\\ &\quad + \frac{1}{4\tau}\left(\big\|\nabla e_\tau^{\varepsilon,k+1}\big\|_{L^2}^2 - \big\|\nabla e_\tau^{\varepsilon,k-1}\big\|_{L^2}^2\right)\\ &= \big(\nabla R^{\varepsilon,k}, \nabla\delta_t e_\tau^{\varepsilon,k}\big) - \big(\nabla\left(G_n^{\varepsilon}(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_n^{\varepsilon}(U^{\varepsilon,k+1}, U^{\varepsilon,k-1})\right), \nabla\delta_t e_\tau^{\varepsilon,k}\big). \end{aligned} \tag{5.55}$$

Now, let us estimate the two terms at the right of (5.55). Similar to the estimate of (5.29), we can easily conclude

$$\begin{aligned} \big|\big(\nabla R^{\varepsilon,k}, \nabla\delta_t e_\tau^{\varepsilon,k}\big)\big| &\le \frac{1}{2}\|\nabla R^{\varepsilon,k}\|_{L^2}^2 + \frac{1}{2}\big\|\nabla\delta_t e_\tau^{\varepsilon,k}\big\|_{L^2}^2\\ &\le C\tau^3 Q^k + \frac{1}{2}\big\|\nabla\delta_t e_\tau^{\varepsilon,k}\big\|_{L^2}^2, \end{aligned} \tag{5.56}$$

where

$$Q^k = \int_{t_{k-1}}^{t_{k+1}} \left(\|u^\varepsilon_{tttt}\|^2_{H^1} + \|u^\varepsilon_{tt}\|^2_{H^2} + \|\partial_{tt}(|u^\varepsilon|^2)\|^2_{H^1} \right) dt.$$

Denote $\eta^{\varepsilon,k}_\tau(\theta) = \theta\ |U^{\varepsilon,k+1}|^2 + (1-\theta)\ |U^{\varepsilon,k-1}|^2$. Using the Cauchy-Schwarz inequality, Young's inequality, (3.7) and (5.30), we get

$$\begin{aligned}
&\left|\left(\nabla\left(G^\varepsilon_n(u^{\varepsilon,k+1},u^{\varepsilon,k-1}) - G^\varepsilon_n(U^{\varepsilon,k+1},U^{\varepsilon,k-1})\right), \nabla\delta_t e^{\varepsilon,k}_\tau\right)\right| \\
\le\ & \frac{1}{2}\left\|\nabla\left(G^\varepsilon_n(u^{\varepsilon,k+1},u^{\varepsilon,k-1}) - G^\varepsilon_n(U^{\varepsilon,k+1},U^{\varepsilon,k-1})\right)\right\|^2_{L^2} + \frac{1}{2}\|\nabla\delta_t e^{\varepsilon,k}_\tau\|^2_{L^2} \\
\le\ & \frac{1}{2}\Bigg\| \underbrace{\int_0^1 (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}(\theta)\right)\nabla\left(\eta^{\varepsilon,k}(\theta)\right)d\theta\ \widehat{u^{\varepsilon,k}} - \int_0^1 (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}_\tau(\theta)\right)\nabla\left(\eta^{\varepsilon,k}_\tau(\theta)\right)d\theta\widehat{U^{\varepsilon,k}}}_{=:\mathbf{VII}} \\
&\quad + \underbrace{\int_0^1 f^\varepsilon_n\left(\eta^{\varepsilon,k}(\theta)\right)d\theta\ \nabla\widehat{u^{\varepsilon,k}} - \int_0^1 f^\varepsilon_n\left(\eta^{\varepsilon,k}_\tau(\theta)\right)d\theta\ \nabla\widehat{U^{\varepsilon,k}}}_{=:\mathbf{VIII}} \Bigg\|^2_{L^2} + \frac{1}{2}\|\nabla\delta_t e^{\varepsilon,k}_\tau\|^2_{L^2}.
\end{aligned} \tag{5.57}$$

For **VII**, we need to make the following deformations:

$$\begin{aligned}
\mathbf{VII} \le & \left|\int_0^1 \left[(f^\varepsilon_n)'\left(\eta^{\varepsilon,k}(\theta)\right) - (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}_\tau(\theta)\right)\right]\nabla\eta^{\varepsilon,k}(\theta)d\theta\ \widehat{u^{\varepsilon,k}}\right| \\
& + \left|\int_0^1 (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}_\tau(\theta)\right)\nabla\eta^{\varepsilon,k}(\theta)d\theta\left(\widehat{u^{\varepsilon,k}} - \widehat{U^{\varepsilon,k}}\right)\right| \\
& + \left|\int_0^1 (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}_\tau(\theta)\right)\left(\nabla\eta^{\varepsilon,k}(\theta) - \nabla\eta^{\varepsilon,k}_\tau(\theta)\right)d\theta\ \widehat{U^{\varepsilon,k}}\right| := \sum_{j=1}^3 E_j.
\end{aligned} \tag{5.58}$$

Assume that $f^\varepsilon_n \in C^2([0,+\infty))$. Utilizing the differential mean value theorem, (4.1), (4.2) and (5.30), we derive

$$\begin{aligned}
E_1 \le & \left|\int_0^1 \left[(f^\varepsilon_n)'\left(\eta^{\varepsilon,k}(\theta)\right) - (f^\varepsilon_n)'\left(\eta^{\varepsilon,k}_\tau(\theta)\right)\right]\nabla\eta^{\varepsilon,k}(\theta)d\theta\ \widehat{u^{\varepsilon,k}}\right| \\
\le & \|u^\varepsilon\|_{L^\infty}\left|\int_0^1 (f^\varepsilon_n)''\left(\xi_2(\theta)\right)\left(\eta^{\varepsilon,k}(\theta) - \eta^{\varepsilon,k}_\tau(\theta)\right)\nabla\eta^{\varepsilon,k}(\theta)d\theta\right| \\
\le & \|u^\varepsilon\|_{L^\infty}\max_\theta\left|(f^\varepsilon_n)''\left(\xi_2(\theta)\right)\right|\left|\int_0^1 \left(\eta^{\varepsilon,k}(\theta) - \eta^{\varepsilon,k}_\tau(\theta)\right)\left(\theta\nabla|u^{\varepsilon,k+1}|^2 + (1-\theta)\nabla|u^{\varepsilon,k-1}|^2\right)d\theta\right| \\
\le & \|u^\varepsilon\|_{L^\infty}\|\nabla u^\varepsilon\|_{L^\infty}\max_\theta\left|(f^\varepsilon_n)''\left(\xi_2(\theta)\right)\right| \\
& \times\left|\int_0^1 \Big[\theta\left(|u^{\varepsilon,k+1}|^2 - |U^{\varepsilon,k+1}|^2\right) + (1-\theta)\left(|u^{\varepsilon,k-1}|^2 - |U^{\varepsilon,k-1}|^2\right)\Big]d\theta\right| \\
\le & \frac{1}{4}\|u^\varepsilon\|_{L^\infty}\|\nabla u^\varepsilon\|_{L^\infty}\max_\theta\left|(f^\varepsilon_n)''\left(\xi_2(\theta)\right)\right|\left||u^{\varepsilon,k+1}|^2 - |U^{\varepsilon,k+1}|^2 + |u^{\varepsilon,k-1}|^2 - |U^{\varepsilon,k-1}|^2\right| \\
\le & \frac{1}{2}K_0\|u^\varepsilon\|_{L^\infty}\|\nabla u^\varepsilon\|_{L^\infty}\max_\theta\left|(f^\varepsilon_n)''\left(\xi_2(\theta)\right)\right|\left(\left|e^{\varepsilon,k+1}_\tau\right| + \left|e^{\varepsilon,k-1}_\tau\right|\right) \\
\le & C\left(\left|e^{\varepsilon,k+1}_\tau\right| + \left|e^{\varepsilon,k-1}_\tau\right|\right),
\end{aligned} \tag{5.59}$$

where $\xi_2(\theta)$ is a bounded function between $\eta^{\varepsilon,k}(\theta)$ and $\eta_\tau^{\varepsilon,k}(\theta)$. From (4.1) and (5.30), we have

$$\begin{aligned}
E_2 &\le \left| \int_0^1 (f_n^\varepsilon)' \left(\eta_\tau^{\varepsilon,k}(\theta)\right) \nabla\eta^{\varepsilon,k}(\theta) d\theta \big(\widehat{u^{\varepsilon,k}} - \widehat{U^{\varepsilon,k}}\big) \right| \\
&\le \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left| \int_0^1 \nabla\eta^{\varepsilon,k}(\theta) d\theta \big(\widehat{u^{\varepsilon,k}} - \widehat{U^{\varepsilon,k}}\big) \right| \\
&\le \frac{1}{4} \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left| \left(\nabla|u^{\varepsilon,k+1}|^2 + \nabla|u^{\varepsilon,k-1}|^2\right) \left(e_\tau^{\varepsilon,k+1} + e_\tau^{\varepsilon,k-1}\right) \right| \\
&\le \frac{1}{2} \|u^\varepsilon\|_{L^\infty} \ \|\nabla u^\varepsilon\|_{L^\infty} \ \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| \right) \\
&\le C \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| \right). \qquad (5.60)
\end{aligned}$$

Similar to the estimate of E_2, we get

$$\begin{aligned}
E_3 &\le \left| \int_0^1 (f_n^\varepsilon)' \left(\eta_\tau^{\varepsilon,k}(\theta)\right) \left(\nabla\eta^{\varepsilon,k}(\theta) - \nabla\eta_\tau^{\varepsilon,k}(\theta)\right) d\theta \widehat{U^{\varepsilon,k}} \right| \\
&\le \|U^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left| \int_0^1 \left(\nabla\eta^{\varepsilon,k}(\theta) - \nabla\eta_\tau^{\varepsilon,k}(\theta)\right) d\theta \right| \\
&\le \frac{1}{2} \|U^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left| \left(\nabla|u^{\varepsilon,k+1}|^2 - \nabla|U^{\varepsilon,k+1}|^2\right) + \left(\nabla|u^{\varepsilon,k-1}|^2 - \nabla|U^{\varepsilon,k-1}|^2\right) \right| \\
&\le \frac{1}{2} \|U^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \Big| \left(2|u^{\varepsilon,k+1}| \ \nabla|u^{\varepsilon,k+1}| - 2|U^{\varepsilon,k+1}|\nabla|U^{\varepsilon,k+1}|\right) \\
&\qquad + \left(2|u^{\varepsilon,k-1}|\nabla|u^{\varepsilon,k-1}| - 2|U^{\varepsilon,k-1}|\nabla|U^{\varepsilon,k-1}|\right) \Big| \\
&\le K_0 \|\nabla u^\varepsilon\|_{L^\infty} \|U^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\eta_\tau^{\varepsilon,k}(\theta)\big)\right| \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|\nabla e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| + \left|\nabla e_\tau^{\varepsilon,k-1}\right| \right) \\
&\le C \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| + \left|\nabla e_\tau^{\varepsilon,k+1}\right| + \left|\nabla e_\tau^{\varepsilon,k-1}\right| \right). \qquad (5.61)
\end{aligned}$$

Substituting (5.59)-(5.61) into (5.58), we obtain

$$\mathbf{VII} \le C \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| + \left|\nabla e_\tau^{\varepsilon,k+1}\right| + \left|\nabla e_\tau^{\varepsilon,k-1}\right| \right). \qquad (5.62)$$

Utilizing (5.12) and (5.30), we have

$$\begin{aligned}
\mathbf{VIII} &\le \left| \int_0^1 \left[f_n^\varepsilon \left(\eta^{\varepsilon,k}(\theta)\right) - f_n^\varepsilon \left(\eta_\tau^{\varepsilon,k}(\theta)\right) \right] d\theta \nabla\widehat{u^{\varepsilon,k}} + \int_0^1 f_n^\varepsilon \left(\eta_\tau^{\varepsilon,k}(\theta)\right) d\theta \big(\nabla\widehat{e_\tau^{\varepsilon,k}}\big) \right| \\
&\le \left| \int_0^1 (f_n^\varepsilon)' \left(\xi_3(\theta)\right) \left(\eta^{\varepsilon,k}(\theta) - \eta_\tau^{\varepsilon,k}(\theta)\right) d\theta \ \nabla\widehat{u^{\varepsilon,k}} \right| + C \left|\nabla e_\tau^{\varepsilon,k+1} + \nabla e_\tau^{\varepsilon,k-1}\right| \\
&\le \frac{1}{2} \|\nabla u^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\xi_3(\theta)\big)\right| \left| |u^{\varepsilon,k+1}|^2 - |U^{\varepsilon,k+1}|^2 + |u^{\varepsilon,k-1}|^2 - |U^{\varepsilon,k-1}|^2 \right| \\
&\quad + C \left|\nabla e_\tau^{\varepsilon,k+1} + \nabla e_\tau^{\varepsilon,k-1}\right| \\
&\le K_0 \|\nabla u^\varepsilon\|_{L^\infty} \max_\theta \left|(f_n^\varepsilon)'\big(\xi_3(\theta)\big)\right| \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| \right) + C \left|\nabla e_\tau^{\varepsilon,k+1} + \nabla e_\tau^{\varepsilon,k-1}\right| \\
&\le C \left(\left|e_\tau^{\varepsilon,k+1}\right| + \left|e_\tau^{\varepsilon,k-1}\right| + \left|\nabla e_\tau^{\varepsilon,k+1}\right| + \left|\nabla e_\tau^{\varepsilon,k-1}\right| \right), \qquad (5.63)
\end{aligned}$$

where $\xi_3(\theta)$ is a bounded function between $\eta^{\varepsilon,k}(\theta)$ and $\eta_\tau^{\varepsilon,k}(\theta)$. Substituting (5.62) and (5.63) into (5.57), we derive

$$\begin{aligned}
&\left| \left(\nabla \left(G_n^\varepsilon(u^{\varepsilon,k+1}, u^{\varepsilon,k-1}) - G_n^\varepsilon(U^{\varepsilon,k+1}, U^{\varepsilon,k-1}) \right), \nabla\delta_t e_\tau^{\varepsilon,k} \right) \right| \\
\le\ & C \left(\left\|e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2 + \left\|e_\tau^{\varepsilon,k-1}\right\|_{L^2}^2 + \left\|\nabla e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2 + \left\|\nabla e_\tau^{\varepsilon,k-1}\right\|_{L^2}^2 \right) + \frac{1}{2} \left\|\nabla\delta_t e_\tau^{\varepsilon,k}\right\|_{L^2}^2. \qquad (5.64)
\end{aligned}$$

Furthermore, substitute (5.56) and (5.64) into (5.55), and replace k by i, and sum up from 1 to k,

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|\nabla\delta_t^- e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\nabla\delta_t^- e_\tau^{\varepsilon,1}\right\|_{L^2}^2\right)+\frac{1}{4\tau}\left(\left\|\Delta e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\Delta e_\tau^{\varepsilon,1}\right\|_{L^2}^2+\left\|\Delta e_\tau^{\varepsilon,k}\right\|_{L^2}^2\right)\\&\quad+\frac{1}{4\tau}\left(\left\|\nabla e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\nabla e_\tau^{\varepsilon,1}\right\|_{L^2}^2+\left\|\nabla e_\tau^{\varepsilon,k}\right\|_{L^2}^2\right)\\&\le C\tau^3\sum_{i=1}^{k}Q^i+C\sum_{i=0}^{k+1}\left(\left\|e_\tau^{\varepsilon,i}\right\|_{L^2}^2+\left\|\nabla e_\tau^{\varepsilon,i}\right\|_{L^2}^2+\left\|\nabla\delta_t^- e_\tau^{\varepsilon,i}\right\|_{L^2}^2\right).\end{aligned}\tag{5.65}$$

By utilizing (4.1), (5.54) and Poincaré inequality, we obtain

$$\begin{aligned}&\left\|\nabla\delta_t^- e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2+\left\|\Delta e_\tau^{\varepsilon,k+1}\right\|_{L^2}^2\\&\le 2\left\|\nabla\delta_t^- e_\tau^{\varepsilon,1}\right\|_{L^2}^2+C\tau^4+C\tau\sum_{i=0}^{k+1}\left(\left\|\Delta e_\tau^{\varepsilon,i}\right\|_{L^2}^2+\left\|\nabla\delta_t^- e_\tau^{\varepsilon,i}\right\|_{L^2}^2\right).\end{aligned}\tag{5.66}$$

Next, according to the discrete Grönwall inequality, there exists positive constants τ_5^* and C_3^* such that when $\tau\le\tau_5^*$, we have from (5.66) that

$$\left\|\nabla\delta_t^- e_\tau^{\varepsilon,k+1}\right\|_{L^2}+\left\|\Delta e_\tau^{\varepsilon,k+1}\right\|_{L^2}\le C_3^*\tau^2,\quad 1\le k\le N-1.\tag{5.67}$$

Furthermore, combining (5.11) and (5.67), we have the following series of conclusions for $0\le k\le N$:

$$\begin{aligned}&\left\|\Delta e_\tau^{\varepsilon,k}\right\|_{H^2}\le\left\|e_\tau^{\varepsilon,k}\right\|_{L^2}+\left\|\nabla e_\tau^{\varepsilon,k}\right\|_{L^2}+\left\|\Delta e_\tau^{\varepsilon,k}\right\|_{L^2}\le C_3^*\tau^2,\\&\left\|\delta_t U^{\varepsilon,k}\right\|_{H^2}\le\left\|\delta_t u^{\varepsilon,k}\right\|_{H^2}+\left\|\delta_t e_\tau^{\varepsilon,k}\right\|_{H^2}\le\left\|\delta_t u^{\varepsilon,k}\right\|_{H^2}+C\tau\le C_4^*,\\&\left\|\delta_t^2 U^{\varepsilon,k}\right\|_{H^2}\le\left\|\delta_t^2 u^{\varepsilon,k}\right\|_{H^2}+\left\|\delta_t^2 e_\tau^{\varepsilon,k}\right\|_{H^2}\le\left\|\delta_t^2 u^{\varepsilon,k}\right\|_{H^2}+4C\le C_4^*.\end{aligned}$$

Therefore, taking $\tau_3=\min\{\tau_2,\tau_5^*\}$, when $\tau\le\tau_3$, the proof of Lemma 5.2 is complete. □

Remark 5.1. Since n in (2.3) is an arbitrary constant, and $n\ge 2$, we can ensure that $f_n^\varepsilon\in C^2([0,+\infty))$ when we take $n=3$. In other words, we use the piecewise quartic polynomials to approximate $F(\rho)$ near the origin to obtain the desired convergence results.

Remark 5.2. In this part, we provide the existence and uniqueness proof of the solutions to the auxiliary problem (5.13), and the proof process is slightly different from that of reference [21]. Since this proof process is very complicated, we have not shown the details here. For simplify of presentation, we only give a general analytical framework:

1. Using the Brouwer's fixed point theorem, show the existence and uniqueness of the solutions for the auxiliary problem (5.13) in the following finite-dimensional space:

$$X_N:=\{\phi_m,1\le m\le N\},$$

   where $\{\phi_m,m\in\mathbb{N}\}$ denotes a countable basis of $H_0^1(\Omega)$.

2. Prove the H^1-norm boundedness of the solution in $H_0^1(\Omega)$ as Lemma 5.1.

3. Using Rellich embedding theorem and Vitali's theorem, prove the strong convergence of every terms of the infinite-dimensional system, and thus the existence of the solutions of the auxiliary problem (5.13) in $H_0^1(\Omega)$ is obtained.

4. The uniqueness of the solutions for the auxiliary problem (5.13) can be derived similar as the error analysis in Section 5.1.

Remark 5.3. From the proof process of Lemmas 5.1 and 5.2, it can be seen that the auxiliary problem (5.13) and the time discrete system (5.1) are equivalent, so the existence and uniqueness of the solutions of time discrete system (5.1) is also naturally established.

5.2. Error estimates for the fully discrete scheme

In this section, we will give the L^∞-norm estimates for the numerical solutions $U_h^{\varepsilon,k}$.

Lemma 5.3. *Assume that u^ε is the solution of the system (2.6) satisfying (4.1), and the fully discrete system (3.5)-(3.6) has unique solutions $U_h^{\varepsilon,k}, 0 \le k \le N$. Then, there exists $\tau_4 > 0$ and $h_1^* > 0$ such that when $\tau \le \tau_4$ and $h \le h_1^*$, there hold*

$$\left\|\nabla\left(R_h U^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} \le C_5^*(\tau^2 + h^2), \quad \left\|U_h^{\varepsilon,k}\right\|_{L^\infty} \le K_0,$$

where C_5^* is a positive constant independent of τ and h.

Proof. Similar to the error analysis of the time discrete system, we need the following auxiliary problems based on cut-off techniques for $k = 1, 2, \ldots, N-1$,

$$\begin{aligned}&\left(\delta_t^2 U_{h,A}^{\varepsilon,k}, \omega_h\right) + \left(\nabla\widehat{U_{h,A}^{\varepsilon,k}}, \nabla\omega_h\right) + \left(\widehat{U_{h,A}^{\varepsilon,k}}, \omega_h\right)\\ &\quad + \left(G_{n,A}^\varepsilon\left(U_{h,A}^{\varepsilon,k+1}, U_{h,A}^{\varepsilon,k-1}\right), \omega_h\right) = 0, \quad \forall\, \omega_h \in W_h \end{aligned} \tag{5.68}$$

with $U_{h,A}^{\varepsilon,0} = R_h\phi_0$ and

$$U_{h,A}^{\varepsilon,1} = R_h\left(\phi_0 + \tau\phi_1 + \frac{\tau^2}{2}\left(\Delta\phi_0 - \phi_0 - \phi_0 f_n^\varepsilon\left(|\phi_0|^2\right)\right)\right).$$

Define

$$\vartheta_{h,A}^{\varepsilon,k} = R_h U^{\varepsilon,k} - U_{h,A}^{\varepsilon,k}, \quad 0 \le k \le N. \tag{5.69}$$

By the definition of $U^{\varepsilon,0}, U^{\varepsilon,1}$ and $U_{h,A}^{\varepsilon,0}$, $U_{h,A}^{\varepsilon,1}$, we have

$$\vartheta_{h,A}^{\varepsilon,0} = R_h U^{\varepsilon,0} - U_{h,A}^{\varepsilon,0} = 0, \quad \vartheta_{h,A}^{\varepsilon,1} = R_h U^{\varepsilon,1} - U_{h,A}^{\varepsilon,1} = 0. \tag{5.70}$$

Subtracting (5.68) from (5.1) with $\omega_h = \delta_t\vartheta_{h,A}^{\varepsilon,k}$, we get the error equation for $k \ge 2$,

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2 - \left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2\right) + \frac{1}{4\tau}\left(\left\|\nabla\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2 - \left\|\nabla\vartheta_{h,A}^{\varepsilon,k-1}\right\|_{L^2}^2\right)\\ &\quad + \frac{1}{4\tau}\left(\left\|\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2 - \left\|\vartheta_{h,A}^{\varepsilon,k-1}\right\|_{L^2}^2\right)\\ &= -\left(\delta_t^2\left(U^{\varepsilon,k} - R_h U^{\varepsilon,k}\right), \delta_t\vartheta_{h,A}^{\varepsilon,k}\right) - \left(\widehat{U^{\varepsilon,k}} - R_h\widehat{U^{\varepsilon,k}}, \delta_t\vartheta_{h,A}^{\varepsilon,k}\right)\\ &\quad - \left(G_n^\varepsilon\left(U^{\varepsilon,k+1}, U^{\varepsilon,k-1}\right) - G_{n,A}^\varepsilon\left(U_{h,A}^{\varepsilon,k+1}, U_{h,A}^{\varepsilon,k-1}\right), \delta_t\vartheta_{h,A}^{\varepsilon,k}\right).\end{aligned} \tag{5.71}$$

By utilizing the Cauchy-Schwarz inequality, Young's inequality, (3.2) and (5.52), we derive

$$\begin{aligned}\left|\left(\delta_t^2\left(U^{\varepsilon,k} - R_h U^{\varepsilon,k}\right), \delta_t\vartheta_{h,A}^{\varepsilon,k}\right)\right| &\le Ch^4\left\|\delta_t^2 U^{\varepsilon,k}\right\|_{H^2}^2 + \frac{1}{2}\left\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2\\ &\le Ch^4 + \frac{1}{2}\left\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2,\end{aligned} \tag{5.72}$$

$$\begin{aligned}\left|\left(\widehat{U^{\varepsilon,k}} - R_h\widehat{U^{\varepsilon,k}}, \delta_t\vartheta_{h,A}^{\varepsilon,k}\right)\right| &\le Ch^4\|\widehat{U^{\varepsilon,k}}\|_{H^2}^2 + \frac{1}{2}\left\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2\\ &\le Ch^4 + \frac{1}{2}\left\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2.\end{aligned} \tag{5.73}$$

In order to estimate the third term of the right-hand side of (5.71), we need to do the following processing:

$$\begin{aligned}
&\Big|\Big(G_n^\varepsilon(U^{\varepsilon,k+1},U^{\varepsilon,k-1})-G_{n,A}^\varepsilon\big(U_{h,A}^{\varepsilon,k+1},U_{h,A}^{\varepsilon,k-1}\big),\delta_t\vartheta_{h,A}^{\varepsilon,k}\Big)\Big|\\
&\le\frac12\Big\|G_n^\varepsilon(U^{\varepsilon,k+1},U^{\varepsilon,k-1})-G_{n,A}^\varepsilon\big(U_{h,A}^{\varepsilon,k+1},U_{h,A}^{\varepsilon,k-1}\big)\Big\|_{L^2}^2+\frac12\big\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\big\|_{L^2}^2\\
&\le\frac12\Big\|\int_0^1 f_n^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)d\theta\,\underbrace{\big(\widehat{U^{\varepsilon,k}}-g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)\big)}_{=:\mathbf{IX}}\\
&\qquad+\underbrace{\int_0^1\Big[f_n^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)-f_{n,A}^\varepsilon\big(\eta_{h,A}^{\varepsilon,k}(\theta)\big)\Big]d\theta}_{=:\mathbf{X}}\,g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)\Big\|_{L^2}^2+\frac12\big\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\big\|_{L^2}^2,
\end{aligned}\tag{5.74}$$

where

$$\eta_{h,A}^{\varepsilon,k}(\theta)=\theta\big|U_{h,A}^{\varepsilon,k+1}\big|^2+(1-\theta)\big|U_{h,A}^{\varepsilon,k-1}\big|^2.$$

Using the estimates methods similar to **III** and **VI**, we have

$$\Big|\int_0^1 f_n^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)d\theta\Big|\le\|f_n^\varepsilon\|_{L^\infty},\quad\big|g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)\big|\le 2K_0^2.\tag{5.75}$$

Due to $\|U^{\varepsilon,k}\|_{L^\infty}\le K_0$, we can obtain by the definition of $g_{n,A}^\varepsilon$ and $f_{n,A}^\varepsilon$

$$g_{n,A}^\varepsilon\big(\widehat{U^{\varepsilon,k}}\big)=\widehat{U^{\varepsilon,k}},\quad f_{n,A}^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)=f_n^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big).\tag{5.76}$$

Based on (5.76), (5.9), (5.10) and (5.69), we deduce

$$\begin{aligned}
\mathbf{IX}&\le\big|\widehat{U^{\varepsilon,k}}-g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)\big|=\big|g_{n,A}^\varepsilon\big(\widehat{U^{\varepsilon,k}}\big)-g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)\big|\le C_{K_0}\big|\widehat{U^{\varepsilon,k}}-\widehat{U_{h,A}^{\varepsilon,k}}\big|\\
&\le\frac12C_{K_0}\Big(\big|U^{\varepsilon,k+1}-R_hU^{\varepsilon,k+1}\big|+\big|U^{\varepsilon,k-1}-R_hU^{\varepsilon,k-1}\big|+\big|\vartheta_{h,A}^{\varepsilon,k+1}\big|+\big|\vartheta_{h,A}^{\varepsilon,k-1}\big|\Big),
\end{aligned}\tag{5.77}$$

$$\begin{aligned}
\mathbf{X}&\le\Big|\int_0^1\Big[f_n^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)-f_{n,A}^\varepsilon\big(\eta_{h,A}^{\varepsilon,k}(\theta)\big)\Big]d\theta\Big|=\Big|\int_0^1\Big[f_{n,A}^\varepsilon\big(\eta_\tau^{\varepsilon,k}(\theta)\big)-f_{n,A}^\varepsilon\big(\eta_{h,A}^{\varepsilon,k}(\theta)\big)\Big]d\theta\Big|\\
&\le C_{K_0}\int_0^1\Big|\sqrt{\eta_\tau^{\varepsilon,k}(\theta)}-\sqrt{\eta_{h,A}^{\varepsilon,k}(\theta)}\Big|d\theta\\
&\le C_{K_0}\int_0^1\Big(\sqrt{\theta}\big|U^{\varepsilon,k+1}-U_{h,A}^{\varepsilon,k+1}\big|+\sqrt{1-\theta}\big|U^{\varepsilon,k-1}-U_{h,A}^{\varepsilon,k-1}\big|\Big)\,d\theta\\
&\le\frac12C_{K_0}\Big(\big|U^{\varepsilon,k+1}-R_hU^{\varepsilon,k+1}\big|+\big|U^{\varepsilon,k-1}-R_hU^{\varepsilon,k-1}\big|+\big|\vartheta_{h,A}^{\varepsilon,k+1}\big|+\big|\vartheta_{h,A}^{\varepsilon,k-1}\big|\Big).
\end{aligned}\tag{5.78}$$

Substituting (5.75), (5.77) and (5.78) into (5.74), and then using (3.2) and (5.12), we get

$$\begin{aligned}
&\Big|\Big(G_n^\varepsilon(U^{\varepsilon,k+1},U^{\varepsilon,k-1})-G_{n,A}^\varepsilon\big(U_{h,A}^{\varepsilon,k+1},U_{h,A}^{\varepsilon,k-1}\big),\delta_t\vartheta_{h,A}^{\varepsilon,k}\Big)\Big|\\
&\le C\Big(\big\|U^{\varepsilon,k+1}-R_hU^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|U^{\varepsilon,k-1}-R_hU^{\varepsilon,k-1}\big\|_{L^2}^2+\big\|\vartheta_{h,A}^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|\vartheta_{h,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)\\
&\quad+\frac12\big\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\big\|_{L^2}^2\le Ch^4+C\Big(\big\|\vartheta_{h,A}^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|\vartheta_{h,A}^{\varepsilon,k-1}\big\|_{L^2}^2\Big)+\frac12\big\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\big\|_{L^2}^2.
\end{aligned}\tag{5.79}$$

Then, based on (5.72), (5.73) and (5.79), the following equation can be obtained from (5.71) that:

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2\right)+\frac{1}{4\tau}\left(\left\|\nabla\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\nabla\vartheta_{h,A}^{\varepsilon,k-1}\right\|_{L^2}^2\right)\\&\quad+\frac{1}{4\tau}\left(\left\|\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2-\left\|\vartheta_{h,A}^{\varepsilon,k-1}\right\|_{L^2}^2\right)\\&\le Ch^4+C\left(\left\|\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2+\left\|\vartheta_{h,A}^{\varepsilon,k-1}\right\|_{L^2}^2\right)+\frac{3}{2}\left\|\delta_t\vartheta_{h,A}^{\varepsilon,k}\right\|_{L^2}^2.\end{aligned}$$

Replacing k by i in the above inequality, and summing up from 1 to k, and then utilizing $\vartheta_{h,A}^{\varepsilon,0}=0, \vartheta_{h,A}^{\varepsilon,1}=0$ and Poincaré inequality, we have

$$\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2+\left\|\nabla\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}^2\le Ch^4+C\tau\sum_{i=1}^{k+1}\left(\left\|\nabla\vartheta_{h,A}^{\varepsilon,i}\right\|_{L^2}^2+\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,i}\right\|_{L^2}^2\right). \tag{5.80}$$

Thus, by discrete Grönwall inequality, there exists constants $\tau_6^*\ge 0$ and $C_5^*\ge 0$ such that

$$\left\|\delta_t^-\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}+\left\|\nabla\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}\le C_5^*\ h^2,\quad 1\le k\le N-1, \tag{5.81}$$

when $\tau\le\tau_6^*$. Furthermore, we can derive the following estimates result:

$$\begin{aligned}\left\|U_{h,A}^{\varepsilon,k+1}\right\|_{L^\infty}&\le\left\|\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^\infty}+\left\|R_hU^{\varepsilon,k+1}-U^{\varepsilon,k+1}\right\|_{L^\infty}+\|U^{\varepsilon,k+1}-u^{\varepsilon,k+1}\|_{L^\infty}+\|u^{\varepsilon,k+1}\|_{L^\infty}\\&\le Ch^{-\frac{d}{2}}\left\|\vartheta_{h,A}^{\varepsilon,k+1}\right\|_{L^2}+\left\|R_hU^{\varepsilon,k+1}-U^{\varepsilon,k+1}\right\|_{W^{1,4}}+C\left\|e_\tau^{\varepsilon,k+1}\right\|_{H^2}+\|u^{\varepsilon,k+1}\|_{L^\infty}\\&\le Ch^{-\frac{d}{2}}h^2+Ch^{1-\frac{d}{4}}\|U^{\varepsilon,k+1}\|_{H^2}+C\tau^2+\|u^{\varepsilon,k+1}\|_{L^\infty}\\&\le Ch^{\frac{1}{4}}+C\tau^2+\|u^{\varepsilon,k+1}\|_{L^\infty}\\&\le 1+\|u^{\varepsilon,k+1}\|_{L^\infty}\le K_0,\end{aligned} \tag{5.82}$$

when $h\le h_1^*=(2C)^{-4}$ and $\tau\le\tau_7^*=(2C)^{-1/2}$. At the same time, it implies

$$f_{n,A}^\varepsilon\big(\eta_{h,A}^{\varepsilon,k}(\theta)\big)=f_n^\varepsilon\big(\eta_{h,A}^{\varepsilon,k}(\theta)\big),\quad g_{n,A}^\varepsilon\big(\widehat{U_{h,A}^{\varepsilon,k}}\big)=\widehat{U_{h,A}^{\varepsilon,k}}. \tag{5.83}$$

Therefore, (3.5) and (5.68) are equivalent, that are, $U_{h,A}^{\varepsilon,k}=U_h^{\varepsilon,k}$ and

$$\begin{aligned}&\left\|\delta_t^-\left(R_hU^{\varepsilon,k}-U_h^{\varepsilon,k}\right)\right\|_{L^2}+\left\|\nabla\big(R_hU^{\varepsilon,k}-U_h^{\varepsilon,k}\big)\right\|_{L^2}\le C_5^*h^2,\\&\left\|U_h^{\varepsilon,k}\right\|_{L^\infty}\le K_0,\quad 0\le k\le N.\end{aligned} \tag{5.84}$$

Taking $\tau\le\tau_4=\min\{\tau_3,\tau_6^*,\tau_7^*\}$ and $h\le h_1^*$, the proof of Lemma 5.3 is complete. □

Remark 5.4. Under the assumption of Lemmas 5.1-5.3, the solutions of fully discrete scheme (3.5)-(3.6) with $r=1$ satisfying

$$\begin{aligned}&\left\|u^{\varepsilon,k}-U_h^{\varepsilon,k}\right\|_{L^2}\le\left\|e_\tau^{\varepsilon,k}\right\|_{L^2}+\left\|U^{\varepsilon,k}-R_hU^{\varepsilon,k}\right\|_{L^2}+\left\|R_hU^{\varepsilon,k}-U_h^{\varepsilon,k}\right\|_{L^2}\le C(\tau^2+h^2),\\&\left\|\nabla(u^{\varepsilon,k}-U_h^{\varepsilon,k})\right\|_{L^2}\le\left\|\nabla e_\tau^{\varepsilon,k}\right\|_{L^2}+\left\|\nabla(U^{\varepsilon,k}-R_hU^{\varepsilon,k})\right\|_{L^2}+\left\|\nabla(R_hU^{\varepsilon,k}-U_h^{\varepsilon,k})\right\|_{L^2}\\&\qquad\qquad\le C(\tau^2+h).\end{aligned}$$

5.3. Proof of Theorem 4.1

In this section, we will give the proof of Theorem 4.1 based on the series of analyses in Sections 5.1 and 5.2. The unconditional optimal error estimates for the fully discrete scheme under L^2-norm and H^1-seminorm will be demonstrated and it plays an important role in the proof of Theorem 4.2. For convenience, we define $\varsigma^{\varepsilon,k}=R_hu^{\varepsilon,k}-U_h^{\varepsilon,k}$. Obviously, we notice that

$$\varsigma^{\varepsilon,0}=0. \tag{5.85}$$

Subtracting (3.5) from (5.4), the error equation is obtained

$$\begin{aligned}&\big(\delta_t^2\varsigma^{\varepsilon,k},\omega_h\big)+\big(\nabla\widehat{\varsigma^{\varepsilon,k}},\nabla\omega_h\big)+\big(\widehat{\varsigma^{\varepsilon,k}},\omega_h\big)\\ =&-\big(\delta_t^2\big(u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big),\omega_h\big)-\big(\widehat{u^{\varepsilon,k}}-R_h\widehat{u^{\varepsilon,k}},\omega_h\big)+\big(R^{\varepsilon,k},\omega_h\big)\\ &-\big(G_n^\varepsilon(u^{\varepsilon,k+1},u^{\varepsilon,k-1})-G_n^\varepsilon\big(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\big),\omega_h\big),\quad 1\le k\le N-1,\end{aligned} \tag{5.86}$$

where $R^{\varepsilon,k}$ is defined in (5.5). Letting $\omega_h=\delta_t\varsigma^{\varepsilon,k}$ into (5.86), we get

$$\begin{aligned}&\frac{1}{2\tau}\Big(\big\|\delta_t^-\varsigma^{\varepsilon,k+1}\big\|_{L^2}^2-\big\|\delta_t^-\varsigma^{\varepsilon,k}\big\|_{L^2}^2\Big)+\frac{1}{4\tau}\big(\|\nabla\varsigma^{\varepsilon,k+1}\|_{L^2}^2-\|\nabla\varsigma^{\varepsilon,k-1}\|_{L^2}^2\big)\\ &\quad+\frac{1}{4\tau}\big(\|\varsigma^{\varepsilon,k+1}\|_{L^2}^2-\|\varsigma^{\varepsilon,k-1}\|_{L^2}^2\big)\\ &=-\big(\widehat{u^{\varepsilon,k}}-R_h\widehat{u^{\varepsilon,k}},\delta_t\varsigma^{\varepsilon,k}\big)-\big(\delta_t^2\big(u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big),\delta_t\varsigma^{\varepsilon,k}\big)+\big(R^{\varepsilon,k},\delta_t\varsigma^{\varepsilon,k}\big)\\ &\quad-\big(G_n^\varepsilon(u^{\varepsilon,k+1},u^{\varepsilon,k-1})-G_n^\varepsilon\big(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\big),\delta_t\varsigma^{\varepsilon,k}\big)\\ &:=\sum_{i=1}^4 I_i,\quad 1\le k\le N-1.\end{aligned} \tag{5.87}$$

By using the Cauchy-Schwarz inequality, Young's inequality, (5.5), (3.2) and (4.1), we get

$$I_1\le\big|-\big(\widehat{u^{\varepsilon,k}}-R_h\widehat{u^{\varepsilon,k}},\delta_t\varsigma^{\varepsilon,k}\big)\big|\le Ch^{2r+2}+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2, \tag{5.88}$$

$$\begin{aligned}I_2&\le\big|-\big(\delta_t^2\big(u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big),\delta_t\varsigma^{\varepsilon,k}\big)\big|\\ &\le C\big\|\delta_t^2\big(u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big)\big\|_{L^2}^2+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\ &\le Ch^{2r+2}\int_{-1}^1(1-|s|)\big\|u_{tt}^\varepsilon(\mathbf{x},\tau s+t_n)\big\|_{H^{r+1}}^2ds+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\ &\le Ch^{2r+2}+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2,\end{aligned} \tag{5.89}$$

$$I_3\le\big|\big(R^{\varepsilon,k},\delta_t\varsigma^{\varepsilon,k}\big)\big|\le C\|R^{\varepsilon,k}\|_{L^2}^2+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\le C\tau^4+\frac14\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2. \tag{5.90}$$

Next, we estimate the last term on the right hand side of (5.87). According to the Cauchy-Schwarz inequality, Young's inequality, the L^∞-norm boundedness of $U_h^{\varepsilon,k}$

$$f_n^\varepsilon\in C^{n-1}\big([0,+\infty)\big),\quad n\ge 2,$$

we derive

$$I_4\le\big|-\big(G_n^\varepsilon(u^{\varepsilon,k+1},u^{\varepsilon,k-1})-G_n^\varepsilon\big(U_h^{\varepsilon,k+1},U_h^{\varepsilon,k-1}\big),\delta_t\varsigma^{\varepsilon,k}\big)\big|$$

$$
\begin{aligned}
&\le C\Big\|\int_0^1 f_n^\varepsilon\big(\eta^{\varepsilon,k}(\theta)\big)d\theta\widehat{u^{\varepsilon,k}}-\int_0^1 f_n^\varepsilon\big(\eta_h^{\varepsilon,k}(\theta)\big)d\theta\widehat{U_h^{\varepsilon,k}}\Big\|_{L^2}^2+\frac{1}{4}\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\
&\le C\Big\|\int_0^1 \big(f_n^\varepsilon\big(\eta^{\varepsilon,k}(\theta)\big)-f_n^\varepsilon\big(\eta_h^{\varepsilon,k}(\theta)\big)\big)d\theta\ \widehat{u^{\varepsilon,k}}+\int_0^1 f_n^\varepsilon\big(\eta_h^{\varepsilon,k}(\theta)\big)d\theta\big(\widehat{u^{\varepsilon,k}}-\widehat{U_h^{\varepsilon,k}}\big)\Big\|_{L^2}^2+\frac{1}{4}\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\
&\le C\Big\|\int_0^1 (f_n^\varepsilon)'\big(\xi_4(\theta)\big)\big(\eta^{\varepsilon,k}(\theta)-\eta_h^{\varepsilon,k}(\theta)\big)d\theta\widehat{u^{\varepsilon,k}}+\int_0^1 f_n^\varepsilon\big(\eta_h^{\varepsilon,k}(\theta)\big)d\theta\big(\widehat{u^{\varepsilon,k}}-\widehat{U_h^{\varepsilon,k}}\big)\Big\|_{L^2}^2+\frac{1}{4}\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\
&\le C\left(\max_\theta\big|(f_n^\varepsilon)'\big(\xi_4(\theta)\big)\big|\ \|u^{\varepsilon,k}\|_{L^\infty}+\max_\theta\big|f_n^\varepsilon\big(\eta_h^{\varepsilon,k}(\theta)\big)\big|\right)\\
&\quad\times\Big(\big\|u^{\varepsilon,k+1}-U_h^{\varepsilon,k+1}\big\|_{L^2}^2+\big\|u^{\varepsilon,k-1}-U_h^{\varepsilon,k-1}\big\|_{L^2}^2\Big)+\frac{1}{4}\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2\\
&\le Ch^{2r+2}+C\|\varsigma^{\varepsilon,k+1}\|_{L^2}^2+C\|\varsigma^{\varepsilon,k-1}\|_{L^2}^2+\frac{1}{4}\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2,
\end{aligned}
\tag{5.91}
$$

where

$$
\eta_h^{\varepsilon,k}(\theta)=\theta\big|U_h^{\varepsilon,k+1}\big|^2+(1-\theta)\big|U_h^{\varepsilon,k-1}\big|^2,
$$

and $\xi_4(\theta)$ is a bounded function between $\eta^{\varepsilon,k}(\theta)$ and $\eta_h^{\varepsilon,k}(\theta)$. Substituting (5.88)-(5.91) into (5.87), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\Big(\big\|\delta_t^-\varsigma^{\varepsilon,k+1}\big\|_{L^2}^2-\big\|\delta_t^-\varsigma^{\varepsilon,k}\big\|_{L^2}^2\Big)+\frac{1}{4\tau}\left(\|\nabla\varsigma^{\varepsilon,k+1}\|_{L^2}^2-\|\nabla\varsigma^{\varepsilon,k-1}\|_{L^2}^2\right)\\
&\quad+\frac{1}{4\tau}\left(\|\varsigma^{\varepsilon,k+1}\|_{L^2}^2-\|\varsigma^{\varepsilon,k-1}\|_{L^2}^2\right)\\
&\le Ch^{2r+2}+C\tau^4+C\|\varsigma^{\varepsilon,k+1}\|_{L^2}^2+C\|\varsigma^{\varepsilon,k-1}\|_{L^2}^2+\big\|\delta_t\varsigma^{\varepsilon,k}\big\|_{L^2}^2.
\end{aligned}
\tag{5.92}
$$

Denote

$$
\begin{aligned}
W^{\varepsilon,k}=&\big\|\delta_t^-\varsigma^{\varepsilon,k+1}\big\|_{L^2}^2+\frac{1}{2}\left(\|\nabla\varsigma^{\varepsilon,k+1}\|_{L^2}^2+\|\nabla\varsigma^{\varepsilon,k}\|_{L^2}^2\right)\\
&+\frac{1}{2}\left(\|\varsigma^{\varepsilon,k+1}\|_{L^2}^2+\|\varsigma^{\varepsilon,k}\|_{L^2}^2\right),\quad 1\le k\le N-1.
\end{aligned}
\tag{5.93}
$$

From (5.92) and (5.93), the following inequality is achieved:

$$
W^{\varepsilon,k}-W^{\varepsilon,k-1}\le C\tau h^{2r+2}+C\tau^5+C\tau(W^{\varepsilon,k}+W^{\varepsilon,k-1}).\tag{5.94}
$$

Then replacing k by i in (5.94), summing the result from 1 to k, we derive

$$
W^{\varepsilon,k}\le W^{\varepsilon,0}+Ch^{2r+2}+C\tau^4+C\tau\sum_{i=1}^k(W^{\varepsilon,i}+W^{\varepsilon,i-1}).\tag{5.95}
$$

Applying the discrete Grönwall inequality to (5.95) gives the result

$$
W^{\varepsilon,k}\le Ch^{2r+2}+C\tau^4,\tag{5.96}
$$

where $\tau\le\tau_8^*$. From the definition of $W^{\varepsilon,k}$, we obtain

$$
\|\nabla\varsigma^{\varepsilon,k+1}\|_{L^2}+\|\varsigma^{\varepsilon,k+1}\|_{L^2}\le Ch^{r+1}+C\tau^2,\quad 0\le k\le N-1.\tag{5.97}
$$

Thus, by the triangle inequality, (3.2), (5.85) and (5.97), the optimal error estimates are deduced

$$
\begin{aligned}
&\big\|u^{\varepsilon,k}-U_h^{\varepsilon,k}\big\|_{L^2}\le\big\|u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big\|_{L^2}+\|\varsigma^{\varepsilon,k}\|_{L^2}\le C(\tau^2+h^{r+1}),\\
&\big\|\nabla\big(u^{\varepsilon,k}-U_h^{\varepsilon,k}\big)\big\|_{L^2}\le\big\|\nabla\big(u^{\varepsilon,k}-R_hu^{\varepsilon,k}\big)\big\|_{L^2}+\|\nabla\varsigma^{\varepsilon,k}\|_{L^2}\le C(\tau^2+h^r),\quad 0\le k\le N.
\end{aligned}
\tag{5.98}
$$

Taking $\tau\le\tau_1=\min\{\tau_4,\tau_8^*\}$ and $h_1\le h_1^*$, the proof of Theorem 4.1 is complete. □

5.4. Proof of Theorems 4.2 and 4.3

In this section, we need to clarify that the two Theorems 4.2 and 4.3 are derived based on quasi-uniform rectangular partitions and bilinear finite elements ($r = 1$) for Ω. The specific proof process of Theorem 4.2 is as follows:

$$\begin{aligned}\left\|\nabla\left(I_h u^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} &\le \left\|\nabla\left(I_h u^{\varepsilon,k} - R_h u^{\varepsilon,k}\right)\right\|_{L^2} + \left\|\nabla\left(R_h u^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} \\ &\le Ch^2 \, \|u^{\varepsilon,k}\|_{H^3} + \|\nabla\varsigma^{\varepsilon,k}\|_{L^2} \\ &\le Ch^2 \, \|u^{\varepsilon,k}\|_{H^3} + C(\tau^2 + h^2) \\ &\le C(\tau^2 + h^2), \end{aligned} \tag{5.99}$$

where using the triangle inequality, (3.3) and (5.97). The proof of Theorem 4.2 is complete. □

Next, we introduce the interpolated postprocessing operator I_{2h} structured by [32], which possesses the following four properties:

$$\begin{aligned} &I_{2h}u^\varepsilon \mid_{\widetilde{K}} \in Q_2(\widetilde{K}), && \forall\, u^\varepsilon \in C(\widetilde{K}), \\ &I_{2h} I_h u^\varepsilon = I_{2h} u^\varepsilon, && \forall\, u^\varepsilon \in C(\widetilde{K}), \\ &\left\|I_{2h}u^\varepsilon - u^\varepsilon\right\|_{H^1} \le Ch^2 \|u^\varepsilon\|_{H^3}, && \forall\, u^\varepsilon \in H^3(\Omega), \\ &\|I_{2h}\omega_h\|_{H^1} \le C\|\omega_h\|_{H^1}, && \forall\, \omega_h \in S^h(\Omega), \end{aligned}$$

where $\widetilde{K}$ is the element formed by merging four adjacent elements on $\mathcal{T}_h$, Q_2 is a biquadratic polynomial space, $C(\widetilde{K})$ is continuous function space on $\widetilde{K}$, and $S^h(\Omega)$ is the finite element space. Based on Theorem 4.2 and the interpolated postprocessing operator I_{2h}, we derive the Theorem 4.3.

$$\begin{aligned}\left\|\nabla\left(u^{\varepsilon,k} - I_{2h}U_h^{\varepsilon,k}\right)\right\|_{L^2} &\le \left\|\nabla\left(u^{\varepsilon,k} - I_{2h}I_h u^{\varepsilon,k}\right)\right\|_{L^2} + \left\|\nabla\left(I_{2h}I_h u^{\varepsilon,k} - I_{2h}U_h^{\varepsilon,k}\right)\right\|_{L^2} \\ &\le \left\|\nabla\left(u^{\varepsilon,k} - I_{2h}u^{\varepsilon,k}\right)\right\|_{L^2} + \left\|\nabla I_{2h}\left(I_h u^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} \\ &\le Ch^2\|u^{\varepsilon,k}\|_{H^3} + C\left\|\nabla\left(I_h u^{\varepsilon,k} - U_h^{\varepsilon,k}\right)\right\|_{L^2} \\ &\le Ch^2\|u^{\varepsilon,k}\|_{H^3} + C(\tau^2 + h^2) \\ &\le C(\tau^2 + h^2). \end{aligned} \tag{5.100}$$

Therefore, the high-accuracy convergence result under H^1-seminorm of the finite element solutions is obtained by using interpolated postprocessing technique. The proof of Theorem 4.3 is complete. □

Remark 5.5. By the numerical results in [47], we learned that three regularized models in Section 2 converge linearly to LogKGE under L^2-norm, L^∞-norm and H^1-norm, with a convergence order of $O(\varepsilon)$. Based on the above conclusions, triangle inequality and Theorems 4.1-4.3, we can obtain

$$\begin{aligned} &\left\|u^k - U_h^{\varepsilon,k}\right\|_{L^2} \le C(\tau^2 + h^2 + \varepsilon), \\ &\left\|\nabla\left(I_h u^k - U_h^{\varepsilon,k}\right)\right\|_{L^2} \le C(\tau^2 + h^2 + \varepsilon), \\ &\left\|\nabla\left(u^k - I_{2h}U_h^{\varepsilon,k}\right)\right\|_{L^2} \le C(\tau^2 + h^2 + \varepsilon). \end{aligned}$$

When ε is small enough or $\varepsilon \lesssim \tau^2$ and $\varepsilon \lesssim h^2$, the energy-conservative Crank-Nicolson FEM for LogKGE (1.1) has the following unconditional optimal and high-accuracy convergence results:

$$\begin{aligned} &\left\|u^k - U_h^{\varepsilon,k}\right\|_{L^2} \le C(\tau^2 + h^2), \quad \left\|\nabla\left(I_h u^k - U_h^{\varepsilon,k}\right)\right\|_{L^2} \le C(\tau^2 + h^2), \\ &\left\|\nabla\left(u^k - I_{2h}U_h^{\varepsilon,k}\right)\right\|_{L^2} \le C(\tau^2 + h^2). \end{aligned}$$

6. Numerical Results

In this section, we will provide some numerical examples to confirm our theoretical results. For the purpose of comparison, the energy conservative Crank-Nicolson finite element numerical scheme proposed in Section 3 is applied to three regularized models to test energy conservation, unconditional optimal and high-accuracy convergence. Denote $\{U_h^{\varepsilon,k}\,|\,0 \le k \le N\}$ as the numerical solutions for the LogKGE of time division τ and space division h at time T.

Example 6.1. Consider the following LogKGE ($\lambda = 1$):

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \lambda u(\mathbf{x},t) f\left(|u(\mathbf{x},t)|^2\right) = 0, \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{6.1a}$$

$$u(\mathbf{x},0) = \sin(\pi x)\sin(\pi y), \quad u_t(\mathbf{x},0) = 0, \quad \mathbf{x} \in \Omega, \tag{6.1b}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times [0,T], \tag{6.1c}$$

where

$$f\left(|u(\mathbf{x},t)|^2\right) = \ln\left(|u(\mathbf{x},t)|^2\right),$$

and $\Omega = [0,1] \times [0,1]$. The corresponding three regularized models with a small regularized parameter ($0 < \varepsilon \ll 1$) can be summarized as

$$\begin{aligned}
&u_{tt}^{\varepsilon}(\mathbf{x},t) - \Delta u^{\varepsilon}(\mathbf{x},t) + u^{\varepsilon}(\mathbf{x},t) + \lambda u^{\varepsilon}(\mathbf{x},t) f_{reg}^{\varepsilon}\left(|u^{\varepsilon}(\mathbf{x},t)|^2\right) = 0, && (\mathbf{x},t) \in \Omega \times (0,T],\\
&u^{\varepsilon}(\mathbf{x},0) = \sin(\pi x)\sin(\pi y), \quad u_t^{\varepsilon}(\mathbf{x},0) = 0, && \mathbf{x} \in \Omega,\\
&u^{\varepsilon}(\mathbf{x},t) = 0, && (\mathbf{x},t) \in \partial\Omega \times [0,T],
\end{aligned}$$

where f_{reg}^{ε} represents three forms $\widetilde{f^{\varepsilon}}, \widehat{f^{\varepsilon}}$ and f_n^{ε}.

Take $n = 3$ and $\varepsilon =$ 1e-08. Define error and convergence order [51]

$$e(\tau,h) = \max_{0\le k\le N}\left|U_h^{\varepsilon,k}(\tau,h) - U_{\frac{h}{2}}^{\varepsilon,2k}\left(\frac{\tau}{2},\frac{h}{2}\right)\right|, \quad Order(\tau,h) = \log_2\frac{\|e(\tau,h)\|}{\|e(\tau/2,h/2)\|}.$$

For this example, we mainly conduct the following tests:

- We first verify the error estimates under L^2-norm and L^∞-norm, as well as the convergence orders for the Crank-Nicolson FEM with three different nonlinear terms. Errors and convergence orders at $T = 1$ are shown in Tables 6.1-6.3. From the data in tables, we see that there are only slight differences of $\|e(\tau,h)\|_{L^2}$ and $\|e(\tau,h)\|_{L^\infty}$ for three different regularized models. This is because our ε is small enough to be negligible, we get

$$\widetilde{f^{\varepsilon}}(\rho) = \widehat{f^{\varepsilon}}(\rho) = f_n^{\varepsilon}(\rho), \quad \rho > 0.$$

  In addition, the convergence orders of $\|e(\tau,h)\|_{L^2}$ and $\|e(\tau,h)\|_{L^\infty}$ are close to 2, which are accordance with the theoretical results.

- Next, we test the discrete energy conservation law for the Crank-Nicolson FEM of three different nonlinear terms fixed $h = 1/16$. The values of discrete energy $E_h^{\varepsilon}(t)$ and its relative error $\Delta E_h^{\varepsilon}(t)$ are provided in Tables 6.4-6.6, where $\Delta E_h^{\varepsilon}(t)$ is defined by

$$\Delta E_h^{\varepsilon}(t) = \frac{\left|E_h^{\varepsilon}(t) - E_h^{\varepsilon}(0)\right|}{E_h^{\varepsilon}(0)}.$$

  Furthermore, we also plot $E_h^{\varepsilon}(t)$ and $\Delta E_h^{\varepsilon}(t)$ in Figs. 6.1-6.2.

Table 6.1: Errors and convergence orders at $T = 1$ for $\widetilde{f^{\varepsilon}}$ (Example 6.1).

h	τ	$\|e(\tau,h)\|_{L^2}$	$Order(\tau,h)$	$\|e(\tau,h)\|_{L^\infty}$	$Order(\tau,h)$
1/8	1/16	1.7636766504e-02	1.9092	3.6333017120e-02	1.9307
1/16	1/32	4.6957790812e-03	1.9608	9.5305403668e-03	1.9532
1/32	1/64	1.2062621591e-03	1.9825	2.4610757068e-03	1.9870
1/64	1/128	3.0525038502e-04	*	6.2084334130e-04	*
1/128	1/256	*	*	*	*

Table 6.2: Errors and convergence orders at $T = 1$ for $\widehat{f^{\varepsilon}}$ (Example 6.1).

h	τ	$\|e(\tau,h)\|_{L^2}$	$Order(\tau,h)$	$\|e(\tau,h)\|_{L^\infty}$	$Order(\tau,h)$
1/8	1/16	1.7636766424e-02	1.9092	3.6333016982e-02	1.9307
1/16	1/32	4.6957790461e-03	1.9608	9.5305403291e-03	1.9532
1/32	1/64	1.2062621441e-03	1.9825	2.4610756768e-03	1.9870
1/64	1/128	3.0525037787e-04	*	6.2084332838e-04	*
1/128	1/256	*	*	*	*

Table 6.3: Errors and convergence orders at $T = 1$ for f_n^{ε} (Example 6.1).

h	τ	$\|e(\tau,h)\|_{L^2}$	$Order(\tau,h)$	$\|e(\tau,h)\|_{L^\infty}$	$Order(\tau,h)$
1/8	1/16	1.7636766424e-02	1.9092	3.6333016982e-02	1.9307
1/16	1/32	4.6957790461e-03	1.9608	9.5305403291e-03	1.9532
1/32	1/64	1.2062621441e-03	1.9825	2.4610756768e-03	1.9870
1/64	1/128	3.0525037786e-04	*	6.2084332836e-04	*
1/128	1/256	*	*	*	*

Table 6.4: The values of discrete energy and its relative error for $\widetilde{f^{\varepsilon}}$ (Example 6.1).

T	$E_h^{\varepsilon}(t)$	$\Delta E_h^{\varepsilon}(t)$
0	2.377502005986851e+02	0
5	2.377502005986806e+02	1.876847367252277e-14
10	2.377502005986738e+02	4.769822290023303e-14
15	2.377502005986769e+02	3.466788130593379e-14
20	2.377502005986860e+02	3.705876967186025e-15

Table 6.5: The values of discrete energy and its relative error for $\widehat{f^{\varepsilon}}$ (Example 6.1).

T	$E_h^{\varepsilon}(t)$	$\Delta E_h^{\varepsilon}(t)$
0	2.377501997885252e+02	0
5	2.377501997885268e+02	6.933576284813681e-15
10	2.377501997885267e+02	6.335854191295260e-15
15	2.377501997885291e+02	1.649712978110841e-14
20	2.377501997885426e+02	7.328072866535838e-14

Table 6.6: The values of discrete energy and its relative error for f_n^ε (Example 6.1).

T	$E_h^\varepsilon(t)$	$\Delta E_h^\varepsilon(t)$
0	2.377501997885250e+02	0
5	2.377501997885240e+02	4.303599073332631e-15
10	2.377501997885222e+02	1.195444187036842e-14
15	2.377501997885263e+02	5.259954422962105e-15
20	2.377501997885369e+02	4.985002259943632e-14

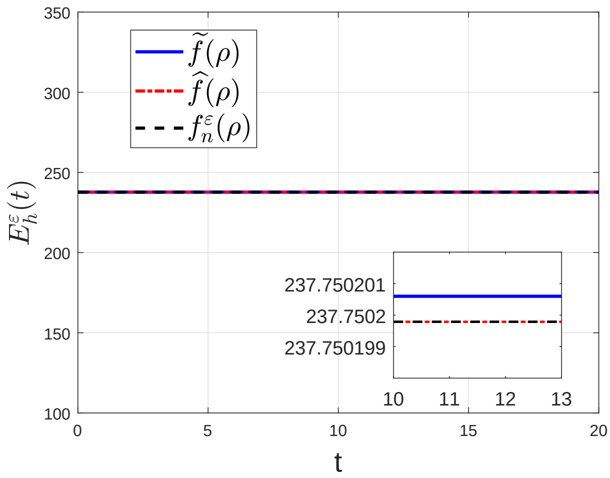

Fig. 6.1. The discrete energy conservation of Crank-Nicolson FEM with three different nonlinear term (Example 6.1).

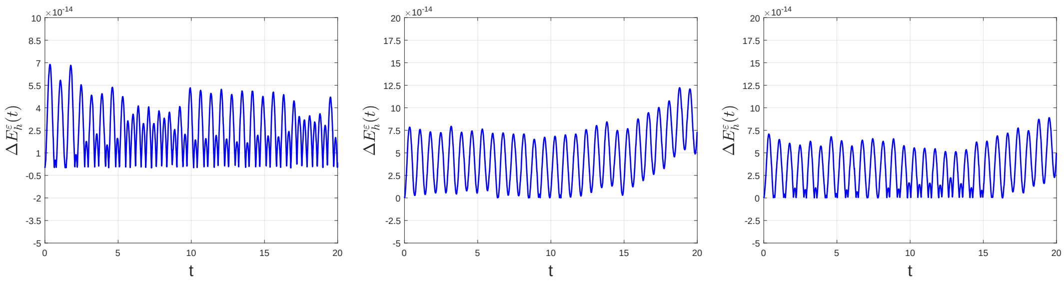

Fig. 6.2. The relative errors of discrete energy for $\widetilde{f}, \widehat{f}$ and f_n^ε (Example 6.1).

Example 6.2. Consider LogKGE with the following exact solution ($\lambda = 1$):

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \lambda u(\mathbf{x},t) f\big(|u(\mathbf{x},t)|^2\big) = g(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{6.2a}$$

$$u(\mathbf{x},0) = \phi_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = \phi_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{6.2b}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times [0,T], \tag{6.2c}$$

where

$$f\big(|u(\mathbf{x},t)|^2\big) = \ln(|u(\mathbf{x},t)|^2),$$

$\Omega = [0, 1] \times [0, 1]$ and the function $g(\mathbf{x}, t)$ is determined by the given solution

$$u(\mathbf{x}, t) = e^{-t} \sin(\pi x) \sin(\pi y).$$

The corresponding three regularized models with a small regularized parameter ($0 < \varepsilon \ll 1$) are presented as

$$\begin{aligned}
&u^{\varepsilon}_{tt}(\mathbf{x}, t) - \Delta u^{\varepsilon}(\mathbf{x}, t) + u^{\varepsilon}(\mathbf{x}, t) + \lambda u^{\varepsilon}(\mathbf{x}, t) f^{\varepsilon}_{reg}\left(|u^{\varepsilon}(\mathbf{x}, t)|^2\right) = g^{\varepsilon}(\mathbf{x}, t), && (\mathbf{x}, t) \in \Omega \times (0, T],\\
&u^{\varepsilon}(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \quad u^{\varepsilon}_t(\mathbf{x}, 0) = \phi_1(\mathbf{x}), && \mathbf{x} \in \Omega,\\
&u^{\varepsilon}(\mathbf{x}, t) = 0, && (\mathbf{x}, t) \in \partial\Omega \times [0, T],
\end{aligned}$$

where f^{ε}_{reg} means the same as in Example 6.1.

In the calculation process, let $n = 3, T = 1$ and $\varepsilon =$ 1e-08. We employ the Crank-Nicolson FEM with three different nonlinear terms to (6.2) to verify the unconditional optimal and high-accuracy convergence results. For this example, we make the following works:

- The unconditional optimal error estimates of L^2-norm, high-accuracy results, and convergence orders of the numerical scheme with three different nonlinear terms are shown in Fig. 6.3. Moreover, we observe the convergence orders of $\|u^k - U_h^{\varepsilon,k}\|_{L^2}$, $\|I_h u^k - U_h^{\varepsilon,k}\|_{H^1}$ and $\|u^k - I_{2h} U_h^{\varepsilon,k}\|_{H^1}$ in both time and space are close to 2, which are consistent with our theoretical results.
- Let $\tau^2 = h/2$, we check the error estimates and convergence orders of $\|u^k - U_h^{\varepsilon,k}\|_{H^1}$ with three different nonlinear terms. We see the convergence orders in time are close 2 and in space are close 1, as exhibited in Fig. 6.4. These are accordance with our theoretical results.

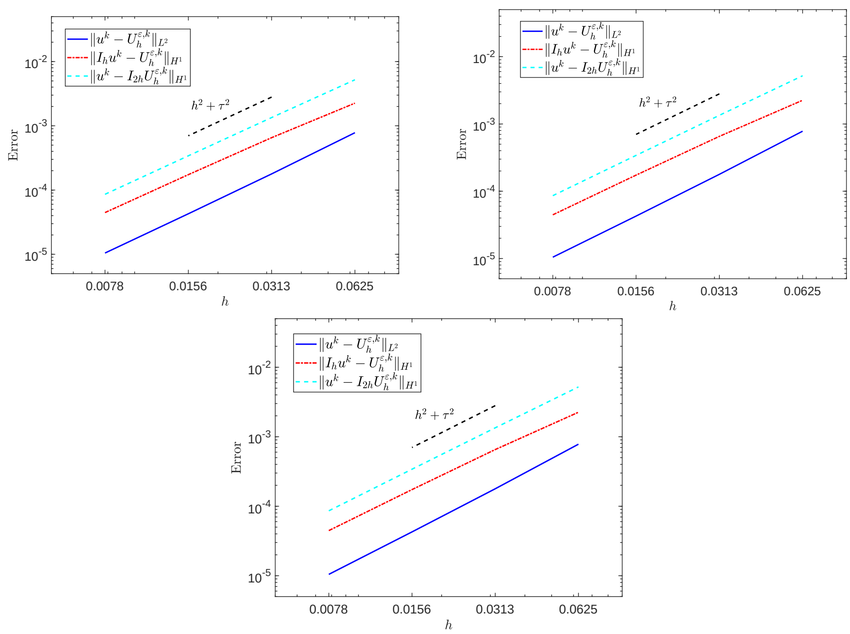

Fig. 6.3. The error estimates and high-accuracy convergence results for $\widetilde{f}, \widehat{f}$ and f^{ε}_n (Example 6.2).

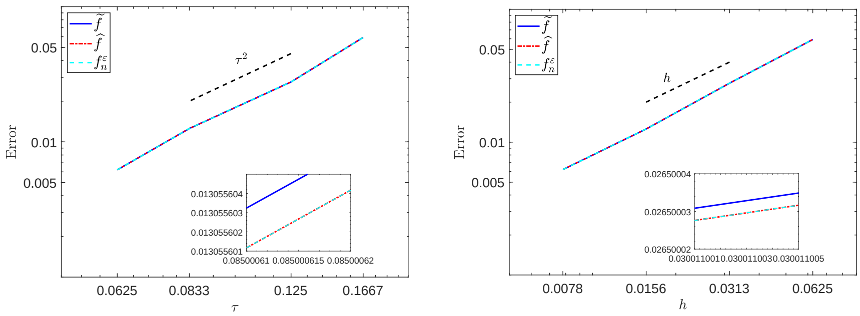

Fig. 6.4. The error estimates and convergence orders in time and space of $\|u^k - U_h^{\varepsilon,k}\|_{H^1}$ for $\widetilde{f}, \widehat{f}$ and f_n^ε (Example 6.2).

7. Conclusions

In this paper, we first proposed three regularized models for LogKGE. Then a fully implicit energy-conservative Crank-Nicolson Galerkin FEM was designed for one of the regularized models, namely ERLogKGE. By innovative combination of the cut-off function technique and the time-space error splitting technique, we strictly proved the discrete energy conservation law, unconditional optimal and high-accuracy convergence results of the numerical scheme. Finally, two numerical experiments which both test three different nonlinear terms were provided to verify the correctness and effectiveness of our theoretical results.

Acknowledgements. The work is supported by the China Postdoctoral Science Foundation (Grant No. 2023T160589), by the Cultivation Foundation of Zhengzhou University (Grant No. JC23153003), by the National Natural Science Foundation of China (Grant Nos. 11801527, 11971416), by the Natural Science Foundation of Henan Province (Grant No. 222300420256), by the Training Plan of Young Backbone Teachers in Colleges of Henan Province (Grant No. 2020GGJS230), by the Program for Innovative Research Team (in Science and Technology) in University of Henan Province (Grant No. 23IRTSTHN018) and by the Academic Degrees & Graduate Education Reform Project of Henan Province (Grant No. 2021SJGLX224Y).

References

[1] G.D. Akrivis, V.A. Dougalis, and O.A. Karakashian,On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation, *Numer. Math.*, **59** (1991), 31–53.

[2] L. Alzaleq and V. Manoranjan, Exact traveling waves for the Klein-Gordon equation with different logarithmic nonlinearities, *Eur. Phys. J. Plus.*, **136** (2021), 313.

[3] W. Bao and Y. Cai, Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator, *SIAM J. Numer. Anal.*, **50**:2 (2012), 492–521.

[4] W. Bao and Y. Cai, Optimal error estimates of finite difference methods for the Gross-Pitaevskii equation with angular momentum rotation, *Math. Comput.*, **82**:281 (2013), 99–128.

[5] W. Bao, R. Carles, C. Su, and Q. Tang, Error estimates of a regularized finite difference method for the logarithmic Schrödinger equation, *SIAM J. Numer. Anal.*, **57**:2 (2019), 657–680.

[6] W. Bao, R. Carles, C. Su, and Q. Tang, Regularized numerical methods for the logarithmic Schrödinger equation, *Numer. Math.*, **143**:2 (2019), 461–487.

[7] W. Bao, R. Carles, C. Su, and Q. Tang, Error estimates of local energy regularization for the logarithmic Schrödinger equation, *Math. Models Methods Appl. Sci.*, **32**:01 (2022), 101–136.

[8] W. Bao and X. Dong, Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime, *Numer. Math.*, **120**:2 (2012), 189–229.
[9] W. Bao, X. Dong and X. Zhao, An exponential wave integrator sine pseudospectral method for the Klein-Gordon-Zakharov system, *SIAM J. Sci. Comput.*, **35** (2013), A2903–A2927.
[10] W. Bao, Y. Feng, and C. Su, Uniform error bounds of a time-splitting spectral method for the long-time dynamics of the nonlinear Klein-Gordon equation with weak nonlinearity, *Math. Comput.*, **91** (2022), 811–842.
[11] W. Bao, Y. Feng, and W. Yi, Long time error analysis of finite difference time domain methods for the nonlinear Klein-Gordon equation with weak nonlinearity, *Commun. Comput. Phys.*, **26**:5 (2019), 1307–1334.
[12] W. Bao and X. Zhao, Comparison of numerical methods for the nonlinear Klein-Gordon equation in the nonrelativistic limit regime, *J. Comput. Phys.*, **398** (2019), 108886.
[13] K. Bartkowski and P. Górka, One-dimensional Klein-Gordon equation with logarithmic nonlinearities, *J. Phys. A: Math. Theor.*, **41**:35 (2008), 355201.
[14] G. Berikelashvili, O. Jokhadze, S. Kharibegashvili, and B. Midodashvili, Finite difference solution of a nonlinear Klein-Gordon equation with an external source, *Math. Comput.*, **80**:274 (2010), 847–862.
[15] I. Bialynicki-Birula and J. Mycielski, Wave equations with logarithmic nonlinearities, *Bull. Acad. Polon Sci., Ser. Sci. Math. Astron. Phys.*, **23**:4 (1975), 461–466.
[16] I. Bialynicki-Birula and J. Mycielski, Nonlinear wave mechanics, *Ann. Phys.*, **100**:1-2 (1976), 62–93.
[17] W. Cao and B. Guo, Fourier collocation method for solving nonlinear Klein-Gordon equation, *J. Comput. Phys.*, **108**:2 (1993), 296–305.
[18] L. Cherfils, A. Miranville, and S. Zelik, The Cahn-Hilliard equation with logarithmic potentials, *Milan J. Math.*, **79**:2 (2011), 561–596.
[19] M. Dahlby and B. Owren, A general framework for deriving integral preserving numerical methods for PDEs, *SIAM J. Sci. Comput.*, **33**:5 (2011), 2318–2340.
[20] C.M. Elliott and H. Garcke, On the Cahn-Hilliard equation with degenerate mobility, *SIAM J. Math. Anal.*, **27** (1996), 404–423.
[21] P. Henning and D. Peterseim, Crank-Nicolson Galerkin approximations to nonlinear Schrödinger equations with rough potentials, *Math. Models Methods Appl. Sci.*, **27**:11 (2017), 2147–2184.
[22] G. James and D. Pelinovsky, Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, *Proc. Math. Phys. Eng. Sci.*, **470**:2165 (2014), 20130462.
[23] M. Křížek and P. Neittaanmäki, On superconvergence techniques, *Acta Appl. Math.*, **9**:3 (1987), 175–198.
[24] H. Lévy, J. Bergé, and J.P. Uzan, Solving nonlinear Klein-Gordon equations on unbounded domains via the finite element method, *Phys. Rev. D*, **106**:12 (2022), 124021.
[25] B. Li and W. Sun, Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations, *Int J. Numer. Anal. Model.*, **10**:3 (2012), 622–633.
[26] B. Li and W. Sun, Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media, *SIAM J. Numer. Anal.*, **51**:4 (2013), 1959–1977.
[27] H. Li, X. Zhao, and Y. Hu, Numerical solution of the regularized logarithmic Schrödinger equation on unbounded domains, *Appl. Numer. Math.*, **140** (2019), 91–103.
[28] M. Li, Cut-off error splitting technique for conservative nonconforming VEM for N-coupled nonlinear Schrödinger–Boussinesq equations, *J. Sci. Comput.*, **93**:3 (2022), 86.
[29] M. Li, D. Shi, and J. Wang, Unconditional superconvergence analysis of a linearized Crank-Nicolson Galerkin FEM for generalized Ginzburg-Landau equation, *Comput. Math. Appl.*, **79**:8 (2020), 2411–2425.
[30] S. Li and L. Vu-Quoc, Finite difference calculus invariant structure of a class of algorithms for the

nonlinear Klein-Gordon equation, *SIAM J. Numer. Anal.*, **32**:6 (1995), 1839–1875.

[31] Q. Lin, Global error expansion and superconvergence for higher order interpolation of finite elements, *J. Comput. Math.*, **10** (1992), 286–289.

[32] Q. Lin and J. Lin, *Finite Element Methods: Accuracy and Improvement*, Science Press, 2007.

[33] Q. Lin, T. Lu, and S. Shen, Maximum norm estimate, extrapolation and optimal point of stresses for finite element methods oh strongly regular triangulation, *J. Comput. Math.*, **1**:4 (1983), 376–383.

[34] Q. Lin and Y. Yang, The finite element interpolated correction method for elliptic eigenvalue problems, *Math. Numer. Sin.*, **14**:3 (1992), 615–626.

[35] F. Natali and J.E. Cardoso, Orbital stability of periodic standing waves for the logarithmic Klein-Gordon equation, *J. Math. Anal. Appl.*, **484**:2 (2020), 123723.

[36] J.T. Oden and J.N. Reddy, Note on an approximate method for computing consistent conjugate stresses in elastic finite elements, *Int J. Numer. Meth. Eng.*, **6**:1 (1973), 55–61.

[37] G. Rosen, Dilatation covariance and exact solutions in local relativistic field theories, *Phys. Rev.*, **183**:5 (1969), 1186–1188.

[38] J.M. Sanz-Serna, Methods for the numerical solution of the nonlinear Schrödinger equation, *Math. Comput.*, **43**:167 (1984), 21–27.

[39] J.M. Sanz-Serna and J.G. Verwer, Conerservative and nonconservative schemes for the solution of the nonlinear Schrödinger equation, *IMA J. Numer. Anal.*, **6**:1 (1986), 25–42.

[40] D. Shi and H. Zhang, Unconditional optimal error estimates and superconvergence analysis of energy-preserving FEM for general nonlinear Schrödinger equation with wave operator, *Comput. Math. Appl.*, **128**:3 (2022), 79–95.

[41] W. Strauss and L. Vazquez, Numerical solution of a nonlinear Klein-Gordon equation, *J. Comput. Phys.*, **28**:2 (1978), 271–278.

[42] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, in: *Springer Series in Computational Mathematics*, Vol. 25, Springer, 2006.

[43] J. Wang, Unconditional stability and convergence of Crank-Nicolson Galerkin FEMs for a nonlinear Schrödinger-Helmholtz system, *Numer. Math.*, **139** (2018), 479–503.

[44] Q. Wang and D. Cheng, Numerical solution of damped nonlinear Klein-Gordon equations using variational method and finite element approach, *Appl. Math. Comput.*, **162**:1 (2005), 381–401.

[45] A.M. Wazwaz, Gaussian solitary waves for the logarithmic-KdV and the logarithmic-KP equations, *Phys. Scr.*, **89**:1 (2014), 252–268.

[46] A. Wazwaz, Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities, *Nonlinear Dyn.*, **83**:1-2 (2016), 591–596.

[47] J. Yan, H. Zhang, X. Qian, X. Chen, and S. Song, A novel regularized model for the logarithmic Klein-Gordon equation, *Appl. Numer. Math.*, **176** (2022), 19–37.

[48] J. Yan, H. Zhang, X. Qian, and S. Song, Regularized finite difference methods for the logarithmic Klein-Gordon equation, *East Asian J. Appl. Math.*, **11**:1 (2021), 119–142.

[49] J. Yan, H. Zhang, X. Qian, and S. Song, Two regularized energy-preserving finite difference methods for the logarithmic Klein-Gordon equation, *J. Comput. Appl. Math.*, **393** (2021), 113478.

[50] L. Zhang, Convergence of a conservative difference scheme for a class of Klein-Gordon-Schrödinger equations in one space dimension, *Appl. Math. Comput.*, **163**:1 (2005), 343–355.

[51] X. Zhao, Z. Sun, and Z. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, *SIAM J. Sci. Comput.*, **36**:6 (2014), A2865–A2886.