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ERROR ANALYSIS OF VIRTUAL ELEMENT METHODS FOR THE TIME-DEPENDENT POISSON-NERNST-PLANCK EQUATIONS*

Ying Yang
School of Mathematics and Computational Science, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Center for Applied Mathematics of Guangxi (GUET), Guilin University of Electronic Technology, Guilin 541004, China
Email: yangying@lsec.cc.ac.cn
Ya Liu
School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, China
Email: 979504374@qq.com
Yang Liu and Shi Shu[1)]
School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China
Email: liuyang@smail.xtu.edu.cn, shushi@xtu.edu.cn

Abstract

We discuss and analyze the virtual element method on general polygonal meshes for the time-dependent Poisson-Nernst-Planck (PNP) equations, which are a nonlinear coupled system widely used in semiconductors and ion channels. After presenting the semi-discrete scheme, the optimal H^1 norm error estimates are presented for the time-dependent PNP equations, which are based on some error estimates of a virtual element energy projection. The Gummel iteration is used to decouple and linearize the PNP equations and the error analysis is also given for the iteration of fully discrete virtual element approximation. The numerical experiment on different polygonal meshes verifies the theoretical convergence results and shows the efficiency of the virtual element method.

Mathematics subject classification: 65N15, 65N30, 35K61.
Key words: Virtual element method, Error estimate, Poisson-Nernst-Planck equations, Polygonal meshes, Energy projection, Gummel iteration.

1. Introduction

The virtual element method (VEM) could be seen as a deformation of the classical mimetic finite difference method, which was originally proposed in [5] as a generalization of the finite element method (FEM). This method is applicable to general polygon/polyhedral grids even including the multiply-connected or non-convex polygon grids, and hence has low requirements on grid quality. The VEM, in comparison to the traditional FEM, does not require an explicit expression of the discrete basis functions. In addition to that, it only needs to define the appropriate degrees of freedom to convert the discrete formulation into the matrix form. Thanks to its applicability and simplicity, the VEM has been applied to many equations, for instance, the second-order elliptic equation [8], the parabolic equations [1, 46], hyperbolic equation [45], the Stokes equations [4,11,17], the elasticity problems [6,24] and the plate bending problem [16], etc.

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1) Corresponding author

Here, we consider the time-dependent Poisson-Nernst-Planck (PNP) equations. The classic PNP equations are a coupled nonlinear system of partial differential equations, which consist of the electrostatic Poisson equation and the Nernst-Planck equation. The coupled nonlinear system was originally derived by Nernst [37] and Planck [38] and has been widely applied in semiconductors [36, 47], biological ion channels [22, 43] and electrochemical systems [35, 39, 48].

Because of the high nonlinearity and strong coupling, it is difficult to find the analytic solution for PNP equations. Many numerical methods were developed to find the approximate solutions, for instance, finite volume methods [12, 18], finite difference methods [23, 30] and FEMs [33, 49], etc. The FEM has been applied to PNP equations for many years and it is popular because of its flexibility and adaptability in dealing with the irregular interface. In recent years, some work on convergence analysis of FEM has emerged. We presented some error bounds in [51] for a piecewise finite element approximation to the steady-state PNP equations describing the electrodiffusion of ions in a solvated biomolecular system. In [44], the authors discussed a priori error estimates of FEM for the time-dependent PNP equations, where the optimal error estimates are obtained in $L^\infty(H^1)$ and $L^2(H^1)$ norms and the suboptimal error estimate is obtained in the $L^\infty(L^2)$ norm. An optimal L^2 norm error estimate of the FEM to a linearized backward Euler scheme for the time-dependent PNP equations has been obtained in [26]. Recently, we presented a decoupling two-grid FEM for the time-dependent PNP equations in [42]. This method costs less computational time and remains the same order of accuracy compared with the FEM combined with the Gummel iteration. The optimal L^2 error estimate for the classic nonlinear backward Euler scheme was also presented in [42] with a generic regularity assumption of the solution. In [52], we studied the superconvergent gradient recovery based on the finite element approximation for the strong nonlinear PNP equations. The superconvergence results are successfully applied to improve the efficiency of the external iteration in the computation of a practical ion channel problem. The a posteriori error estimates and adaptive FEM for the steady-state PNP equations are studied in [29, 41].

In this paper, the main purpose is to provide the a priori error analysis for the virtual element discretization of the time-dependent PNP equations. First, we design a suitable virtual element discretization scheme for the equations. Compared with the finite element discretization for PNP equations, it can be used on very general polygonal meshes, so the requirements for mesh quality are lower. It could be more suitable for PNP practical problems with extremely irregular interfaces, for example ion channel problems. Then, we present the a priori error analysis for the VEM. We focus mainly on the error estimates for the semi-discrete system. The suboptimal L^2 norm and the optimal H^1 norm error estimates with k-th ($k \geq 1$) order virtual element are presented for semi-discrete virtual element approximations. After that, a fully discrete virtual element scheme is given for the PNP equations. Considering the coupling and nonlinearity of the fully discrete system, the Gummel iteration is applied to decouple and linearize it. This iteration is a commonly used decoupling method for solving PNP equations, see e.g. [13, 28, 34]. Here we introduce the Gummel iteration of the fully discrete virtual element approximation and present the error analysis for it. The suboptimal L^2 norm error estimates are obtained for the Gummel iteration of the VEM for PNP equations.

From a mathematical point of view, the PNP equations consist of a linear elliptic (Poisson) equation and two nonlinear parabolic (NP) equations. We follow the frame of convergence analysis in [32] to present the error estimate for the elliptic equation. Some arguments in [44] are used in the analysis of the nonlinear parabolic equation. Compared with these relevant work, we have some own characteristics in the analysis. For example, although our scheme was motivated by our work [32] in which a VEM is provided for steady-state PNP equations, the frame of most of the analysis is quite different, since the NP equation considered in this paper is a parabolic equation, while it is an elliptic one in [32]. The error results are also different in two aspects. Firstly, in [32], since the L^2 norm error estimates of concentrations is difficult to obtain, the H^1 norm error estimates depend on the L^2 norm error estimates. In this paper, we present the suboptimal L^2 error estimates of concentrations, then the optimal error estimates in the H^1 norm are obtained. Secondly, the error estimates in [32] require a regularity assumption of the virtual element solution, i.e. $\phi_h \in W^{1,\infty}(\Omega)$, which is not easy to prove. In this paper, to avoid using this assumption, we show the boundness of the L^2 projection, see Lemma 3.2. Based on the result, the optimal error estimates of the virtual element approximation in the H^1 norm are obtained without the regularity assumption of the discrete solution. In addition, although we follow the definition of the FEM energy projection in [44] to define the VEM energy projection, the detailed proof of the existence and uniqueness of the VEM energy projection is given in this paper, while the existence or uniqueness of the FEM energy projection is not discussed in [44]. The proof is not trivial, in which some detailed discussions of the nonlinear form need to be given, see Lemmas 3.4-3.5.

Recently, we note that a fully coupled and energy-stable VEM was proposed and analyzed for the coupled PNP and Navier-Stokes equations in [21], (the preprint of which was submitted to arXiv at a similar time as ours, see [20, 50]). The existence and uniqueness of the virtual element approximation are presented and the optimal error estimates are derived for fully discrete scheme in the L^2 and H^1 norms, respectively. However, this error analysis is based on an assumption of the initial value of the original solution, which requires the initial data is less than a fixed constant (generally much smaller than one), see [21, Theorem 5.2, Eq. (5.55)]. It is usually difficult to satisfy this condition for the practical PNP equations. This assumption is not needed in the error analysis of this paper.

The rest of this paper is organised as follows. In Section 2, we introduce the time-dependent PNP equations and present the corresponding weak form. The virtual element space and the corresponding semi-discrete virtual element approximation are also discussed in this section. In Section 3, the convergence analysis in the L^2 and H^1 norms for the semi-discrete scheme is derived. Section 4 introduces the Gummel iteration of the fully discrete scheme and presents the convergence analysis of the iteration in the L^2 norm. To confirm the efficiency of the proposed methods and verify the accuracy of the theoretical analysis, a numerical example is given in Section 5. Finally, some conclusions are made in Section 6.

2. Continuous and Discrete Problems

In this section, we introduce the time-dependent PNP equations and discuss the semi-discrete virtual element discretization.

2.1. Time-dependent Poisson-Nernst-Planck equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be the Lipschitz continuous boundary of Ω. We consider the following time-dependent PNP equations (cf. [26, 44]):

$$\begin{cases} p_t^i - \nabla \cdot (\nabla p^i + q^i p^i \nabla \phi) = F^i & \text{in } \Omega \quad \text{for} \quad t \in (0,T], \quad i = 1,2, \\ -\Delta\phi - \sum_{i=1}^{2} q^i p^i = f & \text{in } \Omega \quad \text{for} \quad t \in (0,T] \end{cases} \tag{2.1}$$

with the homogeneous Dirichlet boundary conditions

$$\begin{cases} \phi = 0 & \text{on } \partial\Omega \quad \text{for} \quad t \in (0, T], \\ p^i = 0 & \text{on } \partial\Omega \quad \text{for} \quad t \in (0, T], \end{cases} \tag{2.2}$$

where $p^i, i = 1, 2$, denotes the concentration of the i-th ionic species, $p^i_t = \partial p^i/\partial t$, ϕ represents the electrostatic potential, the constant q^i corresponds to the charge of the species i, f and F^i are the reaction source terms, $p^i(\cdot, 0) := p^i_0$, $\phi(\cdot, 0) := \phi_0$, $f(\cdot, 0) := f_0$, and $F^i(\cdot, 0) := F^i_0$ denote the initial data.

For any $u, v, \psi, u^1, u^2 \in H^1_0(\Omega)$, define

$$a(u, v) = (\nabla u, \nabla v), \quad b_i(u, \psi, v) = (q^i u \nabla\psi, \nabla v), \quad \tilde{b}(u^1, u^2, v) = \Big(-\sum_{i=1}^{2} q^i u^i, v \Big). \tag{2.3}$$

The weak formulation of (2.1)-(2.2) is given as: Find $p^i \in L^2(0, T; H^1_0(\Omega)), i = 1, 2$, and $\phi \in L^2(0, T; H^1_0(\Omega))$ such that

$$\begin{cases} \big(p^i_t(t), v\big) + a\big(p^i(t), v\big) + b_i\big(p^i(t), \phi(t), v\big) = \big(F^i(t), v\big), & \forall\, v \in H^1_0(\Omega), \quad i = 1, 2, \\ a\big(\phi(t), w\big) + \tilde{b}\big(p^1(t), p^2(t), w\big) = \big(f(t), w\big), & \forall\, w \in H^1_0(\Omega). \end{cases} \tag{2.4}$$

The existence and uniqueness of the solution for (2.4) have been shown in [25] for

$$F_i = R(p^1, p^2) = r(p^1, p^2)(1 - p^1 p^2)$$

with a Lipschitzian function $r : R^2_+ \to R_+$. In this case, system (2.1)-(2.2) describes the transport of mobile carriers in a semiconductor device, $R(p^1, p^2)$ is the net recombination rate and p^1 and p^2 represent the densities of mobile holes and electrons, respectively (see [25]).

In the rest of the paper, we follow the standard notations for Sobolev spaces $W^{s,p}(\Omega)$ with $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$ denote the norm and the seminorm (cf. [2, 15]), respectively. For $p = 2$, the notations $H^s(\Omega) = W^{s,2}(\Omega)$ and $H^1_0(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For simplicity, denote by $\|\cdot\|_s = \|\cdot\|_{W^{s,2}(\Omega)}$, $\|\cdot\|_0 = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{0,\infty} = \|\cdot\|_{L^\infty(\Omega)}$. We adopt $(\cdot, \cdot)$ to denote the standard L^2-inner product.

2.2. The VEM semi-discrete scheme

In this subsection, we present local and global virtual element space and outline the semi-discrete virtual element formulation of (2.4). Some lemmas useful in the sequel are also introduced here. Let $\{\mathcal{T}_h\}$ be a family of decompositions of Ω into elements E with $h_E = \text{diam}(E)$ and $h = \max\{h_E : E \in \mathcal{T}_h\}$. On each element E, we suppose $a^E(\cdot, \cdot), b^E_i(\cdot, \cdot, \cdot)$ and $\tilde{b}^E(\cdot, \cdot, \cdot)$ are the restrictions of the corresponding forms defined in (2.3) on E. Following [3, 5], we make the following assumption for the mesh $\mathcal{T}_h$.

Assumption 2.1. *Every element E is star-shaped with respect to a ball of radius greater than γh_E and the distance between any two vertices of E is greater than or equal to ch_E, where γ and c are uniform positive constants.*

Next, let us begin with defining the virtual element space. For any integer $k \geq 0$, denote by $\mathbb{P}_k(D)$ the space of polynomial functions with the total degree up to k living on D. Following [8], for every element $E \in \mathcal{T}_h$, we introduce a local space

$$\tilde{Q}^k_h(E) = \big\{v \in H^1(E) : v|_{\partial E} \in C^0(\partial E),\ v|_e \in \mathbb{P}_k(e),\ \forall\, e \subset \partial E,\ \Delta v \in \mathbb{P}_k(E)\big\}.$$

The projection operator $\Pi_k^\nabla : \tilde{Q}_h^k(E) \to \mathbb{P}_k(E)$ is given by

$$\big(\nabla\big(\Pi_k^\nabla v_h - v_h\big), \nabla q\big)_E = 0, \quad \int_{\partial E} \big(\Pi_k^\nabla v_h - v_h\big) ds = 0, \quad \forall\, q \in \mathbb{P}_k(E). \tag{2.5}$$

Then we define the following local virtual space:

$$Q_h^k(E) = \big\{v_h \in \tilde{Q}_h^k(E) : \big(v_h - \Pi_k^\nabla v_h, q\big)_E = 0,\ \forall\, q \in \big(\mathbb{P}_k/\mathbb{P}_{k-2}(E)\big)\big\},$$

where $\mathbb{P}_k/\mathbb{P}_{k-2}(E)$ denotes the polynomials in $\mathbb{P}_k(E)$ which are $L^2(E)$ orthogonal to $\mathbb{P}_{k-2}(E)$. For $v_h \in Q_h^k(E)$, we define the following local degrees of freedom (cf. [3]):

(D1) The values of v_h at the vertices of E.

(D2) For $k > 1$, the edge moments $\int_e v_h p_{k-2} ds,\ p_{k-2} \in \mathbb{P}_{k-2}(e)$, on each edge e of E.

(D3) For $k > 1$, the internal moments $\int_E v_h p_{k-2} dx,\ p_{k-2} \in \mathbb{P}_{k-2}(E)$.

Finally, we can define the global virtual element space as follows:

$$Q_h^k = \big\{v \in H_0^1(\Omega) : v|_E \in Q_h^k(E),\ \ \forall\, E \in \mathcal{T}_h\big\}.$$

Let now $\Pi_k^0 : Q_h^k(E) \to \mathbb{P}_k(E)$ be the L^2 projection operator defined by

$$\big(v_h - \Pi_k^0 v_h, q\big)_E = 0, \quad \forall\, q \in \mathbb{P}_k(E).$$

Then the following approximation properties can be obtained (cf. [8]):

$$\big\|\Pi_k^0 v - v\big\|_{m,E} \le C h_E^{s-m} |v|_{s,E}, \quad m, s \in \mathbb{N}, \quad m \le s \le k+1, \quad \forall\, v \in H^s(E), \tag{2.6}$$

$$\big\|\Pi_k^0 v\big\|_{m,E} \le C\|v\|_{m,E}, \qquad m \in \mathbb{N}, \quad m \le k+1, \qquad \forall\, v \in H^m(E). \tag{2.7}$$

It is shown in [19] that under Assumption 2.1 for the mesh $\mathcal{T}_h$, for any element $E \in \mathcal{T}_h$, there is a virtual triangulation $\mathcal{T}_E$ of E such that $\mathcal{T}_E$ is uniformly shape regular and quasi-uniform. Then, from the inverse estimate and (2.7), there holds

$$\big\|\Pi_k^0 v\big\|_{0,\infty,E} \le C\|v\|_{0,\infty,E}, \quad \forall\, v \in L^\infty(E), \tag{2.8}$$

$$\|q_k\|_{0,\infty,E} \le C h_E^{-1} \|q_k\|_{0,E}, \quad \forall\, q_k \in \mathbb{P}_k(E), \tag{2.9}$$

where we have used the element area $|E| \le C h_E^2$ (see [32, Section 2.3]).

Assume $v_I \in Q_h^k$ is the interpolant of v, which shares the value of the degrees of freedom with v. According to [10, 14], v_I satisfies

$$\|v - v_I\|_{1,E} \le C h_E^k \|v\|_{k+1,E}, \quad \forall\, v \in H^{k+1}(E). \tag{2.10}$$

And for any $v \in H^k(E)$ there exists a $v_\pi \in \mathbb{P}_k(E)$ such that [5]:

$$\|v - v_\pi\|_{0,E} + h_E\|\nabla v - \nabla v_\pi\|_{0,E} \le C h_E^k \|\nabla v\|_{k,E}. \tag{2.11}$$

For any $u_h, v_h, \psi_h, u_h^1, u_h^2$ on each element E, we construct the local forms

$$
\begin{aligned}
&m_h^E(u_h, v_h) = \int_E [\Pi_k^0 u_h] \cdot [\Pi_k^0 v_h] dx + S_m^E\big((I-\Pi_k^0)u_h, (I-\Pi_k^0)v_h\big)_E,\\
&a_h^E(u_h, v_h) = \int_E [\Pi_{k-1}^0 \nabla u_h] \cdot [\Pi_{k-1}^0 \nabla v_h] dx + S_a^E\big((I-\Pi_k^\nabla)u_h, (I-\Pi_k^\nabla)v_h\big),\\
&b_{i,h}^E(u_h, \psi_h, v_h) = \int_E q^i [\Pi_{k-1}^0 u_h][\Pi_k^0 \nabla \psi_h] \cdot [\Pi_{k-1}^0 \nabla v_h] dx, \quad i = 1,2,\\
&\tilde{b}_h^E(u_h^1, u_h^2, v_h) = -\int_E \left[\Pi_{k-1}^0 \left(\sum_{i=1}^2 q^i u_h^i\right)\right] [\Pi_{k-1}^0 v_h] dx,\\
&\big(F_h^i, v\big)_E = \int_E F^i \Pi_k^0 v_h dx, \quad i = 1,2,\\
&(f_h, v)_E = \int_E f \Pi_k^0 v_h dx,
\end{aligned}
$$

where the symmetric bilinear forms $S_m^E(\cdot\,,\cdot)$ and $S_a^E(\cdot\,,\cdot) : Q_h^k(E) \times Q_h^k(E) \to \mathbb{R}$ satisfy

$$
\begin{aligned}
&\alpha_1 (v_h, v_h)_E \le S_m^E(v_h, v_h) \le \alpha_2 (v_h, v_h)_E, \qquad \forall\, v_h \in Q_h^k(E) \quad \text{with} \quad \Pi_k^0 v_h = 0,\\
&\beta_1 a^E(v_h, v_h) \le S_a^E(v_h, v_h) \le \beta_2 a^E(v_h, v_h), \quad \forall\, v_h \in Q_h^k(E) \quad \text{with} \quad \Pi_k^\nabla v_h = 0,
\end{aligned} \tag{2.12}
$$

respectively, for four positive constants $\alpha_1, \alpha_2, \beta_1$ and β_2. Moreover, define the discrete forms

$$
\begin{aligned}
&a_h(u_h, v_h) := \sum_E a_h^E(u_h, v_h), \qquad b_{i,h}(u_h, \psi_h, v_h) := \sum_E b_{i,h}^E(u_h, \psi_h, v_h), \quad i = 1,2, &(2.13)\\
&m_h(u_h, v_h) := \sum_E m_h^E(u_h, v_h), \quad \tilde{b}_h(u_h^1, u_h^2, v) := \sum_E \tilde{b}_h^E(u_h^1, u_h^2, v_h),\\
&(f_h, v_h) := \sum_E (f_h, v_h)_E, \qquad (F_h^i, v_h) := \sum_E \big(F_h^i, v_h\big)_E, \qquad i = 1,2.
\end{aligned}
$$

Then the semi-discrete virtual element formulation corresponding to (2.4) reads as: Find $p_h^i \in L^2(0,T,Q_h^k)$ with $p_{h,t}^i \in L^2(0,T,Q_h^k)$, $i = 1,2$ and $\phi_h \in L^2(0,T,Q_h^k)$ such that

$$
\begin{cases}
m_h(p_{h,t}^i, v_h) + a_h(p_h^i, v_h) + b_{i,h}(p_h^i, \phi_h, v_h) = (F_h^i, v_h), & \forall\, v_h \in Q_h^k \quad \text{for a.e. } t \text{ in } (0,T),\\
a_h(\phi_h, w_h) + \tilde{b}_h(p_h^1, p_h^2, w_h) = (f_h, w_h), & \forall\, w_h \in Q_h^k \quad \text{for a.e. } t \text{ in } (0,T)
\end{cases} \tag{2.14}
$$

with the initial condition $p_h^i(0) := p_{h,0}^i, i = 1,2$ and $\phi_h(0) := \phi_{h,0}$ given by the interpolation of $p_0^i, i = 1,2$ and ϕ_0, respectively.

We employ the backward Euler scheme for the approximation of time derivative. Let

$$
p_h^{i,n} := p_h^i(\cdot, t_n), \quad n = 0,1,\dots,N, \quad t_n = n\tau, \quad \tau = T/N.
$$

The fully discrete form corresponding to (2.14) reads as find $p_h^{i,n}$, $\phi_h^n \in Q_h^k$ such that

$$
\begin{cases}
m_h\left(\dfrac{p_h^{i,n} - p_h^{i,n-1}}{\tau}, v_h\right) + a_h(p_h^{i,n}, v_h)\\
\quad + b_{i,h}(p_h^{i,n}, \phi_h^n, v_h) = (F_h^{i,n}, v_h), & \forall\, v_h \in Q_h^k \quad \text{for} \quad n = 1,\dots,N,\\
a_h(\phi_h^n, w_h) + \tilde{b}_h(p_h^{1,n}, p_h^{2,n}, w_h) = (f_h^n, w_h), & \forall\, w_h \in Q_h^k \quad \text{for} \quad n = 1,\dots,N,\\
p_h^{i,0} = p_{h,0}^i, \quad \phi_h^0 = \phi_{h,0}.
\end{cases} \tag{2.15}
$$

The wellposedness of (2.15) can be proved by a similar approach shown in [21], since the PNP equations considered in this paper is part of the equations in [21].

Next, we shall introduce some lemmas which are used in the error analysis for the virtual element solution. From [8], the bilinear form $a_h^E(\cdot\,,\cdot)$ satisfies the following stability property and consistency property.

Lemma 2.1 ([8], Stability). *There exist two positive constants C_0 and C_1 independent of h and E such that*

$$C_0 a^E(v_h, v_h) \le a_h^E(v_h, v_h) \le C_1 a^E(v_h, v_h), \quad \forall\, v_h \in Q_h^k(E). \tag{2.16}$$

From (2.16), it is easy to obtain

$$a_h^E(u_h, v_h) \le C\|\nabla u_h\|_{0,E}\|\nabla v_h\|_{0,E}, \quad \forall\, u_h, v_h \in Q_h^E, \tag{2.17}$$

$$a_h^E(u_h, u_h) \ge C\|\nabla u_h\|_{0,E}^2. \tag{2.18}$$

Lemma 2.2 (K-Consistency). *For any $q \in \mathbb{P}_k(E)$ and $v_h \in Q_h^k(E)$ such that*

$$a_h^E(q, v_h) - a^E(q, v_h) = 0. \tag{2.19}$$

Proof. It is easy to get the result from the definitions of Π_k^∇ and Π_{k-1}^0. □

From [46], the bilinear $m_h^E(\cdot\,,\cdot)$ satisfies the following consistency property and stability property.

Lemma 2.3 ([46], K-Consistency). *For all $\chi \in \mathbb{P}_k(E)$ and $v_h \in Q_h^k(E)$, there holds*

$$m_h^E(\chi, v_h) = (\chi, v_h)_E. \tag{2.20}$$

(Stability) There exits two positive constants C_* and C^* independent of h and E such that

$$C_*(v_h, v_h)_E \le m_h^E(v_h, v_h) \le C^*(v_h, v_h)_E, \quad \forall\, v_h \in Q_h^k(E). \tag{2.21}$$

From the stability conditions, it is easy to get the continuity of m_h

$$m_h(u_h, v_h) \le C\|u_h\|_0\|v_h\|_0, \quad \forall\, u_h, v_h \in Q_h^k(E). \tag{2.22}$$

The following lemma will be used to estimate the difference between the continuous form and the discrete bilinear form, see the proof of Lemmas 3.5 and 3.7.

Lemma 2.4 ([8]). *For any $u, v \in H^1(E)$, if $\kappa \in L^\infty(E)$ and $\lambda \in [L^\infty(E)]^2$, then we have the estimate*

$$\begin{aligned}
&\left|(\lambda u, \nabla v)_E - \left(\lambda \Pi_{k-1}^0 u, \Pi_{k-1}^0 \nabla v\right)_E\right| \\
&\le \left\|\lambda u - \Pi_{k-1}^0(\lambda u)\right\|_{0,E}\left\|\nabla v - \Pi_{k-1}^0 \nabla v\right\|_{0,E} \\
&\quad + \left\|\lambda \cdot \nabla v - \Pi_{k-1}^0(\lambda \cdot \nabla v)\right\|_{0,E}\left\|u - \Pi_{k-1}^0 u\right\|_{0,E} \\
&\quad + C_\lambda\left\|u - \Pi_{k-1}^0 u\right\|_{0,E}\left\|\nabla v - \Pi_{k-1}^0 \nabla v\right\|_{0,E}.
\end{aligned} \tag{2.23}$$

It is easy to get Lemmas 2.5 and 2.6 by following the arguments in the proof [32, Lemma 2.7 and Theorem 3.1, Eq. (3.5)], respectively.

Lemma 2.5. *Suppose $w \in W^{k+1,\infty}(E)$ and $w_h \in Q_h^k(E)$. For any $u \in L^\infty(E) \cap H^k(E)$ and $u_h \in Q_h^k(E)$, there holds*

$$b_i^E(u, w, v_h) - b_{i,h}^E(u_h, w_h, v_h) \le C\big(h_E^k(\|w\|_{k+1,E} + \|u\|_{k,E}) + \|\nabla w_h - \nabla w\|_{0,E} + \big\|\Pi_k^0 \nabla w_h\big\|_{0,\infty,E}\|u_h - u\|_{0,E}\big)\|\nabla v_h\|_{0,E}, \quad \forall v_h \in Q_h^k(E).$$

Lemma 2.6. *Suppose v_h and $u_h^i \in Q_h^k(E)$, $i = 1, 2$. There holds*

$$\big|\tilde{b}_h\big(u_h^1, u_h^2, v_h\big) - \tilde{b}\big(u^1, u^2, v_h\big)\big| \le C\left(h^{k+1}\sum_{i=1}^2 \|u^i\|_k |v_h|_1 + \sum_{i=1}^2 \big\|u_h^i - u^i\big\|_0 \|v_h\|_0\right).$$

3. Error Estimates for Semi-discrete Case

In this section, we present the a priori error analysis for the semi-discrete system (2.14). The main results are the optimal H^1 norm error estimates, see Theorem 3.2 in Section 3.3, which are based on some error estimates of the energy projection (see Lemmas 3.7-3.9) in Section 3.2, the L^2 norm error estimate in Theorem 3.1 in Section 3.1 and Lemma 3.10 in Section 3.3. The existence and uniqueness of the energy projection solution are presented in Lemma 3.6, which needs to apply the discrete inf-sup condition (see Lemma 3.5). Next, we first deduce the error estimate in the L^2 norm. Then we give some error estimates of an energy projection. After that, we show the error estimates in the H^1 norm.

3.1. Error estimates for semi-discrete case in the L^2 norm

In this subsection, we present the a priori error estimates in the L^2 norm for the semi-discrete system (2.14). First, following the arguments in [32, Theorem 3.1], it is easy to give the error bound of $\phi(t) - \phi_h(t)$ in the H^1 norm as follows.

Lemma 3.1 (cf. [32]). *Let (ϕ, p^i) and (ϕ_h, p_h^i) be the solutions of (2.4) and (2.14), respectively. Then for all $t \in (0, T]$, the following estimation holds:*

$$\|\phi(t) - \phi_h(t)\|_1 \le C\left(h^k\left(\|f\|_k + \sum_{i=1}^2 \|p^i\|_k + \|\phi\|_{k+1}\right) + \sum_{i=1}^2 \big\|p_h^i - p^i\big\|_0\right).$$

The following lemma shall be used to present the error estimate in the L^2 norm.

Lemma 3.2. *Suppose (ϕ, p^i) and (ϕ_h, p_h^i) are the solutions of (2.4) and (2.14), respectively, and $\phi \in L^\infty(0, T; H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega))$, p^i and $f \in L^\infty(0, T; H^k(\Omega))$. If the decomposition $\mathcal{T}_h$ is quasi-uniform, then there holds*

$$\big\|\Pi_k^0 \nabla \phi_h\big\|_{0,\infty,E} \le Ch^{-1}\sum_{i=1}^2 \big\|p^i - p_h^i\big\|_0 + C.$$

Proof. From (2.9), we get

$$\begin{aligned}\big\|\Pi_k^0 \nabla \phi_h\big\|_{0,\infty,E} &\le \big\|\Pi_k^0 \nabla \phi_h - \Pi_k^0 \nabla \phi\big\|_{0,\infty,E} + \big\|\Pi_k^0 \nabla \phi\big\|_{0,\infty,E} \\ &\le Ch_E^{-1}\big\|\Pi_k^0 \nabla \phi_h - \Pi_k^0 \nabla \phi\big\|_{0,E} + C_1 \\ &\le Ch_E^{-1}\|\nabla \phi - \nabla \phi_h\|_{0,E} + C.\end{aligned}$$

Since $\mathcal{T}_h$ is quasi-uniform, we have $h \le Ch_E$ for any $E \in \mathcal{T}_h$. Then from Lemma 3.1, we get

$$\begin{aligned}\left\|\Pi_k^0 \nabla \phi_h\right\|_{0,\infty,E} &\le Ch^{-1}\|\nabla\phi - \nabla\phi_h\|_0 + C\\ &\le Ch^{-1}\left(h^k + \sum_{i=1}^{2}\left\|p^i - p_h^i\right\|_0\right) + C\\ &\le Ch^{-1}\sum_{i=1}^{2}\left\|p^i - p_h^i\right\|_0 + C.\end{aligned}$$

This completes the proof of this lemma. □

Next, we derive error estimates for p^i in the L^2 norm. Assume

$$\begin{aligned}&p^i \in L^\infty\big(0,T;H^{k+1}(\Omega)\cap L^\infty(\Omega)\big), \quad p_t^i \in L^\infty\big(0,T;H^{k+1}(\Omega)\big), \quad i=1,2,\\ &\phi \in L^\infty\big(0,T;H^{k+1}(\Omega)\cap W^{k+1,\infty}(\Omega)\big).\end{aligned} \tag{3.1}$$

We also suppose

$$f \in L^\infty\big(0,T;H^k(\Omega)\big), \quad F^i \in L^\infty\big(0,T;H^{k+1}(\Omega)\big). \tag{3.2}$$

Theorem 3.1. *Suppose the decomposition $\mathcal{T}_h$ is quasi-uniform. Let (ϕ, p^i) and (ϕ_h, p_h^i) be the solutions of (2.4) and (2.14), respectively, and set $p_{h,0}^i := (p_0^i)_I$, the interpolant function of the initial value of p_0^i in Q_h^k. Assume (3.1) and (3.2) holds, then for $t \in (0,T]$, there holds*

$$\sum_{i=1}^{2}\left\|p_h^i(t) - p^i(t)\right\|_0 \le Ch^k.$$

Proof. Decompose the error as follows:

$$p_h^i(t) - p^i(t) = \big(p_h^i(t) - \Pi_k^0 p^i(t)\big) + \big(\Pi_k^0 p^i(t) - p^i(t)\big) =: \upsilon^i(t) + \varrho^i(t),$$

which are then estimated separately. For the second term $\varrho^i(t)$, it is easy to get

$$\sum_E \|\varrho^i(t)\|_{0,E} = \sum_E \left\|\Pi_k^0 p^i(t) - p^i(t)\right\|_{0,E} \le Ch^{k+1}|p^i(t)|_{k+1}. \tag{3.3}$$

Now, we proceed the estimate for $\upsilon^i(t)$. For any $v_h \in Q_h^k$, an application of (2.4) together with (2.14) yields

$$\begin{aligned}&\sum_E \big(m_h^E\big(\upsilon_t^i(t), v_h\big) + a_h^E\big(\upsilon^i(t), v_h\big)\big)\\ &= \big(m_h\big(p_{h,t}^i(t), v_h\big) + a_h\big(p_h^i(t), v_h\big)\big) - \sum_E m_h^E\left(\frac{d}{dt}\Pi_k^0 p^i(t), v_h\right) - \sum_E a_h^E\big(\Pi_k^0 p^i(t), v_h\big)\\ &\quad + a\big(p^i(t), v_h\big) - a\big(p^i(t), v_h\big)\\ &= \big(\big(F_h^i(t), v_h\big) - b_{i,h}\big(p_h^i(t), \phi_h(t), v_h\big)\big) - \sum_E m_h^E\big(\Pi_k^0 p_t^i(t), v_h\big) - \sum_E a_h^E\big(\Pi_k^0 p^i(t), v_h\big)\\ &\quad + a\big(p^i(t), v_h\big) - \big(\big(F^i(t), v_h\big) - \big(p_t^i(t), v_h\big) - b_i\big(p^i(t), \phi(t), v_h\big)\big)\\ &= \big(F_h^i(t) - F^i(t), v_h\big) + \sum_E \big(\big(p_t^i(t), v_h\big)_E - m_h^E\big(\Pi_k^0 p_t^i(t), v_h\big)\big)\\ &\quad + \sum_E \big(a^E\big(p^i(t), v_h\big) - a_h^E\big(\Pi_k^0 p^i(t), v_h\big)\big) + \big(b_i\big(p^i(t), \phi(t), v_h\big) - b_{i,h}\big(p_h^i(t), \phi_h(t), v_h\big)\big)\\ &=: H_1 + H_2 + H_3 + H_4.\end{aligned} \tag{3.4}$$

The first term can be estimated as follows:

$$H_1 = \big(F_h^i(t) - F^i(t), v_h\big) = \sum_E \big(\Pi_k^0 F^i(t) - F^i(t), v_h\big) \le Ch^{k+1}\|F^i\|_{k+1}\|v_h\|_0. \tag{3.5}$$

The second term can be bounded by the consistency property (2.20)

$$\begin{aligned} H_2 &= \sum_E \Big(\big(p_t^i(t), v_h\big)_E - m_h^E\big(\Pi_k^0 p_t^i(t), v_h\big)\Big) \\ &= \sum_E \Big(\big(p_t^i(t), v_h\big)_E - \big(\Pi_k^0 p_t^i(t), v_h\big)\Big) \\ &\le C\sum_E \big\|p_t^i(t) - \Pi_k^0 p_t^i(t)\big\|_{0,E}\|v_h\|_{0,E} \\ &\le Ch^{k+1}\big\|p_t^i(t)\big\|_{k+1}\|v_h\|_0. \end{aligned} \tag{3.6}$$

From Lemma 2.2, we can express the third term $a(p^i(t), v_h) - a_h(\Pi_k^0 p^i(t), v_h)$ as

$$\begin{aligned} H_3 &= \sum_E \Big(a^E\big(p^i(t), v_h\big) - a_h^E\big(\Pi_k^0 p^i(t), v_h\big)\Big) \\ &= \sum_E \Big(a^E\big(p^i(t), v_h\big) - a^E\big(\Pi_k^0 p^i(t), v_h\big)\Big) \\ &\le Ch^k\|p^i(t)\|_{k+1}\|\nabla v_h\|_0. \end{aligned} \tag{3.7}$$

For the fourth term, from Lemmas 2.5, 3.1 and 3.2, we have

$$\begin{aligned} H_4 &= b_i\big(p^i(t), \phi(t), v_h\big) - b_{i,h}\big(p_h^i(t), \phi_h(t), v_h\big) \\ &= \sum_E \Big(b_i^E\big(p^i(t), \phi(t), v_h\big) - b_{i,h}^E\big(p_h^i(t), \phi_h(t), v_h\big)\Big) \\ &\le \sum_E \Big(h_E^k + \|\nabla\phi(t) - \nabla\phi_h(t)\|_{0,E} + \big\|\Pi_k^0\nabla\phi_h\big\|_{0,\infty,E}\big\|p^i(t) - p_h^i(t)\big\|_{0,E}\Big)\|\nabla v_h\|_{0,E} \\ &\le C\left(h^k + \sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0 + h^{-1}\left(\sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0\right)^2\right)\|\nabla v_h\|_0. \end{aligned} \tag{3.8}$$

Setting $v_h = \upsilon^i(t)$ in (3.4) and using (3.5)-(3.8), we get

$$\begin{aligned} &\sum_E \Big(m_h^E\big(\upsilon_t^i(t), \upsilon^i(t)\big) + a_h^E\big(\upsilon^i(t), \upsilon^i(t)\big)\Big) \\ &\le Ch^{k+1}\|\upsilon^i(t)\|_0 + C\left(h^k + \sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0 + h^{-1}\left(\sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0\right)^2\right)\|\nabla\upsilon^i(t)\|_0, \end{aligned}$$

where

$$\|\upsilon^i(t)\|_0 := \sum_E \|\upsilon^i(t)\|_{0,E}, \quad \|\nabla\upsilon^i(t)\|_0 := \sum_E \|\nabla\upsilon^i(t)\|_{0,E}$$

for simplicity. Then, using (2.18), we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\upsilon^i(t)\|_0^2 + C_0\|\nabla\upsilon^i(t)\|_0^2 \\ &\le Ch^{k+1}\|\upsilon^i(t)\|_0 + C\left(h^k + \sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0 + h^{-1}\left(\sum_{i=1}^2 \big\|p_h^i(t) - p^i(t)\big\|_0\right)^2\right)\|\nabla\upsilon^i(t)\|_0 \\ &\le C\left(h^{2k} + \left(\sum_{i=1}^2 \|\upsilon^i(t)\|_0\right)^2 + h^{-2}\left(\sum_{i=1}^2 \|\upsilon^i(t)\|_0\right)^2 \sum_{i=1}^2 \|\upsilon^i(t)\|_0^2\right) + \frac{C_0}{2}\|\nabla\upsilon^i(t)\|_0^2. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|\upsilon^i(t)\|_0^2 \leq C\left(h^{2k} + \left(\sum_{i=1}^{2}\|\upsilon^i(t)\|_0\right)^2 + h^{-2}\left(\sum_{i=1}^{2}\|\upsilon^i(t)\|_0\right)^2\sum_{i=1}^{2}\|\upsilon^i(t)\|_0^2\right). \tag{3.9}$$

Next, following the arguments in [44] we shall show $h^{-1}\|\upsilon^i(t)\|_0 \leq C$ by induction. First, from (2.10) and (3.3) we have

$$\begin{aligned} h^{-1}\|\upsilon^i(0)\|_0 &= h^{-1}\sum_E \left\|p_{h,0}^i - \Pi_k^0 p_0^i\right\|_{0,E} \\ &\leq h^{-1}\left(\left\|p_{h,0}^i - p_0^i\right\|_0 + \sum_E \|\varrho^i(0)\|_{0,E}\right) \leq Ch^{k+1}\left\|p_0^i\right\|_{k+1} \leq C. \end{aligned}$$

Then, assume $h^{-1}\|\upsilon^i(t)\|_0 \leq C$ holds for $t \in [0, T_0], T_0 < T$. From (3.9), we get

$$\frac{d}{dt}\|\upsilon^i(t)\|_0^2 \leq Ch^{2k} + C\sum_{i=1}^{2}\|\upsilon^i(t)\|_0^2.$$

Integrating the above from 0 to t, the following inequality holds:

$$\|\upsilon^i(t)\|_0^2 \leq \|\upsilon^i(0)\|_0^2 + Ch^{2k} + C\int_0^t \sum_{i=1}^{2}\|\upsilon^i(s)\|_0^2 ds.$$

Summing up for the index i, we get

$$\sum_{i=1}^{2}\|\upsilon^i(t)\|_0^2 \leq \sum_{i=1}^{2}\|\upsilon^i(0)\|_0^2 + Ch^{2k} + C\int_0^t \sum_{i=1}^{2}\|\upsilon^i(s)\|_0^2 ds.$$

By using Gronwall's inequality, we deduce that

$$\sum_{i=1}^{2}\|\upsilon^i(t)\|_0^2 \leq \sum_{i=1}^{2}\|\upsilon^i(0)\|_0^2 + Ch^{2k}.$$

That is

$$\sum_{i=1}^{2}\|\upsilon^i(t)\|_0 \leq \sum_{i=1}^{2}\|\upsilon^i(0)\|_0 + Ch^{k}.$$

From (2.10), the term $\sum_{i=1}^{2}\|\upsilon^i(0)\|_0$ can be estimated as follows:

$$\begin{aligned} \sum_{i=1}^{2}\|\upsilon^i(0)\|_0 &= \sum_{i=1}^{2}\left\|p_{h,0}^i - \Pi_k^0 p_0^i\right\|_0 \leq \sum_{i=1}^{2}\left(\left\|p_{h,0}^i - p_0^i\right\|_0 + \left\|p_0^i - \Pi_k^0 p_0^i\right\|_0\right) \\ &\leq \sum_{i=1}^{2}\left(\left\|(p_0^i)_I - p_0^i\right\|_0 + Ch^{k+1}|p_0^i|_{k+1}\right) \leq Ch^{k+1}\left|p_0^i\right|_{k+1}. \end{aligned}$$

Hence, we get if the assumption $h^{-1}\|\upsilon^i(t)\|_0 \leq C$ holds for $t \in [0, T_0], T_0 < T$ holds, then

$$\sum_{i=1}^{2}\|\upsilon^i(t)\|_0 \leq Ch^k, \quad t \in [0, T_0]. \tag{3.10}$$

It yields

$$h^{-1}\|\upsilon^i(t)\|_0 \le Ch^{k-1} \le C, \quad k \ge 1, \quad t \in [0, T_0].$$

Since $h^{-1}\|\upsilon^i(t)\|_0$ is a continuous function with respect to $t \in [0,T]$, from the uniform continuity with time, there is a constant $\delta > 0$ such that $h^{-1}\|\upsilon^i(t)\|_0 \le C$ holds for $t \in [0, T_0 + \delta]$. Since $[0,T]$ is a finite interval, we have

$$h^{-1}\|\upsilon^i(t)\|_0 \le C, \quad t \in [0,T].$$

Thus, from (3.10), we have

$$\sum_{i=1}^{2} \|\upsilon^i(t)\|_0 \le Ch^k, \quad t \in [0,T]. \tag{3.11}$$

Combining the estimates for $\upsilon^i(t)$ and $\varrho^i(t)$, we get

$$\sum_{i=1}^{2} \left\|p_h^i(t) - p^i(t)\right\|_0 \le \sum_{i=1}^{2} \left(\|\varrho^i(t)\|_0 + \|\upsilon^i(t)\|_0\right) \le Ch^k.$$

We complete the proof. □

As mentioned in Introduction, the optimal L^2 norm error estimates of the finite element approximation were obtained for time-dependent PNP equations in [26, 42], in which a special energy projection is used. Since this energy projection is a coupled nonlinear one, the existence and uniqueness of it are not easy to show. Similar difficulties need be dealt with in the error analysis for the VEM, if a similar special energy projection is applied. In order to present a complete a priori error analysis for the VEM, we use the L^2 projection instead of the coupled energy projection, a suboptimal L^2 norm error estimate is presented as a tradeoff.

3.2. Error estimates of the energy projection

In this subsection, we deduce the error estimates of the energy projection, which shall be used in the a priori error estimates in the H^1 norm for the semi-discrete system (2.14) in the next subsection. Define the energy projection $R_h : H_0^1(\Omega) \to Q_h^k$ satisfying: For any $u, \phi \in H_0^1(\Omega)$,

$$a_h(R_h u, v_h) + b_{i,h}(R_h u, \phi, v_h) = a(u, v_h) + b_i(u, \phi, v_h), \quad \forall\, v_h \in Q_h^k, \tag{3.12}$$

where $a(\cdot\,,\cdot), b_i(\cdot\,,\cdot\,,\cdot)$ and $a_h(\cdot\,,\cdot), b_{i,h}(\cdot\,,\cdot\,,\cdot)$ are defined in (2.3) and (2.13), respectively. In order to present the existence and uniqueness of the solution of (3.12), we need to show some lemmas, see Lemmas 3.3-3.5. First the error estimate of the L^2 projection in the L^∞ norm on arbitrary polygon element E is presented as follows, which shall be used in Lemma 3.5 later.

Lemma 3.3. *If $w \in W^{k,\infty}(E) \cap H^{k+1}(E)$, then there holds*

$$\left\|\Pi_k^0 w - w\right\|_{0,\infty,E} \le Ch_E^k\left(\|w\|_{k,\infty,E} + \|w\|_{k+1,E}\right).$$

Proof. Let $I_h :\ H^1(E) \to S^h(E)$ be a piecewise polynomial interpolant, where $S^h(E)$ is the k-th degree finite element space defined on the element E. From (2.9), we have

$$\begin{aligned}\left\|\Pi_k^0 w - I_h w\right\|_{0,\infty,E} &\le Ch_E^{-1}\left\|\Pi_k^0 w - I_h w\right\|_{0,E}\\ &\le Ch_E^{-1}\left(\left\|\Pi_k^0 w - w\right\|_{0,E} + \|w - I_h w\|_{0,E}\right)\\ &\le Ch_E^k\|w\|_{k+1,E}.\end{aligned}$$

Then

$$\begin{aligned}\left\|\Pi_k^0 w-w\right\|_{0,\infty,E} &\leq \left\|\Pi_k^0 w-I_h w\right\|_{0,\infty,E}+\|I_h w-w\|_{0,\infty,E}\\ &\leq Ch_E^k\big(\|w\|_{k+1,E}+\|w\|_{k,\infty,E}\big),\end{aligned}$$

which finishes the proof of this lemma. □

For any $u,w,v\in H_0^1(\Omega)$, set

$$B_i(u,w,v)=a(u,v)+b_i(u,w,v),$$

where $a(\cdot,\cdot), b_i(\cdot,\cdot,\cdot)$ are defined as (2.3).

Lemma 3.4. *Suppose $w\in W^{2,\infty}(\Omega)$. There exists a positive constants C_B such that*

$$\sup_{v\in H_0^1(\Omega)}\frac{B_i(u,w,v)}{\|v\|_1}\geq C_B\|u\|_1,\quad i=1,2,\quad \forall\, u\in H_0^1(\Omega). \tag{3.13}$$

Proof. Consider the following problem:

$$\begin{cases} Lu=-\nabla\cdot(\nabla u+q^i\nabla w u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{3.14}$$

If $w\in W^{2,\infty}(\Omega)$, then there exists a unique solution to (3.14) and there holds (cf. [40, Chapter 5])

$$\|u\|_1\leq C\|f\|_{-1}. \tag{3.15}$$

By using (3.15), we have (cf. [8,32])

$$\sup_{v\in H_0^1(\Omega)}\frac{B_i(u,w,v)}{\|v\|_1}=\sup_{v\in H_0^1(\Omega)}\frac{\langle Lu,v\rangle}{\|v\|_1}=\|Lu\|_{-1}=\|f\|_{-1}\geq C_B\|u\|_1,\quad \forall\, u\in H_0^1(\Omega).$$

This completes the proof. □

For any $u\in H_0^1(\Omega), u_h$ and $w_h\in Q_h^k$, set

$$B_{i,h}(u_h,w,v_h)=a_h(u_h,v_h)+b_{i,h}(u_h,w,v_h),$$

where $a_h(\cdot,\cdot)$ and $b_{i,h}(\cdot,\cdot,\cdot)$ are defined in (2.13). Then the definition of the energy projection (3.12) can be written as

$$B_{i,h}(R_h u,\phi,v_h)=B_i(u,\phi,v_h),\quad \forall\, v_h\in Q_h^k. \tag{3.16}$$

If $\phi\in W^{1,\infty}(\Omega)$, then the form $B_{i,h}$ is bounded, i.e.

$$B_{i,h}(u_h,\phi,v_h)\leq C\|u_h\|_1\|v_h\|_1. \tag{3.17}$$

From Lemma 3.4, we can get the following lemma for the discrete form $B_{i,h}$.

Lemma 3.5. *Suppose $\phi\in W^{2,\infty}(E)\cap H^3(E)\cap Q_h^k$. There exist an $h^*>0$ and a constant $\tilde{C}_B>0$ such that for all $h\leq h^*$,*

$$\sup_{v_h\in Q_h^k}\frac{B_{i,h}(u_h,\phi,v_h)}{\|v_h\|_1}\geq \tilde{C}_B\|u_h\|_1,\quad \forall\, u_h\in Q_h^k.$$

Proof. From (3.13), there exists $v \in H_0^1(\Omega)$ such that

$$B_i(u, \phi, v) \geq C_B\|u\|_1\|v\|_1, \quad \forall\, u \in H_0^1(\Omega). \tag{3.18}$$

From [8, Lemma 5.6], for any $v \in H_0^1(\Omega)$, there exists a $v_h \in Q_h^k$ such that

$$a_h(v_h, u_h) = a(v, u_h), \quad \forall\, u_h \in Q_h^k, \tag{3.19}$$

and

$$h\|v_h - v\|_1 + \|v_h - v\|_0 \leq Ch\|v\|_1, \quad \|v_h\|_1 \leq C\|v\|_1. \tag{3.20}$$

Then we have

$$\begin{aligned}
B_{i,h}(u_h, \phi, v_h) &= a_h(u_h, v_h) + b_{i,h}(u_h, \phi, v_h)\\
&= a(u_h, v) + b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v) + b_i(u_h, \phi, v)\\
&= B_i(u_h, \phi, v) + b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v)\\
&\geq C_B\|u_h\|_1\|v\|_1 - |b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v)| \quad (\text{by } (3.18))\\
&= C_B\|u_h\|_1\|v\|_1 - |b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v_h) + b_i(u_h, \phi, v_h - v)|.
\end{aligned} \tag{3.21}$$

Next, we estimate $b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v_h)$ and $b_i(u_h, \phi, v_h - v)$, respectively.

$$\begin{aligned}
&b_{i,h}(u_h, \phi, v_h) - b_i(u_h, \phi, v_h)\\
&= \sum_E \left\{ \left(q^i \Pi_{k-1}^0 u_h \Pi_k^0 \nabla\phi, \Pi_{k-1}^0 \nabla v_h\right)_E - \left(q^i u_h \nabla\phi, \nabla v_h\right)_E \right\}\\
&= \sum_E \left\{ \left(q^i u_h \left(\Pi_k^0 \nabla\phi - \nabla\phi\right), \nabla v_h\right)_E + \left(\left(q^i \Pi_{k-1}^0 u_h \Pi_k^0 \nabla\phi, \Pi_{k-1}^0 \nabla v_h\right)_E - \left(q^i u_h \Pi_k^0 \nabla\phi, \nabla v_h\right)_E\right) \right\}\\
&= \sum_E \{I_1 + I_2\}.
\end{aligned} \tag{3.22}$$

From Lemma 3.3, we have

$$\begin{aligned}
I_1 &= \left(q^i u_h \left(\Pi_k^0 \nabla\phi - \nabla\phi\right), \nabla v_h\right)_E\\
&= \left(q^i \left(\Pi_{k-1}^0 u_h - u_h\right)\nabla\phi + \Pi_{k-1}^0 u_h \left(\Pi_k^0 \nabla\phi - \nabla\phi\right) + \left(u_h - \Pi_{k-1}^0 u_h\right)\Pi_k^0 \nabla\phi, \nabla v_h\right)_E\\
&\leq C \left(\left\|\Pi_{k-1}^0 u_h - u_h\right\|_{0,E} \|\nabla\phi\|_{0,\infty,E} + \left\|\Pi_{k-1}^0 u_h\right\|_{0,E} \left\|\Pi_k^0 \nabla\phi - \nabla\phi\right\|_{0,\infty,E} \right) \|v_h\|_{1,E}\\
&\leq Ch_E\|u_h\|_{1,E}\|v_h\|_{1,E}.
\end{aligned} \tag{3.23}$$

Setting $\beta = q^i \Pi_k^0 \nabla\phi$, then from (2.6)-(2.8) and (2.23), we get

$$\begin{aligned}
I_2 &= \left(q^i \Pi_{k-1}^0 u_h \Pi_k^0 \nabla\phi, \Pi_{k-1}^0 \nabla v_h\right)_E - \left(q^i u_h \Pi_k^0 \nabla\phi, \nabla v_h\right)_E\\
&\leq C \left(\left\|\beta u_h - \Pi_{k-1}^0(\beta u_h)\right\|_{0,E} + h_E\|u_h\|_{1,E} \right) \|v_h\|_{1,E}.
\end{aligned}$$

Taking $\hat\beta = q^i \nabla\phi$, and using Lemma 3.3, then we deduce that

$$\begin{aligned}
\left\|\beta u_h - \Pi_{k-1}^0(\beta u_h)\right\|_{0,E} &\leq \left\|(\beta - \hat\beta)u_h\right\|_{0,E}\\
&\quad + \left\|\hat\beta u_h - \Pi_{k-1}^0(\hat\beta u_h)\right\|_{0,E} + \left\|\Pi_{k-1}^0(\hat\beta u) - \Pi_{k-1}^0(\beta u)\right\|_{0,E}\\
&\leq C\left\|(\beta - \hat\beta)u_h\right\|_{0,E} + Ch_E\|\nabla\phi u_h\|_{1,E}\\
&\leq \|\beta - \hat\beta\|_{0,\infty,E}\|u_h\|_{1,E} + Ch_E\|\nabla\phi u_h\|_{1,E}\\
&\leq Ch_E\|u_h\|_{1,E}.
\end{aligned} \tag{3.24}$$

Hence,

$$I_2 \le Ch_E\|u_h\|_{1,E}\|v_h\|_{1,E}. \tag{3.25}$$

Inserting (3.23) and (3.25) into (3.22), we get

$$b_{i,h}(u_h,\phi,v_h) - b_i(u_h,\phi,v_h) \le \tilde{C}_1 h\|u_h\|_1\|v_h\|_1 \le C_1 h\|u_h\|_1\|v\|_1. \tag{3.26}$$

Now, we estimate the term $b_i(u_h,\phi,v_h-v)$. There holds

$$\begin{aligned} b_i(u_h,\phi,v_h-v) &= \big(q^i u_h\nabla\phi, \nabla(v_h-v)\big) \\ &= -\big(\mathrm{div}\big(q^i u_h\nabla\phi\big), v_h-v\big) + \int_{\partial\Omega} q^i u_h\nabla\phi\cdot n(v_h-v) \\ &\le C\big\|\mathrm{div}\big(q^i u_h\nabla\phi\big)\big\|_0\|v-v_h\|_0 \\ &\le C_2 h\|u_h\|_1\|v\|_1 \quad \text{(by (3.20))}. \end{aligned} \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.21), it yields

$$B_{i,h}(u_h,\phi,v_h) \ge \frac{1}{2}C_B\|u_h\|_1\|v\|_1 \ge \tilde{C}_B\|u_h\|_1\|v_h\|_1,$$

where

$$h \le h^* = \frac{C_B}{2(C_1+C_2)}.$$

This completes the proof. □

Now we can show the existence and uniqueness of the solution of (3.12) as follows.

Lemma 3.6. *If $\phi \in W^{2,\infty}(E)\cap H^3(E)\cap Q_h^k$ and h is small enough, then there exists a unique solution $R_h u$ satisfying* (3.12).

Proof. It is easy to get the result of Lemma 3.6 by using generalized Lax-Milgram lemma (see [27]), since the discrete form $B_{i,h}$ satisfies all the conditions of generalized Lax-Milgram lemma from (3.17) and Lemma 3.5. □

Next, we shall present some error estimates for the energy projection R_h in Lemmas 3.7-3.9. First, we show the error estimates for R_h in the L^2 and H^1 norms as follows.

Lemma 3.7. *Suppose $u \in H^{k+1}(\Omega)$ and $\phi \in W^{2,\infty}(\Omega)\cap H^{k+2}(\Omega)$. If h is small enough, then the following estimate for the projection R_h holds:*

$$\|R_h u - u\|_0 + h\|\nabla(R_h u - u)\|_0 \le Ch^{k+1}.$$

Proof. Suppose $u_I \in Q_h^k$ is the interpolant to $u \in H_0^1(\Omega)$ and $u_\pi \in P_k(E)$. From (2.16) and (3.12), we have

$$\begin{aligned} C_0\|\nabla(R_h u - u_I)\|_0^2 &= C_0 a(R_h u - u_I, R_h u - u_I) \le a_h(R_h u - u_I, R_h u - u_I) \\ &= a_h(R_h u, R_h u - u_I) - a_h(u_I, R_h u - u_I) \\ &= a(u, R_h u - u_I) + b_i(u,\phi,R_h u - u_I) - b_{i,h}(R_h u,\phi,R_h u - u_I) \\ &\quad - \sum_E \big(a_h^E(u_I - u_\pi, R_h u - u_I) + a_h^E(u_\pi, R_h u - u_I)\big) \\ &= -\sum_E \big(a_h^E(u_I - u_\pi, R_h u - u_I) + a^E(u_\pi - u, R_h u - u_I)\big) \\ &\quad + \big(b_i(u,\phi,R_h u - u_I) - b_{i,h}(R_h u,\phi,R_h u - u_I)\big) \\ &=: B_1 + B_2. \end{aligned} \tag{3.28}$$

Using (2.10), (2.11) and (2.17), it yields

$$\begin{aligned} B_1 &= -\sum_E \big(a_h^E(u_I - u_\pi, R_h u - u_I) + a^E(u_\pi - u, R_h u - u_I)\big) \\ &\le C \sum_E \big(\|\nabla(u_I - u)\|_{0,E} + \|\nabla(u_\pi - u)\|_{0,E}\big)\|\nabla(R_h u - u_I)\|_{0,E} \\ &\le C h_E^k \|\nabla(R_h u - u_I)\|_0. \end{aligned} \tag{3.29}$$

To estimate B_2, first for any $w, v_h \in H^1(\Omega)$ and $u_h \in Q_h^k$ we have

$$\begin{aligned} & b_i^E(u_h, w, v_h) - b_{i,h}^E(u_h, w, v_h) \\ &= \big(q^i u_h \nabla w, \nabla v_h\big)_E - \big(q^i \Pi_{k-1}^0 u_h \Pi_k^0 \nabla w, \Pi_{k-1}^0 \nabla v_h\big)_E \\ &= \big(q^i u_h \big(\nabla w - \Pi_k^0 \nabla w\big), \nabla v_h\big)_E \\ &\quad + \Big(\big(q^i u_h \Pi_k^0 \nabla w, \nabla v_h\big)_E - \big(q^i \Pi_{k-1}^0 u_h \Pi_k^0 \nabla w, \Pi_{k-1}^0 \nabla v_h\big)_E\Big) \\ &= \tilde{B}_{21} + \tilde{B}_{22}. \end{aligned} \tag{3.30}$$

For $w \in W^{1,\infty}(E)$ and $u \in L^\infty(E)$, there holds

$$\begin{aligned} \tilde{B}_{21} &= \big(q^i u_h \big(\nabla w - \Pi_k^0 \nabla w\big), \nabla v_h\big)_E \\ &= \big(q^i \big((u_h - u)\nabla w + u\big(\nabla w - \Pi_k^0 \nabla w\big) + (u - u_h)\Pi_k^0 \nabla w\big), \nabla v_h\big)_E \\ &\le C\Big(\|u - u_h\|_{0,E} \|\nabla w\|_{0,\infty,E} + \|u\|_{0,\infty,E} \big\|\nabla w - \Pi_k^0 \nabla w\big\|_{0,\infty,E}\Big)\|\nabla v_h\|_{0,E} \\ &\le C\big(\|u - u_h\|_{0,E} + h_E^k \|w\|_{k+1,E}\big)\|\nabla v_h\|_{0,E}. \end{aligned} \tag{3.31}$$

And setting $\beta = q^i \Pi_k^0 \nabla w$, from (2.23) we have

$$\begin{aligned} \tilde{B}_{22} &= \big(q^i u_h \Pi_{k-1}^0 \nabla w, \nabla v_h\big)_E - \big(q^i \Pi_{k-1}^0 u_h \Pi_{k-1}^0 \nabla w, \Pi_{k-1}^0 \nabla v_h\big)_E \\ &\le \big\|\beta u_h - \Pi_{k-1}^0(\beta u_h)\big\|_{0,E} \big\|\nabla v_h - \Pi_{k-1}^0 \nabla v_h\big\|_{0,E} \\ &\quad + \big\|\beta \cdot \nabla v_h - \Pi_{k-1}^0(\beta \cdot \nabla v_h)\big\|_{0,E} \big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} \\ &\quad + C_\beta \big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} \big\|\nabla v_h - \Pi_{k-1}^0 \nabla v_h\big\|_{0,E} \\ &\le C\Big(\big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} + \big\|\beta u_h - \Pi_{k-1}^0(\beta u_h)\big\|_{0,E}\Big)\|\nabla v_h\|_{0,E}. \end{aligned} \tag{3.32}$$

Combining (3.31)-(3.32) with (3.30), we have

$$\begin{aligned} & b_i^E(u_h, w, v_h) - b_{i,h}^E(u_h, w, v_h) \\ &\le \Big(\|u - u_h\|_{0,E} + \big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} \\ &\qquad + \big\|\beta u_h - \Pi_{k-1}^0(\beta u_h)\big\|_{0,E} + h_E^k \|w\|_{k+1,E}\Big)\|\nabla v_h\|_{0,E}. \end{aligned} \tag{3.33}$$

Second, taking $u_h = R_h u, w = \phi$ and $v_h = R_h u - u_I$ in (3.33), we have

$$\begin{aligned} & b_i(R_h u, \phi, R_h u - u_I) - b_{i,h}(R_h u, \phi, R_h u - u_I) \\ &\le \sum_E C\Big(h_E^k + \|u - R_h u\|_{0,E} + \big\|R_h u - \Pi_{k-1}^0 R_h u\big\|_{0,E} \\ &\qquad + \big\|\beta R_h u - \Pi_{k-1}^0(\beta R_h u)\big\|_{0,E}\Big)\|\nabla R_h u - u_I\|_{0,E} \\ &\le C\big(\|u - R_h u\|_0 + h^k\big)\|\nabla(R_h u - u_I)\|_0. \end{aligned} \tag{3.34}$$

Hence,

$$\begin{aligned}B_2 &= b_i(u,\phi,R_hu-u_I)-b_{i,h}(R_hu,\phi,R_hu-u_I)\\ &= b_i(u,\phi,R_hu-u_I)-b_i(R_hu,\phi,R_hu-u_I)\\ &\quad + b_i(R_hu,\phi,R_hu-u_I)-b_{i,h}(R_hu,\phi,R_hu-u_I)\\ &\le C\big(\|u-R_hu\|_0+h^k\big)\|\nabla(R_hu-u_I)\|_0. \end{aligned} \tag{3.35}$$

Substituting (3.29) and (3.35) into (3.28), it yields

$$\|\nabla(R_hu-u_I)\|_0 \le C\big(\|u-R_hu\|_0+h^k\big). \tag{3.36}$$

Next, we present the L^2 estimate for R_h-u. Define the adjoint problem as follows:

$$\begin{cases}-\Delta w^i+q^i\nabla\phi\cdot\nabla w^i=u-R_hu, & x\in\Omega,\\ \ \ w^i=0, & x\in\partial\Omega.\end{cases} \tag{3.37}$$

If $\phi\in W^{1,\infty}(\Omega)$, then the regularity result holds (cf. [44])

$$\|w^i\|_2\le C\|u-R_hu\|_0. \tag{3.38}$$

From (3.12) and (3.37), we have

$$\begin{aligned}\|u-R_hu\|_0^2 &= \big(-\Delta w^i,u-R_hu\big)+\big(q^i\nabla\phi\cdot\nabla w^i,u-R_hu\big)\\ &= a\big(u-R_hu,w^i\big)+b_i\big(u-R_hu,\phi,w^i\big)\\ &= a\big(u-R_hu,w^i-w_I^i\big)+a\big(u-R_hu,w_I^i\big)+b_i\big(u-R_hu,\phi,w^i-w_I^i\big)\\ &\quad + b_i\big(u-R_hu,\phi,w_I^i\big)+a_h\big(R_hu,w_I^i\big)-a_h\big(R_hu,w_I^i\big)\\ &\quad + b_{i,h}\big(R_hu,\phi,w_I^i\big)-b_{i,h}\big(R_hu,\phi,w_I^i\big)\\ &= a\big(u-R_hu,w^i-w_I^i\big)+\big(a_h\big(R_hu,w_I^i\big)-a\big(R_hu,w_I^i\big)\big)\\ &\quad + b_i\big(u-R_hu,\phi,w^i-w_I^i\big)+\big(b_{i,h}\big(R_hu,\phi,w_I^i\big)-b_i\big(R_hu,\phi,w_I^i\big)\big)\quad \text{(by (3.12))}\\ &=: D_1+D_2+D_3+D_4. \end{aligned} \tag{3.39}$$

From (3.38), we get

$$\begin{aligned}D_1 &= a\big(u-R_hu,p^i-p_I^i\big)\le C\|\nabla(u-R_hu)\|_0\big\|\nabla\big(w^i-w_I^i\big)\big\|_0\\ &\le Ch\|\nabla(u-R_hu)\|_0\|w^i\|_2\le Ch\|\nabla(u-R_hu)\|_0\|u-R_hu\|_0. \end{aligned} \tag{3.40}$$

From (2.11) and Lemma 2.2, we have

$$\begin{aligned}D_2 &= a_h\big(R_hu,w_I^i\big)-a\big(R_hu,w_I^i\big)\\ &= \sum_E\big\{a_h^E\big(R_hu-u_\pi,w_I^i-\Pi_k^0w^iv\big)-a^E\big(R_hu-u_\pi,w_I^i-\Pi_k^0w^i\big)\big\}\\ &\le \sum_E\|\nabla(R_hu-u_\pi)\|_{0,E}\big\|\nabla\big(w_I^i-\Pi_k^0w^i\big)\big\|_{0,E}\\ &\le C\big(h^k\|u\|_{k+1}+\|\nabla(R_hu-u)\|_0\big)h\|w^i\|_2\\ &\le C\big(h^{k+1}+h\|\nabla(R_hu-u)\|_0\big)\|R_hu-u\|_0. \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} D_3 &= b_i\big(u - R_h u, \phi, w^i - w_I^i\big) \le C\|\nabla(R_h u - u)\|_0 \big\|\nabla\big(w^i - w_I^i\big)\big\|_0 \\ &\le C\|\nabla(R_h u - u)\|_0 h\|w^i\|_2 \le Ch\|\nabla(R_h u - u)\|_0\|R_h u - u\|_0. \end{aligned} \tag{3.42}$$

It remains to estimate D_4. There holds

$$\begin{aligned} D_4 &= b_{i,h}\big(R_h u, \phi, w_I^i\big) - b_i\big(R_h u, \phi, w_I^i\big) \\ &= \sum_E \Big(q^i\big(\Pi_{k-1}^0(R_h u)\Pi_k^0\nabla\phi, \Pi_{k-1}^0\nabla w_I^i\big)_{0,E} - q^i\big(R_h u\nabla\phi, \nabla w_I^i\big)_{0,E}\Big) \\ &= \sum_E \Big(\big(q^i\big(\Pi_{k-1}^0(R_h u)\Pi_k^0\nabla\phi, \Pi_{k-1}^0\nabla w_I^i\big)_E - q^i\big(R_h u\Pi_k^0\nabla\phi, \nabla w_I^i\big)_{0,E}\big) \\ &\qquad + q^i\big(R_h u\big(\Pi_k^0\nabla\phi - \nabla\phi\big), \nabla w_I^i\big)_E\Big) \\ &=: \sum_E (D_{41} + D_{42}). \end{aligned} \tag{3.43}$$

Setting $\tilde\beta = q^i\Pi_k^0\nabla\phi$ and $u_h = R_h u$, from (2.23) it yields

$$\begin{aligned} D_{41} &= q^i\big(\Pi_{k-1}^0(R_h u)\Pi_k^0\nabla\phi, \Pi_{k-1}^0\nabla w_I^i\big)_{0,E} - q^i\big(R_h u\Pi_k^0\nabla\phi, \nabla w_I^i\big)_{0,E} \\ &= \big(\tilde\beta\Pi_{k-1}^0 u_h, \Pi_{k-1}^0\nabla w_I^i\big)_E - \big(\tilde\beta u_h, \nabla w_I^i\big)_E \\ &\le \big\|\tilde\beta\cdot\nabla w_I^i - \Pi_{k-1}^0\big(\tilde\beta\cdot\nabla w_I^i\big)\big\|_{0,E}\big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} \\ &\quad + \big\|\nabla w_I^i - \Pi_{k-1}^0\big(\nabla w_I^i\big)\big\|_{0,E}\big\|\tilde\beta u_h - \Pi_{k-1}^0(\tilde\beta u_h)\big\|_{0,E} \\ &\quad + C\big\|\nabla w_I^i - \Pi_{k-1}^0\nabla w_I^i\big\|_{0,E}\big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E}. \end{aligned} \tag{3.44}$$

Note that

$$\begin{aligned} \big\|u_h - \Pi_{k-1}^0 u_h\big\|_{0,E} &= \big\|R_h u - \Pi_{k-1}^0(R_h u)\big\|_{0,E} \\ &= \|R_h u - u\|_{0,E} + \big\|u - \Pi_{k-1}^0 u\big\|_{0,E} \\ &\quad + \big\|\Pi_{k-1}^0 u - \Pi_{k-1}^0 R_h u\big\|_{0,E} \\ &\le \|R_h u - u\|_{0,E} + Ch_E^k\|u\|_{k,E}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} &\big\|\tilde\beta\cdot\nabla w_I^i - \Pi_{k-1}^0\big(\tilde\beta\cdot\nabla w_I^i\big)\big\|_{0,E} \\ &\le \big\|\tilde\beta\cdot\big(\nabla w_I^i - \nabla w^i\big)\big\|_{0,E} + \big\|\tilde\beta\nabla w^i - \Pi_{k-1}^0(\tilde\beta\cdot\nabla w^i)\big\|_{0,E} \\ &\quad + \big\|\Pi_{k-1}^0\tilde\beta\cdot\big(\nabla w^i - \nabla w_I^i\big)\big\|_{0,E} \\ &\le Ch_E\|w^i\|_{2,E}, \end{aligned} \tag{3.46}$$

$$\begin{aligned} &\big\|\tilde\beta u_h - \Pi_{k-1}^0(\tilde\beta u_h)\big\|_{0,E} \\ &= \big\|\tilde\beta R_h u - \Pi_{k-1}^0(\tilde\beta R_h u)\big\|_{0,E} \\ &\le \|\tilde\beta(R_h u - u)\|_{0,E} + \big\|\tilde\beta u - \Pi_{k-1}^0(\tilde\beta u)\big\|_{0,E} \\ &\quad + \big\|\Pi_{k-1}^0\tilde\beta u - \tilde\beta R_h u\big\|_{0,E} \\ &\le C\big(\|R_h u - u\|_{0,E} + h^k\|u\|_{k,E}\big). \quad \text{(following (3.24))} \end{aligned} \tag{3.47}$$

Combining (3.45)-(3.47) with (3.44), we get

$$\begin{aligned}\sum_E D_{41} &\le C\sum_E h_E\|w^i\|_{2,E}\big(\|R_hu-u\|_{0,E}+h_E^k\|u\|_{k,E}\big)\\ &\le C\big(h\|R_hu-u\|_0^2+h^{k+1}\|R_hu-u\|_0\big).\end{aligned} \tag{3.48}$$

Now, we estimate D_{42}. From Lemma 3.3, it follows that

$$\begin{aligned}\sum_E D_{42} &= \sum_E q^i\big(R_hu(\Pi_k^0\nabla\phi-\nabla\phi),\nabla w_I^i\big)_{0,E}\\ &= \sum_E\Big(q^i\big((R_hu-u)(\Pi_k^0\nabla\phi-\nabla\phi),\nabla w_I^i\big)_{0,E}+q^i\big(u(\Pi_k^0\nabla\phi-\nabla\phi),\nabla w_I^i\big)_{0,E}\Big)\\ &\le C\sum_E\Big(\big(\|R_hu-u\|_{0,E}\big\|\Pi_k^0\nabla\phi-\nabla\phi\big\|_{0,\infty,E}\big\|\nabla w_I^i\big\|_{0,E}\\ &\qquad+\|u\|_{0,\infty,E}\big\|\Pi_k^0\nabla\phi-\nabla\phi\big\|_{0,E}\big)\big\|\nabla w_I^i\big\|_{0,E}\Big)\\ &\le C\big(h\|R_hu-u\|_0^2+h^{k+1}\|R_hu-u\|_0\big).\end{aligned} \tag{3.49}$$

Substituting (3.48) and (3.49) into (3.43), we deduce

$$\begin{aligned}D_4 &= b_{i,h}\big(R_hu,\phi,w_I^i\big)-b_i\big(R_hu,\phi,w_I^i\big)\\ &\le C\big(h\|R_hu-u\|_0^2+h^{k+1}\|R_hu-u\|_0\big).\end{aligned} \tag{3.50}$$

Inserting (3.40)-(3.42) and (3.50) into (3.39), and using (3.36), we have

$$\|u-R_hu\|_0^2\le C\big(h^{k+1}\|R_hu-u\|_0+h\|R_hu-u\|_0^2\big). \tag{3.51}$$

Hence, if h is small enough, then it follows that

$$\|u-R_hu\|_0\le Ch^{k+1}. \tag{3.52}$$

Combining the above inequality with (3.36), it yields

$$\|\nabla(u-R_hu)\|_0\le Ch^k, \tag{3.53}$$

which finishes the proof of this lemma. □

Lemma 3.8. *Suppose $u\in H^{k+1}(\Omega)\cap L^\infty(\Omega)$, $\partial_tu\in H^{k+1}(\Omega)$, $\phi\in W^{2,\infty}(\Omega)\cap H^{k+2}(\Omega)$ and $\partial_t\phi(t)\in W^{1,\infty}(\Omega)$. For h sufficiently small, there holds*

$$\|\nabla\partial_t(R_hu-u)\|_0\le C\big(h^k+\|\partial_t(R_hu-u)\|_0\big). \tag{3.54}$$

Proof. For simplicity, denote by

$$\big\|v-\Pi_k^0u\big\|_s=\sum_E\big\|v-\Pi_k^0u\big\|_{s,E},\quad \big(v,\Pi_k^0v\big)=\sum_E\big(v,\Pi_k^0u\big)_E$$

for any $u,v\in H_0^1(\Omega)$. Since

$$\begin{aligned}\|\nabla\partial_t(R_hu-u)\|_0 &\le \big\|\nabla\partial_t\big(R_hu-\Pi_k^0u\big)\big\|_0+\big\|\nabla\partial_t\big(\Pi_k^0u-u\big)\big\|_0\\ &\le \big\|\nabla\partial_t\big(R_hu-\Pi_k^0u\big)\big\|_0+Ch^k\|\partial_tu\|_{k+1},\end{aligned}$$

it suffices to present $\|\nabla\partial_t(R_hu-\Pi_k^0u)\|_0$. Taking derivative with respecting to t on both sides of (3.12), we have

$$\begin{aligned}&a(\partial_tu,v_h)+a(u,\partial_tv_h)+q^i\big(\partial_t(u\nabla\phi),\nabla v_h\big)+q^i(u\nabla\phi,\partial_t\nabla v_h)\\&=a_h(\partial_tR_hu,v_h)+a_h(R_hu,\partial_tv_h)+q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla v_h\big)\\&\quad+q^i\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi,\partial_t\Pi_{k-1}^0\nabla v_h\big).\end{aligned}\tag{3.55}$$

Setting $v_h=\partial_tv_h$ in (3.12), it follows that

$$a(u,\partial_tv_h)+q^i(u\nabla\phi,\nabla\partial_tv_h)=a_h(R_hu,\partial_tv_h)+q^i\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi,\nabla\partial_tv_h\big).\tag{3.56}$$

Combining (3.55) with (3.56), we get

$$a(\partial_tu,v_h)+q^i\big(\partial_t(u\nabla\phi),\nabla v_h\big)=a_h(\partial_tR_hu,v_h)+q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla v_h\big).\tag{3.57}$$

To estimate $\|\nabla\partial_t(R_hu-\Pi_k^0u)\|_0$, for simplicity, we set $\psi=R_hu-\Pi_k^0u$. Then from (3.57), we have

$$\begin{aligned}C_0\|\nabla\partial_t\psi\|_0^2&=C_0a(\partial_t\psi,\partial_t\psi)\le a_h(\partial_t\psi,\partial_t\psi)\\&=a_h(\partial_tR_hu,\partial_t\psi)-a_h\big(\partial_t\Pi_k^0u,\partial_t\psi\big)\\&=\big\{a(\partial_tu,\partial_t\psi)-a_h\big(\partial_t\Pi_k^0u,\partial_t\psi\big)\big\}\\&\quad+\big\{q^i\big(\partial_t(u\nabla\phi),\nabla\partial_t\psi\big)-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\big\}\quad(\text{by }(3.57))\\&=:\Gamma_1+\Gamma_2.\end{aligned}\tag{3.58}$$

Using (2.19), it is easy to deduce that

$$\begin{aligned}\Gamma_1&=a(\partial_tu,\partial_t\psi)-a_h\big(\partial_t\Pi_k^0u,\partial_t\psi\big)=\sum_Ea^E\big(\partial_tu-\partial_t\Pi_k^0u,\partial_t\psi\big)\\&\le\sum_E\big\|\nabla\big(\partial_tu-\partial_t\Pi_k^0u\big)\big\|_{0,E}\|\nabla\partial_t\psi\|_{0,E}\le Ch^k\|\partial_tu\|_{k+1}\|\nabla\partial_t\psi\|_0\\&\le Ch^{2k}\|\partial_tu\|_{k+1}^2+\epsilon\|\nabla\partial_t\psi\|_0^2.\end{aligned}\tag{3.59}$$

There holds

$$\begin{aligned}\Gamma_2&=q^i(\partial_t(u\nabla\phi),\nabla\partial_t\psi)-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\\&=\big\{q^i\big(\partial_t(u\nabla\phi-R_hu\nabla\phi),\nabla\partial_t\psi\big)\big\}\\&\quad+\big\{q^i\big(\partial_t(R_hu\nabla\phi),\nabla\partial_t\psi\big)-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\big\}\\&\le C(\|\partial_t(u-R_hu)\|_0+\|u-R_hu\|_0)\|\nabla\partial_t\psi\|_0\\&\quad+\big\{q^i\big(\partial_t(R_hu\nabla\phi),\nabla\partial_t\psi\big)-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\big\}.\end{aligned}\tag{3.60}$$

Next, we estimate the last term

$$\begin{aligned}&q^i\big(\partial_t(R_hu\nabla\phi),\nabla\partial_t\psi\big)-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\\&=q^i\big(\partial_t(R_hu)\nabla\phi,\nabla\partial_t\psi\big)+q^i(R_hu\partial_t\nabla\phi,\nabla\partial_t\psi)\\&\quad-q^i\big(\partial_t\big(\Pi_{k-1}^0R_hu\big)\Pi_k^0\nabla\phi,\Pi_{k-1}^0\nabla\partial_t\psi\big)-q^i\big(\Pi_{k-1}^0R_hu\partial_t\big(\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\\&=\sum_E\big\{b_i^E\big(\partial_t(R_hu),\phi,\partial_t\psi\big)-b_{i,h}^E\big(\partial_t(R_hu),\phi,\partial_t\psi\big)\\&\qquad+b_i^E(R_hu,\partial_t\phi,\partial_t\psi)-b_{i,h}^E(R_hu,\partial_t\phi,\partial_t\psi)\big\}.\end{aligned}\tag{3.61}$$

Similarly as the deduction of (3.34), by taking $u_h{=}\partial_t R_h u, w{=}\phi, v_h{=}\partial_t\psi$ and $u_h{=}R_h u, w{=}\partial_t\phi$, $v_h = \partial_t\psi$ in (3.33), respectively, we have

$$\begin{aligned}
&b_i^E\big(\partial_t(R_h u),\phi,\partial_t\psi\big) - b_{i,h}^E\big(\partial_t(R_h u),\phi,\partial_t\psi\big)\\
&\le C\big(\|\partial_t u - \partial_t R_h u\|_{0,E} + h_E^k\|\phi\|_{k+1,E}\big)\|\nabla\partial_t\psi\|_{0,E},\\
&b_i^E(R_h u,\partial_t\phi,\partial_t\psi) - b_{i,h}^E(R_h u,\partial_t\phi,\partial_t\psi)\\
&\le C\big(\|u - R_h u\|_{0,E} + h_E^k\|\partial_t\phi\|_{k+1,E}\big)\|\nabla\partial_t\psi\|_{0,E}.
\end{aligned}$$

Inserting the above two inequality into (3.61) and using Lemma 3.7, we get

$$\begin{aligned}
&q^i\big(\partial_t(R_h u\nabla\phi),\nabla\partial_t\psi\big) - q^i\big(\partial_t\big(\Pi_{k-1}^0 R_h u\Pi_k^0\nabla\phi\big),\Pi_{k-1}^0\nabla\partial_t\psi\big)\\
&\le C\sum_E\big\{\big(\|\partial_t u - \partial_t R_h u\|_{0,E} + \|u - R_h u\|_{0,E} + h^k\big)\|\nabla\partial_t\psi\|_{0,E}\big\}\\
&\le C\big(h^{2k} + \|\partial_t(u - R_h u)\|_0^2\big) + \epsilon\|\nabla\partial_t\psi\|_0^2.
\end{aligned} \tag{3.62}$$

Combining (3.60) with (3.62), and using Lemma 3.7 again, it yields

$$\Gamma_2 \le C\big(h^{2k} + \|\partial_t(u - R_h u)\|_0^2\big) + \epsilon\|\nabla\partial_t\psi\|_0^2. \tag{3.63}$$

At last, substituting (3.59) and (3.63) into (3.58), we have

$$\big\|\nabla\partial_t\big(R_h u - \Pi_k^0 u\big)\big\|_0 = \|\nabla\partial_t\psi\|_0 \le C\big(h^k + \|\partial_t(u - R_h u)\|_0\big).$$

Hence,

$$\|\nabla\partial_t(R_h u - u)\|_0 \le C\big(h^k + \|\partial_t(u - R_h u)\|_0\big).$$

This completes the proof of the lemma. □

Using the similar analysis for Lemmas 3.7 and 3.8, we can get the following result.

Lemma 3.9. *Suppose* $u \in H^{k+1}(\Omega)\cap L^\infty(\Omega), \partial_t u \in H^{k+1}(\Omega), \phi \in W^{2,\infty}(\Omega)\cap H^{k+2}(\Omega)$ *and* $\partial_t\phi(t) \in W^{1,\infty}(\Omega)$. *If* h *is small enough, then we have*

$$\|\partial_t(R_h u - u)\|_0 \le Ch^{k+1}.$$

3.3. Error estimates for semi-discrete case in the H^1 norm

In this subsection, we give a priori error estimate in the H^1 norm for the semi-discrete solution. Assume

$$\begin{aligned}
&p^i,\quad p_t^i \in L^\infty\big(0,T;H^{k+1}(\Omega)\cap L^\infty(\Omega)\big),\quad i=1,2,\\
&\phi \in L^\infty\big(0,T;H^{k+2}(\Omega)\cap W^{2,\infty}(\Omega)\big),\quad \partial_t\phi(t) \in L^\infty\big(0,T;H^{k+1}(\Omega)\cap W^{1,\infty}(\Omega)\big).
\end{aligned} \tag{3.64}$$

We also suppose

$$f \in L^\infty\big(0,T;H^k(\Omega)\big),\quad F^i \in L^\infty\big(0,T;H^{k+1}(\Omega)\big). \tag{3.65}$$

In the following lemma, we present the error bound of $\partial_t(\nabla\phi - \nabla\phi_h)$, which will be used in the H^1 norm error estimates.

Lemma 3.10. *Let (ϕ, p^i) and (ϕ_h, p_h^i) be the solutions of (2.4) and (2.14), respectively. Assume (3.64) and (3.65) hold. If h is small enough, then we have*

$$\|\partial_t(\nabla\phi - \nabla\phi_h)\|_0 \le C\left(h^k + \sum_{i=1}^{2}\left\|\partial_t\big(R_h p^i - p_h^i\big)\right\|_0\right).$$

Proof. For any $t \in (0,T)$, suppose $\tilde{R}_h\phi \in Q_h^k$ is the H^1 projection to $\phi(t)$, satisfying

$$a_h\big(\tilde{R}_h\phi, w_h\big) = a(\phi, w_h), \quad \forall\, w_h \in Q_h^k. \tag{3.66}$$

From (2.4) and (2.14), we have

$$a(\phi, w_h) - a_h(\phi_h, w_h) + \tilde{b}\big(p^1, p^2, w_h\big) - \tilde{b}_h\big(p_h^1, p_h^2, w_h\big) = (f - f_h, w_h).$$

Then, from (3.66), it follows that

$$a_h\big(\tilde{R}_h\phi - \phi_h, w_h\big) = (f - f_h, w_h) + \tilde{b}_h\big(p_h^1, p_h^2, w_h\big) - \tilde{b}\big(p^1, p^2, w_h\big). \tag{3.67}$$

Taking derivative with respect to t on both side of the above and setting $\eta = \tilde{R}_h\phi - \phi_h$, then it yields

$$\begin{aligned}
&a_h(\partial_t\eta, w_h) + a_h(\eta, \partial_t w_h)\\
=&\ \big(\partial_t(f - f_h), w_h\big) + (f - f_h, \partial_t w_h)\\
&+ \tilde{b}_h\big(\partial_t p_h^1, \partial_t p_h^2, w_h\big) + \tilde{b}_h\big(p_h^1, p_h^2, \partial_t w_h\big)\\
&- \tilde{b}\big(\partial_t p^1, \partial_t p^2, w_h\big) - \tilde{b}\big(p^1, p^2, \partial_t w_h\big).
\end{aligned} \tag{3.68}$$

Setting $w_h = \partial_t w_h$ in (3.67), then we get

$$a_h(\eta, \partial_t w_h) = (f - f_h, \partial_t w_h) + \tilde{b}_h\big(p_h^1, p_h^2, \partial_t w_h\big) - \tilde{b}\big(p^1, p^2, \partial_t w_h\big).$$

Inserting the above into (3.68), it follows:

$$a_h(\partial_t\eta, w_h) = \big(\partial_t(f - f_h), w_h\big) + \tilde{b}_h\big(\partial_t p_h^1, \partial_t p_h^2, w_h\big) - \tilde{b}\big(\partial_t p^1, \partial_t p^2, w_h\big).$$

Taking $w_h = \partial_t\eta$ and using Lemma 2.6, we have

$$\begin{aligned}
a_h(\partial_t\eta, \partial_t\eta) &= \big(\partial_t(f - f_h), \partial_t\eta\big) + \tilde{b}_h\big(\partial_t p_h^1, \partial_t p_h^2, \partial_t\eta\big) - \tilde{b}\big(\partial_t p^1, \partial_t p^2, \partial_t\eta\big)\\
&\le \|\partial_t(f - f_h)\|_0\|\partial_t\eta\|_0 + \big|\tilde{b}_h\big(\partial_t p_h^1, \partial_t p_h^2, \partial_t\eta\big) - \tilde{b}\big(\partial_t p^1, \partial_t p^2, \partial_t\eta\big)\big|\\
&\le C\left(h^k + \sum_{i=1}^{2}\left\|\partial_t\big(p^i - p_h^i\big)\right\|_0\right)\|\nabla\partial_t\eta\|_0.
\end{aligned} \tag{3.69}$$

Hence,

$$\begin{aligned}
C_0\|\partial_t\nabla\eta\|_0^2 = C_0\|\nabla\partial_t\eta\|_0^2 &\le a_h(\partial_t\eta, \partial_t\eta)\\
&\le C\left(h^k + \sum_{i=1}^{2}\left\|\partial_t\big(p^i - p_h^i\big)\right\|_0\right)\|\nabla\partial_t\eta\|_0,
\end{aligned}$$

which deduce that

$$\left\|\partial_t\big(\nabla\tilde{R}_h\phi - \nabla\phi_h\big)\right\|_0 \le C\left(h^k + \sum_{i=1}^{2}\left\|\partial_t\big(p^i - p_h^i\big)\right\|_0\right). \tag{3.70}$$

Next, we estimate $\|\partial_t(\nabla\tilde{R}_h\phi-\nabla\phi)\|_0$. First, taking derivative with respect to t on both sides of (3.66), we have

$$a_h\big(\partial_t\tilde{R}_h\phi,w_h\big)+a_h\big(\tilde{R}_h\phi,\partial_t w_h\big)=a(\partial_t\phi,w_h)+a(\phi,\partial_t w_h).$$

Taking $w_h=\partial_t w_h$ in (3.66) and inserting the resulted equation into the above, we get

$$a_h\big(\partial_t\tilde{R}_h\phi,w_h\big)=a(\partial_t\phi,w_h).$$

Then setting

$$w_h=\partial_t\tilde{\eta}=\partial_t\big(\nabla\tilde{R}_h\phi-\Pi_k^0\nabla\phi\big),$$

it follows that

$$\begin{aligned}
C_0\sum_E\|\nabla\partial_t\tilde{\eta}\|_{0,E}^2&\le C_0\sum_E a_h^E(\partial_t\tilde{\eta},\partial_t\tilde{\eta})\\
&=\sum_E\big(a_h^E\big(\partial_t\tilde{R}_h\phi,\partial_t\tilde{\eta}\big)-a_h^E\big(\partial_t\Pi_k^0\phi,\partial_t\tilde{\eta}\big)\big)\\
&=\sum_E\big(a^E(\partial_t\phi,\partial_t\tilde{\eta})-a^E\big(\partial_t\Pi_k^0\phi,\partial_t\tilde{\eta}\big)\big)\\
&\le C\sum_E\big\|\nabla\partial_t\phi-\nabla\Pi_k^0\partial_t\phi\big\|_0\|\nabla\partial_t\tilde{\eta}\|_0\\
&\le Ch^k\|\nabla\partial_t\tilde{\eta}\|_0.
\end{aligned}$$

Hence,

$$\sum_E\big\|\partial_t\big(\nabla\tilde{R}_h\phi-\Pi_k^0\nabla\phi\big)\big\|_{0,E}\le Ch^k.$$

Then

$$\big\|\partial_t\big(\nabla\phi-\nabla\tilde{R}_h\phi\big)\big\|_0\le\sum_E\Big(\big\|\partial_t\big(\nabla\phi-\Pi_k^0\nabla\phi\big)\big\|_{0,E}+\big\|\partial_t\big(\Pi_k^0\nabla\phi-\nabla\tilde{R}_h\phi\big)\big\|_{0,E}\Big)\le Ch^k.$$

Combing the above with (3.70) and using Lemma 3.9, we get

$$\|\partial_t(\nabla\phi-\nabla\phi_h)\|_0\le C\left(h^k+\sum_{i=1}^2\|\partial_t(R_hp^i-p_h^i)\|_0\right).$$

This completes the proof of this lemma. □

Now we can present the error estimates in the H^1 norm.

Theorem 3.2. *Suppose the decomposition $\mathcal{T}_h$ is quasi-uniform. Let (ϕ,p^i) and (ϕ_h,p_h^i) be the solutions of* (2.4) *and* (2.14)*, respectively, and set $p_{h,0}^i:=(p_0^i)_I$, the interpolant function of the initial value of p_0^i in Q_h^k. Suppose* (3.64) *and* (3.65) *hold. For all $t\in(0,T]$ and h sufficiently small, the following estimate holds:*

$$\big\|p^i(t)-p_h^i(t)\big\|_1+\|\phi(t)-\phi_h(t)\|_1\le Ch^k. \tag{3.71}$$

Proof. From (2.4) and (2.14), for any $v_h\in Q_h^k$, we have

$$\begin{aligned}
&\big(p_t^i,v_h\big)-m_h\big(p_{h,t}^i,v_h\big)+a\big(p^i,v_h\big)-a_h\big(p_h^i,v_h\big)\\
&\quad+b_i\big(p^i,\phi_h,v_h\big)-b_{i,h}\big(p_h^i,\phi_h,v_h\big)=\big(F^i,v_h\big)-\big(F_h^i,v_h\big).
\end{aligned}\tag{3.72}$$

Then from (3.12), it follows that

$$\begin{aligned}&\big(p_t^i-\big(R_hp^i\big)_t+\big(R_hp^i\big)_t,v_h\big)+m_h\big(\big(R_hp^i\big)_t-p_{h,t}^i,v_h\big)-m_h\big(\big(R_hp^i\big)_t,v_h\big)\\&\quad+a_h\big(R_hp^i-p_h^i,v_h\big)+b_{i,h}\big(R_hp^i,\phi,v_h\big)-b_{i,h}\big(p_h^i,\phi_h,v_h\big)=\big(F^i-F_h^i,v_h\big).\end{aligned}$$

Let $\theta^i=R_hp^i-p_h^i,\eta^i=p^i-R_hp^i$ and take $v_h=\theta_t^i:=(R_hp^i-p_h^i)_t$ in the above equation. Then it follows that

$$\begin{aligned}&m_h\big(\theta_t^i,\theta_t^i\big)+a_h\big(\theta^i,\theta_t^i\big)\\&=-\big(\eta_t^i,\theta_t^i\big)+\big\{m_h\big(\big(R_hp^i\big)_t,\theta_t^i\big)-\big(\big(R_hp^i\big)_t,\theta_t^i\big)\big\}\\&\quad+\big(F^i-F_h^i,\theta_t^i\big)+\big\{b_{i,h}\big(p_h^i,\phi_h,\theta_t^i\big)-b_{i,h}\big(R_hp^i,\phi,\theta_t^i\big)\big\}\\&=:A_1+A_2+A_3+A_4.\end{aligned}\tag{3.73}$$

Next, we shall estimate $A_i,i=1,2,3,4$, respectively. First, from Lemma 3.9, we get

$$\begin{aligned}A_1&=-\big(\eta_t^i,\theta_t^i\big)\le\big\|p_t^i-\big(R_hp^i\big)_t\big\|_0\big\|\theta_t^i\big\|_0\\&\le Ch^k\big\|\theta_t^i\big\|_0\le Ch^{2k}+\epsilon\big\|\theta_t^i\big\|_0^2.\end{aligned}\tag{3.74}$$

Using (2.20), (2.22), Lemmas 3.7 and 3.9, it yields

$$\begin{aligned}A_2&=m_h\big(\big(R_hp^i\big)_t,\theta_t^i\big)-\big(\big(R_hp^i\big)_t,\theta_t^i\big)\\&=m_h\big(\big(R_hp^i\big)_t-\Pi_k^0\big(R_hp^i\big)_t,\theta_t^i\big)\\&\quad+m_h\big(\big(\Pi_k^0(R_hp^i\big)_t,\theta_t^i\big)-\big(\big(R_hp^i\big)_t,\theta_t^i\big)\\&\le C\big\|\big(R_hp^i\big)_t-\Pi_k^0\big(R_hp^i\big)_t\big\|_0\|\theta_t^i\|_0\quad\text{(by (2.20) and (2.22))}\\&\le Ch^k\big\|\theta_t^i\big\|_0\qquad\text{(by Lemmas 3.7 and 3.9)}\\&\le Ch^{2k}+\epsilon\big\|\theta_t^i\big\|_0^2.\end{aligned}\tag{3.75}$$

For the term A_3, we have

$$\begin{aligned}A_3&=\big(F^i-F_h^i,\theta_t^i\big)\le Ch^{k+1}\|F^i\|_{k+1}\big\|\theta_t^i\big\|_0\\&\le Ch^{2k+2}+\epsilon\big\|\theta_t^i\big\|_0^2.\end{aligned}\tag{3.76}$$

To estimate the term A_4, first we get

$$\begin{aligned}A_4&=b_{i,h}\big(p_h^i,\phi_h,\theta_t^i\big)-b_{i,h}\big(R_hp^i,\phi,\theta_t^i\big)\\&=b_{i,h}\big(p_h^i-R_hp^i,\phi,\theta_t^i\big)+b_{i,h}\big(p^i,\phi_h-\phi,\theta_t^i\big)+b_{i,h}\big(p_h^i-p^i,\phi_h-\phi,\theta_t^i\big)\\&=A_{41}+A_{42}+A_{43}.\end{aligned}\tag{3.77}$$

Next, we shall estimate the terms A_{41},A_{42},A_{41}, respectively.

From Lemma 3.7 and Theorem 3.1, it yields

$$\begin{aligned}A_{41}&=b_{i,h}\big(p_h^i-R_hp^i,\phi,\theta_t^i\big)\\&=\sum_E q^i\big(\Pi_{k-1}^0\theta^i\Pi_k^0\nabla\phi,\Pi_{k-1}^0\nabla\theta_t^i\big)_E\\&=\sum_E q^i\frac{\partial}{\partial_t}\big(\Pi_{k-1}^0\theta^i\Pi_k^0\nabla\phi,\Pi_{k-1}^0\nabla\theta^i\big)_E\end{aligned}$$

$$
\begin{aligned}
&-\sum_E \Big\{q^i\big(\big(\Pi^0_{k-1}\theta^i\big)_t\Pi^0_k\nabla\phi,\Pi^0_{k-1}\nabla\theta^i\big)_E+q^i\big(\Pi^0_{k-1}\theta^i\big(\Pi^0_k\nabla\phi\big)_t,\Pi^0_{k-1}\nabla\theta^i\big)_E\Big\}\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(\theta^i,\phi,\theta^i\big)+\sum_E\Big\{\|\big(\Pi^0_{k-1}\theta^i\big)_t\|_{0,E}\|\Pi^0_k\nabla\phi\|_{0,\infty,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\\
&\qquad+\big\|\Pi^0_{k-1}\theta^i\big\|_{0,E}\big\|\big(\Pi^0_k\nabla\phi\big)_t\big\|_{0,\infty,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\Big\}\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(\theta^i,\phi,\theta^i\big)+C\big(\|\theta^i\|_0^2+\|\nabla\theta^i\|_0^2\big)+\epsilon\big\|\theta^i_t\big\|_0^2\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(\theta^i,\phi,\theta^i\big)+C\big(h^{2k}+\|\nabla\theta^i\|_0^2\big)+\epsilon\big\|\theta^i_t\big\|_0^2\quad\text{(by Lemma 3.7 and Theorem 3.1).}
\end{aligned}\tag{3.78}
$$

Similarly, we get

$$
\begin{aligned}
A_{42}&=b_{i,h}\big(p^i,\phi_h-\phi,\theta^i_t\big)\\
&=\sum_E q^i\big(\Pi^0_{k-1}p^i\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i_t\big)_E\\
&=\sum_E q^i\frac{\partial}{\partial_t}\big(\Pi^0_{k-1}p^i\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i\big)_E\\
&\quad-\sum_E\Big\{q^i\big(\big(\Pi^0_{k-1}p^i\big)_t\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i\big)_E+q^i\big(\Pi^0_{k-1}p^i\big(\Pi^0_k\nabla(\phi_h-\phi)\big)_t,\Pi^0_{k-1}\nabla\theta^i\big)_E\Big\}\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(p^i,\phi_h-\phi,\theta^i\big)+\sum_E\Big\{\big\|\big(\Pi^0_{k-1}p^i\big)_t\big\|_{0,\infty,E}\big\|\Pi^0_k\nabla(\phi_h-\phi)\big\|_{0,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\\
&\qquad+\big\|\Pi^0_{k-1}p^i\big\|_{0,\infty,E}\big\|\big(\Pi^0_k\nabla(\phi_h-\phi)\big)_t\big\|_{0,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\Big\}\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(p^i,\phi_h-\phi,\theta^i\big)+C\big(\|\nabla(\phi_h-\phi)\|_0+\|\nabla(\phi_h-\phi)_t\|_0\big)\|\nabla\theta^i\|_0\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(p^i,\phi_h-\phi,\theta^i\big)+C\big(h^{2k}+\|\nabla\theta^i\|_0^2\big)\\
&\quad+\epsilon\sum_{i=1}^2\big\|\theta^i_t\big\|_0^2,\quad\text{(by Theorem 3.1 and Lemma 3.10)}
\end{aligned}\tag{3.79}
$$

and

$$
\begin{aligned}
A_{43}&=b_{i,h}\big(p^i_h-p^i,\phi_h-\phi,\theta^i_t\big)\\
&=\sum_E q^i\big(\Pi^0_{k-1}\big(p^i_h-p^i\big)\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i_t\big)_E\\
&=\sum_E q^i\frac{\partial}{\partial_t}\big(\Pi^0_{k-1}\big(p^i_h-p^i\big)\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i\big)_E\\
&\quad-\sum_E\Big\{q^i\big(\big(\Pi^0_{k-1}(p^i_h-p^i)\big)_t\Pi^0_k\nabla(\phi_h-\phi),\Pi^0_{k-1}\nabla\theta^i\big)_E\\
&\qquad+q^i\big(\Pi^0_{k-1}\big(p^i_h-p^i\big)\big(\Pi^0_k\nabla(\phi_h-\phi)\big)_t,\Pi^0_{k-1}\nabla\theta^i\big)_E\Big\}\\
&\le\frac{\partial}{\partial_t}b_{i,h}\big(p^i_h-p^i,\phi_h-\phi,\theta^i\big)\\
&\quad+\sum_E\Big\{\big\|\big(\Pi^0_{k-1}\big(p^i-p^i_h\big)\big)_t\big\|_{0,E}\big\|\Pi^0_k\nabla(\phi_h-\phi)\big\|_{0,\infty,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\\
&\qquad+\big\|\Pi^0_{k-1}\big(p^i-p^i_h\big)\big\|_{0,E}\big\|\big(\Pi^0_k\nabla(\phi_h-\phi)\big)_t\big\|_{0,\infty,E}\big\|\Pi^0_{k-1}\nabla\theta^i\big\|_{0,E}\Big\}.
\end{aligned}\tag{3.80}
$$

Note that from Lemma 3.2 and Theorem 3.1, we have

$$\left\|\Pi_k^0\nabla(\phi_h-\phi)\right\|_{0,\infty,E}\le\left\|\Pi_k^0\nabla\phi_h\right\|_{0,E}+\left\|\Pi_k^0\nabla\phi\right\|_{0,E}\le C.$$

Following the arguments in Lemma 3.2 and using Lemma 3.10, it follows that

$$\begin{aligned}\left\|\left(\Pi_k^0\nabla(\phi_h-\phi)\right)_t\right\|_{0,\infty,E}&\le\left\|\Pi_k^0\nabla(\phi_h)_t\right\|_{0,E}+\left\|\Pi_k^0\nabla\phi_t\right\|_{0,E}\\&\le C+Ch^{-1}\|\partial_t(\nabla\phi-\nabla\phi_h)\|_0\\&\le C+Ch^{-1}\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_E\left\|\Pi_{k-1}^0\left(p^i-p_h^i\right)\right\|_{0,E}\left\|\left(\Pi_k^0\nabla(\phi_h-\phi)\right)_t\right\|_{0,\infty,E}\left\|\Pi_{k-1}\nabla\theta^i\right\|_{0,E}\\&\le Ch^{k-1}\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0\|\nabla\theta^i\|_0+Ch^k\|\nabla\theta^i\|_0\\&\le C\left(\|\nabla\theta^i\|_0^2+h^{2k}\right)+\epsilon\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0^2.\end{aligned}$$

Combining the above with (3.80), we get

$$\begin{aligned}A_{43}&=b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta_t^i\right)\\&\le\frac{\partial}{\partial_t}b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta^i\right)+C\left(\|\nabla\theta^i\|_0^2+h^{2k}\right)+\epsilon\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0^2+\epsilon\left\|\left(p^i-p_h^i\right)_t\right\|_0^2\\&\le\frac{\partial}{\partial_t}b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta^i\right)+C\left(\|\nabla\theta^i\|_0^2+h^{2k}\right)+\epsilon\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0^2+2\epsilon\left(\left\|\theta_t^i\right\|_0^2+\left\|\eta_t^i\right\|_0^2\right)\\&\le\frac{\partial}{\partial_t}b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta^i\right)+C\left(h^{2k}+\|\nabla\theta^i\|_0^2\right)+3\epsilon\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0^2,\end{aligned}\tag{3.81}$$

where Lemma 3.9 is used in the last inequality. Inserting (3.78), (3.79) and (3.81) into (3.77), it follows that

$$\begin{aligned}A_4\le&\frac{\partial}{\partial_t}\left(b_{i,h}(\theta^i,\phi,\theta^i)+b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta^i\right)+b_{i,h}\left(p^i,\phi_h-\phi,\theta^i\right)\right)\\&+C\left(\|\nabla\theta^i\|_0^2+h^{2k}\right)+\sum_{i=1}^{2}3\epsilon\left\|\theta_t^i\right\|_0^2.\end{aligned}\tag{3.82}$$

Thus combing (3.73)-(3.76) and (3.82), we have

$$\begin{aligned}\tilde{C}_*\left\|\theta_t^i\right\|_0^2+\frac{1}{2}\left\|\nabla\theta_t^i\right\|_0^2&\le m_h\left(\theta_t^i,\theta_t^i\right)+a_h\left(\theta^i,\theta_t^i\right)\\&\le\frac{\partial}{\partial_t}\left(b_{i,h}(\theta^i,\phi,\theta^i)+b_{i,h}\left(p_h^i-p^i,\phi_h-\phi,\theta^i\right)+b_{i,h}\left(p^i,\phi_h-\phi,\theta^i\right)\right)\\&\quad+C\left(\|\nabla\theta^i\|_0^2+h^{2k}\right)+5\epsilon\sum_{i=1}^{2}\left\|\theta_t^i\right\|_0^2.\end{aligned}$$

Then integrating with respect to t on both sides, it yields

$$\begin{aligned}\|\nabla\theta\|_0^2 &\le C\bigg(b_{i,h}(\theta^i,\phi,\theta^i)+b_{i,h}\big(p_h^i-p^i,\phi_h-\phi,\theta^i\big)\\&\qquad +b_{i,h}\big(p^i,\phi_h-\phi,\theta^i\big)+\int_0^t\|\nabla\theta\|_0^2dt+h^{2k}\bigg)\\&\le C\bigg(\Big(\|\theta^i\|_0\|\nabla\phi\|_{0,\infty}+\big\|p_h^i-p^i\big\|_0\big\|\Pi_k^0(\nabla\phi-\nabla\phi_h)\big\|_{0,\infty}\\&\qquad +\|p^i\|_{0,\infty}\|\nabla\phi-\nabla\phi_h\|_0\Big)\|\nabla\theta^i\|_0+\int_0^t\|\nabla\theta\|_0^2dt+h^{2k}\bigg)\\&\le C\bigg(\int_0^t\|\nabla\theta\|_0^2dt+h^{2k}\bigg)+\epsilon\|\nabla\theta^i\|_0^2.\quad\text{(by Lemmas 3.1 and Theorem 3.1)}\end{aligned}$$

Thus, we have

$$\|\nabla\theta^i\|_0^2\le C\bigg(\int_0^t\|\nabla\theta\|_0^2dt+h^{2k}\bigg).$$

By using the Gronwall inequality, we get

$$\|\nabla\theta^i\|_0^2\le Ch^{2k}.$$

Then using Lemma 3.7, we have

$$\big\|p^i-p_h^i\big\|_1\le Ch^k,$$

which combined with Lemma 3.1 shows

$$\|\phi-\phi_h\|_1\le Ch^k.$$

This completes the proof of the lemma. □

4. Error Estimates of the Gummel Iteration Based on VEM

In this section, we first present the fully discrete virtual element scheme for the PNP equations, and then show the error estimate of the Gummel iteration based on the VEM. As already mentioned in the introduction, in this paper we focus mainly on the error estimates for semi-discrete virtual element approximation, since the error analysis of the time discretization follows a standard procedure. As an example of error analysis for the Gummel iteration of VEM, we here only present the L^2 norm error estimates.

The fully discrete scheme is presented (2.15). Since the system (2.15) is a coupled nonlinear one, we apply the Gummel iteration to decouple and linearize it. The Gummel iteration of VEM (2.15) for PNP equations is presented as follows.

Algorithm 4.1: Gummel iteration of VEM.

Step 1. Give the initial value $p_h^{i,0}, i=1,2, \phi_h^0\in Q_h^k$.

Step 2. Let $p_h^{i,n,0}=p_h^{i,n-1}, i=1,2, \phi_h^{n,0}=\phi_h^{n-1}$ as $n\ge 1$.

Step 3. For $l\ge 1$, compute $p_h^{i,n,l}, i=1,2, \phi_h^{n,l}\in Q_h^k$ such that for any v_h and $w_h\in Q_h^k$,

$$
\begin{cases}
m_h\left(\dfrac{p_h^{i,n,l}}{\tau}, v_h\right) + a_h\big(p_h^{i,n,l}, v_h\big) + b_{i,h}\big(p_h^{i,n,l}, \phi_h^{n,l}, v_h\big) \\
= \big(F_i^n, v_h\big) + m_h\left(\dfrac{p_h^{i,n-1}}{\tau}, v_h\right), \quad i = 1,2, \\
a_h\big(\phi_h^{n,l}, w_h\big) + \tilde{b}_h\big(p_h^{1,n,l-1}, p_h^{2,n,l-1}, w_h\big) = \big(f_h^n, w_h\big).
\end{cases} \tag{4.1}
$$

Step 4. For a fixed tolerance ϵ, stop the iteration if

$$\|p_h^{1,n,l} - p_h^{1,n,l-1}\| + \|p_h^{2,n,l} - p_h^{2,n,l-1}\| + \|\phi_h^{n,l} - \phi_h^{n,l-1}\| \le \epsilon,$$

and set $(p_h^{1,n}, p_h^{2,n}, \phi_h^n) = (p_h^{1,n,l}, p_h^{2,n,l}, \phi_h^{n,l})$. Otherwise set $l \leftarrow l+1$ and goto Step 3 to continue the nonlinear iteration.

Step 5. Time marching: Stop when $n+1 = N$. Otherwise, set $n \leftarrow n+1$, and go to Step 2.

Assume the following regularity properties hold:

$$
\begin{aligned}
&p^{i,n}, \quad p_t^{i,n} \in L^\infty\big(0,T; H^{k+1}(\Omega) \cap L^\infty(\Omega)\big), \quad p_{tt}^{i,n} \in L^\infty\big(0,T; L^2(\Omega)\big), \quad i = 1,2, \\
&\phi^n \in L^\infty\big(0,T; H^{k+2}(\Omega) \cap W^{k+1,\infty}(\Omega)\big), \qquad \partial_t \phi^n \in L^\infty\big(0,T; W^{1,\infty}(\Omega)\big).
\end{aligned} \tag{4.2}
$$

We also suppose

$$f^n \in L^\infty\big(0,T; H^k(\Omega) \cap L^\infty(\Omega)\big), \quad F^{i,n} \in L^\infty\big(0,T; H^{k+1}(\Omega)\big). \tag{4.3}$$

We present the error estimate of $\phi_h^{n,l} - \phi^n$ in the following lemma.

Lemma 4.1. *Let $(\phi^n, p^{i,n})$ and $(\phi_h^{n,l}, p_h^{i,n,l})$ be the solutions of (2.4) and (4.1), respectively, set the initial value $p_h^{i,0} = R_h p^{i,0}$ and assume (4.2)-(4.3) hold. Then for all $n = 1, \dots, N$ and $l \ge 1$, there holds*

$$\|\phi_h^{n,l} - \phi^n\|_1 \le C\left(h^k + \sum_{i=1}^2 \|p_h^{i,n,l-1} - p^{i,n}\|_0\right) \tag{4.4}$$

$$\|\Pi_k^0 \nabla \phi_h^{n,l}\|_{0,\infty,E} \le C\left(1 + \sum_{i=1}^2 \|p_h^{i,n,l-1} - p^{i,n}\|_1\right). \tag{4.5}$$

Proof. The first result (4.4) can be obtained by repeating the arguments in Lemma 3.1. We only need to show (4.5). It is easy to see that $\phi_h^{n,l}$ can be viewed as the virtual element approximation to the solution of the following Poisson equation:

$$-\Delta \phi_h^{n,l} = f_h^{n,l} + \sum_{i=1}^2 q^i p_h^{i,n,l-1} \quad \text{in } \Omega$$

with homogeneous Dirichlet boundary condition. Then, using the maximum norm error estimate of the virtual element solution [31], and from the regularity estimate, Taylor's theorem and the Gagliardo-Nirenberg inequality, we have

$$\|\Pi_k^0 \nabla \phi_h^{n,l}\|_{0,\infty,E} \le C\|\nabla \phi^{n,l}\|_{0,\infty,E}$$

$$
\begin{aligned}
&\le C\left(\|f_h^{n,l}\|_{0,4}+\left\|\sum_{i=1}^{2} q^i p_h^{i,n,l-1}\right\|_{0,4}\right)\\
&\le C\left(\|f^{n,l}\|_{0,\infty}+\sum_{i=1}^{2}\left\|p_h^{i,n,l-1}-p^{i,n}\right\|_{0,4}+\|p^{1,n}-p^{2,n}\|_{0,4}\right)\\
&\le C\left(1+\sum_{i=1}^{2}\left\|p_h^{i,n,l-1}-p^{i,n}\right\|_{1}\right),
\end{aligned}
$$

where the fact $q^2=-C_q q^1$ ($C_q>0$ is a certain constant) is used. This completes the proof of this lemma. □

Now, we proceed to estimate the error $p_h^{i,n,l}-p^{i,n}$ in the L^2 norm.

Theorem 4.1. *Suppose the decomposition $\mathcal{T}_h$ is quasi-uniform. Let $(\phi^n,p^{i,n})$ and $(\phi_h^{n,l},p_h^{i,n,l})$ be the solutions of (2.4) and (4.1), respectively, and set $p_h^{i,0}:=R_h p^{i,0}$. Suppose (4.2)-(4.3) hold and $\tau+h$ is small enough. Then for all $n=1,\dots,N$ and $l\ge 0$, we have*

$$
\sum_{i=1}^{2}\left\|p_h^{i,n,l}-p^{i,n}\right\|_0+\left\|\phi_h^{n,l}-\phi^n\right\|_0\le C(\tau+h^k). \tag{4.6}
$$

Proof. Set

$$
p_h^{i,n,l}-p^{i,n}=\left(p_h^{i,n,l}-R_h p^{i,n}\right)+\left(R_h p^{i,n}-p^{i,n}\right)=:\upsilon^{i,n,l}+\varrho^{i,n},
$$

where $p^{i,n}=p^i(t_n), n=1,2,\dots,N$. From Lemma 3.7, we obtain

$$
\|\varrho^{i,n}\|_0\le Ch^{k+1}. \tag{4.7}
$$

The estimate for $\upsilon^{i,n,l}$ requires more analysis. Denote by

$$
D_\tau u^n=\frac{1}{\tau}(u^n-u^{n-1}).
$$

For all $v_h\in Q_h^k$, from (2.4) and (4.1), there holds

$$
\begin{aligned}
&m_h\left(D_\tau \upsilon^{i,n,l},v_h\right)+a_h\left(\upsilon^{i,n,l},v_h\right)\\
&=\left(F_h^{i,n},v_h\right)-b_{i,h}\left(p_h^{i,n,l},\phi_h^{n,l},v_h\right)-m_h\left(D_\tau R_h p^{i,n},v_h\right)-a_h\left(R_h p^{i,n},v_h\right)\\
&\quad+m_h\left(\frac{1}{\tau}\left(p_h^{i,n-1}-p_h^{i,n-1,l}\right),v_h\right)+a\left(p^{i,n},v_h\right)-a\left(p^{i,n},v_h\right)\\
&=\left(F_h^{i,n},v_h\right)-b_{i,h}\left(p_h^{i,n,l},\phi_h^{n,l},v_h\right)-m_h\left(D_\tau R_h p^{i,n},v_h\right)-a_h\left(R_h p^{i,n},v_h\right)\\
&\quad+a\left(p^{i,n},v_h\right)-\left(F^{i,n},v_h\right)+\left(p_t^{i,n},v_h\right)+b_i\left(p^{i,n},\phi^n,v_h\right)+m_h\left(\frac{1}{\tau}\left(p_h^{i,n-1}-p_h^{i,n-1,l}\right),v_h\right)\\
&=\left(F_h^{i,n}-F^{i,n},v_h\right)+\left(\left(p_t^{i,n},v_h\right)-m_h\left(D_\tau R_h p^{i,n},v_h\right)\right)+\left(a\left(p^{i,n},v_h\right)-a_h\left(R_h p^{i,n},v_h\right)\right)\\
&\quad+\left(b_i\left(p^{i,n},\phi^n,v_h\right)-b_{i,h}\left(p_h^{i,n,l},\phi_h^{n,l},v_h\right)\right)+m_h\left(\frac{1}{\tau}\left(p_h^{i,n-1}-p_h^{i,n-1,l}\right),v_h\right)\\
&=:H_1^n+H_2^n+H_3^n+H_4^{n,l}+H_5^{n,l}.
\end{aligned} \tag{4.8}
$$

From (3.5), we have

$$
H_1^n=\left(F_h^{i,n}-F^{i,n},v_h\right)=\left(\Pi_k^0 F^{i,n}-F^{i,n},v_h\right)\le Ch^{k+1}|F^{i,n}|_{k+1}\|v_h\|_0. \tag{4.9}
$$

In order to bound the second term, adding and subtracting suitable terms, we can obtain

$$\begin{aligned} H_2^n &= \big(p_t^{i,n}, v_h\big) - m_h\big(D_\tau R_h p^{i,n}, v_h\big) \\ &= \Big(\big(p_t^{i,n}, v_h\big) - \big(D_\tau p^{i,n}, v_h\big)\Big) + \sum_E \big(D_\tau\big(p^{i,n} - \Pi_k^0 p^{i,n}\big), v_h\big)_E \\ &\quad + \bigg(\sum_E \big(D_\tau \Pi_k^0 p^{i,n}, v_h\big)_E - m_h^E\big(D_\tau R_h p^{i,n}, v_h\big)\bigg) \\ &=: H_{21} + H_{22} + H_{23}. \end{aligned} \tag{4.10}$$

The estimates for H_{21} to H_{23} can be determined as follows:

$$\begin{aligned} H_{21} &\le \big\|p_t^{i,n} - D_\tau p^{i,n}\big\|_0 \|v_h\|_0 \\ &\le \Big\|\frac{1}{2}\tau \cdot \partial_{tt} p^i(x,\xi)\Big\|_0 \|v_h\|_0, \quad (t^{n-1} < \xi < t^n) \quad \text{(by Taylor's expansion)} \\ &\le C\tau \|v_h\|_0, \\ H_{22} &= \sum_E \big(D_\tau\big(p^{i,n} - \Pi_k^0 p^{i,n}\big), v_h\big) \\ &\le \sum_E \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \big\|\partial_t\big(p^i - \Pi_k^0 p^i\big)(s)\big\|_{0,E} ds \|v_h\|_{0,E} \\ &\le Ch^k \|v_h\|_0. \\ H_{23} &= \sum_E \Big(\big(D_\tau \Pi_k^0 p^{i,n}, v_h\big)_E - m_h^E\big(D_\tau R_h p^{i,n}, v_h\big)\Big) \\ &= \sum_E \Big(m_h^E\big(D_\tau \Pi_k^0 p^{i,n} - D_\tau R_h p^{i,n}, v_h\big)\Big) \qquad \text{(by (2.20))} \\ &\le \sum_E \big\|D_\tau\big(\Pi_k^0 p^{i,n} - p^{i,n}\big)\big\|_{0,E} \|v_h\|_{0,E} + \big\|D_\tau\big(p^{i,n} - R_h p^{i,n}\big)\big\|_0 \|v_h\|_0 \\ &\le \sum_E \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \big\|\partial_t\big(\Pi_k^0 p^{i,n} - p^{i,n}\big)(s)\big\|_0 ds \|v_h\|_{0,E} \\ &\quad + \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \big\|\partial_t\big(p^i - R_h p^i\big)(s)\big\|_0 ds \|v_h\|_0 \\ &\le Ch^k \|v_h\|_0. \qquad \text{(by Lemma 3.9).} \end{aligned}$$

Hence,

$$H_2^n = \big(p_t^{i,n}, v_h\big) - m_h\big(D_\tau R_h p^{i,n}, v_h\big) \le C(\tau + h^k)\|v_h\|_0. \tag{4.11}$$

Using (2.17) and Lemma 3.7, we have

$$\begin{aligned} H_3^n &= a\big(p^{i,n}, v_h\big) - a_h\big(R_h p^{i,n}, v_h\big) \\ &= a\big(p^{i,n}, v_h\big) - a_h\big(p^{i,n}, v_h\big) + a_h\big(p^{i,n} - R_h p^{i,n}, v_h\big) \\ &\le Ch^k \|\nabla v_h\|_0. \end{aligned} \tag{4.12}$$

From Lemmas 2.5 and 4.1, we can express the fourth term as follows:

$$H_4^{n,l} = b_i\big(p^{i,n}, \phi^n, v_h\big) - b_{i,h}\big(p_h^{i,n,l}, \phi_h^{n,l}, v_h\big)$$

$$\leq C\Bigg(h^k + \sum_{i=1}^{2} \big\|p_h^{i,n,l-1} - p^{i,n}\big\|_0 \\ + \Bigg(1 + \sum_{i=1}^{2} \big\|p_h^{i,n,l-1} - p^{i,n}\big\|_1\Bigg) \big\|p_h^{i,n,l} - p^{i,n}\big\|_0\Bigg) \|\nabla v_h\|_0. \tag{4.13}$$

There holds

$$\begin{aligned} H_5^{n,l} &= m_h\left(\frac{1}{\tau}\big(p_h^{i,n-1} - p_h^{i,n-1,l}\big), v_h\right) \\ &= m_h\left(\frac{1}{\tau}\big(p_h^{i,n-1} - R_h p^{i,n-1}\big) + \frac{1}{\tau}\big(R_h p^{i,n-1} - p_h^{i,n-1,l}\big), v_h\right) \\ &\leq C\frac{1}{\tau}\big(\|\hat{v}^{i,n-1}\|_0 + \|v^{i,n-1,l}\|_0\big)\|v_h\|_0, \end{aligned} \tag{4.14}$$

where

$$\hat{v}^{i,n-1} = p_h^{i,n-1} - R_h p^{i,n-1}.$$

Collecting the estimation of (4.9), (4.11)-(4.14) into (4.8) and taking $v_h = v^{i,n,l}$, we get

$$\begin{aligned} & m_h\left(\frac{1}{\tau}(v^{i,n,l} - v^{i,n-1,l}), v^{i,n,l}\right) + a_h(v^{i,n,l}, v^{i,n,l}) \\ &= H_1^n + H_2^n + H_3^n + H_4^{n,l} + H_5^{n,l} \\ &\leq C\big(h^{k+1}|F^{i,n}|_{k+1} + \tau + h^k\big)\|v^{i,n,l}\|_0 \\ &\quad + C\left(h^k + \sum_{i=1}^{2} \big\|p_h^{i,n,l-1} - p^{i,n}\big\|_0 + \left(1 + \sum_{i=1}^{2} \|\nabla\big(p_h^{i,n,l-1} - p^{i,n}\big)\|_0\right) \big\|p_h^{i,n,l} - p^{i,n}\big\|_0\right) \|\nabla v^{i,n,l}\|_0 \\ &\quad + \frac{C}{\tau}\big(\|\hat{v}^{i,n-1}\|_0 + \|v^{i,n-1,l}\|_0\big)\|v^{i,n,l}\|_0 \\ &\leq C\left(\tau + h^k + \sum_{i=1}^{2} \big\|p_h^{i,n,l-1} - p^{i,n}\big\|_0 + \left(1 + \sum_{i=1}^{2} \big\|\nabla\big(p_h^{i,n,l-1} - p^{i,n}\big)\big\|_0\right) \big\|p_h^{i,n,l} - p^{i,n}\big\|_0\right) \|\nabla v^{i,n,l}\|_0 \\ &\quad + \frac{C}{\tau}\big(\|\hat{v}^{i,n-1}\|_0 + \|v^{i,n-1,l}\|_0\big)\|v^{i,n,l}\|_0, \end{aligned} \tag{4.15}$$

where $\|v^{i,n,l}\|_0 \leq C\|\nabla v^{i,n,l}\|_0$ is used which is due to the Poincaré inequality, and C is dependent of $|F^{i,n}|_{k+1}$. Note that

$$\begin{aligned} \frac{1}{2\tau}\big(\|v^{i,n,l}\|_0^2 - \|v^{i,n-1,l}\|_0^2\big) &= \frac{1}{\tau}\|v^{i,n,l}\|_0^2 - \frac{1}{2\tau}\big(\|v^{i,n-1,l}\|_0^2 + \|v^{i,n,l}\|_0^2\big) \\ &\leq \frac{1}{\tau} m_h(v^{i,n,l}, v^{i,n,l}) - \frac{1}{\tau}\|v^{i,n,l}\|_0\|v^{i,n-1,l}\|_0 \\ &\leq \frac{1}{\tau} m_h(v^{i,n,l}, v^{i,n,l}) - \frac{1}{\tau} m_h(v^{i,n-1,l}, v^{i,n,l}). \end{aligned}$$

Combining the above with (4.15), we get

$$\begin{aligned} & \frac{1}{2\tau}\big(\|v^{i,n,l}\|_0^2 - \|v^{i,n-1,l}\|_0^2\big) + \|\nabla v^{i,n,l}\|_0^2 \\ &\leq m_h\left(\frac{1}{\tau}(v^{i,n,l} - v^{i,n-1,l}), v^{i,n,l}\right) + a_h(v^{i,n,l}, v^{i,n,l}) \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\tau+h^k+\sum_{i=1}^{2}\left\|p_h^{i,n,l-1}-p^{i,n}\right\|_0+\left(1+\sum_{i=1}^{2}\left\|\nabla\left(p_h^{i,n,l-1}-p^{i,n}\right)\right\|_0\right)\left\|p_h^{i,n,l}-p^{i,n}\right\|_0\right)\|\nabla\upsilon^{i,n,l}\|_0\\
&\quad+\frac{C}{\tau}\left(\|\hat{\upsilon}^{i,n-1}\|_0+\|\upsilon^{i,n-1,l}\|_0\right)\|\upsilon^{i,n,l}\|_0\\
&\le C\left(\tau+h^k+\sum_{i=1}^{2}\|\upsilon^{i,n,l-1}\|_0+\sum_{i=1}^{2}\|\upsilon^{i,n,l}\|_0+\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0\|\upsilon^{i,n,l}\|_0\right.\\
&\qquad\left.+h^k\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0\right)\|\nabla\upsilon^{i,n,l}\|_0\\
&\quad+\frac{C}{\tau}\left(\|\hat{\upsilon}^{i,n-1}\|_0+\|\upsilon^{i,n-1,l}\|_0\right)\|\upsilon^{i,n,l}\|_0\quad\text{(by (4.7))}\\
&\le C\left(\tau^2+h^{2k}+\sum_{i=1}^{2}\|\upsilon^{i,n,l-1}\|_0^2+\sum_{i=1}^{2}\|\upsilon^{i,n,l}\|_0^2+h^{2k}\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0^2\right.\\
&\qquad\left.+\left(\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0\|\upsilon^{i,n,l}\|_0\right)^2+\epsilon\|\nabla\upsilon^{i,n,l}\|_0^2\right)\\
&\quad+\frac{C}{\tau}\left(\|\hat{\upsilon}^{i,n-1}\|_0+\|\upsilon^{i,n-1,l}\|_0\right)\|\upsilon^{i,n,l}\|_0.
\end{aligned}
\tag{4.16}
$$

Then, it is easy to get

$$
\begin{aligned}
\|\upsilon^{i,n,l}\|_0^2\le C&\left(\tau^3+\tau h^{2k}+\tau\sum_{i=1}^{2}\|\upsilon^{i,n,l-1}\|_0^2+\tau\sum_{i=1}^{2}\|\upsilon^{i,n,l}\|_0^2+\tau\left(\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0\|\upsilon^{i,n,l}\|_0\right)^2\right.\\
&\left.+\tau h^{2k}\sum_{i=1}^{2}\|\nabla\upsilon^{i,n,l-1}\|_0^2+\|\hat{\upsilon}^{i,n-1}\|_0^2+\|\upsilon^{i,n-1,l}\|_0^2\right)+\epsilon\|\upsilon^{i,n,l}\|_0^2.
\end{aligned}
$$

Thus, if h is small enough such that $C\tau h^{2k}\ll 1$, then

$$
\begin{aligned}
\|\upsilon^{i,n,l}\|_0^2\le C&\left(\tau^3+\tau h^{2k}+\tau\sum_{i=1}^{2}\left(\|\upsilon^{i,n,l-1}\|_0+\|\nabla\upsilon^{i,n,l-1}\|_0\|\upsilon^{i,n,l}\|_0+h^k\|\nabla\upsilon^{i,n,l-1}\|_0\right)^2\right.\\
&\left.+\|\hat{\upsilon}^{i,n-1}\|_0^2+\|\upsilon^{i,n-1,l}\|_0^2\right).
\end{aligned}
\tag{4.17}
$$

Next, we shall use the mathematical induction to show

$$
\|\upsilon^{i,n,l}\|_0\le C(\tau+h^k),\quad n=1,\ldots,N,\quad l\ge 0.\tag{4.18}
$$

The idea is the mathematical induction will be used respectively for the index n and l. This means we first fixed the index $n=1$ and show (4.18) holds for any $l\ge 0$ by the mathematical induction. Then, we assume (4.18) holds for the case $n=J$, $J=1,2,\ldots N-1$, $l\ge 0$ and prove it also holds for the case $n=J+1$, where the mathematical induction will be used again for the varying index l.

First, there holds

$$
\begin{aligned}
\|\upsilon^{i,1,0}\|_0&=\left\|p_h^{i,1,0}-R_hp^{i,1}\right\|_0\\
&\le\left\|p_h^{i,1,0}-p^{i,0}\right\|_0+\|p^{i,0}-p^{i,1}\|_0+\left\|p^{i,1}-R_hp^{i,1}\right\|_0.
\end{aligned}
$$

From Step 2 in Algorithm 4.1, we know $p_h^{i,1,0} = p_h^{i,0}$, and using the Taylor's theorem and Lemma 3.7, we get

$$\|\upsilon^{i,1,0}\|_0 \le C(\tau + h^k). \tag{4.19}$$

Similarly,

$$\|\nabla\upsilon^{i,1,0}\|_0 \le C(\tau + h^k) \le C. \tag{4.20}$$

From (4.17), we have

$$\|\upsilon^{i,1,1}\|_0^2 \le C\Bigg(\tau^3 + \tau h^{2k} + \tau\sum_{i=1}^{2}\big(\|\upsilon^{i,1,0}\|_0 + \|\nabla\upsilon^{i,1,0}\|_0\|\upsilon^{i,1,1}\|_0 + h^k\|\nabla\upsilon^{i,1,0}\|_0\big)^2 + \|\hat{\upsilon}^{i,0}\|_0^2 + \|\upsilon^{i,0,1}\|_0^2\Bigg).$$

Noting that

$$\hat{\upsilon}^{i,0} = p_h^{i,0} - R_h p^{i,0} = 0, \quad \upsilon^{i,0,1} = p_h^{i,0,1} - R_h p^{i,0} = p_h^{i,0} - R_h p^{i,0} = 0,$$

and assuming $C(\tau + h^k) \ll 1$, it follows:

$$\|\upsilon^{i,1,1}\|_0 \le C(\tau + h^k). \tag{4.21}$$

Similarly, combining (4.16) with (4.19)-(4.21), we get

$$\|\nabla\upsilon^{i,1,1}\|_0 \le C. \tag{4.22}$$

Now assume

$$\|\upsilon^{i,1,r}\|_0 \le C(\tau + h^k), \quad \|\nabla\upsilon^{i,1,r}\|_0 \le C$$

holds for $r \ge 1$ and we shall show

$$\|\upsilon^{i,1,r+1}\|_0 \le C(\tau + h^k), \quad \|\nabla\upsilon^{i,1,r+1}\|_0 \le C.$$

By repeatedly using (4.17), it yields

$$\|\upsilon^{i,1,r+1}\|_0 \le C(\tau + h^k), \tag{4.23}$$

where

$$\upsilon^{i,0,r+1} = p_h^{i,0,r+1} - R_h p^{i,0} = p_h^{i,0} - R_h p^{i,0} = 0$$

is used. Hence, (4.18) holds for $n = 1$ and $l \ge 0$. Similar to the deduction of (4.22) and using (4.23), we have

$$\|\nabla\upsilon^{i,1,r+1}\|_0 \le C. \tag{4.24}$$

Next, assume

$$\|\upsilon^{i,J,l}\|_0 \le C(\tau + h^k), \quad \|\nabla\upsilon^{i,J,l}\|_0 \le C, \quad J = 1, 2, \dots N-1, \quad l \ge 0, \tag{4.25}$$

and we will prove $\|\upsilon^{i,J+1,l}\|_0 \le C(\tau + h^k)$. First, similar to the proof of (4.19), we get

$$\begin{aligned}\|\upsilon^{i,J+1,0}\|_0 &= \big\|p_h^{i,J+1,0} - R_h p^{i,J+1}\big\|_0 = \big\|p_h^{i,J} - R_h p^{i,J+1}\big\|_0 \\ &\le \big\|p_h^{i,J} - R_h p^{i,J}\big\|_0 + \big\|R_h p^{i,J} - p^{i,J}\big\|_0 + \|p^{i,J} - p^{i,J+1}\|_0 + \big\|p^{i,J+1} - R_h p^{i,J+1}\big\|_0.\end{aligned}$$

Note that from Step 4 in Algorithm 4.1, $p_h^{i,J} = p_h^{i,J,l}$ is the numerical approximation at time t_J. Hence, using the assumption (4.25), the Taylor's theorem and Lemma 3.7, it yields

$$\|\upsilon^{i,J+1,0}\|_0 \leq C(\tau + h^k), \quad \|\nabla \upsilon^{i,J+1,0}\|_0 \leq C.$$

Then, similar to the deduction of (4.21) and (4.22), we have

$$\|\upsilon^{i,J+1,1}\|_0 \leq C(\tau + h^k), \quad \|\nabla \upsilon^{i,J+1,1}\|_0 \leq C, \tag{4.26}$$

where

$$\|\hat{\upsilon}^{i,J}\|_0 = \left\|p_h^{i,J} - R_h p^{i,J}\right\|_0 \leq C(\tau + h^k)$$

is used. Now suppose

$$\|\upsilon^{i,J+1,r}\|_0 \leq C(\tau + h^k), \quad \|\nabla \upsilon^{i,J+1,r}\|_0 \leq C, \quad r \geq 1. \tag{4.27}$$

By repeated application of (4.17), and using the assumptions (4.25) and (4.27), we get

$$\|\upsilon^{i,J+1,r+1}\|_0 \leq C(\tau + h^k).$$

Thus, (4.18) holds for $n = 1, 2, \ldots, N$, $l \geq 0$. Then we can easily obtain

$$\begin{aligned}\sum_{i=1}^{2} \left\|p_h^{i,n,l} - p^{i,n}\right\|_0 &\leq \sum_{i=1}^{2} \left(\|\upsilon^{i,n,l}\|_0 + \|\varrho^{i,n}\|_0\right) \\ &\leq C(\tau + h^k). \end{aligned} \tag{4.28}$$

Using the similar analysis for Lemma 3.1 and from (4.28), we have

$$\|\phi_h^n - \phi^n\|_0 \leq \left\|\phi_h^n - \phi^n\right\|_1 \leq C(\tau + h^k). \tag{4.29}$$

This completes the proof. □

5. Numerical Results

In this section, we report a numerical example to test the practical performance of the virtual element method for solving (2.1). The implement of the numerical experiment is based on [7]. All the computations are carried out in Fortran 90 on the computer with CPU-2.90 GHz (Intel (R) Core (TM) i5-10400F), RAM-16 GB.

Example 5.1. Consider problem (2.1)-(2.2) and the right-hand side functions are determined from the exact solution (cf. [44])

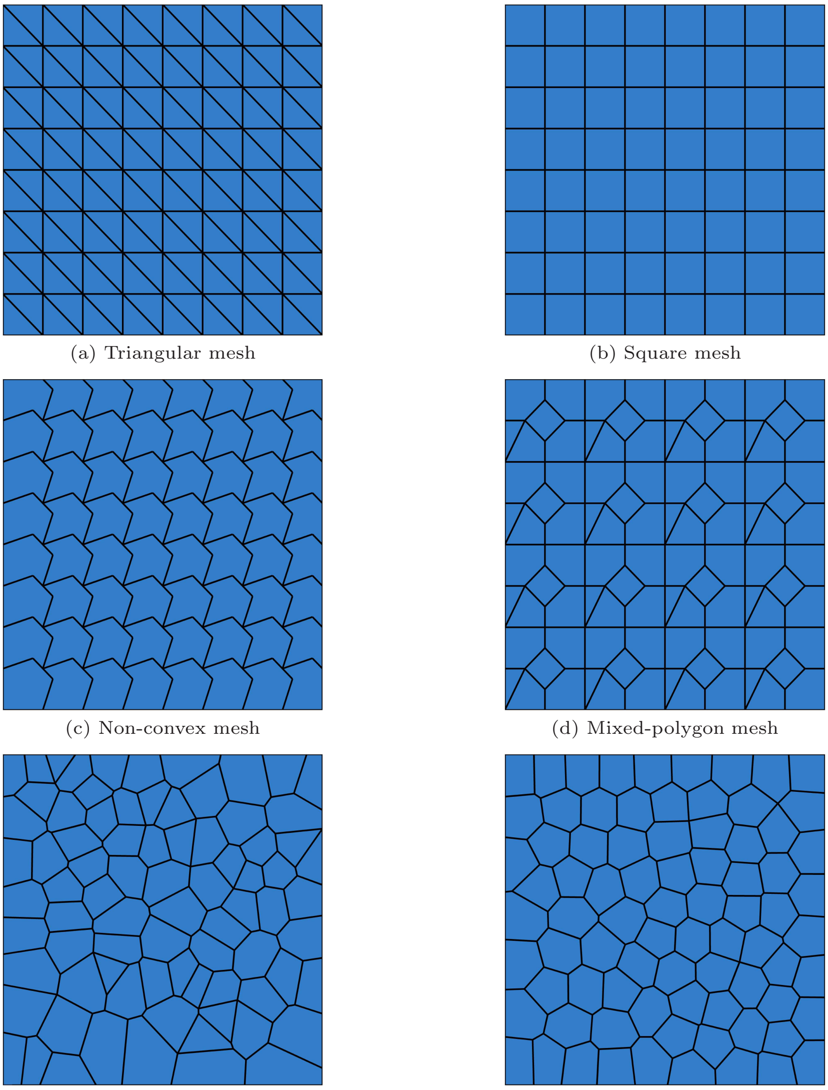
$$\begin{cases} \phi(t,x,y) = (1 - e^{-t}) \sin(\pi x) \sin(\pi y), \\ p^1(t,x,y) = \sin t \sin(2\pi x) \sin(2\pi y), \\ p^2(t,x,y) = \sin(2t) \sin(3\pi x) \sin(3\pi y). \end{cases} \tag{5.1}$$

In the test we consider the time interval [0, 1], the computational domain $\Omega = [0,1] \times [0,1]$ with $T = 1$ and time step $\tau = h^2$, and the charge $q^1 = 1$ and $q^2 = -1$. The rectangular domain is discretized with several different types of polygonal meshes, viz., triangle, square, non-convex polygons, mixed-polygon, random Voronoi and smooth Voronoi meshes, which are shown in Figs. 5.1(a)-5.1(f), respectively.

As the VEM solution can not be known explicitly inside the elements, the convergence of VEM is evaluated through the relative L^2 norm and H^1 semi-norm using the projection operator Π_k^∇ onto $\mathbb{P}_k$, that is

$$e_{L^2} := \sqrt{\sum_{E\in\mathcal{T}^h} \left\|u - \Pi_k^\nabla u_h\right\|_{0,E}^2}, \quad e_{H^1} := \sqrt{\sum_{E\in\mathcal{T}^h} \left\|\nabla\left(u - \Pi_k^\nabla u_h\right)\right\|_{0,E}^2},$$

where u is corresponding the exact solution ϕ, p^1 or p^2 in (5.1), u_h represents the VEM solution with the order $k = 1, \Pi_k^\nabla$ is defined by (2.5). To display the convergence results in the figures, we shall use the mesh-size parameter h which is measured in following ratios (cf. [9]):



(a) Triangular mesh (b) Square mesh

(c) Non-convex mesh (d) Mixed-polygon mesh

(e) Random Voronoi mesh (f) Smooth Voronoi mesh

Fig. 5.1. Six types of polygonal meshes.

$$h = \left(\frac{|\Omega|}{N_E} \right)^{\frac{1}{2}}, \tag{5.2}$$

where N_E is the number of polygons. The convergences of the errors in L^2 and H^1 norms at $t = 1.0$ are displayed in Figs. 5.2-5.7, which infers that the convergence orders approximate the second-order and first-order in L^2 norm and H^1 norm, respectively. Figs. 5.2-5.7 show that the numerical result matches with the theoretical results established in Section 4. The results in Figs. 5.2-5.7 are based on the charge values $q^1 = 1, q^2 = -1$ in (2.1), which is the most common case. Fig. 5.8 displays the numerical result of different charge values, which indicates the VEM is also efficient for the values of some other charge values.

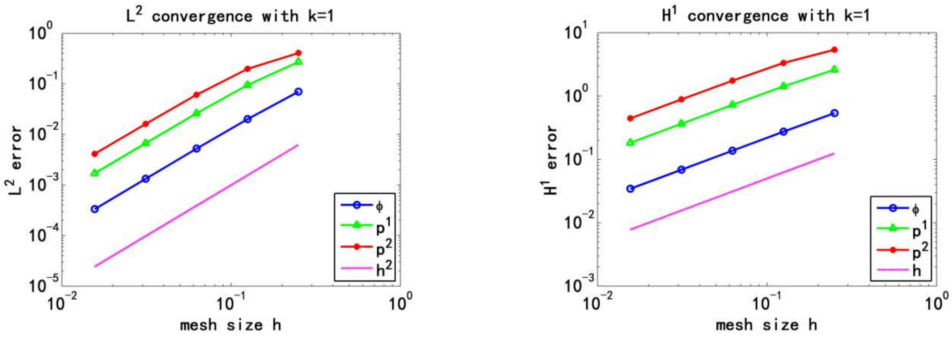


Fig. 5.2. h-convergence on triangular mesh with $t = 1.0$.

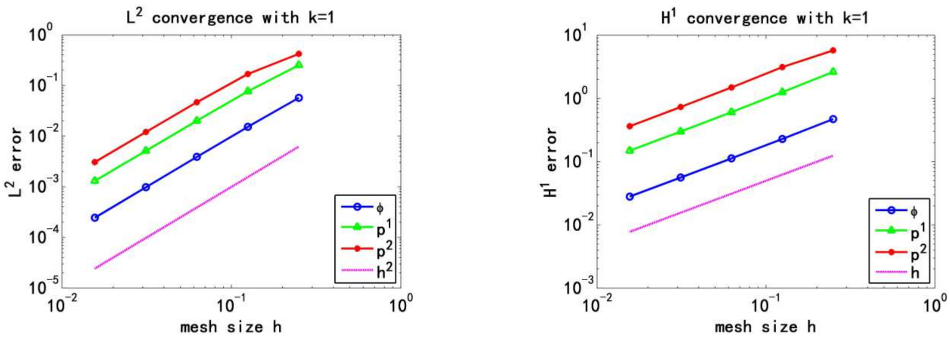


Fig. 5.3. h-convergence on square mesh with $t = 1.0$.

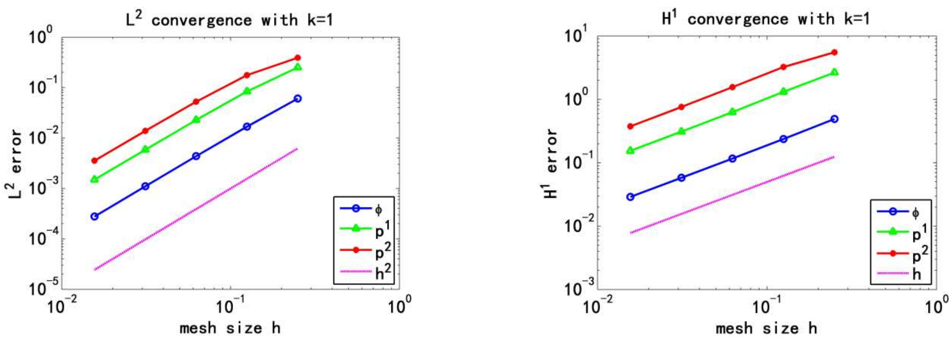


Fig. 5.4. h-convergence on non-convex mesh with $t = 1.0$.

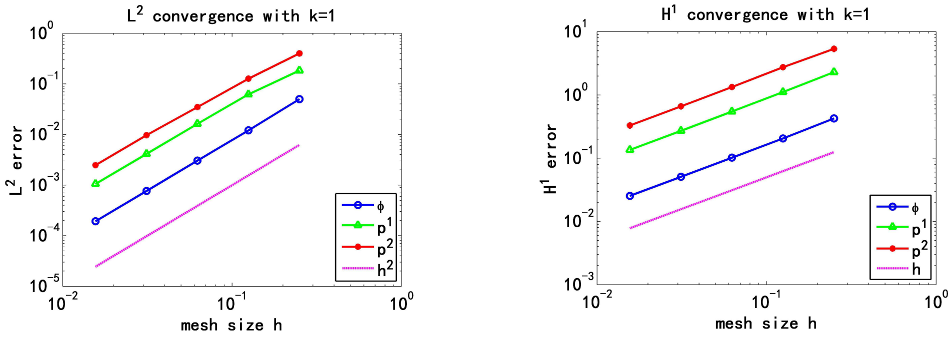


Fig. 5.5. h-convergence on mixed-polygon mesh with $t = 1.0$.

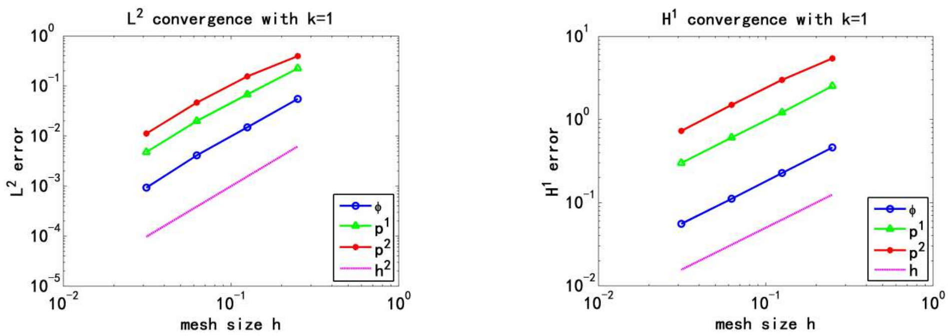


Fig. 5.6. h-convergence on random Voronoi mesh with $t = 1.0$.

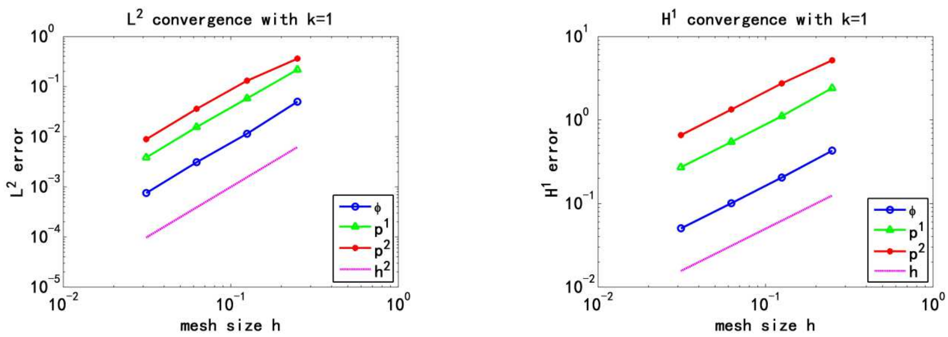
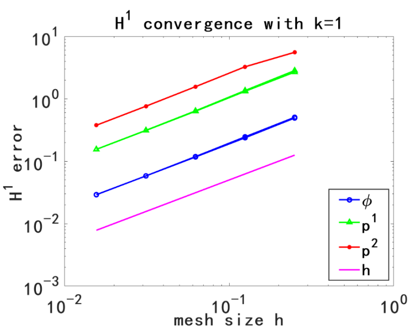
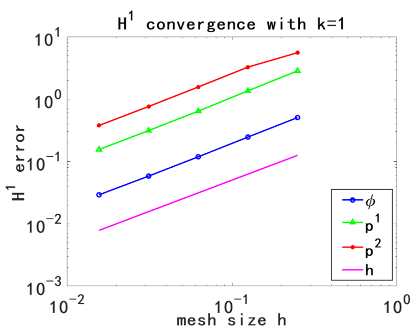


Fig. 5.7. h-convergence on smooth Voronoi mesh with $t = 1.0$.



(a) The charge $q^1 = 2, q^2 = -1$



(b) The charge $q^1 = 10, q^2 = -1$

Fig. 5.8. h-convergence on non-convex mesh with $t = 1.0$.

6. Conclusion

In this paper, we study VEMs to approximate the solutions of the time-dependent PNP equations on general polygonal meshes. We derive the a priori error estimates for semi-discrete and the Gummel iteration of the fully discrete schemes, respectively. The numerical errors have been conducted to show that the convergence orders agree with the theoretical results well. This method is expected to be extended to more complex PNP models, for example, PNP equations in semiconductor devices and three-dimensional ion channel.

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