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An Extended Courant Element on a Polytope with Application in Approximating an Obstacle Problem

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Abstract. An extended Courant element is constructed on an n dimensional polytope K, which reduces to the usual Courant element when K is a simplex. The set of the degrees of freedom consists of function values at all vertices of K, while the shape function space P_K is formed by repeatedly using the harmonic extension from lower dimensional face to higher dimensional face. Several fundamental estimates are derived on this element under reasonable geometric assumptions. Moreover, the weak maximum principle holds for any function in P_K, which enables us to use the element for approximating an obstacle problem in three dimensions. The corresponding optimal error estimate in H^1-norm is also established. Numerical results are reported to verify theoretical findings.

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Key words: Extended Courant element, virtual element, quasi-elliptic projection operators, obstacle problem.

1. Introduction

The finite element method (FEM) is a type of handy and effective numerical methods for solving various industrial and engineering problems. Historically, the first finite element was proposed by Courant [16], which is now called the Courant element. In this case, a finite element function is a continuous piecewise linear function associated with a triangular mesh. However, only until the 1960s, Argyris, Clough, Zienkiewicz et al. re-discovered the element and used it to study structure analysis in engineering. The terminology finite element method was first raised in Clough's paper [15]. During the same period, the Chinese former mathematician Feng also proposed and analyzed the

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finite element method independently, which was named by him as the finite difference scheme based on variational principle (cf. [20]). We refer to [3, 10, 14, 34] and references therein for details about the comprehensive introduction of history, mathematical theories and applications of FEMs.

Following Ciarlet's convention (cf. [14]), a finite element is a triple $(K, \mathcal{P}_K, \mathcal{N}_K)$. Here, $K \subset \mathbb{R}^n$ (with n as a positive integer) is a bounded set with nonempty interior and piecewise smooth boundary, $\mathcal{P}_K$ is a finite-dimensional space of functions on K and $\mathcal{N}_K$ is a set of degrees of freedom (Dofs).

Let K be a triangle. The simplest choice of the Dofs $\mathcal{N}_K$ is the evaluation at all vertices of K. Then the Courant element can be represented by $(K, \mathbb{P}_1(K), \mathcal{N}_K)$. If K is a tetrahedron, the Courant element can be naturally extended to three dimensions as $(K, \mathbb{P}_1(K), \mathcal{N}_K)$ with $\mathcal{N}_K$ involving the evaluation at all vertices of K.

An interesting problem is how to extend the Courant element to a general polytope K in $\mathbb{R}^n$ with $n \geq 2$ a positive integer. In some sense, the most recently developed virtual element method (VEM) offered an answer to this issue (cf. [1, 4, 6, 9]). In fact, if K is a polygon, a finite element $(K, V_1(K), \mathcal{N}_K)$ was introduced in [4], where

$$V_1(K) = \big\{ v \in H^1(K) : \Delta v = 0, \; v|_{\partial K} \in V_1(\partial K) \big\}, \tag{1.1}$$

$$V_1(\partial K) = \big\{ v \in C(\partial K) : \; v|_e \in V_1(e) = \mathbb{P}_1(e), \; \forall\, e \subset \partial K \big\}, \tag{1.2}$$

while the set of Dofs consists of the function values at all vertices of K. If K is a triangle, this finite element is nothing but the Courant element. However, for a general polygon K, its shape function is implicitly defined, so this finite element is named as virtual element. The similar analogue is devised in three dimensions (cf. [1,28]), where the Dofs $\mathcal{N}_K$ also consist of function values at all vertices, and the corresponding shape function space $\overline{V}_1(K)$ is obtained by the harmonic extension from a boundary function belonging to a boundary space $W_1(\partial K)$ using the enhancement technique. Here,

$$\begin{aligned}
\overline{V}_1(K) &= \big\{ v \in H^1(K) \cap C(\overline{K}) : \Delta v = 0, \; v|_{\partial K} \in W_1(\partial K) \big\}, \\
W_1(\partial K) &= \big\{ v \in C(\partial K) : \; v|_F \in W_1(F), \; \forall\, F \subset \partial K \big\}, \\
W_1(F) &= \big\{ v|_F \in \widetilde{V}_1(F) : \; (v, m_F)_F = \big(\Pi_F^{\nabla} v, m_F\big)_F, \; \forall\, m_F \in \mathbb{M}_1(F) \big\}, \\
\widetilde{V}_1(F) &= \big\{ v \in H^1(F) \cap C(\overline{F}) : \Delta_F v \in \mathbb{P}_1(F), \; v|_{\partial F} \in V_1(\partial F) \big\},
\end{aligned}$$

$\Pi_F^{\nabla} v$ is the standard elliptic projection, $\mathbb{M}_1(K)$ is the set of all scaled monomial on a domain K with degree up to 1 and $V_1(\partial F)$ is defined as in (1.2). In addition, applying the enhancement technique to the shape function space on K, one can get another virtual element $(K, W_1(K), \mathcal{N}_K)$ (cf. [1, 5, 8, 12, 21, 28]), where

$$\begin{aligned}
W_1(K) &= \big\{ v \in \widetilde{V}_1(K) : (v, m_K)_K = \big(\Pi_K^{\nabla} v, m_K\big)_K, \; \forall\, m_K \in \mathbb{M}_1(K) \big\}, \\
\widetilde{V}_1(K) &= \big\{ v \in H^1(K) \cap C(\overline{K}) : \Delta v \in \mathbb{P}_1(K), \; v|_{\partial K} \in W_1(\partial K) \big\}.
\end{aligned}$$

After some direct manipulation, one can find that the above two virtual elements also become the standard Courant element when K is a simplex. That means, all of the above elements can be viewed as the extended Courant element on a polytope in two or three dimensions. It is mentioned that Chen *et al.* [12] have proposed and studied an enhanced H^m-conforming virtual element of any degree k with $k \geq m$ in arbitrary dimension technically.

In this paper, we are devoted to proposing and analyzing an extended Courant element which is a direct extension of the virtual element (1.1) in two dimensions. In other words, for an n-dimensional polytope K, the set of Dofs $\mathcal{N}_K$ consists of function values at all vertices of K as before, but the shape function space P_K is obtained by harmonic extension from lower dimensional surface to higher dimensional surface recursively without exploiting the enhancement technique. Though the construction of this element is very simple, the disadvantage is that it is impossible to construct a computable elliptic projection operator in a standard way. Fortunately, one can construct a family of computable quasi-elliptic projection operators (defined in (2.3)-(2.4)) for later requirement. Furthermore, the weak maximum principle (cf. [23, Theorem 8.1]) holds for any function in P_K which implies its use in numerical solution of obstacle problems in three dimensions.

Mathematically, applying a result in [2], we show the previously proposed triple $(K, P_K, \mathcal{N}_K)$ is well-defined. Under some geometric assumptions motivated by [11,13] and following some ideas in [27,28], we derive some fundamental estimates involving inverse estimate, norm equivalence and error estimates for the quasi-elliptic projection. As an application, we use the above element to numerically solve an obstacle problem (see (4.1)) (cf. [24,26,29]). As far as we know, Wang and Wei [32] considered the numerical solution of this problem in two dimensions using the virtual element (1.1) and derived the optimal error estimates. Later on, Feng *et al.* [18] proposed a virtual element method related to (1.1) for solving a frictionless unilateral contact problem in two dimensions and derived its optimal error estimates based on a Céa-type inequality. Here, using the quasi-elliptic operator and following some ideas in [1,4,19], we devise a modified non-consistent VEM (see (4.2)) for the obstacle problem (4.1) in three dimensions. To develop the error analysis, we first derive a new Céa-type inequality for the method (4.2). Then we introduce two interpolation operators and derive their error estimates. Combining all the previous results leads to the optimal error estimate of (4.2) in H^1-norm. Finally, we report some numerical results related to different meshes to verify the theoretical findings.

The rest of this paper is organized as follows. In Section 2, after introducing some notations and the mesh assumptions we need, we define an extended Courant element in n-dimensional ($n \geq 2$) case and introduce a family of quasi-elliptic projection operators for later requirement. In Section 3, we develop some fundamental estimates for the extended Courant element in detail. In Section 4, we construct a non-consistent VEM for the obstacle problem in three dimensions and establish its error estimate. Some numerical results are reported in the last section, corresponding to three types of meshes in three dimensions.

2. An extended Courant element space

2.1. Notations and mesh assumptions

Given a bounded polytope $D \subset \mathbb{R}^n$ $(n \geq 1)$, denote by h_D its diameter. For a non-negative integer s, let $H^s(D)$ be the standard Sobolev space on D, equipped with the norm $\|\cdot\|_{s,D}$ and the semi-norm $|\cdot|_{s,D}$. Let $H_0^s(D)$ denote the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{s,D}$. Set $[H]^n = H \otimes \mathbb{R}^n$ for any function space H. In addition, introduce the following infinity semi-norm:

$$|v|_{s,\infty,D} = \max_{|\boldsymbol{\alpha}|=s} \operatorname*{ess\,sup}_{\boldsymbol{x}\in D} \big|D_w^{\boldsymbol{\alpha}} v\big|,$$

whenever the right-hand side in the above equation makes sense. Here, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of non-negative integers with $|\boldsymbol{\alpha}| = \sum_i \alpha_i$, and $D_w^{\boldsymbol{\alpha}} v$ means the standard weak derivative of v corresponding to the tuple $\boldsymbol{\alpha}$. $(\cdot,\cdot)_D$ symbolizes the standard L^2-inner product on D and the subscript is omitted when $D = \Omega$. As usual, for a non-negative integer $m, \mathbb{P}_m(D)$ denotes the set of all polynomials in D with the total degree no more than m.

For simplicity of presentation, we will omit the infinitesimal volume element for an integral notation when there is no confusion caused. And for any $v \in H^1(D)$, let

$$A_D(v) = \frac{1}{|D|}\int_D v, \quad A_{\partial D}(v) = \frac{1}{|\partial D|}\int_{\partial D} v,$$

where $|D|$ and $|\partial D|$ are the measures of D and ∂D, respectively. Moreover, for any two quantities a and b, use $a \lesssim b$ to indicate $a \leq Cb$ with a hidden positive constant C independent of h_K and K, which may depend on the dimension n and the chunkiness parameter of the domain. In addition, $a \eqsim b$ means $a \lesssim b \lesssim a$.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded polytopal domain with the boundary $\partial\Omega$. Let $\mathcal{T}_h$ be a partition of Ω into nonoverlapping polytopes, with K as a universal element and $h = \max_{K\in\mathcal{T}_h} h_K$. For $r = 1, 2, \ldots, n$, denote by $\mathcal{F}_h^r$ (respectively $\mathcal{F}_{hi}^r$) the set of all (respectively interior) $(n-r)$-dimensional faces of the partition $\mathcal{T}_h$ and $\mathcal{F}_{hb}^r = \mathcal{F}_h^r \backslash \mathcal{F}_{hi}^r$. It is worth noting that the superscript r stands for the codimension of an $(n-r)$-dimensional face F. For any $K \in \mathcal{T}_h$, let $\mathcal{F}^r(K)$ be the set of all $(n-r)$-dimensional faces of K. In particular, $\mathcal{F}_h^0 = \mathcal{T}_h$ and $\mathcal{F}^0(K) = \{K\}$. Similarly, for any face $F \in \mathcal{F}_h^r$ and $s = 1, 2, \ldots, r$, denote by $\mathcal{F}^s(F)$ the set of $(n-r-s)$-dimensional faces of $\overline{F}$. For any vertex δ in $\mathcal{T}_h$, write $\mathcal{T}(\delta)$ as the set consisting of all elements in $\mathcal{T}_h$ with δ as one vertex; for any $K \in \mathcal{T}_h$, write $\mathcal{T}(K)$ as the set consisting of all elements in $\mathcal{T}_h$ which have non-empty intersection with ∂K. As usual, #S means the number of elements in a finite set S. For any $F \in \mathcal{F}^r(K)$ with $1 \leq r \leq n-1$, denote its mutually perpendicular unit normal vectors by $\nu_{F,1}, \cdots, \nu_{F,r}$. Rewrite $\nu_{F,1}$ as ν_F when $r = 1$. As usual, define the surface gradient on F as

$$\nabla_F v = \nabla v - \sum_{i=1}^{r} \frac{\partial v}{\partial \nu_{F,i}} \nu_{F,i},$$

in other words, $\nabla_F v$ is the orthogonal projection of ∇v to the face F. Thereby, define $\Delta_F v = \nabla_F \cdot \nabla_F v$. For each $e \in \mathcal{F}^1(F)$, let $\nu_{F,e}$ be its unit outward normal vector related to F and define $\partial_{\nu_{F,e}} v = \nabla_F v \cdot \nu_{F,e}$.

For later theoretical analysis, we will always require that the polytopal domain K satisfies the following assumptions:

A1. Each polytope $K \in \mathcal{T}_h$ is star-shaped with a ball B_K contained in K and each face $F \in \mathcal{F}_h^r$ with $r = 1, \ldots, n-2$ is star-shape with a ball B_F contained in F. The chunkiness parameters $\gamma_K = h_K/\rho_K$ and $\gamma_F = h_F/\rho_F$ are uniformly bounded, where ρ_K (respectively ρ_F) is the radius of B_K (respectively B_F).

A2. For any $K \in \mathcal{T}_h$, there exists a constant $\eta > 0$ such that $h_K \leq \eta h_F$ for any $F \in \mathcal{F}^r(K)$ with $r = 1, \ldots, n-2$.

2.2. An extended Courant element in higher dimensions

Let $K \in \mathcal{T}_h$ be an n-dimensional ($n \geq 2$) polytope. Since each face $e \in \mathcal{F}^{n-2}(K)$ can be viewed as a two-dimensional polygon equipped with a local orthonormal basis, one can naturally define a virtual element space $V_1(e)$ through (1.1) and (1.2). Then, for each face $F \in \mathcal{F}^r(K)$ with $1 \leq r \leq n-2$, by the harmonic extension of functions on its $(n-r-1)$-dimensional submanifold $e \in \mathcal{F}^1(F)$, define the virtual element space on F as

$$
\begin{aligned}
V_1(F) &= \big\{ v \in H^1(F) : \Delta_F v = 0,\, v|_{\partial F} \in V_1(\partial F) \big\}, \\
V_1(\partial F) &= \big\{ v \in C(\partial F) : v|_e \in V_1(e),\, \forall\, e \in \mathcal{F}^1(F) \big\}.
\end{aligned}
$$

Denote by $\mathcal{V}(F) = \mathcal{F}^{n-r}(F)$ the set of all vertices of F. The related Dofs $\mathcal{N}_F$ are given as follows:

$$
v(\zeta), \quad \forall\, \zeta \in \mathcal{V}(F). \tag{2.1}
$$

Note that $V_1(e) = \mathbb{P}_1(e)$ with $e \in \mathcal{F}^{n-1}(K)$. Then, we can construct the following shape function space recursively:

$$
\begin{aligned}
V_1(K) &= \big\{ v \in H^1(K) : \Delta v = 0,\, v|_{\partial K} \in V_1(\partial K) \big\}, \\
V_1(\partial K) &= \big\{ v \in C(\partial K) : v|_F \in V_1(F),\, \forall\, F \in \mathcal{F}^1(K) \big\}.
\end{aligned}
$$

Obviously, $\mathbb{P}_1(F) \subset V_1(F)$ for any $F \in \mathcal{F}^r(K)$ with $0 \leq r \leq n-2$. The triple $(K, V_1(K), \mathcal{N}_K)$ is exactly the extended Courant element we intend to construct on K. To show the element is well-defined, we first recall some results in [2].

Definition 2.1. *A domain $D \subset \mathbb{R}^n$ $(n \geq 2)$ is said to satisfy the exterior cone condition at ζ relative to a ball $B_r(\zeta)$ if $D^c \cap B_r(\zeta)$ contains a circular cone. In other words, there exists a constant $\kappa > 0$ and a unit vector η such that*

$$
\{x \in B_r(\zeta) : \eta \cdot (x - \zeta) > 0 \ \textit{and} \ |\eta \cdot (x - \zeta)| < \kappa |x - \zeta|\} \subset D^c,
$$

where $B_r(\zeta)$ stands for the usual Euclidean ball in $\mathbb{R}^n$ and D^c stands for the complementary set of D. We say that the domain D satisfies the exterior cone condition if it satisfies the exterior cone condition at each $\zeta \in \partial D$ with respect to some ball $B_{r_\zeta}(\zeta)$ and $\kappa_\zeta > 0$.

Theorem 2.1 ([2, Theorem 11.3]). *Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain that satisfies the exterior cone condition. Moreover, assume that $g \in C(\partial D) \cap H^{1/2}(\partial D)$. Then there exists a unique exact solution $v \in C(\overline{D}) \cap H^1(D)$ satisfying*

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ v = g & \text{on } \partial D. \end{cases} \tag{2.2}$$

According to the Definition 2.1, one can show that any bounded star-shaped polytope satisfies the exterior cone condition.

Lemma 2.1. *If a bounded polytope $K \subset \mathbb{R}^n$ is star-shaped with a ball B_K contained in K, where $n \geq 2$ is an arbitrary integer, then it must satisfy the exterior cone condition.*

Proof. As illustrated in Fig. 1 in two dimensions, the bounded polytope K is star-shaped with respect to a ball B_K centered at x_K. For any $\zeta \in \partial K$, let ℓ_ζ be the straight line passing through ζ and x_K. It is evident that all straight lines connecting ζ and a point in B_K form two cones G_1 and G_2, which have the same vertex ζ and the axis ℓ_ζ. Here, $B_K \subset G_1$. We next show $G_2 \subset K^c$. Otherwise, there exists a certain point $y \in G_2$ such that $y \in K$. According to the definition of G_2, we know that ζ must lie in the line segment $\overline{yz}$ for a certain point $z \in B_K$. However, by the star-shaped property of K, $\overline{yz}$ must lie inside the domain K. This leads to a contradiction. Hence, $G_2 \subset K^c$, and we can derive a required circular cone $G_2 \cap B_r(\zeta)$ for any $r > 0$. The proof is complete. $\square$

Making use of the above lemma, we are able to prove a key result given below.

Lemma 2.2. *For each $F \in \mathcal{F}^r(K)$ with $0 \leq r \leq n-2$, any function in $V_1(F)$ is continuous. Moreover, the element $(F, V_1(F), \mathcal{N}_F)$ is well-defined, or equivalently, $\dim V_1(F) = \#\mathcal{N}_F = \#\mathcal{V}(F)$, and any function in $V_1(F)$ can be uniquely determined by the Dofs* (2.1).

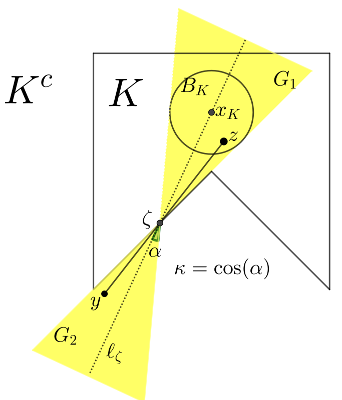


Figure 1: The schema of two dimensional case.

Proof. Since each $F \in \mathcal{F}^r(K)$ is star-shaped by assumption **A1**, it satisfies the exterior cone condition by Lemma 2.1 and thus Theorem 2.1 holds over F. As given in [4], the virtual element defined above is well-posed for $r = n-2$ and its shape functions are continuous by Theorem 2.1. Assume the element $(e, V_1(e), \mathcal{N}_e)$ is well-defined for $e \in \mathcal{F}^l(K)$ with $1 \le l \le n-2$, in other words, any function in $V_1(F)$ can be uniquely determined by the Dofs (2.1) and we can find a set of nodal basis functions $\{\varphi_\zeta^e\}_{\zeta\in\mathcal{V}(e)} \subset V_1(e)$. Next, we are going to prove $\mathrm{dim}V_1(F) = \#\mathcal{N}_F$ for $F \in \mathcal{F}^{l-1}(K)$.

We first show $\mathrm{dim}V_1(F) \ge \#\mathcal{N}_F$. In fact, it suffices to verify that there exists a set of linearly independent functions $\{\varphi_\zeta^F\}_{\zeta\in\mathcal{V}(F)}$ belonging to $V_1(F)$. For any $\zeta\in\mathcal{V}(F)$, let

$$\varphi_\zeta^{\partial F}|_e = \varphi_\zeta^e, \quad \forall\, e \in \mathcal{F}^1(F) \subset \mathcal{F}^l(K), \quad \zeta \in \mathcal{V}(e),$$

and $\varphi_\zeta^{\partial F}|_{\partial F\setminus\overline{\{e\in\mathcal{F}^1(F),\,\zeta\in\mathcal{V}(e)\}}} = 0$. It is easy to know that for $e_1, e_2 \in \mathcal{F}^1(F)$ sharing a common face $\hat{e} \in \mathcal{F}^1(e_1) \cap \mathcal{F}^1(e_2)$, $\varphi_\zeta^{e_1}|_{\hat{e}} = \varphi_\zeta^{e_2}|_{\hat{e}}$, which indicates $\varphi_\zeta^{\partial F}$ is well-defined and $\varphi_\zeta^{\partial F} \in V_1(\partial F) \subset C(\partial F) \cap H^1(\partial F)$. Setting g as $\varphi_\zeta^{\partial F}$ and D as F in (2.2), we have by Theorem 2.1 that there is a function $\varphi_\zeta^F \in C(\overline{F}) \cap H^1(F)$ and $\Delta_F \varphi_\zeta^F = 0$. Thus, $\{\varphi_\zeta^F\}_{\zeta\in\mathcal{V}(F)}$ is a set of linearly independent functions in $C(\overline{F}) \cap V_1(F)$ which indicates $\dim V_1(F) \ge \#\{\varphi_\zeta^F\}_{\zeta\in\mathcal{V}(F)} = \#\mathcal{N}_F$.

On the other hand, if $v \in V_1(F)$ satisfying $v(\zeta) = 0$ for any vertex $\zeta \in \mathcal{V}(F)$, it is evident that $v|_e = 0$ for any $e \in \mathcal{F}^1(F)$. Due to the definition of $V_1(F)$, it turns out that v is a solution of the following boundary value problem:

$$\begin{cases} \Delta_F v = 0 & \text{in } F, \\ v = 0 & \text{on } \partial F, \end{cases}$$

which combined with Theorem 2.1 yields $v = 0$. This implies $\dim V_1(F) \le \#\mathcal{N}_F$.

Moreover, due to the fact that $\{\varphi_\zeta^F\}_{\zeta\in\mathcal{V}(F)}$ is a set of continuous basis functions in $V_1(F)$, we find $V_1(F)$ is a subset of $C(\overline{F})$. Hence, the proof is completed by mathematical induction. ☐

For any function $v \in V_1(F)$, the standard elliptic projection $\Pi_F^\nabla v$ is given by

$$\begin{cases} \left(\nabla_F \Pi_F^\nabla v, \nabla_F p\right)_F = (\nabla_F v, \nabla_F p)_F = \sum\limits_{e\in\mathcal{F}^1(F)} (v, \partial_{\nu_{F,e}} p)_e, \quad \forall\, p \in \mathbb{P}_1(F), \\ \left(\Pi_F^\nabla v, 1\right)_{\partial F} = (v, 1)_{\partial F}. \end{cases}$$

However, for all $F \in \mathcal{F}^r(K)$ with $0 \le r \le n-3$, since the moments of v on any $(n-r-1)$ -dimensional faces are unknown, we cannot determine the right-hand side quantities on the above equations with respect to the Dofs of v. This means such construction is uncomputable. Hence, we turn to introduce a set of quasi-elliptic projection operators $\{\widehat{\Pi}_F^\nabla\}$ as follows. Set $\widehat{\Pi}_e^\nabla v = \Pi_e^\nabla v$ for any $v \in V_1(e)$ with $e \in \mathcal{F}^{n-2}(K)$ since $\Pi_e^\nabla v$ is computable and for any $v \in V_1(F)$, let $\widehat{\Pi}_F^{\nabla} v$ be the solution of

$$\begin{cases} \big(\nabla_F \widehat{\Pi}_F^{\nabla} v, \nabla_F p\big)_F = \sum_{e\in\mathcal{F}^1(F)} \big(\widehat{\Pi}_e^{\nabla} v, \partial_{\nu_{F,e}} p\big)_e, \quad \forall p \in \mathbb{P}_1(F), & (2.3)\\ \big(\widehat{\Pi}_F^{\nabla} v, 1\big)_{\partial F} = \sum_{e\in\mathcal{F}^1(F)} \big(\widehat{\Pi}_e^{\nabla} v, 1\big)_e, & (2.4)\end{cases}$$

where $F \in \mathcal{F}^r(K)$ with $0 \le r \le n-3$. Also, it is easy to check $\widehat{\Pi}_F^{\nabla} p = p$ for $p \in \mathbb{P}_1(F)$.

3. Some fundamental estimates

At the beginning, let us recall some basic results for later requirement. As given in [11, 13] there holds

$$\|v\|_{0,\partial K} \lesssim \varepsilon h_K^{\frac{1}{2}} |v|_{1,K} + C(\varepsilon) h_K^{-\frac{1}{2}} \|v\|_{0,K}, \quad \forall v \in H^1(K), \tag{3.1}$$

where $\varepsilon > 0$ is any given constant and $C(\varepsilon)$ is a positive constant depending on ε.

Using the scaling argument as given in [11], we can obtain the following lemma.

Lemma 3.1. *The following results hold true:*

(a) The scaled Poincaré-Friedrichs inequality

$$\|v\|_{0,K} \lesssim h_K |v|_{1,K} + h_K^{1-\frac{n}{2}} \left| \int_{\partial K} v \right|, \quad \forall v \in H^1(K).$$

(b) If $n \ge 3$, there holds

$$|v|_{\frac{1}{2},\partial K} \lesssim h_K^{\frac{1}{2}} |v|_{1,\partial K}, \quad \forall v \in H^1(\partial K). \tag{3.2}$$

In addition, for any $v \in H^{1/2}(\partial K)$, there exists a function $\tilde{v} \in H^1(K)$ such that

$$\tilde{v}|_{\partial K} = v|_{\partial K}, \quad |\tilde{v}|_{1,K} \lesssim |v|_{\frac{1}{2},\partial K}. \tag{3.3}$$

Proof. We first derive the scaled Poincaré-Friedrichs inequality. Owing to the geometric assumptions on K, it follows from the Dupont-Scott theory that

$$\|v - A_K(v)\|_{0,K} = \inf_{w\in\mathbb{P}_0(K)} \|v - w\|_{0,K} \lesssim h_K |v|_{1,K}, \quad \forall v \in H^1(K). \tag{3.4}$$

This along with (3.1) and the Cauchy-Schwarz inequality gives

$$\begin{aligned}\|v\|_{0,K} &\lesssim \|v - A_K(v)\|_{0,K} + \|A_K(v)\|_{0,K}\\ &\lesssim h_K |v|_{1,K} + h_K^{\frac{n}{2}} \big|A_{\partial K}\big(A_K(v)\big)\big|\\ &\lesssim h_K |v|_{1,K} + h_K^{\frac{1}{2}} \|v - A_K(v)\|_{0,\partial K} + h_K^{\frac{n}{2}} |A_{\partial K}(v)|\\ &\lesssim h_K |v|_{1,K} + \|v - A_K(v)\|_{0,K} + h_K^{\frac{n}{2}} |A_{\partial K}(v)|\\ &\lesssim h_K |v|_{1,K} + h_K^{1-\frac{n}{2}} \left| \int_{\partial K} v \right|.\end{aligned}$$

On the other hand, if $n \geq 3$, noting that K is star-shaped with a ball B_K with the radius $\rho_K \eqsim h_K$, we have from [31] that there exists a Lipschitz homeomorphism $\phi : B_K \to K$ such that $|\phi|_{1,\infty,B_K}$ and $|\phi^{-1}|_{1,\infty,B_K}$ are uniformly bounded with respect to h_K. Hence, we easily have

$$|v|_{1,\partial K} \eqsim |\hat{v}|_{1,\partial B_K}, \quad |v|_{\frac{1}{2},\partial K} \eqsim |\hat{v}|_{\frac{1}{2},\partial B_K}, \tag{3.5}$$

where $\hat{v} = v \circ \phi$ and $v = \hat{v} \circ \phi^{-1}$. For any $\hat{v} \in H^1(\partial B_K)$ with $v \in H^1(\partial K)$, following the arguments for deriving (2.16) in [11], we have

$$\begin{aligned} |\hat{v}|_{\frac{1}{2},\partial B_K} &= |\hat{v} - A_{\partial B_K}(\hat{v})|_{\frac{1}{2},\partial B_K} \\ &\lesssim h_K^{-\frac{1}{2}} \|\hat{v} - A_{\partial B_K}(\hat{v})\|_{0,\partial B_K} + h_K^{\frac{1}{2}} |\hat{v}|_{1,\partial B_K} \\ &\lesssim h_K^{\frac{1}{2}} |\hat{v}|_{1,\partial B_K}, \end{aligned}$$

which along with (3.5) yields (3.2). It is evident that the last result (3.3) can be obtained by using the inverse trace theorem in smooth domains [33, Theorem 8.8] and the similar arguments for proving (3.2). ☐

3.1. Inverse estimate

In the subsection, we focus on deriving the inverse estimate on the extended Courant element by mathematical induction.

Theorem 3.1. *There holds*

$$|v|_{1,F} \lesssim h_F^{-1} \|v\|_{0,F}, \quad \forall v \in V_1(F), \quad F \in \mathcal{F}^r(K)$$

with $0 \leq r \leq n-2$.

Proof. We prove the result recursively with respect to r. If $r = n-2$, the result naturally holds from [13, Theorem 3.6]. Assume it is true for $r = l$ with $1 \leq l \leq n-2$, in other words, there holds

$$|v|_{1,e} \lesssim h_e^{-1} \|v\|_{0,e}, \quad \forall v \in V_1(e), \quad e \in \mathcal{F}^l(K). \tag{3.6}$$

We want to prove the above result still holds for $r = l-1$. For any $v \in V_1(F)$ with $F \in \mathcal{F}^{l-1}(K)$, let $\tilde{v}$ be the function satisfying (3.3). Since $\Delta v = 0$, by the principle of energy minimization for harmonic functions and Lemma 3.1(b), we have

$$|v|_{1,F} \lesssim |\tilde{v}|_{1,F} \lesssim |v|_{\frac{1}{2},\partial F} \lesssim h_F^{\frac{1}{2}} |v|_{1,\partial F}. \tag{3.7}$$

Then, it follows from (3.1) and (3.6) that

$$|v|_{1,F} \lesssim h_F^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{F}^1(F)} |v|_{1,e}^2 \Big)^{\frac{1}{2}} \lesssim h_F^{-\frac{1}{2}} \|v\|_{0,\partial F} \lesssim \varepsilon |v|_{1,F} + C(\varepsilon) h_F^{-1} \|v\|_{0,F}. \tag{3.8}$$

Applying the absorbing technique to (3.8), we have

$$|v|_{1,F} \lesssim h_F^{-1}\|v\|_{0,F}, \quad \forall v \in V_1(F), \quad F \in \mathcal{F}^{l-1}(K).$$

Consequently, we derive the desired inequality by the principle of mathematical induction. □

3.2. Norm equivalence

In this subsection, we will establish the norm equivalence between the L^2-norm (or H^1-seminorm) of a function on the extended Courant element and the ℓ^2-norm of its Dofs. We will also derive the H^1 stability of the quasi-elliptic projection defined by (2.3)-(2.4).

Theorem 3.2. *For all $v \in V_1(F)$ with $F \in \mathcal{F}^r(K), 0 \le r \le n-2$, there holds*

$$\|v\|_{0,F} \eqsim h_F^{\frac{n-r}{2}}\Big(\sum_{\zeta\in\mathcal{V}(F)}|v(\zeta)|^2\Big)^{\frac{1}{2}}. \tag{3.9}$$

Proof. The inequality holds for $r = n-2$ by [13, Theorem 4.5]. Suppose (3.9) holds for $r = l$ with $1 \le l \le n-2$ and it is sufficient to show

$$\|v\|_{0,F} \eqsim h_F^{\frac{n-l+1}{2}}\Big(\sum_{\zeta\in\mathcal{V}(F)}|v(\zeta)|^2\Big)^{\frac{1}{2}}, \quad \forall v \in V_1(F), \quad F \in \mathcal{F}^{l-1}(K). \tag{3.10}$$

By (3.1) and the inverse estimate in Theorem 3.1,

$$\begin{aligned} h_F^{\frac{n-l+1}{2}}\Big(\sum_{\zeta\in\mathcal{V}(F)}|v(\zeta)|^2\Big)^{\frac{1}{2}} &\lesssim h_F^{\frac{n-l+1}{2}}\Big(\sum_{e\in\mathcal{F}^1(F)}\sum_{\zeta\in\mathcal{V}(e)}|v(\zeta)|^2\Big)^{\frac{1}{2}} \\ &\lesssim h_F^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{F}^1(F)}\|v\|_{0,e}^2\Big)^{\frac{1}{2}} = h_F^{\frac{1}{2}}\|v\|_{0,\partial F} \\ &\lesssim h_F|v|_{1,F} + \|v\|_{0,F} \lesssim \|v\|_{0,F}. \end{aligned}$$

Furthermore, employing the similar arguments for deriving (3.7) and (3.8), there exists a function $\tilde{v}$ such that

$$\tilde{v}|_{\partial F} = v|_{\partial F}, \quad |v|_{1,F} \lesssim |\tilde{v}|_{1,F} \lesssim h_F^{-\frac{1}{2}}\|v\|_{0,\partial F}.$$

Therefore, combining the Poincaré-Friedrichs inequality and the norm equivalence for functions in $V_1(e)$ with $e \in \mathcal{F}^l(K)$, we achieve the upper bound of (3.10)

$$\|v\|_{0,F} \lesssim h_F|v|_{1,F} + h_F^{\frac{1}{2}}\|v\|_{0,\partial F} \lesssim h_F^{\frac{1}{2}}\|v\|_{0,\partial F} \lesssim h_F^{\frac{n-l+1}{2}}\Big(\sum_{\zeta\in\mathcal{V}(F)}|v(\zeta)|^2\Big)^{\frac{1}{2}}.$$

Thus, the proof is completed by mathematical induction. □

Theorem 3.3. *For each $F \in \mathcal{F}^r(K)$, $0 \le r \le n-2$, the quasi-elliptic projection operator $\widehat{\Pi}_F^\nabla$ has the following H^1 stability:*

$$\big|\widehat{\Pi}_F^\nabla v\big|_{1,F} \lesssim |v|_{1,F}, \quad \forall v \in V_1(F). \tag{3.11}$$

Moreover, there holds

$$h_F\big|v - \widehat{\Pi}_F^\nabla v\big|_{1,F} \approx h_F^{\frac{n-r}{2}} \Big(\sum_{\zeta \in \mathcal{V}(F)} \big|v(\zeta) - \widehat{\Pi}_F^\nabla v(\zeta)\big|^2 \Big)^{\frac{1}{2}}, \quad \forall v \in V_1(F). \tag{3.12}$$

Proof. Since we choose the standard elliptic projection operator when $r = n-2$, the H^1 stability and the norm equivalence (3.12) hold from [13]. Assume the projection operator $\widehat{\Pi}_e^\nabla$ possesses the H^1 stability and there holds

$$h_e\big|v - \widehat{\Pi}_e^\nabla v\big|_{1,e} \approx h_e^{\frac{n-l}{2}} \Big(\sum_{\zeta \in \mathcal{V}(e)} \big|v(\zeta) - \widehat{\Pi}_e^\nabla v(\zeta)\big|^2 \Big)^{\frac{1}{2}} \tag{3.13}$$

for $e \in \mathcal{F}^l(K)$ with $1 \le l \le n-2$. We want to prove these results still hold for $r = l-1$. We first show the operator $\widehat{\Pi}_F^\nabla$ has the H^1 stability for $F \in \mathcal{F}^{l-1}(K)$. Recalling the definitions of Π_F^∇ and $\widehat{\Pi}_F^\nabla$, perform an integration by parts to get

$$\begin{aligned}
\big|\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big|_{1,F}^2 &= \sum_{e \in \mathcal{F}^1(F)} \big(\partial_{\nu_{F,e}}\big(\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big), \widehat{\Pi}_e^\nabla v - v\big)_e \\
&\lesssim \Big(\sum_{e \in \mathcal{F}^1(F)} \big\|\partial_{\nu_{F,e}}\big(\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big)\big\|_{0,e}^2 \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\sum_{e \in \mathcal{F}^1(F)} \big\|\widehat{\Pi}_e^\nabla v - v\big\|_{0,e}^2 \Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.14}$$

It follows from the inverse estimate of polynomials that

$$\begin{aligned}
&\sum_{e \in \mathcal{F}^1(F)} \big\|\partial_{\nu_{F,e}}\big(\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big)\big\|_{0,e}^2 \\
\lesssim &\sum_{e \in \mathcal{F}^1(F)} |e| \big\|\partial_{\nu_{F,e}}\big(\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big)\big\|_{0,\infty,e}^2 \\
\lesssim &\, |\partial F| \big\|\nabla_F\big(\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big)\big\|_{0,\infty,F}^2 \\
\lesssim &\, h_F^{n-l} \big|\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big|_{1,\infty,F}^2 \\
\lesssim &\, h_F^{-1} \big|\widehat{\Pi}_F^\nabla v - \Pi_F^\nabla v\big|_{1,F}^2.
\end{aligned} \tag{3.15}$$

We also have by the norm equivalence in Theorem 3.2 and (3.13) that for all $e \in \mathcal{F}^1(F)$,

$$\begin{aligned}
\big\|\widehat{\Pi}_e^\nabla v - v\big\|_{0,e}^2 &\lesssim h_e^{n-l} \sum_{\zeta \in \mathcal{V}(e)} \big|v(\zeta) - \widehat{\Pi}_e^\nabla v(\zeta)\big|^2 \\
&\lesssim h_e^2 \big|\widehat{\Pi}_e^\nabla v - v\big|_{1,e}^2.
\end{aligned} \tag{3.16}$$

Thus, inserting (3.15)-(3.16) into (3.14) and using the H^1 stability of $\widehat{\Pi}_e^{\nabla}$ and Π_F^{∇}, we know

$$\begin{aligned}\big|\widehat{\Pi}_F^{\nabla}v\big|_{1,F} &\lesssim \big|\widehat{\Pi}_F^{\nabla}v-\Pi_F^{\nabla}v\big|_{1,F}+\big|\Pi_F^{\nabla}v\big|_{1,F}\\ &\lesssim h_F^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{F}^1(F)}|\widehat{\Pi}_e^{\nabla}v-v|_{1,e}^2\bigg)^{\frac{1}{2}}+|v|_{1,F}\\ &\lesssim h_F^{\frac{1}{2}}|v|_{1,\partial F}+|v|_{1,F}. \end{aligned}\tag{3.17}$$

In addition, since $(v-A_F(v))|_e\in V_1(e)$, we have by (3.1), (3.17) and the inverse estimate in Theorem 3.1 that

$$\begin{aligned}\big|\widehat{\Pi}_F^{\nabla}v\big|_{1,F} &\lesssim h_F^{\frac{1}{2}}|v-A_F(v)|_{1,\partial F}+|v|_{1,F}\\ &\lesssim h_F^{-\frac{1}{2}}\|v-A_F(v)\|_{0,\partial F}+|v|_{1,F}\\ &\lesssim h_F^{-1}\|v-A_F(v)\|_{0,F}+|v|_{1,F}. \end{aligned}\tag{3.18}$$

Employing the similar arguments for deriving (3.4), we easily have

$$\|v-A_F(v)\|_{0,F}\lesssim h_F|v|_{1,F},\quad \forall v\in H^1(F).\tag{3.19}$$

The combination of (3.18) and (3.19) leads to (3.11) for $F\in\mathcal{F}^{l-1}(K)$ directly.

It remains to prove (3.12) for $F\in\mathcal{F}^{l-1}(K)$. From the Poincaré-Friedrichs inequality, (2.4) and the norm equivalence in Theorem 3.2, it follows that

$$\begin{aligned}&h_F^{\frac{n-l+1}{2}}\bigg(\sum_{\zeta\in\mathcal{V}(F)}\big|v(\zeta)-\widehat{\Pi}_F^{\nabla}v(\zeta)\big|^2\bigg)^{\frac{1}{2}}\\ &\lesssim \big\|v-\widehat{\Pi}_F^{\nabla}v\big\|_{0,F}\lesssim h_F\big|v-\widehat{\Pi}_F^{\nabla}v\big|_{1,F}+h_F^{\frac{1-n+l}{2}}\bigg|\int_{\partial F}\big(v-\widehat{\Pi}_F^{\nabla}v\big)\bigg|\\ &\lesssim h_F\big|v-\widehat{\Pi}_F^{\nabla}v\big|_{1,F}+h_F^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{F}^1(F)}\big\|v-\widehat{\Pi}_e^{\nabla}v\big\|_{0,e}^2\bigg)^{\frac{1}{2}}. \end{aligned}\tag{3.20}$$

Since $\widehat{\Pi}_e^{\nabla}(\widehat{\Pi}_F^{\nabla}v)|_e=(\widehat{\Pi}_F^{\nabla}v)|_e$ for any $v\in V_1(F)$, it follows from (3.16) and the H^1 stability of $\widehat{\Pi}_e^{\nabla}$ that

$$\begin{aligned}h_F^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{F}^1(F)}\big\|v-\widehat{\Pi}_e^{\nabla}v\big\|_{0,e}^2\bigg)^{\frac{1}{2}} &\lesssim h_F^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{F}^1(F)}h_e^2\big|\big(v-\widehat{\Pi}_F^{\nabla}v\big)-\widehat{\Pi}_e^{\nabla}\big(v-\widehat{\Pi}_F^{\nabla}v\big)\big|_{1,e}^2\bigg)^{\frac{1}{2}}\\ &\lesssim h_F^{\frac{3}{2}}\big|v-\widehat{\Pi}_F^{\nabla}v\big|_{1,\partial F}. \end{aligned}$$

Then, applying the similar arguments for deriving (3.18), we further have

$$h_F^{\frac{1}{2}}\bigg(\sum_{e\in\mathcal{F}^1(F)}\big\|v-\widehat{\Pi}_e^{\nabla}v\big\|_{0,e}^2\bigg)^{\frac{1}{2}}\lesssim h_F^{\frac{3}{2}}\big|v-\widehat{\Pi}_F^{\nabla}v-A_F\big(v-\widehat{\Pi}_F^{\nabla}v\big)\big|_{1,\partial F}$$

$$\lesssim h_F^{\frac{1}{2}} \big\| v - \widehat{\Pi}_F^{\nabla} v - A_F\big(v - \widehat{\Pi}_F^{\nabla} v\big) \big\|_{0,\partial F}$$
$$\lesssim \big\| v - \widehat{\Pi}_F^{\nabla} v - A_F\big(v - \widehat{\Pi}_F^{\nabla} v\big) \big\|_{0,F}.$$

This together with (3.19) and (3.20) gives rise to the lower bound of (3.12) for $r = l-1$ directly.

The upper bound of (3.12) for $r = l-1$ can be derived by the inverse inequality in Theorem 3.1 and the norm equivalence in Theorem 3.2 easily

$$h_F \big| v - \widehat{\Pi}_F^{\nabla} v \big|_{1,F} \lesssim \big\| v - \widehat{\Pi}_F^{\nabla} v \big\|_{0,F} \lesssim h_F^{\frac{n-r}{2}} \bigg(\sum_{\zeta \in \mathcal{V}(F)} \big| v(\zeta) - \widehat{\Pi}_F^{\nabla} v(\zeta) \big|^2 \bigg)^{\frac{1}{2}}.$$

Therefore, the proof is completed by mathematical induction. □

The following result gives the error estimate for the quasi-elliptic operators.

Lemma 3.2. *For any $F \in \mathcal{F}^r(K)$ with $0 \le r \le n-2$, there holds*

$$\big| v - \widehat{\Pi}_F^{\nabla} v \big|_{t,F} \lesssim h_F^{m-t} |v|_{m,F}, \quad \forall v \in V_1(F), \quad 0 \le t \le m \le 1,$$

where $\widehat{\Pi}_F^{\nabla} v$ is defined in (2.3)-(2.4).

Proof. It follows from (3.9), (3.11) and (3.12) that

$$\big\| v - \widehat{\Pi}_F^{\nabla} v \big\|_{0,F} \lesssim h_F \big| v - \widehat{\Pi}_F^{\nabla} v \big|_{1,F} \lesssim h_F |v|_{1,F}.$$

Moreover, by the inverse estimate in Theorem 3.1, we see that

$$\big\| v - \widehat{\Pi}_F^{\nabla} v \big\|_{0,F} \lesssim h_F |v|_{1,F} \lesssim \|v\|_{0,F}.$$

The proof is complete. □

4. A non-consistent VEM for the obstacle problem in three dimensions

In this section, we propose a non-consistent VEM in the lowest-order case for the simplest elliptic obstacle problem in three dimensions.

4.1. An obstacle problem and its discretization

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain. Let $\phi \in H^1(\Omega) \cap C(\overline{\Omega})$ with $\phi \le 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$. Then the obstacle problem we want to study is to find $u \in X := \{v \in H_0^1(\Omega) : v \ge \phi \text{ a.e. in } \Omega\}$ such that

$$a(u, v-u) \ge \langle f, v-u \rangle, \quad \forall v \in X, \tag{4.1}$$

where

$$a(u, v-u) := (\nabla u, \nabla v - \nabla u), \quad \langle f, v-u\rangle = (f, v-u).$$

Notice that for a given polyhedron partition $\mathcal{T}_h$ of Ω satisfying the mesh assumptions **A1**-**A2**, the bilinear form can be decomposed as

$$a(u, v-u) = \sum_{K\in\mathcal{T}_h} a^K(u, v-u) = \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla v - \nabla u)_K.$$

The unique solvability of problem (4.1) has been shown in [30].

Imitating the discrete form in [4, 7], set

$$a_h^K(v,w) := a^K\big(\widehat{\Pi}_K^\nabla v, \widehat{\Pi}_K^\nabla w\big) + S^K\big(v - \widehat{\Pi}_K^\nabla v, w - \widehat{\Pi}_K^\nabla w\big),$$

and the stabilization term reads as

$$S^K\big(v - \widehat{\Pi}_K^\nabla v, w - \widehat{\Pi}_K^\nabla w\big) = h_K^{n-2} \sum_{\zeta\in\mathcal{V}(K)} \big[\big(v - \widehat{\Pi}_K^\nabla v\big)\big(w - \widehat{\Pi}_K^\nabla w\big)\big](\zeta).$$

Define

$$\begin{aligned} V_h &= \big\{v \in H^1(\Omega)\cap C(\overline{\Omega}) : v|_K \in V_1(K),\ \forall\, K \in \mathcal{T}_h\big\}, \\ X_h &= \big\{v \in V_h \cap H_0^1(\Omega) : v(\delta) \geq \phi(\delta),\ \forall\, \delta \in \mathcal{V}^i(\mathcal{T}_h) \text{ and } v(\delta) = 0,\ \forall\, \delta \in \mathcal{V}^\partial(\mathcal{T}_h)\big\}. \end{aligned}$$

Then, a virtual element method for solving (4.1) is to find $u_h \in X_h$ such that

$$a_h(u_h, v_h - u_h) \geq \langle f_h, v_h - u_h\rangle, \quad \forall v_h \in X_h, \tag{4.2}$$

where

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} a_h^K(u_h, v_h), \quad \langle f_h, v_h\rangle = \sum_{K\in\mathcal{T}_h} \big(f, \widehat{\Pi}_K^\nabla v_h\big)_K.$$

Obviously, $a_h^K(v,w) = a^K(\widehat{\Pi}_K^\nabla v, w)$ for $w \in \mathbb{P}_1(K)$ and the following stability holds.

Lemma 4.1. *There holds*

$$a^K(v_h, v_h) \lesssim a_h^K(v_h, v_h) \lesssim a^K(v_h, v_h)$$

for all $v_h \in V_1(K)$.

Proof. By the norm equivalence in Theorem 3.3, for all $v_h \in V_1(K)$,

$$\begin{aligned} & a^K\big(v_h - \widehat{\Pi}_K^\nabla v_h, v_h - \widehat{\Pi}_K^\nabla v_h\big) \\ \lesssim\ & S^K\big(v_h - \widehat{\Pi}_K^\nabla v_h, v_h - \widehat{\Pi}_K^\nabla v_h\big) \\ \lesssim\ & a^K\big(v_h - \widehat{\Pi}_K^\nabla v_h, v_h - \widehat{\Pi}_K^\nabla v_h\big). \end{aligned}$$

Then, combining the Cauchy-Schwarz inequality, the definitions of $a^K(v_h, v_h)$ and Lemma 3.2, we can readily derive the desired result. ☐

4.2. Some interpolation operators

We begin with an interpolation operator $I_K : H^1(K) \cap C(\overline{K}) \to V(K)$, which is determined by the Dofs (2.1) (cf. [4])

$$I_K v(\zeta) = v(\zeta), \quad \forall \zeta \in \mathcal{V}(K), \tag{4.3}$$

and denote the related elementwise defined global operator by I_h. For any $v \in X \cap C(\overline{\Omega})$, one can easily know that $I_h v \in X_h$. By (3.4), the norm equivalence in Theorem 3.2, the inverse inequality in Theorem 3.1 and Lemma 3.2, it follows that

$$\|v - I_K v\|_{0,K} + h_K |v - I_K v|_{1,K} \lesssim h_K^2 |v|_{2,K}, \tag{4.4}$$

$$\left\|v - \widehat{\Pi}_K^{\nabla} I_K v\right\|_{0,K} + h_K \left|v - \widehat{\Pi}_K^{\nabla} I_K v\right|_{1,K} \lesssim h_K^2 |v|_{2,K}. \tag{4.5}$$

For further analysis, we construct a weak interpolation operator $Q_h : H^1(\Omega) \to V_h$ with the help of the standard L^2 orthogonal projection operator $\Pi_K^1 : H^1(K) \to \mathbb{P}_1(K)$. The Dofs of $Q_h v$ are defined by

$$Q_h v(\zeta) = \frac{1}{\#\mathcal{T}(\zeta)} \sum_{K \in \mathcal{T}(\zeta)} \Pi_h^1 v|_K(\zeta), \quad \forall \zeta \in \mathcal{F}_h^3,$$

where $\Pi_h^1 \cdot |_K := \Pi_K^1 \cdot$.

Theorem 4.1. *For any $v \in H^1(\Omega)$ and $K \in \mathcal{T}_h$, the following error estimates hold:*

$$h_K^{-1} \|v - Q_h v\|_{0,K} + |v - Q_h v|_{1,K} \lesssim \sum_{K' \in \mathcal{T}(K)} |v|_{1,K'}, \tag{4.6}$$

$$h_K^{-1} \left\|v - \widehat{\Pi}_K^{\nabla} Q_h v\right\|_{0,K} + \left|v - \widehat{\Pi}_K^{\nabla} Q_h v\right|_{1,K} \lesssim \sum_{K' \in \mathcal{T}(K)} |v|_{1,K'}. \tag{4.7}$$

Proof. Write $[\![v]\!]|_F = v|_{K_1} - v|_{K_2}$, where $F \in \mathcal{F}_{hi}^1$ is a two dimensional face shared by two adjacent elements K_1 and K_2 which are labelled in a certain way. Since $(\Pi_h^1 v - Q_h v)|_K \in V_1(K)$, it follows from the norm equivalence in Theorem 3.2 and the scaling argument that

$$\begin{aligned}
\left\|\Pi_h^1 v - Q_h v\right\|_{0,K}^2 &\lesssim h_K^3 \sum_{\zeta \in \mathcal{V}(K)} \frac{1}{\#\mathcal{T}(\zeta)} \sum_{K' \in \mathcal{T}(\zeta)} \left|\Pi_K^1 v(\zeta) - \Pi_{K'}^1 v(\zeta)\right|^2 \\
&\lesssim h_K^3 \sum_{\zeta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}_{hi}^1, \zeta \in \mathcal{V}(F)} \left\| [\![\Pi_h^1 v]\!] |_F \right\|_{\infty,F}^2 \\
&\lesssim h_K \sum_{\zeta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}_{hi}^1, \zeta \in \mathcal{V}(F)} \left\| [\![\Pi_h^1 v]\!] |_F \right\|_{0,F}^2.
\end{aligned}$$

This along with the triangle inequality and (3.1) yields

$$\|v - Q_h v\|_{0,K} \lesssim \left\|v - \Pi_K^1 v\right\|_{0,K} + \sum_{K' \in \mathcal{T}(K)} h_{K'}^{\frac{1}{2}} \left\|v - \Pi_{K'}^1 v\right\|_{0,\partial K'}$$

$$\lesssim \sum_{K'\in\mathcal{T}(K)} \left(\left\| v - \Pi^1_{K'} v \right\|_{0,K'} + h_{K'} \left| v - \Pi^1_{K'} v \right|_{1,K'} \right).$$

By the triangle inequality and the inverse estimate in Theorem 3.1,

$$h_K |v - Q_h v|_{1,K} \lesssim h_K \left| v - \Pi^1_K v \right|_{1,K} + \left\| \Pi^1_K v - Q_h v \right\|_{0,K}.$$

Therefore, combining the last two inequalities and the standard error estimates for Π^1_K leads to the first estimate (4.6).

Moreover, a direct manipulation shows

$$\left\| v - \widehat{\Pi}^{\nabla}_K Q_h v \right\|_{m,K} \le \left\| v - \Pi^1_K v \right\|_{m,K} + \left\| \widehat{\Pi}^{\nabla}_K \left(\Pi^1_K v - Q_h v \right) \right\|_{m,K}, \quad m = 0, 1.$$

Hence, the estimate (4.7) can be derived by (4.6) and Lemma 3.2 combined with the standard error estimates for Π^1_K. ☐

When there is no confusion caused, we will use Q_h to denote its vector analogue for simplicity.

4.3. The a priori error estimate

We next derive a Céa-type inequality for our non-consistent VEM following some ideas in [17, 18]. From now on, we simply write $|\cdot|_{1,h} = \sum_{K\in\mathcal{T}_h} |\cdot|_{1,K}$.

Lemma 4.2. *Let u and u_h be the solution of* (4.1) *and* (4.2)*, respectively. Then, for any $v \in X, u_I \in X_h$ and piecewise linear polynomial u_π, there holds*

$$|u - u_h|_1 \lesssim |u - u_I|_1 + |u - u_\pi|_{1,h} + \|f - f_h\|_{X_h^*} + E_h + |R_u(u_I, u)|^{\frac{1}{2}} + |R_u(v, u_h)|^{\frac{1}{2}},$$

where

$$E_h = \sup_{w\in X_h} \left| \sum_{K\in\mathcal{T}_h} a^K\left(u, \widehat{\Pi}^{\nabla}_K w\right) - a(u, w) \right| \Big/ |w|_1,$$

$$R_u(w_1, w_2) = a(u, w_1 - w_2) - (f, w_1 - w_2), \quad \forall w_1, w_2 \in H^1(\Omega),$$

and X_h^* denotes the dual space of X_h.

Proof. Let $\varepsilon = u_I - u_h$. We have by Lemma 4.1 and (4.2) that

$$\begin{aligned} |\varepsilon|_1^2 &\lesssim a_h(\varepsilon, \varepsilon) = a_h(u_I, \varepsilon) - a_h(u_h, \varepsilon) \le a_h(u_I, \varepsilon) - \langle f_h, \varepsilon \rangle \\ &= \sum_{K\in\mathcal{T}_h} a_h^K(u_I - u_\pi, \varepsilon) + \sum_{K\in\mathcal{T}_h} a^K\left(u_\pi - u, \widehat{\Pi}^{\nabla}_K \varepsilon\right) + (f - f_h, \varepsilon) \\ &\quad + \sum_{K\in\mathcal{T}_h} a^K\left(u, \widehat{\Pi}^{\nabla}_K \varepsilon\right) - a(u, \varepsilon) + a(u, \varepsilon) - (f, \varepsilon). \end{aligned} \tag{4.8}$$

Employing a simple manipulation and using (4.1), one easily gets

$$\begin{aligned} a(u,\varepsilon)-(f,\varepsilon) &= R_u(u_I,u)+R_u(v,u_h)+a(u,u-v)-(f,u-v) \\ &\le R_u(u_I,u)+R_u(v,u_h) \end{aligned} \tag{4.9}$$

for any $v \in X$. Hence, plugging (4.9) into (4.8) and using (3.12), we have

$$\begin{aligned} |\varepsilon|_1^2 \le C\bigg(&|u-u_I|_1+|u-u_\pi|_{1,h}+\|f-f_h\|_{X_h^*}+\frac{\sum_{K\in\mathcal{T}_h} a^K\big(u,\widehat{\Pi}_K^\nabla\varepsilon\big)-a(u,\varepsilon)}{|\varepsilon|_1}\bigg)|\varepsilon|_1 \\ &+|R_u(u_I,u)|+|R_u(v,u_h)| \end{aligned} \tag{4.10}$$

with a generic constant C independent of h_K and K. Finally, applying Young's inequality and the absorbing technique to (4.10), one has the upper bound for error $u-u_h$ by the triangle inequality. □

Remark 4.1. Observing that the k-consistency

$$a_h^K(p,v_h)=a^K(p,v_h), \quad \forall p\in\mathbb{P}_1(K)$$

does not hold for all $v_h \in X_h$, we cannot directly use the Céa-type inequality in [18, Theorem 2.4] for error analysis of the method (4.2).

Lemma 4.3. *There holds*

$$E_h \lesssim h|u|_2.$$

Proof. Perform an integration by parts to get

$$a^K\big(u,\widehat{\Pi}_K^\nabla w\big)-a^K(u,w)=\sum_{F\in\mathcal{F}^1(K)}\big(\nabla u\cdot\nu_F,\widehat{\Pi}_K^\nabla w-w\big)_F, \quad \forall w\in X_h, \quad K\in\mathcal{T}_h. \tag{4.11}$$

Since $\widehat{\Pi}_F^\nabla w|_F = w|_F = 0$ for $F\in\mathcal{F}_{hb}^1$, we have for any $K\in\mathcal{T}_h$ that

$$\begin{aligned} &\sum_{F\in\mathcal{F}^1(K)}\big(\nabla u\cdot\nu_F, w-\widehat{\Pi}_K^\nabla w\big)_F \\ =&\sum_{F\in\mathcal{F}^1(K)}\big(\nabla u\cdot\nu_F,\widehat{\Pi}_F^\nabla w-\widehat{\Pi}_K^\nabla w\big)_F \\ &+\sum_{F\in\mathcal{F}^1(K)\cap\mathcal{F}_{hi}^1}\big(\nabla u\cdot\nu_F, w-\widehat{\Pi}_F^\nabla w\big)_F. \end{aligned} \tag{4.12}$$

Let $p=A_K(\nabla u)\cdot\boldsymbol{x}\in\mathbb{P}_1(K)$ with $\nabla u=[\partial_1 u,\partial_2 u,\partial_3 u]^\intercal$ and

$$A_K(\nabla u)=[A_K(\partial_1 u),A_K(\partial_2 u),A_K(\partial_3 u)]^\intercal.$$

Then, we have by (2.3) that

$$0=\sum_{F\in\mathcal{F}^1(K)}\big(\partial_{\nu_F}p,\widehat{\Pi}_F^\nabla w-\widehat{\Pi}_K^\nabla w\big)_F=\sum_{F\in\mathcal{F}^1(K)}\big(A_K(\nabla u)\cdot\nu_F,\widehat{\Pi}_F^\nabla w-\widehat{\Pi}_K^\nabla w\big)_F,$$

which together with (4.11) and (4.12) yields

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} a^K\big(u,\widehat{\Pi}_K^{\nabla}w\big)-a(u,w)\\
=&\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^1(K)}\big(\big(\nabla u-A_K(\nabla u)\big)\cdot\nu_F,\widehat{\Pi}_F^{\nabla}w-\widehat{\Pi}_K^{\nabla}w\big)_F\\
&+\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^1(K)\cap\mathcal{F}_{hi}^1}\big((\nabla u-Q_h\nabla u)\cdot\nu_F,w-\widehat{\Pi}_F^{\nabla}w\big)_F. \qquad (4.13)
\end{aligned}
$$

Moreover, in view of the definition of the quasi-elliptic operators $\widehat{\Pi}_F^{\nabla}$ and Lemma 3.2, we have

$$
\big|v-\widehat{\Pi}_F^{\nabla}v\big|_{m,F}=\big|\big(v-\widehat{\Pi}_K^{\nabla}v\big)-\widehat{\Pi}_F^{\nabla}\big(v-\widehat{\Pi}_K^{\nabla}v\big)\big|_{m,F}\lesssim\big|v-\widehat{\Pi}_K^{\nabla}v\big|_{m,F}, \qquad (4.14)
$$

$$
\big|\widehat{\Pi}_K^{\nabla}v-\widehat{\Pi}_F^{\nabla}v\big|_{m,F}=\big|\widehat{\Pi}_F^{\nabla}\big(\widehat{\Pi}_K^{\nabla}v-v\big)\big|_{m,F}\lesssim\big|v-\widehat{\Pi}_K^{\nabla}v\big|_{m,F} \qquad (4.15)
$$

with $v\in V_1(K)$ and $m=0,1$. Thus, combining (4.13)-(4.15) and (3.1), one has

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} a^K\big(u,\widehat{\Pi}_K^{\nabla}w\big)-a(u,w)\\
\lesssim&\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^1(K)}\Big(\big\|\big(\nabla u-A_K(\nabla u)\big)\cdot\nu_F\big\|_{0,F}+\|(\nabla u-Q_h\nabla u)\cdot\nu_F\|_{0,F}\Big)\big\|w-\widehat{\Pi}_K^{\nabla}w\big\|_{0,F}\\
\lesssim&\sum_{K\in\mathcal{T}_h}\big(\|\nabla u-A_K(\nabla u)\|_{0,K}+h_K|\nabla u|_{1,K}\big)\Big(h_K^{-1}\big\|w-\widehat{\Pi}_K^{\nabla}w\big\|_{0,K}+\big|w-\widehat{\Pi}_K^{\nabla}w\big|_{1,K}\Big)\\
&+\sum_{K\in\mathcal{T}_h}\big(\|\nabla u-Q_h\nabla u\|_{0,K}+h_K|\nabla u-Q_h\nabla u|_{1,K}\big)\Big(h_K^{-1}\big\|w-\widehat{\Pi}_K^{\nabla}w\big\|_{0,K}+\big|w-\widehat{\Pi}_K^{\nabla}w\big|_{1,K}\Big).
\end{aligned}
$$

This along with Lemma 3.2, (3.4) and (4.6) yields

$$
E_h\lesssim\sup_{w\in X_h}\frac{h|\nabla u|_1|w|_1}{|w|_1}\lesssim h|u|_2. \qquad (4.16)
$$

The proof is complete. ☐

Lemma 4.4. *Assume $u\in X\cap H^2(\Omega)$. There exist $v_0\in X$ and $u_I\in X_h$ such that*

$$
|R_u(u_I,u)|+|R_u(v,u_h)|\lesssim h^2\|\Delta u+f\|_0(|u|_2+|\phi|_2).
$$

Proof. By integration by parts, we see that

$$
\begin{aligned}
&|R_u(u_I,u)|+|R_u(v,u_h)|\\
=&|(\Delta u+f,u_I-u)|+|(\Delta u+f,v-u_h)|\\
\le&\|\Delta u+f\|_0\Big(\inf_{u_I\in X_h}\|u_I-u\|_0+\inf_{v\in X}\|v-u_h\|_0\Big). \qquad (4.17)
\end{aligned}
$$

We first bound the error $\inf_{v\in X}\|v-u_h\|_0$. Let $v_0=\max\{u_h,\phi\}\in H^1(\Omega)$. Obviously, $v_0\geq\phi$ and $v_0\in X$. In fact, since $\phi\leq 0$ on $\partial\Omega$ and $u_h\in H_0^1(\Omega)$, there holds $v_0|_{\partial\Omega}=0$. By $u_h(\zeta)\geq\phi(\zeta)$ for all $\zeta\in\mathcal{F}_h^3$ and (4.3), we find

$$u_h(\zeta)-I_h\phi(\zeta)\geq 0,\quad\forall\zeta\in\mathcal{F}_h^3. \tag{4.18}$$

Due to the fact that $u_h|_{\partial F}, I_h\phi|_{\partial F}\in\mathbb{P}_1(\partial F)$ for any $F\in\mathcal{F}_h^1$ and by (4.18), it is easy to see that

$$(u_h-I_h\phi)|_{\partial F}\geq 0,$$

which, together with the weak maximum principle of the harmonic equation (cf. [23, Theorem 8.1]), gives rise to

$$(u_h-I_h\phi)|_F\geq 0.$$

Then, using this maximum principle again, we know

$$(u_h-I_h\phi)|_K\geq 0,\quad\forall K\in\mathcal{T}_h. \tag{4.19}$$

Define

$$\Omega^*=\{x\in\Omega:\ u(x)\leq\phi(x)\}.$$

By (4.19), we can immediately deduce that $v_0=u_h$ on $\Omega\backslash\Omega^*$ and

$$\|v_0-u_h\|_0^2=\int_{\Omega^*}(\phi-u_h)^2\leq\int_{\Omega^*}(\phi-I_h\phi)^2. \tag{4.20}$$

Take u_I as I_hu in (4.17) and the proof is completed by (4.4) and (4.20). ☐

Remark 4.2. When checking the above proof, we can find that our estimate on the term $\inf_{v\in X_h}\|v-u_h\|_0$ relies heavily on the weak maximum principle of harmonic equations. However, the functions in the enhanced virtual element space $\overline{V}_1(K)$ or $W_1(K)$ may not be harmonic. So we cannot derive the similar estimates for the other enhanced VEMs for problem (4.1) in view of the above arguments.

Theorem 4.2. *Let $u\in X\cap H^2(\Omega)$ and $u_h\in X_h$ be the solution of* (4.1) *and* (4.2), *respectively. Then, there holds*

$$|u-u_h|_1+\sum_{K\in\mathcal{T}_h}\left|u-\widehat{\Pi}_K^{\nabla}u_h\right|_{1,K}\lesssim h,$$

where the hidden constant is independent of h and u_h.

Proof. Using the triangle inequality and Lemma 3.2, we find

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\left|u-\widehat{\Pi}_K^{\nabla}u_h\right|_{1,K}&\lesssim\sum_{K\in\mathcal{T}_h}\left|u-u_\pi+\widehat{\Pi}_K^{\nabla}(u_\pi-u_h)\right|_{1,K}\\&\lesssim|u-u_\pi|_{1,h}+|u_\pi-u_h|_{1,h}\\&\lesssim|u-u_\pi|_{1,h}+|u-u_h|_1,\end{aligned}$$

which along with Lemmas 4.2-4.4 implies

$$\begin{aligned}&|u-u_h|_1+\sum_{K\in\mathcal{T}_h}\big|u-\widehat{\Pi}_K^{\nabla}u_h\big|_{1,K}\\ \lesssim\,& |u-u_I|_1+|u-u_\pi|_{1,h}+\|f-f_h\|_{X_h^*}+h|u|_2\\ &+h\|\Delta u+f\|_0^{\frac{1}{2}}|u|_2^{\frac{1}{2}}+h\|\Delta u+f\|_0^{\frac{1}{2}}|\phi|_2^{\frac{1}{2}}.\end{aligned}\tag{4.21}$$

By Lemma 3.2,

$$\|f-f_h\|_{X_h^*}\lesssim\sup_{v\in X_h}\frac{\sum_{K\in\mathcal{T}_h}\big(f,v-\widehat{\Pi}_K^{\nabla}v\big)_K}{|v|_1}\lesssim h\|f\|_0.\tag{4.22}$$

Let $u_I=I_hu$ and $u_\pi=\Pi_h^1u$. Then, the desired result can be derived by (4.4), (4.21), (4.22) and the standard error estimates of L^2 projection. □

5. Numerical experiments

In this section, we report some numerical results for a problem discussed in [22].

Example 5.1. Let $\Omega=(0,1)^3\subset\mathbb{R}^3$. We choose the force function f as

$$f(x,y,z)=\begin{cases}-4\big(2r^2+3\big(r^2-r_0^2\big)\big), & \forall r>r_0,\\ -8r_0^2\big(1-r^2+r_0^2\big), & \forall r\le r_0,\end{cases}$$

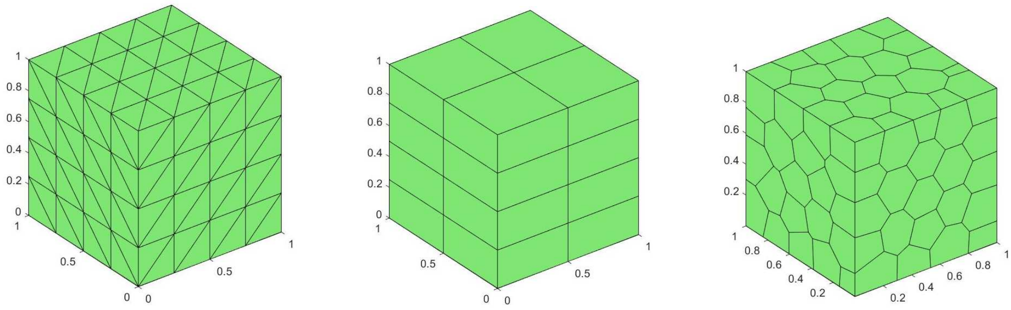
where $r=(x_1^2+x_2^2+x_3^2)^{1/2}$ and $r_0=0.7$. The Dirichlet boundary condition is determined by the exact solution

$$u(x,y,z)=\big(\max\big\{r^2-r_0^2,0\big\}\big)^2,$$

and the obstacle function is chosen as $\phi\equiv 0$. We solve the discrete obstacle problem by a primal-dual active set strategy in [25] on the tetrahedral mesh (Fig. 2(a)), the cuboid mesh (Fig. 2(b)) and the Centroidal Voronoi Tessellation (CVT) polyhedral mesh (Fig. 2(c)), respectively. Define

$$\begin{aligned}\mathrm{ErrH1}&=\Big(\sum_{K\in\mathcal{T}_h}\big\|u-\widehat{\Pi}_K^{\nabla}u_h\big\|_{1,K}\Big)^{\frac{1}{2}},\\ \mathrm{ErrL2}&=\Big(\sum_{K\in\mathcal{T}_h}\big\|u-\widehat{\Pi}_K^{\nabla}u_h\big\|_{0,K}\Big)^{\frac{1}{2}}.\end{aligned}$$

Since the shape functions are not explicit, we display the discrete errors $\mathrm{ErrH1}$ and $\mathrm{ErrL2}$ for three types of meshes with different mesh sizes in Tables 1-3, respectively.



(a) Tetrahedral mesh (b) Cuboid mesh (c) CVT polyhedral mesh

Figure 2: Different types of meshes.

From Table 1, we can observe that $\mathrm{ErrH1} = \mathcal{O}(h^{1.00})$ and $\mathrm{ErrL2} = \mathcal{O}(h^{2.00})$ for the tetrahedral mesh, from Table 2 $\mathrm{ErrH1} = \mathcal{O}(h^{1.00})$ and $\mathrm{ErrL2} = \mathcal{O}(h^{2.00})$ for the cuboid mesh, while from Table 3 $\mathrm{ErrH1} = \mathcal{O}(h^{1.00})$ and $\mathrm{ErrL2} = \mathcal{O}(h^{2.17})$ for the CVT polyhedral mesh.

To sum up the above numerical results, we can draw some conclusions for the proposed numerical method as follows:

1. It has linear convergence rate in the H^1 norm, which is in good agreement with the theoretical result in Theorem 4.2. Also, we find it has quadratic convergence rate in L^2 norm numerically.

2. It is flexible in choosing geometric meshes, i.e. one can adopt a geometric mesh for discretization in terms of the shape of the solution domain while handling the error analysis in a unified way. This is also the advantage of the general VEMs.

Table 1: The performance of Example 5.1 on the tetrahedral mesh.

h	0.275	0.138	0.069	0.034	0.017	Rate
ErrH1	2.747e+00	1.431e+00	7.235e-01	3.628e-01	1.815e-01	1.00
ErrL2	5.091e-01	1.299e-01	3.256e-02	8.150e-03	2.039e-03	2.00

Table 2: The performance of Example 5.1 on the cuboid mesh.

h	0.397	0.198	0.099	0.050	0.025	Rate
ErrH1	1.502e+00	7.723e-01	3.884e-01	1.944e-01	9.722e-02	1.00
ErrL2	2.746e-01	6.803e-02	1.671e-02	4.146e-03	1.035e-03	2.00

Table 3: The performance of Example 5.1 on the CVT polyhedral mesh.

h	0.250	0.165	0.131	0.087	0.061	Rate
ErrH1	8.963e-01	5.926e-01	4.685e-01	3.145e-01	2.225e-01	1.00
ErrL2	9.281e-02	3.874e-02	2.500e-02	1.201e-02	5.657e-03	2.17

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