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The Regularized Global GMERR Method for Solving Large-Scale Linear Discrete Ill-Posed Problems

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Abstract. For the large-scale linear discrete ill-posed problems with multiple right-hand sides, the global Krylov subspace iterative methods have received a lot of attention. In this paper, we analyze the regularizing properties of the global generalized minimum error method (GMERR), and develop a regularized global GMERR method for solving linear discrete ill-posed problems with multiple right-hand sides. The efficiency of the proposed method is confirmed by the numerical experiments on test matrices.

AMS subject classifications: 65F10, 65F22
Key words: Linear discrete ill-posed problems, multiple right-hand sides, global GMERR method, regularizing properties.

1. Introduction

We are interested in approximate solutions of the large-scale linear discrete ill-posed problems

$$Ax^{(j)} = b^{(j)}, \quad j = 1, 2, \ldots, s, \tag{1.1}$$

which have the same coefficient matrix $A \in \mathbf{R}^{n\times n}$ and different right-hand sides $b^{(j)}$, $j = 1, 2, \ldots, s$. These problems arise in real-world applications, including electromagnetic wave scattering problem [45], pattern classification [12], image restoration [36], and dimensionality reduction [52] and so on. The difficulty in solving the problems is that the coefficient matrix A is nonsingular, but ill-conditioned with its singular values decaying to zero with increasing index without a noticeable gap and the right-hand vectors $b^{(j)} = \hat{b}^{(j)} + e^{(j)} \in \mathbf{R}^n$, $j = 1, 2, \ldots, s$ are assumed to be contaminated by unknown error-free vectors $\hat{b}^{(j)}$ and unknown measurement errors or noises $e^{(j)}$. Writing

$$X = \left[x^{(1)}, x^{(2)}, \ldots, x^{(s)}\right], \quad B = \left[b^{(1)}, b^{(2)}, \ldots, b^{(s)}\right],$$
$$\hat{B} = \left[\hat{b}^{(1)}, \hat{b}^{(2)}, \ldots, \hat{b}^{(s)}\right], \quad E = \left[e^{(1)}, e^{(2)}, \ldots, e^{(s)}\right],$$

we refer to E as the error or noise matrix. Then $B = \hat{B} + E$, and (1.1) can be written as

$$AX = B. \tag{1.2}$$

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The unavailable linear system of equations

$$AX = \hat{B} \tag{1.3}$$

is assumed to be consistent — i.e. the right-hand side $\hat{B}$ is in the range of A. However, it is possible that the available error-contaminated right-hand side B is outside of the range of A. In fact, we are interested in the minimum Frobenius norm solution of (1.3), which will be denoted by $\hat{X} \in \mathbf{R}^{n\times s}$.

Let $\|E\|_F$ denote the Frobenius norm of the noise matrix E, and the bound of the norm of the noise matrix E be $\delta > 0$, i.e.

$$\|E\|_F \leq \delta. \tag{1.4}$$

We consider how to determine a meaningful approximation of $\hat{X}$ by computing a proper approximate solution of (1.2).

If the coefficient matrix A is well-conditioned, the linear system (1.2) may be solved by direct or iterative methods [2, 18], including matrix splitting iteration methods [2] and Krylov subspace methods [1, 18]. However, since the matrix splitting iteration methods cannot be directly applied to large-scale linear discrete ill-posed problems, more attention has been paid to Krylov subspace methods. Due to the ill-conditioning of the matrix A and the presence of a noise E in B, the native solution to problem (1.2) does not furnish a useful approximation of the true solution to problem (1.3). Therefore, regularization methods must be used to extract as good an approximation to the true solution as possible. Two of the most popular regularization methods are Tikhonov regularization method [37, 47] and truncated singular value decomposition regularization method (TSVD) [20–22], which are either computationally unfeasible or extremely time-consuming for large-scale problems. So iterative regularization has received considerable attention.

There are a number of Krylov subspace methods for solving large-scale linear discrete ill-posed problems with a single right-hand side

$$Ax = b, \quad b \in \mathbf{R}^n. \tag{1.5}$$

When A is symmetric positive definite, Gilyazov [17] and Hanke [19] developed the conjugate gradient (CG) method for solving the ill-posed problem (1.5). Plato [42] analyzed the regularizing properties of CG. Using the Lanczos tridiagonalization, Paige and Saunders [41] developed an MINRES method for solving the linear systems with symmetric indefinite matrix A. Hanke [19], Kilmer and Stewart [34], Jensen and Hansen [29], Huang and Jia [26] analyzed regularizing effects of MINRES and showed its semi-convergence. When A is nonsymmetric, Björck [6] presented the CGLS algorithm, which implicitly applies CG to the normal equation $A^TAx = A^Tb$. Hanke [19] studied the regularizing properties of CGLS. Based on the Lanczos bidiagonalization, Paige and Saunders [40] proposed the LSQR algorithm, which is mathematically equivalent to CGLS. The regularizing effects of the LSQR algorithm are analyzed in [25]. Fong and Saunders [14] developed an LSMR algorithm, which is equivalent to the MINRES method applied to the normal equation $A^TAx = A^Tb$. Jia [30] analyzed the regularization properties of the LSMR algorithm. The GMRES method [44] is a popular iterative method for solving large linear

systems of equations. For quaternion linear systems arising from color image denoising and three-dimensional signal processing, Jia and Ng [31] developed a quaternion generalized minimal residual method (QGMRES), and discussed its convergence. Calvetti *et al*. [9] studied regularizing properties of the GMRES method. Using the generalized minimal error (GMERR) method [43, 50], Wang and Dai [49] developed a regularized GMERR method for solving large-scale linear discrete ill-posed problem (1.5). However, the regularizing properties of the GMERR method have not been analyzed. For iterative regularization, selecting a good stopping rule is a crucial task. In order to overcome semi-convergence behavior and to avoid selecting a regularization parameter a priori, some hybrid methods based on LSQR, LSMR, GMRES and Tikhonov regularization have been proposed — cf. Refs. [6, 7, 10, 11, 13, 16, 23, 32, 33, 35, 38, 39].

For the large-scale linear discrete ill-posed problems with multiple right-hand sides, one approach for solving (1.2) is to apply the methods mentioned above to each ill-posed system (1.1) separately. However, in this case, the information gathered during the solution of each ill-posed system is not used. Therefore, we are interested in the methods, which can share information and converge faster than the methods solving every ill-posed system individually. Block and global Krylov subspace methods are good choices for solving large-scale linear discrete ill-posed problems with multiple right-hand sides. Based on the block Lanczos process, Bentbib *et al*. [3] developed a block Lanczos method for solving linear discrete ill-posed problems with multiple right-hand sides. However, the block Lanczos process may break down. Using the global Lanczos process [28] and the global Golub-Kahan bidiagonalization [48], Bentbib *et al*. [4, 5] developed a global Lanczos method and a global Golub-Kahan bidiagonalization method for linear ill-posed matrix equations. Based on the global Arnoldi process [27] and its generalization, Bouhamidi *et al*. [8] and Huang *et al*. [24] respectively developed the generalized global Arnoldi method and the global Arnoldi-Tikhonov regularization method for linear ill-posed matrix equations. Zhang and Dai [51] analyzed the regularizing properties of the global GMRES method, and proposed a regularized global GMRES method for solving linear discrete ill-posed problem (1.2). These block and global Krylov subspace methods for solving linear discrete ill-posed problem (1.2) are all based on the residual norm minimization. But for linear discrete ill-posed problems, even if the residual norm of an approximate solution is small, the approximate solution is not necessarily close to the true solution. Ideally, iterative regularization for solving linear discrete ill-posed problem (1.2) should be designed based on error norm minimization. Zheng [53] proposed a global GMERR method for solving linear systems with multiple right-hand sides based on the error norm minimization. In this paper, we analyze the regularizing properties of the global GMERR method, and propose a regularized global GMERR method for solving large-scale linear discrete ill-posed problem (1.2).

Let us first introduce the notation used in this paper. For a matrix $P \in \mathbf{R}^{m\times n}$, we denote by P^{T} and P^{+} the transpose and the Moore-Penrose pseudo-inverse of the matrix P, respectively. The Frobenius inner product of matrices $X, Y \in \mathbf{R}^{n\times s}$ is defined by

$$\langle X, Y\rangle_F = \mathrm{tr}(X^T Y),$$

where $\mathrm{tr}(Z)$ denotes the trace of a square matrix Z. The corresponding Frobenius norm

is denoted by $\|X\|_F$, i.e. $\|X\|_F = \langle X, X\rangle_F^{1/2}$. Besides, $K_m(A, V)$ refers to the matrix Krylov subspace $\text{span}\{V, AV, \ldots, A^{m-1}V\}$ generated by the matrices $A \in \mathbf{R}^{n\times n}$ and $V \in \mathbf{R}^{n\times s}$, i.e. every $Z \in K_m(A, V)$ has the form

$$Z = \sum_{i=1}^{m} \alpha_i A^{i-1} V.$$

Two matrices $X, Y \in \mathbf{R}^{n\times s}$ are F-orthogonal, denoted by $X \perp_F Y$, if $\langle X, Y\rangle_F = 0$. Furthermore, the matrices $V_1, V_2, \ldots, V_m$ are said to be F-orthonormal if

$$\langle V_i, V_j\rangle_F = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

In particular, if $s = 1$, then the Frobenius inner product and the Frobenius norm reduce to the standard inner product $\langle\cdot,\cdot\rangle$ and the Euclidean vector norm $\|\cdot\|_2$ in $\mathbf{R}^n$, respectively. Note that $\|\cdot\|_2$ also denotes the associated induced matrix norm. $\text{cond}(P) = \|P\|_2\|P^+\|_2$ denotes the condition number of a matrix $P \in \mathbf{R}^{m\times n}$.

The operator vec transforms matrix $P = (p_{i,j}) \in \mathbf{R}^{m\times n}$ into the following vector:

$$\text{vec}(P) = (p_{1,1}, p_{2,1}, \ldots, p_{m,1}, p_{1,2}, p_{2,2}, \ldots, p_{m,2}, \ldots, p_{m,n})^\intercal \in \mathbf{R}^{mn}.$$

The inverse operator math transforms the vector $\text{vec}(P) \in \mathbf{R}^{mn}$ into the associated matrix $P = (p_{i,j}) \in \mathbf{R}^{m\times n}$, i.e. $\text{math}(\text{vec}(P)) = P$. Obviously, the matrix equation (1.2) is equivalent to the following linear system:

$$(I_s \otimes A)\text{vec}(X) = \text{vec}(B), \tag{1.6}$$

where $\otimes$ is the Kronecker product of two matrices, and I_s denotes the identity matrix of order s.

The paper is organized as follows. In Section 2, we review briefly the global GMERR method and analyze its regularizing properties. In Section 3, we present the regularized global GMERR method for solving large-scale linear ill-posed problem (1.2). In Section 4, we report some numerical experiments on test matrices to show the efficiency of the proposed method. Finally, we give some concluding remarks in Section 5.

2. Global GMERR Method and Its Regularizing Properties

The global GMERR algorithm is based on the global Arnoldi process which constructs an F-orthonormal basis $\{V_i\}_{i=1}^k$ of the matrix Krylov subspace $K_k(A^\intercal, V)$. The global Arnoldi process may be described as follows — cf. [27].

Algorithm 2.1 Global Arnoldi Process for Constructing an F-orthonormal Basis $\{V_i\}_{i=1}^k$ of $K_k(A^\intercal, V)$.

1: **Input:** $A \in \mathbf{R}^{n\times n}$, $V \in \mathbf{R}^{n\times s}$, and a positive integer k.
2: **Output:** Upper Hessenberg matrix $\tilde{H}_k = (h_{i,j}) \in \mathbf{R}^{(k+1)\times k}$, F-orthonormal basis $\{V_i\}_{i=1}^k$ of $K_k(A^\intercal, V)$.

Set $F_1 = V$, $V_1 = F_1/\|F_1\|_F$.
for $j = 1, 2, \ldots, k$ **do**
  $F_{j+1} = A^T V_j$;
  for $i = 1, 2, \ldots, j$ **do**
    $h_{i,j} = \langle F_{j+1}, V_i \rangle_F$;
    $F_{j+1} = F_{j+1} - h_{i,j} V_i$;
  end for
  Compute $h_{j+1,j} = \|F_{j+1}\|_F$.
  if $h_{j+1,j} > 0$ **then**
    $V_{j+1} = F_{j+1}/h_{j+1,j}$;
  else
    exit.
  end if
end for

Let $\mathcal{V}_k = [V_1, V_2, \ldots, V_k] \in \mathbf{R}^{n \times ks}$, $\mathcal{V}_{k+1} = [\mathcal{V}_k, V_{k+1}] \in \mathbf{R}^{n \times (k+1)s}$ and $\tilde{H}_k \in \mathbf{R}^{(k+1)\times k}$ be the upper Hessenberg matrix whose nonzero entries $h_{i,j}$ are generated by Algorithm 2.1. A matrix $H_k \in \mathbf{R}^{k\times k}$ is obtained by deleting the last row in $\tilde{H}_k$. The j-th column of the matrix H_k is denoted by $H_{.,j}$. If the matrix Krylov subspace $K_k(A^\intercal, V)$ satisfies

$$\dim K_k\big(A^\intercal, V)\big) = k,$$

then $F_{j+1} \neq 0$, $j = 1, 2, \ldots, k-1$ and Algorithm 2.1 does not break down.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)^\intercal \in \mathbf{R}^k$, then we set

$$\mathcal{V}_k * \alpha := \sum_{i=1}^{k} \alpha_i V_i,$$

and note that

$$\mathcal{V}_k * H_k = \big[\mathcal{V}_k * H_{.,1}, \mathcal{V}_k * H_{.,2}, \ldots, \mathcal{V}_k * H_{.,k}\big] = \mathcal{V}_k (H_k \otimes I_s).$$

The following relations are easily verified:

$$\begin{aligned} &\mathcal{V}_k * \alpha = \mathcal{V}_k(\alpha \otimes I_s), && A(\mathcal{V}_k * \alpha) = (A\mathcal{V}_k) * \alpha, \\ &\mathcal{V}_k * (\alpha + \beta) = \mathcal{V}_k * \alpha + \mathcal{V}_k * \beta, && (\mathcal{V}_k * H_k) * \alpha = \mathcal{V}_k * (H_k \alpha), \end{aligned}$$

where $\alpha, \beta \in \mathbf{R}^k$.

Let us recall the following result about the global Arnoldi process.

Lemma 2.1 (cf. Jbilou *et al.* [27]). *If the global Arnoldi process does not break down, then:*

1. $\{V_i\}_{i=1}^k$ *is an F-orthonormal basis of the matrix Krylov subspace* $K_k(A^\intercal, V)$.
2. *For any vector* $\alpha \in \mathbf{R}^k$, $\|\mathcal{V}_k * \alpha\|_F = \|\alpha\|_2$.

3. The following relation holds:

$$A^{\intercal}\mathcal{V}_k = \mathcal{V}_k * H_k + W_{k+1} = \mathcal{V}_{k+1} * \tilde{H}_k = \mathcal{V}_{k+1}(\tilde{H}_k \otimes I_s),$$

where

$$W_{k+1} = [0_{n\times s}, \ldots, 0_{n\times s}, F_{k+1}] = h_{k+1,k}[0_{n\times s}, \ldots, 0_{n\times s}, V_{k+1}].$$

Suppose that $X_0 \in \mathbf{R}^{n\times s}$ is an initial approximation to the solution X of the linear system (1.2). Then the associated residual and initial error are $R_0 = B - AX_0$ and $\varepsilon_0 = X - X_0$, respectively. At the k-th iterate, the global GMERR method tries to find a correction Z_k in the matrix Krylov subspace $K_k(A^T, A^T R_0) = A^T K_k(A^T, R_0)$ such that the k-th approximate solution $X_k = X_0 + Z_k$ satisfies the following error minimization problem:

$$\|\varepsilon_k\|_F = \|X - X_k\|_F = \min_{Z\in A^T K_k(A^T,R_0)} \|X - X_0 - Z\|_F = \min_{z\in A^T K_k(A^T,R_0)} \|\varepsilon_0 - Z\|_F,$$

or, equivalently,

$$Z_k \in A^T K_k(A^T, R_0), \quad R_k \perp_F K_k(A^T, R_0), \tag{2.1}$$

where $\varepsilon_k = X - X_k$ and $R_k = B - AX_k = R_0 - AZ_k$. Then the k-th approximate solution X_k generated by the global GMERR method and corresponding residual may be expressed as

$$X_k = X_0 + Z_k = X_0 + A^T \mathcal{V}_k * y_k, \quad R_k = R_0 - AA^{\intercal}\mathcal{V}_k * y_k, \tag{2.2}$$

where $y_k \in \mathbf{R}^k$. It follows from $V_1 = R_0/\|R_0\|_F$, Lemma 2.1, the Eqs. (2.1) and (2.2) that y_k satisfies the following linear system:

$$\bar{H}_k y_k = \beta e_1^{(k)}, \tag{2.3}$$

where $\beta = \|R_0\|_F$, $e_1^{(k)}$ denotes the first column vector of I_k and

$$\bar{H}_k = \begin{pmatrix} \mathrm{tr}(V_1^T AA^T V_1) & \mathrm{tr}(V_1^T AA^T V_2) & \cdots & \mathrm{tr}(V_1^T AA^T V_k) \\ \mathrm{tr}(V_2^T AA^T V_1) & \mathrm{tr}(V_2^T AA^T V_2) & \cdots & \mathrm{tr}(V_2^T AA^T V_k) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{tr}(V_k^T AA^T V_1) & \mathrm{tr}(V_k^T AA^T V_2) & \cdots & \mathrm{tr}(V_k^T AA^T V_k) \end{pmatrix}. \tag{2.4}$$

Theorem 2.1. *If $\{V_j\}_{j=1}^k$ are the F-orthonormal basis of $K_k(A^{\intercal}, R_0)$ generated by Algorithm* 2.1*, then the matrix $\bar{H}_k$ defined by* (2.4) *is nonsingular.*

Proof. Since $\{V_j\}_{j=1}^k$ are the F-orthonormal basis of $K_k(A^{\intercal}, R_0)$, then $\{V_j\}_{j=1}^k$ are linearly independent under the product $*$. The matrix $\bar{H}_k$ is the Gram matrix of $V_1, V_2, \ldots, V_k$ under the Frobenius inner product with the weight $AA^{\intercal}$, so $\bar{H}_k$ is nonsingular. □

It follows from Theorem 2.1 that the small-scale problem (2.3) has a unique solution $y_k = \beta(\bar{H}_k)^{-1} e_1^{(k)}$ and can be solved by direct methods. Therefore, the k-th approximate solution X_k of the linear system (1.2) can be given by $X_k = X_0 + A^T \mathcal{V}_k * y_k$. The global

GMERR method for computing an approximate solution X_k of the linear system (1.2) may be summarized as Algorithm 2.2— cf. [53].

Algorithm 2.2 The Global GMERR Algorithm for Solving $AX = B$.

1: **Input:** $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times s}$ and an initial guess $X_0 \in \mathbf{R}^{n\times s}$.
2: **Output:** Approximate solution X_k of the linear system (1.2).
3: Compute $R_0 = B - AX_0$.
4: Construct the F-orthonormal basis $\{V_i\}_{i=1}^k$ of $K_k(A^\intercal, R_0)$ by using Algorithm 2.1.
5: Find the solution y_k of the linear system (2.3) and compute $X_k = X_0 + A^T \mathscr{V}_k * y_k$.

If $s = 1$, the global GMERR algorithm reduces the GMERR method.

Now we analyze the regularizing properties of the global GMERR method. For convenience, we assume that the matrix E satisfies (1.4), and the matrices A and $\hat{B}$ in (1.3) are scaled so that

$$\|A\|_F \le 1, \quad \delta < \|\hat{B}\|_F \le 1, \tag{2.5}$$

and for all iterative methods, we choose $X_0 = \mathbf{0}$ and $\hat{X}_0 = \mathbf{0}$ as the initial approximate solutions to the linear systems (1.2) and (1.3), respectively. The Arnoldi process for constructing an orthonormal basis $\{v_i\}_{i=1}^k$ of the Krylov subspace $K_k(I_s \otimes A^\intercal, \mathrm{vec}(B))$ can be described in Algorithm 2.3.

Algorithm 2.3 Arnoldi Process for Generating an Orthonormal Basis $\{v_i\}_{i=1}^k$ of $K_k((I_s \otimes A^\intercal), \mathrm{vec}(B))$.

Input: $(I_s \otimes A^\intercal) \in \mathbf{R}^{ns\times ns}$, $\mathrm{vec}(B) \in \mathbf{R}^{ns\times 1}$, and a positive integer k.
Output: Upper Hessenberg matrix $\tilde{H}_k = (h_{i,j}) \in \mathbf{R}^{(k+1)\times k}$, orthonormal basis $\{v_i\}_{i=1}^k$ of $K_k((I_s \otimes A^\intercal), \mathrm{vec}(B))$.
Set $f_1 = \mathrm{vec}(B)$, $v_1 = f_1/\|f_1\|_2$.
for $j = 1, 2, \ldots, k$ **do**
  $f_{j+1} = (I_s \otimes A^\intercal)v_j$;
  for $i = 1, 2, \ldots, j$ **do**
    $h_{i,j} = \langle f_{j+1}, v_i\rangle$;
    $f_{j+1} = f_{j+1} - h_{i,j}v_i$;
  end for
  Compute $h_{j+1,j} = \|f_{j+1}\|_2$.
  if $h_{j+1,j} > 0$ **then**
    $v_{j+1} = f_{j+1}/h_{j+1,j}$;
  else
    exit.
  end if
end for

It is easy to verify that the output matrices $\tilde{H}_k$ generated by both Algorithm 2.1 with $V = B$ and Algorithm 2.3 are the same, $v_i = \mathrm{vec}(V_i)$, $f_i = \mathrm{vec}(F_i)$, $i = 1, 2, \ldots, k+1$, and

using the global GMERR method to solve the linear matrix equation (1.2) is equivalent to applying the GMERR method to solve the linear system (1.6).

Let $\bar{\mathcal{V}}_j = [v_1, v_2, \ldots, v_j] \in \mathbf{R}^{ns\times j}$, $j = 1, 2, \ldots, k+1$, and

$$\left(I_s \otimes A^{\intercal}\right)_j = \bar{\mathcal{V}}_{j+1}\tilde{H}_j\left(\bar{\mathcal{V}}_j\right)^{\intercal}, \quad j = 1, 2, \ldots, k,$$

then the matrix $(I_s \otimes A^{\intercal})_j$ can be considered a regularization approximation to $I_s \otimes A^{\intercal}$, and the number of steps in Algorithm 2.3 plays the role of the regularization parameter. It follows from [9, Proposition 2.1] that

$$\|(I_s \otimes A^{\intercal})_j\|_2 = \|\tilde{H}_j\|_2, \quad \|(I_s \otimes A^{\intercal})_j^{+}\|_2 = \|(\tilde{H}_j)^{+}\|_2,$$

and

$$\begin{aligned}
&\operatorname{cond}\left(\tilde{H}_j\right) \le \operatorname{cond}\left(\tilde{H}_{j+1}\right) \le \operatorname{cond}(A^{\intercal}) = \operatorname{cond}(A),\\
&\operatorname{cond}\left[(I_s \otimes A^{\intercal})_j\right] \le \operatorname{cond}\left[(I_s \otimes A^{\intercal})_{j+1}\right] \le \operatorname{cond}(I_s \otimes A^{\intercal}) = \operatorname{cond}(I_s \otimes A).
\end{aligned}$$

It is easily verified that

$$\bar{H}_k = \tilde{H}_k^{\intercal}\tilde{H}_k, \quad \tilde{H}_k^{+} = (\tilde{H}_k^{\intercal}\tilde{H}_k)^{-1}\tilde{H}_k^{\intercal} = \bar{H}_k^{-1}\tilde{H}_k^{\intercal}. \tag{2.6}$$

Then we have

$$\begin{aligned}
&\operatorname{cond}(\bar{H}_j) = \left(\operatorname{cond}(\tilde{H}_j)\right)^2,\\
&\operatorname{cond}(\bar{H}_j) \le \operatorname{cond}(\bar{H}_{j+1}).
\end{aligned} \tag{2.7}$$

Performing $l, l \ge 1$ steps of Algorithm 2.1 with the initial matrix $\hat{V} = \hat{B}$, we get the F-orthonormal basis $\{\hat{V}_j\}_{j=1}^{l}$ of $K_l(A^{\intercal}, \hat{B})$ and the matrices $\hat{F}_j \in \mathbf{R}^{n\times s}$, $1 \le j \le l+1$ satisfying

$$\hat{F}_j \ne 0, \quad 1 \le j \le l, \quad \hat{F}_{l+1} = 0. \tag{2.8}$$

Lemma 2.2. *If l satisfies the condition* (2.8)*, and $\{V_j\}_{j=1}^{l}$ and $\{\hat{V}_j\}_{j=1}^{l}$ are respectively F-orthonormal bases of the matrix Krylov subspaces $K_l(A^{\intercal}, B)$ and $K_l(A^{\intercal}, \hat{B})$ generated by Algorithm* 2.1 *with the inputs (A, B) and $(A, \hat{B})$, then for $1 \le j, i \le l$ one has*

$$\left\|\langle A^T V_j, V_i\rangle_F V_i - \langle A^T \hat{V}_j, \hat{V}_i\rangle_F \hat{V}_i\right\|_F \le \left\|V_j - \hat{V}_j\right\|_F + 2\left\|V_i - \hat{V}_i\right\|_F.$$

Proof. The proof follows from the proof of [51, Lemma 3.1] if we replace A by A^T. □

Let

$$\mu_j = \begin{cases} 1, & j = 1,\\ \dfrac{2j}{\|\hat{F}_{j-1}\|_F}\mu_{j-1} + 2\displaystyle\sum_{i=1}^{j-1}\frac{2}{\|\hat{F}_i\|_F}\mu_i, & 2 \le j \le l+1. \end{cases} \tag{2.9}$$

The constants μ_j are positive and well-defined since (2.8) holds.

Using Lemma 2.2, we obtain the following two theorems whose proofs are similar to the proofs of [51, Theorems 3.1 and 3.2].

Theorem 2.2. *Let the matrices* $\{F_j\}_{j=1}^{l+1}$ *and* $\{V_j\}_{j=1}^{l}$ *be generated by Algorithm* 2.1 *with the input* (A,B)*, and the matrices* $\{\hat{F}_j\}_{j=1}^{l+1}$ *and* $\{\hat{V}_j\}_{j=1}^{l}$ *be generated by Algorithm* 2.1 *with the input* $(A,\hat{B})$*, and*

$$\delta_j = \begin{cases} \delta, & j=1, \\ j\|V_{j-1}-\hat{V}_{j-1}\|_F + 2\sum_{i=1}^{j-1}\|V_i-\hat{V}_i\|_F, & 2\le j\le l+1. \end{cases}$$

Then

$$\begin{aligned} \left\|F_j-\hat{F}_j\right\|_F &\le \delta_j, \qquad 1\le j\le l+1, \\ \left\|V_j-\hat{V}_j\right\|_F &\le \frac{2}{\|\hat{F}_j\|_F}\delta_j, \quad 1\le j\le l, \end{aligned}$$

and

$$\delta_j \le \mu_j\delta, \quad 1\le j\le l+1$$

with the constants μ_j *defined in* (2.9).

Theorem 2.3. *Let* $\mu_j, 1\le j\le l$ *be the constants* (2.9) *and*

$$\hat{\delta} = \min_{1\le j\le l}\left\{\frac{\|\hat{F}_j\|_F}{2\mu_j}\right\}. \tag{2.10}$$

Then $\dim[K_l(A^{\mathsf{T}},B)] = l$ *for all* $B\in\mathbf{R}^{n\times s}$ *provided* $\|B-\hat{B}\|_F\le\hat{\delta}$.

Theorems 2.2 and 2.3 yield the following result.

Theorem 2.4. *Let* δ *satisfy* $0\le\delta\le\hat{\delta}$ *where* $\hat{\delta}$ *is defined by* (2.10). *Let* $\mathcal{V}_k=[V_1,V_2,\ldots,V_k]$ *and* $\hat{\mathcal{V}}_k=[\hat{V}_1,\hat{V}_2,\ldots,\hat{V}_k]$ *be generated by applying* k $k\le l$ *steps of Algorithm* 2.1 *with the inputs* (A,B) *and* $(A,\hat{B})$*, respectively, and let the matrix* $B\in\mathbf{R}^{n\times s}$ *satisfy* (1.4). *Then*

$$\left\|\mathcal{V}_k-\hat{\mathcal{V}}_k\right\|_F \le \beta_k\delta, \quad \sum_{j=1}^{k}\|V_j-\hat{V}_j\|_F\le\gamma_k\delta, \quad 1\le k\le l, \tag{2.11}$$

where

$$\beta_k = 2\left(\sum_{j=1}^{k}\frac{\mu_j^2}{\|\hat{F}_j\|_F^2}\right)^{1/2}, \quad \gamma_k=\sum_{j=1}^{k}\frac{2\mu_j}{\|\hat{F}_j\|_F} \tag{2.12}$$

with the constants μ_j *defined in* (2.9).

Theorem 2.5. *Let* δ *satisfy* $0\le\delta\le\hat{\delta}$ *with* $\hat{\delta}$ *defined in* (2.10), $\tilde{H}_k\in\mathbf{R}^{(k+1)\times k}$ *and* $\tilde{\hat{H}}_k\in\mathbf{R}^{(k+1)\times k}$ *be generated by performing* k, $k\le l$ *steps of Algorithm* 2.1 *with the inputs* (A,B) *and* $(A,\hat{B})$*, respectively, and the matrix* $B\in\mathbf{R}^{n\times s}$ *satisfy* (1.4). *Then we have*

$$\left\|\tilde{H}_k-\tilde{\hat{H}}_k\right\|_F \le \left(2\sqrt{k}\gamma_k+\mu_{k+1}\right)\delta, \quad 1\le k\le l$$

with the constants μ_k *and* γ_k *defined in* (2.9) *and* (2.12)*, respectively.*

Proof. The proof is similar to the proof of [51, Theorem 3.4] and is omitted here. □

In order to estimate the errors of the difference of the Moore-Penrose pseudo-inverses of the matrices $\tilde{H}_k \in \mathbf{R}^{(k+1)\times k}$ and $\tilde{\hat{H}}_k \in \mathbf{R}^{(k+1)\times k}$ generated by performing k, $k \le l$ steps of Algorithm 2.1, we need the following result — cf. [46, Theorems 2.10 and 2.11].

Lemma 2.3 (cf. Sun [46]). *Let $H, R \in \mathbf{R}^{p\times q}$, and $\tilde{H} = H + R$. If $\operatorname{rank}(H) = \operatorname{rank}(\tilde{H})$ and $\eta = \|H^+\|_2\|R\|_2 < 1$, then*

$$\|\tilde{H}^+\|_2 \le \frac{\|H^+\|_2}{1-\eta}.$$

Moreover, if $\operatorname{rank}(H) = \min_{p\ne q}\{p,q\}$, the following inequality holds:

$$\|\tilde{H}^+ - H^+\|_F \le \|H^+\|_2\|\tilde{H}^+\|_2\|R\|_F.$$

Lemma 2.4. *Let $\tilde{H}_k \in \mathbf{R}^{(k+1)\times k}$ and $\tilde{\hat{H}}_k \in \mathbf{R}^{(k+1)\times k}$ be generated by performing k, $k \le l$ steps of Algorithm 2.1 with the inputs (A,B) and $(A,\hat{B})$, respectively, and*

$$\gamma = \max_{1\le k\le l}\left\{\left\|\tilde{\hat{H}}_k^+\right\|_2\left(2\sqrt{k}\gamma_k + \mu_{k+1}\right)\right\}, \quad \tilde{\delta} = \min\left\{\frac{1}{2\gamma}, \hat{\delta}\right\},$$

where $\mu_k, \hat{\delta}$ and γ_k are defined by (2.9), (2.10) and (2.12), respectively. Let δ satisfy $0 < \delta \le \tilde{\delta}$ and the matrix $E \in \mathbf{R}^{n\times s}$ satisfy (1.4). Then

$$\left\|\tilde{H}_k^+\right\|_2 \le 2\left\|\tilde{\hat{H}}_k^+\right\|_2, \quad 1 \le k \le l. \tag{2.13}$$

Moreover, if $F_{k+1} \ne 0$, then the following inequality holds:

$$\left\|\tilde{H}_k^+ - \tilde{\hat{H}}_k^+\right\|_F \le 2\left\|\tilde{\hat{H}}_k^+\right\|_2^2\left(2\sqrt{k}\gamma_k + \mu_{k+1}\right)\delta, \quad 1 \le k \le l.$$

Proof. If k satisfies $1 \le k \le l$, then according to Theorem 2.3, Algorithm 2.1 does not break down and

$$\operatorname{rank}\left(\tilde{H}_k\right) = \operatorname{rank}\left(\tilde{\hat{H}}_k\right). \tag{2.14}$$

By Theorem 2.5, we have

$$\left\|\tilde{\hat{H}}_k^+\right\|_2\left\|\tilde{H}_k - \tilde{\hat{H}}_k\right\|_F \le \left\|\tilde{\hat{H}}_k^+\right\|_2\left(2\sqrt{k}\gamma_k + \mu_{k+1}\right)\delta \le \gamma\tilde{\delta} \le \frac{1}{2}.$$

It follows from Lemma 2.3 and (2.14) that

$$\left\|\tilde{H}_k^+\right\|_2 \le \frac{\|\tilde{\hat{H}}_k^+\|_2}{1 - \|\tilde{\hat{H}}_k^+\|_2\|\tilde{H}_k - \tilde{\hat{H}}_k\|_2} \le \frac{\|\tilde{\hat{H}}_k^+\|_2}{1 - \|\tilde{\hat{H}}_k^+\|_2\|\tilde{H}_k - \tilde{\hat{H}}_k\|_F} \le \frac{\|\tilde{\hat{H}}_k^+\|_2}{1-\gamma\tilde{\delta}} \le 2\left\|\tilde{\hat{H}}_k^+\right\|_2.$$

By Lemma 2.3, (2.13), (2.14) and Theorem 2.5, we have

$$\left\|\tilde{H}_k^+ - \tilde{\hat{H}}_k^+\right\|_F \le 2\left\|\tilde{\hat{H}}_k^+\right\|_2^2\left\|\tilde{H}_k - \tilde{\hat{H}}_k\right\|_F \le 2\left\|\tilde{\hat{H}}_k^+\right\|_2^2\left(2\sqrt{k}\gamma_k + \mu_{k+1}\right)\delta.$$

The proof is complete. □

Let

$$\tau_k = 2\left\|\tilde{\hat{H}}_k^+\right\|_2^2\left(2\sqrt{k}\gamma_k + \mu_{k+1}\right), \quad 1 \le k \le l.$$

Then Lemma 2.4 shows that

$$\left\|\tilde{H}_k^+ - \tilde{\hat{H}}_k^+\right\|_F \le \tau_k\delta, \quad 1 \le k \le l. \tag{2.15}$$

Theorem 2.6. *Let $0 \le \delta \le \tilde{\delta}$ and the matrix $B \in \mathbf{R}^{n\times s}$ satisfy* (1.4). *For $1 \le k \le l$, X_k and $\hat{X}_k$ denote the k-th iterates computed by Algorithm* 2.2 *applied to* (1.2) *and* (1.3) *with initial guesses $X_0 = \mathbf{0}$ and $\hat{X}_0 = \mathbf{0}$, respectively. Then*

$$\left\|X_k - \hat{X}_k\right\|_F \le \sigma_k \delta, \tag{2.16}$$

where

$$\sigma_k = (k+1)(1+\delta)\tau_k + (k+1)\left\|\tilde{\hat{H}}_k^+\right\|_F + \left\|\tilde{\hat{H}}_k^+\right\|_F \gamma_{k+1}.$$

Proof. It follows from the properties of the product $*$, Algorithm 2.2 and (2.6) that

$$\begin{aligned}
X_k &= A^T \mathcal{V}_k * y_k = \left(\mathcal{V}_{k+1} * \tilde{H}_k\right) * y_k \\
&= \mathcal{V}_{k+1} * \left(\tilde{H}_k y_k\right) \\
&= \mathcal{V}_{k+1} * \left(\tilde{H}_k \bar{H}_k^{-1} \|B\|_F e_1^{(k)}\right) \\
&= \sum_{j=1}^{k+1} \left[\left(\tilde{H}_k^T\right)^+ \|B\|_F e_1^{(k)}\right]^T e_j^{(k+1)} V_j, \\
\hat{X}_k &= \hat{\mathcal{V}}_{k+1} * \left(\tilde{\hat{H}}_k \bar{\hat{H}}_k^{-1} \|\hat{B}\|_F e_1^{(k)}\right) \\
&= \sum_{j=1}^{k+1} \left[\left(\tilde{\hat{H}}_k^T\right)^+ \|\hat{B}\|_F e_1^{(k)}\right]^T e_j^{(k+1)} \hat{V}_j,
\end{aligned}$$

where $\bar{\hat{H}}_k = (\mathrm{tr}(\hat{V}_i^T A A^\intercal \hat{V}_j)) \in \mathbf{R}^{k\times k}$. Using (1.4), (2.5), (2.11) and (2.15), we have

$$\begin{aligned}
\left\|X_k - \hat{X}_k\right\|_F &= \left\| \sum_{j=1}^{k+1} \left\{ \left[\left(\tilde{H}_k^T\right)^+ \|B\|_F e_1^{(k)}\right]^T e_j^{(k+1)} V_j - \left[\left(\tilde{\hat{H}}_k^T\right)^+ \|\hat{B}\|_F e_1^{(k)}\right]^T e_j^{(k+1)} \hat{V}_j \right\} \right\|_F \\
&\le \sum_{j=1}^{k+1} \left\| \left\{ \left[\left(\tilde{H}_k^T\right)^+ - \left(\tilde{\hat{H}}_k^T\right)^+ \right] \|B\|_F e_1^{(k)} \right\}^T e_j^{(k+1)} V_j \right\|_F \\
&\quad + \sum_{j=1}^{k+1} \left\| \left[\left(\tilde{\hat{H}}_k^T\right)^+ \left(\|B\|_F - \|\hat{B}\|_F\right) e_1^{(k)}\right]^T e_j^{(k+1)} V_j \right\|_F \\
&\quad + \sum_{j=1}^{k+1} \left\| \left[\left(\tilde{\hat{H}}_k^T\right)^+ \|\hat{B}\|_F e_1^{(k)}\right]^T e_j^{(k+1)} \left(V_j - \hat{V}_j\right) \right\|_F \\
&\le \left[(k+1)(1+\delta)\tau_k + (k+1)\left\|\tilde{\hat{H}}_k^+\right\|_F + \left\|\tilde{\hat{H}}_k^+\right\|_F \gamma_{k+1}\right] \delta.
\end{aligned}$$

The proof is complete. □

Theorem 2.6 shows that the solution X_k is close to $\hat{X}_k$ if the bound δ decreases to zero. The global GMRES method seeks an approximate solution X_k of the linear system (1.2) such that the k-th residual norm $\|R_k\|_F$ is minimized on the matrix Krylov subspace $K_k(A, R_0)$. The decrease of the error norm is closely connected to the decrease of the residual norm

if the linear system (1.2) is well-conditioned. For this case, the global GMRES method is a powerful solution technique. If the linear system (1.2) is ill-conditioned or ill-posed, the residual norm decreases while the error norm may increase [50]. The user may be misled to stop the iteration too early by small residual norm. The global GMERR method, which is based on error norm minimization on the matrix Krylov subspace $A^T K_k(A^\intercal, R_0)$, can overcome this difficulty.

3. A Regularized Global GMERR Method

In this section, we show how to apply the global GMERR method to solve linear discrete ill-posed problems with several right-hand sides, and develop a regularized global GMERR method for solving ill-posed problem (1.2).

Applying the global GMERR method to solve the large-scale linear discrete ill-posed problem (1.2), we can obtain its approximate solution $X_k = X_0 + A^T V_k * y_k$ by seeking a solution y_k of the projection problem (2.3). However, it follows from (2.7) that the condition number of the projection problem (2.3) will increase as the iteration number k increases. In general, the projection problem (2.3) is also ill-posed. Therefore, it is necessary to use regularization method to solve the projection problem (2.3). We apply the TSVD method [20–22] to compute the regularized solution y_k of the projection problem (2.3) because it is small-scale.

Let

$$\bar{H}_k = \bar{U}_k \Sigma_k \bar{V}_k^\intercal,$$

where $\bar{U}_k = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k] \in \mathbf{R}^{k\times k}$ and $\bar{V}_k = [\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k] \in \mathbf{R}^{k\times k}$ are orthogonal, and $\Sigma_k = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbf{R}^{k\times k}$ with the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$. Then the regularized solution y_k of the projection problem (2.3) determined by the TSVD method may be expressed as

$$y_k = \|R_0\|_F \sum_{i=1}^{\tilde{k}} \frac{\bar{u}_i^\intercal e_1^{(k)}}{\sigma_i} \bar{v}_i, \tag{3.1}$$

where the regularization parameter $\tilde{k}$ is determined by the generalized cross-validation (GCV) [22]. Then the k-th approximate solution of the ill-posed problem (1.2) can be obtained by $X_k = X_0 + A^T V_k * y_k$. However, $\|\hat{X} - X_k\|_F$ hardly decreases with the increase of the iteration number k for appropriately large k. Therefore, the key to applying the global GMERR method to solve the ill-posed problem (1.2) is to choose an appropriate stopping rule to terminate the global GMERR iterations as soon as a suitable approximate solution X_k is found.

When the bound δ of the Frobenius norm of the noise matrix E is available, we adopt the following stopping rule based on the discrepancy principle [22].

Stopping Rule 3.1. Let $\delta > 0$ be an available bound for the noise matrix $E, \alpha > 0$ a given positive number and the matrix B satisfy (1.4). Besides, let X_k, $k = 1, 2, \ldots$ denote the iterates computed by applying the global GMERR method to the ill-posed problem (1.2).

Terminate the global GMERR iterations as soon as an iterate X_k satisfies

$$\|B - AX_k\|_F \leq \alpha\delta.$$

The termination index is denoted by k_δ.

Zhang and Dai [51] discussed the influence of the parameter α in Stopping Rule 3.1 on the iteration number and computing time of the regularized global GMRES method and suggested to set $\alpha = 1$.

If the approximate solution X_k satisfies the Stopping Rule 3.1, X_k is taken as a regularized solution of the ill-posed problem (1.2). The regularized global GMERR method for solving the large-scale discrete ill-posed problem (1.2) can be described as follows.

Algorithm 3.1 Regularized Global GMERR Algorithm.

1: **Input:** $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times s}$ and an initial guess $X_0 \in \mathbf{R}^{n\times s}$.
2: **Output:** Regularized solution X_{k_δ} of the linear discrete ill-posed problem (1.2).
3: Compute $R_0 = B - AX_0$, and set $k = 1$.
4: Construct the F-orthonormal basis $\{V_i\}_{i=1}^k$ of $K_k(A^\intercal, R_0)$ and the upper Hessenberg matrix $\bar{H}_k$ by applying Algorithm 2.1 with A and R_0.
5: Compute the regularized solution (3.1) of the projected problem (2.3) by the TSVD method, and compute the k-th approximate solution $X_k = X_0 + A^T\mathcal{V}_k(y_k \otimes I_s)$ of the ill-posed problem (1.2).
6: **if** X_k satisfies the Stopping Rule 3.1 **then**
7: exit;
8: **else**
9: set $k := k + 1$ and go to step 4.
10: **end if**

Theorem 3.1. *Let δ and the matrix $B \in \mathbf{R}^{n\times s}$ satisfy $0 \leq \delta \leq \tilde{\delta}$ and* (1.4)*, respectively, and Algorithm* 3.1 *applied to* (1.2) *and* (1.3) *with the initial guesses $X_0 = \mathbf{0}$ and $\hat{X}_0 = \mathbf{0}$, respectively, using Stopping Rule* 3.1*, determine the approximate solutions X_{k_δ} and $\hat{X}_{k_\delta}$ of the linear systems* (1.2) *and* (1.3)*. Then*

$$\lim_{\delta\to 0} \sup_{\|B-\hat{B}\|_F\leq\delta} \left\|X_{k_\delta} - \hat{X}_{k_\delta}\right\|_F = 0. \tag{3.2}$$

Proof. It follows from (2.16) that (3.2) holds. □

Theorem 3.1 shows that if Algorithm 3.1 seeks the approximate solution X_{k_δ} of the linear system (1.2) in finite iterations, then the whole process can be considered as a regularization method.

4. Experimental Results

In this section, we present the results of numerical experiments aimed to show the effectiveness of the regularized global GMERR method for solving large-scale linear discrete ill-posed problems with multiple right-hand sides. All the numerical experiments

are performed on personal computer (3.4 GHz Intel Core i5, 8 GB 1600 MHZ DDR3) using MATLAB R2020b with double precision. RGGMERR, RGMERR and RGGMRES denote the regularized global GMERR method, regularized GMERR method [49] equipped with Stopping Rule 3.1 to solve the ill-posed problem $AX = B$ for each of the right-hand sides separately and regularized global GMRES method [51], respectively. For all the experiments, the initial guess is $X_0 = \mathbf{0} \in R^{n\times s}$, the exact solution of the ill-posed problems is $\hat{X} = \texttt{ones}(n,s) \in R^{n\times s}$, the error-free right-hand side is $\hat{B} = A\hat{X}$, and the noise matrix is $E = \delta \times \hat{E}/\|\hat{E}\|_F$, $\hat{E} = \texttt{rand}(n,s)$, where function `rand` creates an $n \times s$ random matrix with entries uniformly distributed in $[0,1]$, $\delta = 10^{-2}, 10^{-3}, 10^{-4}$ and $B = \hat{B} + E$. We take also the parameter $\alpha = 1$ in Stopping Rule 3.1 for all the numerical experiments. In the tables, k_δ and CPU denote the termination index and the computing time (in seconds) for an algorithm to obtain an approximate solution of the linear system (1.2), respectively, and Res and Err are defined as follows:

$$\text{Res} = \|B - AX_{k_\delta}\|_F, \quad \text{Err} = \|\hat{X} - X_{k_\delta}\|_F / \|\hat{X}\|_F.$$

Example 4.1. Let H_n denote the Hilbert matrix of order n. Consider the linear discrete ill-posed problem $AX = B$, where $A = H_{5000}$, its condition number is 6.7909×10^{22}, and $B \in \mathbf{R}^{5000\times s}$, s is set to $2, 3, 4$. This is a typical ill-posed problem.

Applying the regularized global GMERR method, the regularized global GMRES method and the regularized GMERR method to solve the ill-posed problem $AX = B$, we obtain the numerical results reported in Table 1. It can be observed from Table 1 that the computing time of these three methods grows with s for the same error level δ and increases when the error level δ decreases for the same s. Both Res and Err of the regularized global GMERR method are almost at the same level as those of the other two methods, which indicates that the regularized global GMERR method is feasible but time-consuming compared with the regularized global GMRES method. While compared with the regularized GMERR method, the regularized global GMERR method has great advantage in computing time.

The relation between the dimension k of the matrix Krylov subspace $K_k(A^\intercal, B)$ and log(Res) for the Hilbert matrix H_{5000}, $s = 3$ and $\delta = 10^{-2}$ is displayed in Fig. 1. We observe

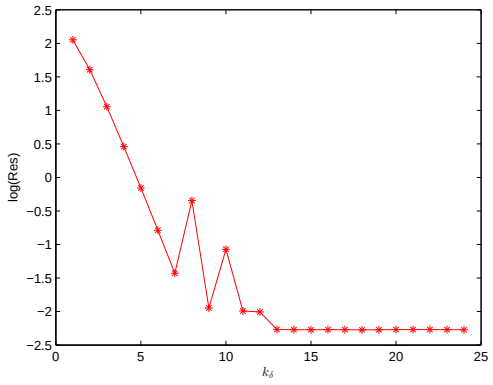

Figure 1: The relation between k_δ and log(Res).

Table 1: Comparison of RGGMERR, RGGMRES and RGMERR for Hilbert.

δ	s	Methods	k_δ	Res	Err	CPU
10^{-2}	2	RGGMERR	12	8.5565e-3	0.0335	3.7134
		RGGMRES	9	9.3192e-3	0.0337	2.1082
		RGMERR	11	9.3877e-3	0.0354	4.2352
	3	RGGMERR	13	9.3452e-3	0.0348	5.0938
		RGGMRES	10	8.9361e-3	0.0322	2.2662
		RGMERR	11	9.8809e-3	0.0366	7.9856
	4	RGGMERR	13	9.2863e-3	0.0199	5.7628
		RGGMRES	10	9.9958e-3	0.0325	2.1798
		RGMERR	11	9.9871e-3	0.0304	10.7382
10^{-3}	2	RGGMERR	15	7.0341e-4	0.0119	6.9790
		RGGMRES	14	5.6557e-4	0.0111	3.4270
		RGMERR	14	8.6328e-4	0.0189	10.3726
	3	RGGMERR	15	6.3321e-4	0.0119	6.9236
		RGGMRES	14	6.0167e-4	0.0111	3.6086
		RGMERR	14	7.3925e-4	0.0241	16.3463
	4	RGGMERR	15	6.6219e-4	0.0120	7.0165
		RGGMRES	14	6.3366e-4	0.0112	3.6221
		RGMERR	14	8.2483e-4	0.0141	20.4536
10^{-4}	2	RGGMERR	17	8.0701e-5	0.0072	8.1510
		RGGMRES	16	7.7874e-5	0.0068	4.1790
		RGMERR	15	9.0438e-5	0.0101	11.5835
	3	RGGMERR	21	9.3777e-5	0.0074	11.6664
		RGGMRES	16	9.2851e-5	0.0070	4.2556
		RGMERR	20	9.9805e-5	0.0098	20.4536
	4	RGGMERR	25	6.7088e-5	0.0043	15.2663
		RGGMRES	17	9.9694e-5	0.0070	4.7271
		RGMERR	23	7.4536e-5	0.0062	27.4836

that the Res hardly decreases when the dimension k of the matrix Krylov subspace $K_k(A^\intercal, B)$ is greater than 13, which coincides with the results in Table 1. Other numerical results also show that the Stopping Rule 3.1 is reliable.

Example 4.2. Consider the linear discrete ill-posed problem $AX = B$, where $A \in \mathbf{R}^{500\times 500}$ is the Heat matrix [21], its condition number is 4.9566×10^{124}, $B \in \mathbf{R}^{500\times s}$, s is set to $2, 3, 4$. This is a severely ill-posed problem.

Apply the regularized global GMERR method, the regularized global GMRES method and the regularized GMERR method to solve the ill-posed problem $AX = B$, the numerical results are displayed in Table 2. It can be observed from Table 2 that the regularized global GMERR method has great advantage in Err compared with the regularized global GMRES method and consumes less computing time than the regularized GMERR method.

Table 2: Comparison of RGGMERR, RGGMRES and RGMERR for Heat.

δ	s	Methods	k_δ	Res	Err	CPU
10^{-2}	2	RGGMERR	7	6.0144e-3	0.0404	0.2366
		RGGMRES	7	6.0378e-3	1.2179	0.2436
		RGMERR	6	8.0834e-3	0.0524	0.3245
	3	RGGMERR	7	6.6424e-3	0.0405	0.2448
		RGGMRES	7	8.5898e-3	1.5039	0.2342
		RGMERR	6	8.4367e-3	0.0544	0.4132
	4	RGGMERR	7	7.0828e-3	0.0405	0.2504
		RGGMRES	7	6.7607e-3	1.7419	0.2471
		RGMERR	7	8.3425e-3	0.0542	0.5846
10^{-3}	2	RGGMERR	11	7.0494e-4	0.0211	0.4063
		RGGMRES	9	7.7550e-4	0.6348	0.3471
		RGMERR	10	9.8954e-4	0.0243	0.5643
	3	RGGMERR	10	8.8340e-4	0.0208	0.3672
		RGGMRES	9	7.1527e-4	0.7821	0.3086
		RGMERR	10	9.5647e-4	0.0234	0.7567
	4	RGGMERR	10	8.9478e-4	0.0211	0.4171
		RGGMRES	9	7.8533e-4	0.9078	0.3500
		RGMERR	9	9.6749e-4	0.0243	1.0043
10^{-4}	2	RGGMERR	12	7.7272e-5	0.0105	0.4678
		RGGMRES	11	7.4956e-5	0.3268	0.4126
		RGMERR	11	8.4674e-5	0.0171	0.6342
	3	RGGMERR	12	8.6041e-5	0.0105	0.4669
		RGGMRES	11	8.3132e-5	0.4039	0.4408
		RGMERR	10	9.5042e-5	0.0154	0.9824
	4	RGGMERR	12	9.4628e-5	0.0106	0.4909
		RGGMRES	11	9.7426e-5	0.4716	0.4452
		RGMERR	11	9.8988e-5	0.0184	1.6632

Example 4.3. Consider the large-scale linear ill-posed problem $AX = B$ with the PRblurdefocus matrix $A \in \mathbf{R}^{4096\times4096}$ arising in image deblurring [15]. Besides, $B \in \mathbf{R}^{4096\times s}$ with s set to $2, 5, 10$. The condition number of the matrix A is 4.1628×10^{16}.

Applying the regularized global GMERR method, the regularized global GMRES method and the regularized GMERR method to solve the ill-posed problem $AX = B$, we obtain the numerical results shown in Table 3. We observe from Table 3 that the Err of the regularized global GMERR method is slightly smaller than those of both the regularized global GMRES method and the regularized GMERR method, while the computing time of the regularized global GMERR method is slightly more than that of the regularized global GMRES method, but less than that of the regularized GMERR method. The effect of image restoration is shown in Fig. 2, which shows that the regularized global GMERR method is efficient for image deblurring problem PRblurdefocus.

Table 3: Comparison of RGGMERR, RGGMRES and RGMERR for PRblurdefocus.

δ	s	Methods	k_δ	Res	Err	CPU
10^{-2}	2	RGGMERR	2	4.8174e-3	1.2330e-4	0.2049
		RGGMRES	3	4.1253e-3	1.6165e-4	0.1699
		RGMERR	2	5.2585e-3	1.5261e-4	0.2548
	5	RGGMERR	2	4.8159e-3	7.8002e-5	0.2052
		RGGMRES	3	4.1214e-3	1.0194e-4	0.1783
		RGMERR	2	5.2458e-3	1.2031e-4	0.6201
	10	RGGMERR	2	4.8355e-3	5.5087e-5	0.2122
		RGGMRES	3	4.1764e-3	7.1605e-5	0.1786
		RGMERR	2	5.2654e-3	1.0245e-4	1.2541
10^{-3}	2	RGGMERR	2	4.7280e-4	1.2357e-5	0.2046
		RGGMRES	3	4.1258e-4	1.6073e-5	0.1719
		RGMERR	2	5.4673e-4	1.7657e-5	0.2897
	5	RGGMERR	2	4.7484e-4	7.8320e-6	0.2118
		RGGMRES	3	4.1316e-4	1.0203e-5	0.1754
		RGMERR	2	6.5857e-4	1.8594e-5	0.7654
	10	RGGMERR	2	4.8461e-4	5.5024e-6	0.2232
		RGGMRES	3	4.0484e-4	7.2061e-6	0.1891
		RGMERR	2	6.4673e-4	9.4364e-6	1.4543
10^{-4}	2	RGGMERR	2	4.7794e-5	1.2345e-6	0.2228
		RGGMRES	3	4.0983e-5	1.6023e-6	0.1767
		RGMERR	2	5.9864e-5	2.5944e-6	0.3042
	5	RGGMERR	2	4.8897e-5	7.7607e-7	0.2200
		RGGMRES	3	4.1392e-5	1.0198e-6	0.1777
		RGMERR	2	6.7855e-5	1.8965e-6	0.8232
	10	RGGMERR	2	4.8100e-5	5.5141e-7	0.2150
		RGGMRES	3	4.1052e-5	7.1536e-7	0.1846
		RGMERR	2	6.7985e-5	1.0233e-6	1.5643

5. Conclusion

For large-scale linear discrete ill-posed problems, iterative solvers have received considerable attention over the decades. Of them, the Krylov iterative solvers are most widely used. However, there are few studies on regularizing properties and regularization of the GMERR method. In this paper, we analyze the regularizing properties of the global GMERR method covering the GMERR method as a special case, develop the regularized global GMERR method for solving large-scale linear discrete ill-posed problems with several right-hand sides, and present some numerical experiments on typical test matrices to demonstrate the effectiveness of the proposed method. The superior performance of the regularized global GMERR method over both the regularized GMERR method and the regularized global GMRES method can be observed from our numerical experiments.

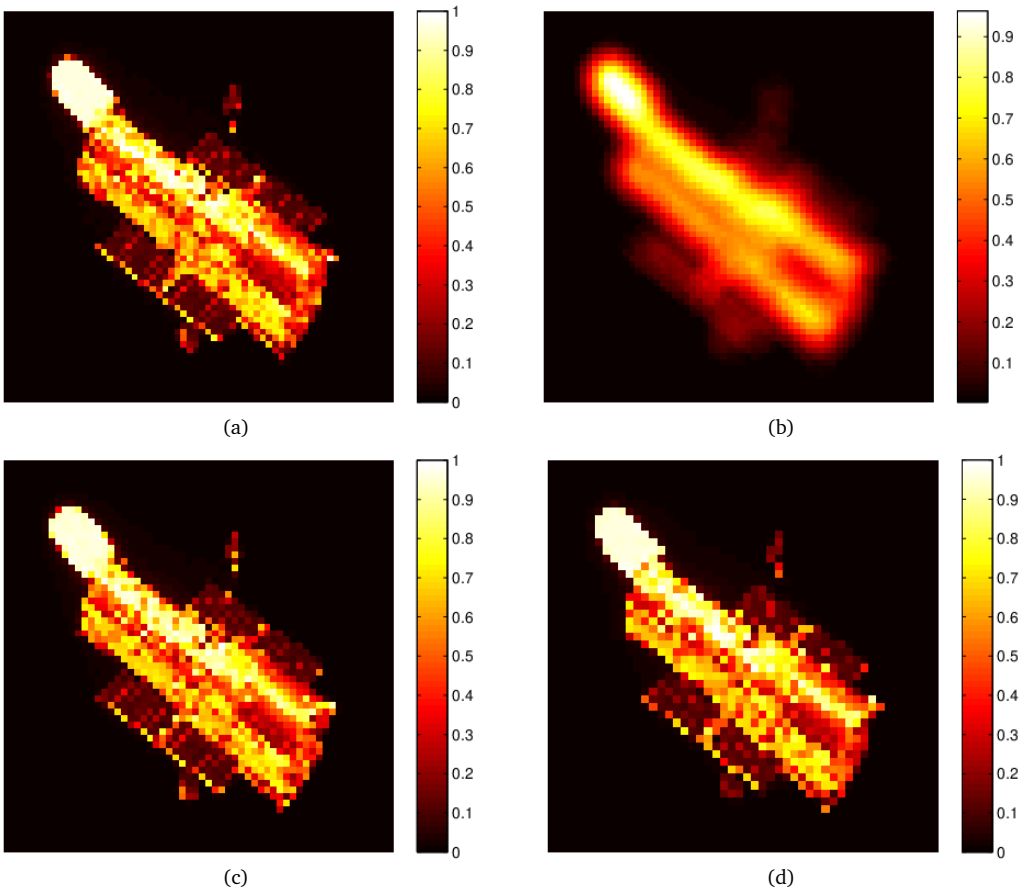

Figure 2: (a) true image (64×64); (b) noised image with $\delta = 1e-2$; (c) restored image by regularized global GMERR method; (d) restored image by regularized global GMRES method.

If the upper bound δ of the noise matrix is unavailable, the generalized cross-validation (GCV) [22] may be used together with the regularized global GMERR method to determine the iteration number as the regularization parameter.

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