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TAMED STOCHASTIC RUNGE-KUTTA-CHEBYSHEV METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-GLOBALLY LIPSCHITZ COEFFICIENTS*

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Abstract

In this paper, we introduce a new class of explicit numerical methods called the tamed stochastic Runge-Kutta-Chebyshev (t-SRKC) methods, which apply the idea of taming to the stochastic Runge-Kutta-Chebyshev (SRKC) methods. The key advantage of our explicit methods is that they can be suitable for stochastic differential equations with non-globally Lipschitz coefficients and stiffness. Under certain non-globally Lipschitz conditions, we study the strong convergence of our methods and prove that the order of strong convergence is 1/2. To show the advantages of our methods, we compare them with some existing explicit methods (including the Euler-Maruyama method, balanced Euler-Maruyama method and two types of SRKC methods) through several numerical examples. The numerical results show that our t-SRKC methods are efficient, especially for stiff stochastic differential equations.

Mathematics subject classification: 65C30, 60H10, 60H35.
Key words: Stochastic differential equation, Non-globally Lipschitz coefficient, Stiffness, Explicit tamed stochastic Runge-Kutta-Chebyshev method, Strong convergence.

1. Introduction

Stochastic differential equations (SDEs) have a wide range of applications in simulating problems across various fields such as biology, chemistry, physics, and economics [6, 16, 21, 22, 28, 29]. Since obtaining analytical solutions of most SDEs directly is often difficult, the development of effective numerical methods for approximating stochastic equations has become the focus of research. Under the classical assumptions that drift and diffusion coefficients meet the global Lipschitz condition and linear growth condition, there have been many significant research results about the numerical solutions of SDEs. The most commonly used methods include Euler-Maruyama (EM) method [18], Milstein method [19], etc.

However, the coefficients of many important SDEs do not satisfy the global Lipschitz condition. This leads to the divergence of traditional explicit numerical methods, see e.g. [7, 20]. To overcome this difficulty, many modified explicit methods for the SDEs with non-globally Lipschitz continuous coefficients have been proposed, see e.g. [11–13, 17]. In particular, the balanced and tamed explicit methods have received much attention from scholars in recent years, for example, the balanced EM method [24], the balanced Milstein method [31], the split-step balanced θ-method [14], the semi-tamed Milstein method [15], the tamed Milstein method [27], and the tamed Runge-Kutta methods [4].

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In fact, the SDEs from application problems may not only have coefficients which do not meet global Lipschitz condition, but may also be stiff, for example, see [9]. Although the modified explicit numerical methods proposed in [13–15, 17, 27, 31] can handle some situations where the coefficients do not satisfy the global Lipschitz condition, they are still not suitable for stiff problems. To handle stiff terms, implicit numerical methods are often used. However, when the dimensions of the equations are high and the problems are complex, it is difficult to implement the implicit methods. This limits the practicality of the implicit methods.

For ordinary differential equations with mild stiffness, the explicit Runge-Kutta-Chebyshev (RKC) methods are relatively suitable for solving such equations because of their long stability domain along the negative real axis, see e.g. [25,26]. Some scholars have extended these methods to stiff SDEs and constructed the corresponding stochastic Runge-Kutta-Chebyshev (SRKC) methods, as shown in [1,3,10]. These SRKC methods are able to inherit the advantage of RKC methods in stability. However, it should be noted that these methods can only ensure strong convergence under the global Lipschitz condition. In fact, when the equation coefficients fail to satisfy the global Lipschitz condition, these methods do not converge in some cases (see the numerical experiments section).

In the present paper, we apply the taming technique to the SRKC methods and propose the tamed stochastic Runge-Kutta-Chebyshev (t-SRKC) methods. Under certain non-globally Lipschitz conditions, the strong convergence of our t-SRKC methods is proved, and the strong convergence order is 1/2. On the one hand, our t-SRKC methods have a wider range of applications than the traditional SRKC methods proposed in [1, 3, 10] because our t-SRKC methods are applicable to some problems with non-globally Lipschitz coefficients. On the other hand, for the stiff SDEs, our t-SRKC methods can avoid the step size restriction problem which the existing balanced or tamed explicit methods (such as the balanced EM method in [24]) suffer from. Finally, the numerical results well verify the advantages of our t-SRKC methods in the above two aspects.

The remaining sections of this paper are structured as follows. In Section 2, we introduce some useful assumptions and conclusions. In Section 3, we propose the t-SRKC methods (3.10). We discuss the boundedness of moments for the t-SRKC methods (3.10) in Section 4. In Section 5, we study the strong convergence of the t-SRKC methods (3.10). Finally, we present our numerical results in Section 6.

2. Preliminary

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_t^W$ be an increasing family of σ-subalgebras of $\mathcal{F}$ induced by $W(t)$ with $0 \le t \le T$, where $W(t) = (W_1(t), \ldots, W_m(t))^\top$ is an m-dimensional standard Wiener process. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. In this paper, we study the following system of Itô SDEs:

$$dX(t) = f\big(t, X(t)\big)dt + \sum_{r=1}^{m} g_r\big(t, X(t)\big)dW_r(t), \quad t \in (t_0, T], \quad X(t_0) = X_0, \tag{2.1}$$

where X, f, g_r are d-dimensional column-vectors. We suppose that the solution $X_{t_0,X_0}(t)$ of (2.1) is well-defined on $[t_0, T]$.

We consider the equidistant discretization $\mathcal{T}_h : t_0 < t_1 < t_2 < \cdots < t_N = T$ of the time interval $[t_0, T]$ with $t_j = t_0 + jh, h = (T - t_0)/N$ for $j = 0, 1, \ldots, N$. For convenience, we use C to represent a generic constant, which may represent different values in different places.

In order to formulate our convergence results for the t-SRKC method (3.10), we introduce the following assumptions.

Assumption 2.1 ([24]). (i) The initial condition satisfies that for all $p \geq 1$,

$$\mathbb{E}|X_0|^{2p} \leq C < \infty. \tag{2.2}$$

(ii) For a sufficiently large constant $p_0 \geq 2$, there is a constant $c_1 \geq 0$ such that for $t \in [t_0, T]$ and $x_1, x_2 \in \mathbb{R}^d$,

$$\big(x_1 - x_2, f(t,x_1) - f(t,x_2)\big) + \frac{2p_0 - 1}{2} \sum_{r=1}^{m} |g_r(t,x_1) - g_r(t,x_2)|^2 \leq c_1 |x_1 - x_2|^2. \tag{2.3}$$

(iii) There exists constants $c_2 \geq 0$ and $\varkappa \geq 1$ such that for $t \in [t_0, T]$ and $x_1, x_2 \in \mathbb{R}^d$,

$$|f(t,x_1) - f(t,x_2)|^2 \leq c_2\big(1 + |x_1|^{2\varkappa-2} + |x_2|^{2\varkappa-2}\big)|x_1 - x_2|^2. \tag{2.4}$$

The above assumptions can guarantee p-th-order bounded moments of the solution [24], i.e.

$$\mathbb{E}|X_{t_0,X_0}(t)|^{2p} < C\big(1 + \mathbb{E}|X_0|^{2p}\big), \quad 1 \leq p \leq p_0 - 1, \quad t \in [t_0, T]. \tag{2.5}$$

For all $t \in [t_0, T]$ and $x \in \mathbb{R}^d$, from Assumption 2.1 and mean value theorem, we have

$$\big(x, f(t,x)\big) + \frac{2p_0 - 3}{2} \sum_{r=1}^{m} |g_r(t,x)|^2 \leq C(1 + |x|^2), \tag{2.6}$$

$$|f(t,x)|^2 \leq C(1 + |x|^{2\varkappa}), \quad \sum_{r=1}^{m} |g_r(t,x)|^2 \leq C(1 + |x|^{2\varkappa}). \tag{2.7}$$

Consider the scalar stochastic Ginzburg-Landau equation in [8]

$$\mathrm{d}X(t) = \left(\left(a_1 + \frac{1}{2}a_2^2\right) X(t) - a_3 X^3(t)\right) dt + a_2 X(t) \mathrm{d}W(t), \quad X(0) = X_0 > 0, \tag{2.8}$$

where the constants $a_1, a_2, a_3 > 0$. We set $t \in [0,1], a_1 = 1$ and $a_2 = 10$. For the case 1, let $X_0 = 8, a_3 = 1$, and for the case 2, consider $X_0 = 1000, a_3 = 1000$. Clearly, for the case 1, the coefficients

$$f(t,X) = 51X - X^3, \quad g(t,X) = 10X.$$

To verify Assumption 2.1(ii), we note that

$$\begin{aligned}
&\big(X_1 - X_2, f(t,X_1) - f(t,X_2)\big) \\
&= (X_1 - X_2)\big(51(X_1 - X_2) - \big(X_1^3 - X_2^3\big)\big) \\
&\leq 51(X_1 - X_2)^2 - (X_1 - X_2)^2\big(X_1^2 + X_1X_2 + X_2^2\big) \\
&\leq 51(X_1 - X_2)^2 - \frac{1}{2}(X_1 - X_2)^2\big(X_1^2 + X_2^2\big) \\
&\leq 51(X_1 - X_2)^2.
\end{aligned}$$

Hence, for $p_0 = 2$,

$$\big(X_1 - X_2, f(t,X_1) - f(t,X_2)\big) + \frac{2p_0 - 1}{2} \sum_{r=1}^{m} |g_r(t,X_1) - g_r(t,X_2)|^2$$

$$\leq 51(X_1 - X_2)^2 + \frac{3}{2}(10X_1 - 10X_2)^2$$
$$= 201(X_1 - X_2)^2.$$

We have hence verified Assumption 2.1(ii) with $p_0 = 2$ and $c_1 = 201$. It is also straightforward to show that Assumption 2.1(iii) is satisfied with $\varkappa = 3$ and $c_2 = 5202$. For the case 2, we can similarly verify that it also satisfies Assumptions 2.1(ii) and 2.1(iii).

It is worth noting that for the case 2, the coefficients of the Eq. (2.8) not only do not meet the non-globally Lipschitz condition but also have stiffness. In this case, the convergence of traditional explicit numerical methods cannot be guaranteed, and the balanced or tamed explicit EM methods face strict step size restrictions (see the numerical experiment Example 6.2). To overcome the above difficulties, we apply taming technique to SRKC methods and construct t-SRKC methods (see Section 3). The results of numerical Example 6.2 show that our t-SRKC methods can effectively solve the case 2.

The Eqs. (2.1) can be written as an integral form

$$X_{t,x}(t+h) = x + \int_t^{t+h} f\big(\theta, X_{t,x}(\theta)\big)d\theta + \int_t^{t+h} \sum_{r=1}^{m} g_r\big(\theta, X_{t,x}(\theta)\big)dW_r(\theta). \tag{2.9}$$

Consider the following general one-step approximation for the Eq. (2.9), i.e.

$$\bar{X}_{t,x}(t+h) = x + B\big(t, x, h, W(t+h) - W(t)\big), \tag{2.10}$$

while B is a mapping depending on t, x, h and the increment of Brownian motion $W(t+h)-W(t)$. Let $X_0 = X(t_0)$, according to the one-step approximation (2.10), we can get the following approximation:

$$X_{n+1} = \bar{X}_{t_n,\bar{X}_n}(t_{n+1}) = X_n + B\big(t_n, X_n, h, W(t_{n+1}) - W(t_n)\big), \quad n = 0, 1, \ldots, N. \tag{2.11}$$

Theorem 2.1 ([24]). *Suppose that*

(i) Assumption 2.1 *holds.*

(ii) The one-step approximation $\bar{X}_{t,x}(t+h)$ from (2.10) *has the following orders of accuracy: For some $p \geq 1$ there exist $\alpha \geq 1, h_0 > 0$ and $C > 0$ such that for arbitrary $t_0 \leq t \leq T - h$, $x \in \mathbb{R}^d$, and all $0 < h \leq h_0$,*

$$\big|\mathbb{E}[X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)]\big| \leq C(1 + |x|^{2\alpha})^{\frac{1}{2}} h^{q_1}, \tag{2.12}$$
$$\big[\mathbb{E}|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)|^{2p}\big]^{\frac{1}{2p}} \leq C(1 + |x|^{2\alpha p})^{\frac{1}{2p}} h^{q_2} \tag{2.13}$$

with

$$q_2 \geq \frac{1}{2}, \quad q_1 \geq q_2 + \frac{1}{2}. \tag{2.14}$$

(iii) The approximation X_n from (2.11) *has p-th-order bounded moments, i.e. for some $p \geq 1$ there are $\beta \geq 1$ and $h_0 > 0$ such that for all $0 < h \leq h_0$ and all $n = 0, 1, \ldots, N$,*

$$\mathbb{E}|X_n|^{2p} < C\big(1 + \mathbb{E}|X_0|^{2p\beta}\big). \tag{2.15}$$

Then for any N and $n = 0, 1, \ldots, N$, the following inequality holds:

$$\big[\mathbb{E}|X_{t_0,X_0}(t_n) - \bar{X}_{t_0,X_0}(t_n)|^{2p}\big]^{\frac{1}{2p}} \leq M\big(1 + \mathbb{E}|X_0|^{2\gamma p}\big)^{\frac{1}{2p}} h^{q_2 - \frac{1}{2}}, \tag{2.16}$$

where $M \geq 0, \gamma \geq 1$ do not depend on h and n, i.e. the order of accuracy of the method (2.11) *is $q = q_2 - 1/2$.*

Using this theorem, we can obtain the convergence order of our method introduced in the next section.

3. Tamed Stochastic Runge-Kutta-Chebyshev Methods

In this section, we will introduce our t-SRKC methods. Let us start with a brief review of the RKC methods. In the context of deterministic differential equations

$$\frac{dX(t)}{dt} = f\big(t, X(t)\big), \quad X(0) = X_0. \tag{3.1}$$

The first-order RKC methods for solving Eqs. (3.1) can be written in the following form (see [25]):

$$\begin{aligned}
&K_0 = X_n,\\
&K_1 = K_0 + \mu_1 h f(t_0, K_0),\\
&K_i = \mu_i h f(t_{i-1}, K_{i-1}) + \nu_i K_{i-1} + \kappa_i K_{i-2}, \quad i = 2, 3, \ldots, s,\\
&X_{n+1} = K_s,
\end{aligned} \tag{3.2}$$

where

$$\mu_1 = \frac{\omega_1}{\omega_0}, \quad \omega_0 = 1 + \frac{\eta}{s^2}, \quad \omega_1 = \frac{T_s(\omega_0)}{T_s'(\omega_0)}, \tag{3.3}$$

η is usually a given small constant, T_s is the first kind of Chebyshev polynomials with degree s, and for all $i = 2, 3, \ldots, s$,

$$\mu_i = \frac{2\omega_1 T_{i-1}(\omega_0)}{T_i(\omega_0)}, \quad \nu_i = \frac{2\omega_0 T_{i-1}(\omega_0)}{T_i(\omega_0)}, \quad \kappa_i = -\frac{T_{i-2}(\omega_0)}{T_i(\omega_0)} = 1 - \nu_i. \tag{3.4}$$

The advantage of the deterministic first-order RKC methods is that the length of the stability domain along the negative real axis is cs^2 (the optimal value of coefficient c is 2), so they are suitable for solving some mild stiff problems by selecting the appropriate s. Due to the stability advantages of deterministic RKC methods, some scholars have extended them to SDEs and constructed SRKC methods, see [1–3, 10]. According to [3], the SRKC methods for (2.1) can be written as

$$\begin{aligned}
&K_0 = X_n,\\
&K_1 = K_0 + \mu_1 h f(t_0, K_0),\\
&K_i = \mu_i h f(t_{i-1}, K_{i-1}) + \nu_i K_{i-1} + \kappa_i K_{i-2}, \quad i = 2, 3, \ldots, s,\\
&X_{n+1} = K_s + \sum_{r=1}^{m} g_r(t_{s-1}, K_{s-1}) \Delta W_r,
\end{aligned} \tag{3.5}$$

where the coefficients $\mu_1, \mu_i, \nu_i, \kappa_i, i = 2, 3, \ldots, s$ are defined by (3.3), (3.4).

It can be seen from the literatures [1–3,10] that the SRKC methods can inherit the stability advantages of RKC methods. However, the convergence of these methods can only be guaranteed under the global Lipschitz condition. When the coefficients do not satisfy the global Lipschitz condition, these methods may fail to converge, and specific details can be seen in the numerical experiment part. To obtain explicit numerical methods for the problems whose coefficients only satisfy Assumption 2.1 and which are stiff, we apply the tamed technique to the SRKC method (3.5) and construct the t-SRKC method (3.6) as follows:

$$
\begin{aligned}
&K_0 = X_n, \\
&K_1 = K_0 + \frac{\mu_1 h f(t_0, K_0)}{1 + |\mu_1 h f(t_0, K_0)|}, \\
&K_i = \frac{\mu_i h f(t_{i-1}, K_{i-1})}{1 + |\mu_i h f(t_{i-1}, K_{i-1})|} + \nu_i K_{i-1} + \kappa_i K_{i-2}, \quad i = 2, 3, \ldots, s, \\
&X_{n+1} = K_s + \frac{\sum_{r=1}^m g_r(t_{s-1}, K_{s-1}) \Delta W_r}{1 + \left| \sum_{r=1}^m g_r(t_{s-1}, K_{s-1}) \Delta W_r \right|},
\end{aligned} \tag{3.6}
$$

where the coefficients $\mu_1, \mu_i, \nu_i, \kappa_i, i = 2, 3, \ldots, s$ are defined by (3.3), (3.4).

Define the following $(s+1) \times (s+1)$ matrix A, and let $A(i, j)$ denote the element in i-th row and j-th column,

$$
A(i,j) = \begin{cases}
0, & i=1,2,\ldots,s+1, \quad j=i, i+1, \ldots, s+1, \\
\mu_{i-1}, & i=2,3,\ldots,s+1, \quad j=i-1, \\
v_{i-1}\mu_{i-2}, & i=3,4,\ldots,s+1, \quad j=i-2, \\
v_{i-1}A(i-1,j) + \kappa_{i-1}A(i-2,j), & i=4,5,\ldots,s+1, \quad j=1,2,\ldots,i-3,
\end{cases} \tag{3.7}
$$

where the coefficients $\mu_1, \mu_2, \nu_2, \mu_i, \nu_i, \kappa_i, i = 3, 4, \ldots, s$ are defined by (3.3), (3.4). After a simple calculation, we have

$$
\sum_{j=1}^{s} A(s+1, j) = 1. \tag{3.8}
$$

Using matrix A, the method (3.5) can be equivalently rewritten in the form

$$
\begin{aligned}
&K_i = X_n + \sum_{j=1}^{i} A(i+1, j) h f(t_{j-1}, K_{j-1}), \quad i = 1, 2, \ldots, s, \\
&X_{n+1} = X_n + \sum_{j=1}^{s} A(s+1, j) h f(t_{j-1}, K_{j-1}) + \sum_{r=1}^{m} g_r(t_{s-1}, K_{s-1}) \Delta W_r.
\end{aligned} \tag{3.9}
$$

Similarly, the method (3.6) can be equivalently rewritten in the form

$$
\begin{aligned}
&K_i = X_n + \sum_{j=1}^{i} \frac{A(i+1, j) h f(t_{j-1}, K_{j-1})}{1 + |\mu_j h f(t_{j-1}, K_{j-1})|}, \quad i = 1, 2, \ldots, s, \\
&X_{n+1} = X_n + \sum_{j=1}^{s} \frac{A(s+1, j) h f(t_{j-1}, K_{j-1})}{1 + |\mu_j h f(t_{j-1}, K_{j-1})|} + \frac{\sum_{r=1}^m g_r(t_{s-1}, K_{s-1}) \Delta W_r}{1 + \sum_{r=1}^m |g_r(t_{s-1}, K_{s-1}) \Delta W_r|}.
\end{aligned} \tag{3.10}
$$

4. Boundedness of Moments for the Tamed Runge-Kutta-Chebyshev Methods

The following theorem shows that the t-SRKC method (3.10) satisfies the moments boundedness conditions.

Theorem 4.1. *Suppose Assumption* 2.1 *holds with sufficiently large* $p_0 \geq 2$. *Then for all* $n = 0, 1, \ldots, N$, *the following inequality holds for* X_n, *i.e.*

$$\mathbb{E}|X_n|^{2p} \leq M\big(1 + \mathbb{E}|X_0|^{2p\beta}\big), \quad 1 \leq p \leq \frac{p_0 - 1}{2G(\varkappa)} - \frac{1}{2}, \tag{4.1}$$

where the constants $\beta \geq 1, M \geq 0$ *are independent of* h, n, *and* $G(\varkappa) = 6\varkappa - 4$.

Proof. Let just consider $0 < h < 1$ and $f(t, X)$ is non-globally Lipschitz ($\varkappa > 1$). M represents different values in different places and is always independent of h, n. According to the method (3.10), we have

$$|X_{n+1}| \leq |X_n| + s + 1 \leq |X_0| + (n+1)(s+1). \tag{4.2}$$

The inequality (4.2) provides a relationship between X_{n+1} and X_n, which is the basis for subsequent analysis.

Inspired by literature [24], for $R > 0$, introduce the events

$$\tilde{\Omega}_{R,n} := \{\omega : |X_l| \leq R,\, l = 0, \ldots, n\}, \tag{4.3}$$

and their complements are denoted as $\tilde{\Lambda}_{R,n}$. We first prove that inequality (4.1) holds for the positive integer $p \geq 1$. Based on (4.3), we have

$$\begin{aligned}
\mathbb{E}\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|X_{n+1}|^{2p} &\leq \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_{n+1}|^{2p} \\
&= \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|(X_{n+1} - X_n) + X_n|^{2p} \\
&\leq \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} \\
&\quad + \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-2}\big[2p(X_n, X_{n+1} - X_n) + p(2p-1)|X_{n+1} - X_n|^2\big] \\
&\quad + M\sum_{l=3}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}|X_{n+1} - X_n|^l.
\end{aligned} \tag{4.4}$$

For the right second term in (4.4), we have

$$\begin{aligned}
&\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-2}\big[2p(X_n, X_{n+1} - X_n) + p(2p-1)|X_{n+1} - X_n|^2\big] \\
&= 2p\mathbb{E}\Bigg(\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-2}\mathbb{E}\Bigg[\Bigg(X_n, \sum_{j=1}^{s}\frac{A(s+1,j)hf(t_{j-1}, K_{j-1})}{1 + |\mu_j hf(t_{j-1}, K_{j-1})|} \\
&\qquad + \frac{\sum_{r=1}^{m} g_r(t_{s-1}, K_{s-1})\Delta W_r}{1 + \sum_{r=1}^{m}|g_r(t_{s-1}, K_{s-1})\Delta W_r|}\Bigg) \\
&\qquad + \frac{2p-1}{2}\Bigg|\sum_{j=1}^{s}\frac{A(s+1,j)hf(t_{j-1}, K_{j-1})}{1 + |\mu_j hf(t_{j-1}, K_{j-1})|} \\
&\qquad + \frac{\sum_{r=1}^{m} g_r(t_{s-1}, K_{s-1})\Delta W_r}{1 + \sum_{r=1}^{m}|g_r(t_{s-1}, K_{s-1})\Delta W_r|}\Bigg|^2\Bigg|\mathcal{F}_{t_n}\Bigg]\Bigg).
\end{aligned} \tag{4.5}$$

To make it easier to describe, let

$$J := \chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg[\Bigg(X_n, \sum_{j=1}^{s}\frac{A(s+1,j)hf(t_{j-1}, K_{j-1})}{1 + |\mu_j hf(t_{j-1}, K_{j-1})|} + \frac{\sum_{r=1}^{m} g_r(t_{s-1}, K_{s-1})\Delta W_r}{1 + \sum_{r=1}^{m}|g_r(t_{s-1}, K_{s-1})\Delta W_r|}\Bigg)$$

$$+\frac{2p-1}{2}\Bigg|\sum_{j=1}^{s}\frac{A(s+1,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|}$$
$$+\frac{\sum_{r=1}^{m}g_r(t_{s-1},K_{s-1})\Delta W_r}{1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|}\Bigg|^2\Bigg|\mathcal{F}_{t_n}\Bigg].$$

Similar to the [24, Eqs. (3.8), (3.9)], it is not difficult to show that

$$\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\left[\frac{\sum_{r=1}^{m}g_r(t_{s-1},K_{s-1})\Delta W_r}{1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|}\Bigg|\mathcal{F}_{t_n}\right]=0, \tag{4.6}$$
$$\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\left[\frac{g_r(t_{s-1},K_{s-1})\Delta W_r g_l(t_{s-1},K_{s-1})\Delta W_l}{\big(1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|\big)^2}\Bigg|\mathcal{F}_{t_n}\right]=0,\quad r\neq l. \tag{4.7}$$

Using the Eqs. (4.6), (4.7), we have

$$\begin{aligned}
J\le{}&\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg[\sum_{j=1}^{s}\frac{\big(X_n,A(s+1,j)hf(t_{j-1},K_{j-1})\big)}{1+|\mu_j hf(t_{j-1},K_{j-1})|}+\frac{2p-1}{2}\frac{\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|^2}{\big(1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|\big)^2}\\
&+\frac{2p-1}{2}\Bigg|\sum_{j=1}^{s}A(s+1,j)hf(t_{j-1},K_{j-1})\Bigg|^2\Bigg|\mathcal{F}_{t_n}\Bigg]\\
={}&\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg[\sum_{j=1}^{s-1}\frac{\big(X_n,A(s+1,j)hf(t_{j-1},K_{j-1})\big)}{1+|\mu_j hf(t_{j-1},K_{j-1})|}\\
&+\frac{\big(X_n-K_{s-1},A(s+1,s)hf(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}\\
&+\frac{\big(A(s+1,s)-1\big)h\big(K_{s-1},f(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}+\frac{h\big(K_{s-1},f(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}\\
&+\frac{2p-1}{2}\frac{h\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^2}{1+|\mu_s hf(t_{s-1},K_{s-1})|}-\frac{2p-1}{2}\frac{h\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^2}{1+|\mu_s hf(t_{s-1},K_{s-1})|}\\
&+\frac{2p-1}{2}\frac{\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|^2}{\big(1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|\big)^2}\\
&+\frac{2p-1}{2}\Bigg|\sum_{j=1}^{s}A(s+1,j)hf(t_{j-1},K_{j-1})\Bigg|^2\Bigg|\mathcal{F}_{t_n}\Bigg]\\
={}&J_1+J_2+J_3,
\end{aligned}\tag{4.8}$$

where

$$\begin{aligned}
J_1={}&\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg(\sum_{j=1}^{s-1}\frac{\big(X_n,A(s+1,j)hf(t_{j-1},K_{j-1})\big)}{1+|\mu_j hf(t_{j-1},K_{j-1})|}\\
&+\frac{\big(X_n-K_{s-1},A(s+1,s)hf(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}\Bigg|\mathcal{F}_{t_n}\Bigg),\\
J_2={}&\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg(\frac{\big(A(s+1,s)-1\big)h\big(K_{s-1},f(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}\\
&+\frac{h\big(K_{s-1},f(t_{s-1},K_{s-1})\big)}{1+|\mu_s hf(t_{s-1},K_{s-1})|}+\frac{2p-1}{2}\frac{h\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^2}{1+|\mu_s hf(t_{s-1},K_{s-1})|}
\end{aligned}$$

$$+\frac{2p-1}{2}\sum_{j=1}^{s}|A(s+1,j)hf(t_{j-1},K_{j-1})|^2\bigg|\mathcal{F}_{t_n}\Bigg),$$

$$J_3=\chi_{\tilde{\Omega}_{R,n}}\mathbb{E}\Bigg(-\frac{2p-1}{2}\frac{h\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^2}{1+|\mu_s hf(t_{s-1},K_{s-1})|}+\frac{2p-1}{2}\frac{\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|^2}{(1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|)^2}\bigg|\mathcal{F}_{t_n}\Bigg).$$

Next, we will estimate J_1, J_2, J_3 one by one.

Let

$$R=R(h)=h^{-\frac{1}{G(\varkappa)}},\quad G(\varkappa)\geq 2\varkappa-2. \tag{4.9}$$

We will prove that if the condition (4.9) is satisfied, then the following inequalities hold:

$$\chi_{\tilde{\Omega}_{R,n}}\big(h|K_i|^2\big)\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big),\quad i=1,2,\ldots,s-1, \tag{4.10}$$

$$\chi_{\tilde{\Omega}_{R,n}}\big(h\big(X_n,f(t_i,K_i)\big)\big)\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big). \tag{4.11}$$

In fact, if the condition (4.9) holds, then we have

$$\begin{aligned}\chi_{\tilde{\Omega}_{R,n}}\big(h|X_n|^{2\varkappa}\big)&=\chi_{\tilde{\Omega}_{R,n}}\big(h|X_n|^{2+2\varkappa-2}\big)\leq\chi_{\tilde{\Omega}_{R,n}}\big(|X_n|^2\big),\\ \chi_{\tilde{\Omega}_{R,n}}\big(h^2|X_n|^{2\varkappa}\big)&=\chi_{\tilde{\Omega}_{R,n}}\big(h^2|X_n|^{2+2\varkappa-2}\big)\leq\chi_{\tilde{\Omega}_{R,n}}\big(|X_n|^2\big).\end{aligned} \tag{4.12}$$

Using $p_0\geq 2$, (2.6), and for $x\in\mathbb{R}^d$, we can get

$$\big(x,f(t,x)\big)\leq C(1+|x|^2). \tag{4.13}$$

From the definition of the explicit t-SRKC methods, the inequalities (2.7), (4.12) and (4.13), we have

$$\begin{aligned}\chi_{\tilde{\Omega}_{R,n}}(h|K_1|^2)&=\chi_{\tilde{\Omega}_{R,n}}\Bigg(h\Bigg(K_0+\frac{\mu_1 hf(t_0,K_0)}{1+|\mu_1 hf(t_0,K_0)|},K_0+\frac{\mu_1 hf(t_0,K_0)}{1+|\mu_1 hf(t_0,K_0)|}\Bigg)\Bigg)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(h|K_0|^2+\big|2\mu_1 h^2\big(K_0,f(t_0,K_0)\big)\big|+\mu_1^2h^3|f(t_0,K_0)|^2\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(h|X_n|^2+Mh^2\big(1+|X_n|^2\big)+Mh^3\big(1+|X_n|^{2\varkappa}\big)\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh+Mh|X_n|^2+Mh^3|X_n|^{2\varkappa}\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big).\end{aligned} \tag{4.14}$$

Using (2.7) and (4.12)-(4.14), we have

$$\begin{aligned}\chi_{\tilde{\Omega}_{R,n}}\big(h\big(X_n,f(t_1,K_1)\big)\big)&=\chi_{\tilde{\Omega}_{R,n}}\Bigg(h\Bigg(K_1-\frac{\mu_1 hf(t_0,K_0)}{1+|\mu_1 hf(t_0,K_0)|},f(t_1,K_1)\Bigg)\Bigg)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|K_1|^2\big)+Mh^2|f(t_0,K_0)|\,|f(t_1,K_1)|\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|K_1|^2\big)+Mh^2\big(1+|X_n|^{\varkappa}+|K_1|^{\varkappa}+|X_n|^{\varkappa}|K_1|^{\varkappa}\big)\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh+Mh|X_n|^2+Mh^2|X_n|^{2\varkappa}\big)\\ &\leq\chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big).\end{aligned} \tag{4.15}$$

Similarly, based on (2.7), (3.10) and (4.12)-(4.15), we have

$$\begin{aligned}&\chi_{\tilde{\Omega}_{R,n}}\big(h|K_2|^2\big)\\ &=\chi_{\tilde{\Omega}_{R,n}}\Bigg(h\Bigg(X_n+\sum_{j=1}^{2}\frac{A(3,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|},X_n+\sum_{j=1}^{2}\frac{A(3,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|}\Bigg)\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h|X_n|^2 + Mh^2\sum_{j=1}^{2}\big|\big(X_n, f(t_{j-1},K_{j-1})\big)\big| + Mh^3\sum_{j=1}^{2}|f(t_{j-1},K_{j-1})|^2\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h|X_n|^2 + Mh^2\big(1+|X_n|^2\big) + Mh^3\sum_{j=1}^{2}\big(1+|K_{j-1}|^{2\varkappa}\big)\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\big(Mh + Mh|X_n|^2 + Mh^3|X_n|^{2\varkappa}\big)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big),
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&\chi_{\tilde{\Omega}_{R,n}}\big(h\big(X_n, f(t_2,K_2)\big)\big)\\
&= \chi_{\tilde{\Omega}_{R,n}}\left(h\left(K_2 - \sum_{j=1}^{2}\frac{A(3,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|}, f(t_2,K_2)\right)\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h\big(K_2, f(t_2,K_2)\big) + Mh^2\sum_{j=1}^{2}|f(t_{j-1},K_{j-1})|\,|f(t_2,K_2)|\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\Big(Mh\big(1+|K_2|^2\big) + Mh^2\big(\big(1+|X_n|^{\varkappa}\big)+\big(1+|K_1|^{\varkappa}\big)\big)\big(1+|K_2|^{\varkappa}\big)\Big)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\big(Mh + Mh|X_n|^2 + Mh^2|X_n|^{2\varkappa}\big)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\big(Mh\big(1+|X_n|^2\big)\big).
\end{aligned}
\tag{4.17}
$$

For $i = 3, 4, \ldots, s-1$, by some simple and similar calculations, we can obtain

$$
\begin{aligned}
\chi_{\tilde{\Omega}_{R,n}}(h|K_i|^2) &= \chi_{\tilde{\Omega}_{R,n}}\left(h\left(X_n + \sum_{j=1}^{i}\frac{A(i+1,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|},\right.\right.\\
&\qquad\left.\left. X_n + \sum_{j=1}^{i}\frac{A(i+1,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|}\right)\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h|X_n|^2 + 2\sum_{j=1}^{i}A(i+1,j)h^2\big(X_n, f(t_{j-1},K_{j-1})\big)\right.\\
&\qquad\left. + \sum_{j=1}^{i}A(i+1,j)^2h^3|f(t_{j-1},K_{j-1})|^2\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h|X_n|^2 + Mh^2\big(1+|X_n|^2\big) + Mh^3\left(1+\sum_{j=1}^{i}|K_{j-1}|^{2\varkappa}\right)\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\big(Mh + Mh|X_n|^2 + Mh^3|X_n|^{2\varkappa}\big),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
\chi_{\tilde{\Omega}_{R,n}}\big(h\big(X_n, f(t_i,K_i)\big)\big) &= \chi_{\tilde{\Omega}_{R,n}}\left(h\left(K_i - \sum_{j=1}^{i}\frac{A(i+1,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|}, f(t_i,K_i)\right)\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\left(h\big(K_i, f(t_i,K_i)\big) + Mh^2\sum_{j=1}^{i}|f(t_{j-1},K_{j-1})||f(t_i,K_i)|\right)\\
&\le \chi_{\tilde{\Omega}_{R,n}}\Big(Mh\big(1+|K_i|^2\big) + Mh^2\Big(\big(1+|X_n|^{\varkappa}\big)+\big(1+|K_1|^{\varkappa}\big)+\cdots\\
&\qquad + \big(1+|K_{i-1}|^{\varkappa}\big)\Big)\big(1+|K_i|^{\varkappa}\big)\Big)
\end{aligned}
$$

$$\leq \chi_{\tilde{\Omega}_{R,n}}\big(Mh + Mh|X_n|^2 + Mh^2|X_n|^{2\varkappa}\big). \tag{4.19}$$

Combining (4.12), (4.18) and (4.19), we know that the inequalities (4.10) and (4.11) are fulfilled.

Now, based on (2.7), (4.10), (4.11) and (4.13), we have

$$\begin{aligned}
J_1 &\leq \chi_{\tilde{\Omega}_{R,n}} M\mathbb{E}\left(h\big(1+|X_n|^2\big) + h^2 \sum_{j=1}^{s-1} |f(t_{j-1},K_{j-1})|\,|f(t_{s-1},K_{s-1})| \middle| \mathcal{F}_{t_n}\right)\\
&\leq \chi_{\tilde{\Omega}_{R,n}} M\left(h + h|X_n|^2 + h^2\left(1+\sum_{j=1}^{s-1}|K_{j-1}|^{\varkappa} + |K_{s-1}|^{\varkappa} + \sum_{j=1}^{s-1}|K_{j-1}|^{\varkappa}|K_{s-1}|^{\varkappa}\right)\right)\\
&\leq \chi_{\tilde{\Omega}_{R,n}} Mh\big(1+|X_n|^2 + h|X_n|^{2\varkappa}\big).
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
J_2 &\leq \chi_{\tilde{\Omega}_{R,n}} Mh\left(1+|K_{s-1}|^2 + h\left(1+\sum_{j=1}^{s}|K_{j-1}|^{2\varkappa}\right)\right)\\
&\leq \chi_{\tilde{\Omega}_{R,n}} Mh\big(1+|X_n|^2 + h|X_n|^{2\varkappa}\big).
\end{aligned}$$

For J_3, we have

$$\begin{aligned}
J_3 = \chi_{\tilde{\Omega}_{R,n}} \mathbb{E}\Bigg(& \frac{2p-1}{2} h \sum_{r=1}^{m} |g_r(t_{s-1},K_{s-1})|^2\\
&\times \left(\frac{\big|\Delta W_r/h^{\frac{1}{2}}\big|^2}{(1+\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\Delta W_r|)^2} - \frac{1}{1+|\mu_s h f(t_{s-1},K_{s-1})|}\right)\Bigg|\mathcal{F}_{t_n}\Bigg).
\end{aligned}$$

Notice

$$\left(1+\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\Delta W_r|\right)^2 > \left(1+\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\Delta W_r|\right).$$

Let $\xi = \Delta W_r/h^{1/2}$. Then we have

$$\begin{aligned}
J_3 \leq \chi_{\tilde{\Omega}_{R,n}} \mathbb{E}\Bigg(& \frac{2p-1}{2} h \sum_{r=1}^{m} |g_r(t_{s-1},K_{s-1})|^2\\
&\times \left(\frac{\xi^2}{1+h^{\frac{1}{2}}\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\xi|} - \frac{1}{1+|\mu_s h f(t_{s-1},K_{s-1})|}\right)\Bigg|\mathcal{F}_{t_n}\Bigg)\\
= \chi_{\tilde{\Omega}_{R,n}} \mathbb{E}\Bigg(&\left(\frac{2p-1}{2} h \sum_{r=1}^{m} |g_r(t_{s-1},K_{s-1})|^2\right)(I_2+I_3)\Bigg|\mathcal{F}_{t_n}\Bigg).
\end{aligned} \tag{4.20}$$

Let

$$\begin{aligned}
I_1 &= 1 + h^{\frac{1}{2}}\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\xi| + |\mu_s h f(t_{s-1},K_{s-1})|\\
&\quad + h^{\frac{1}{2}}\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\xi||\mu_s h f(t_{s-1},K_{s-1})|,\\
I_2 &= \frac{\xi^2-1}{I_1},\\
I_3 &= \frac{\big(\xi^2|\mu_s h f(t_{s-1},K_{s-1})| - h^{\frac{1}{2}}\sum_{r=1}^m |g_r(t_{s-1},K_{s-1})\xi|\big)}{I_1}.
\end{aligned}$$

Then by (2.7) and (4.10), we have

$$\begin{aligned}\mathbb{E}[I_2] &= \mathbb{E}[I_2+\xi^2-1]\\ &= \mathbb{E}\bigg[\frac{\xi^2-1}{I_1}\bigg(h^{\frac{1}{2}}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\xi| + |\mu_s hf(t_{s-1},K_{s-1})|\\ &\qquad + h^{\frac{1}{2}}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\xi||\mu_s hf(t_{s-1},K_{s-1})|\bigg)\bigg]\\ &\le \mathbb{E}\bigg[(\xi^2-1)\bigg(h^{\frac{1}{2}}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\xi| + |\mu_s hf(t_{s-1},K_{s-1})|\\ &\qquad + h^{\frac{1}{2}}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\xi||\mu_s hf(t_{s-1},K_{s-1})|\bigg)\bigg]\\ &\le M\big(h^{\frac{1}{2}}|X_n|^{\varkappa} + h|X_n|^{\varkappa} + h^{\frac{3}{2}}|X_n|^{2\varkappa}\big), \qquad (4.21)\\ \mathbb{E}[I_3] &\le \mathbb{E}\bigg[\xi^2|\mu_s hf(t_{s-1},K_{s-1})| - h^{\frac{1}{2}}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\xi|\bigg]\\ &\le M\big(h^{\frac{1}{2}}|X_n|^{\varkappa} + h|X_n|^{\varkappa}\big). \qquad (4.22)\end{aligned}$$

Combining (2.7), (4.10) and (4.20)-(4.22), we obtain that

$$J_3 \le \chi_{\tilde{\Omega}_{R,n}} Mh\big(1 + h^{\frac{1}{2}}|X_n|^{3\varkappa} + h|X_n|^{3\varkappa} + h^{\frac{3}{2}}|X_n|^{4\varkappa}\big). \tag{4.23}$$

Therefore,

$$J \le J_1 + J_2 + J_3 \le \chi_{\tilde{\Omega}_{R,n}} Mh\big(1 + |X_n|^2 + h^{\frac{1}{2}}|X_n|^{3\varkappa} + h^{\frac{3}{2}}|X_n|^{4\varkappa}\big). \tag{4.24}$$

Next, we will estimate the right third term in (4.4). Combining the inequalities (2.7) and (4.10) yields

$$\begin{aligned}&\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}|X_{n+1}-X_n|^l\\ &\le M\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}\bigg[h^l\sum_{j=1}^{s}|f(t_{j-1},K_{j-1})|^l + \sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^l|\Delta W_r|^l\bigg]\\ &\le M\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}\bigg[h^l\sum_{j=1}^{s}\big(1+|K_{j-1}|^{l\varkappa}\big) + h^{\frac{l}{2}}\big(1+|K_{s-1}|^{l\varkappa}\big)\bigg]\\ &\le M\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}h^{\frac{l}{2}}\big[1+|X_n|^{l\varkappa}\big]. \qquad (4.25)\end{aligned}$$

By (4.4), (4.24), and (4.25), we obtain that

$$\begin{aligned}&\mathbb{E}\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|X_{n+1}|^{2p}\\ &\le \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-2}Mh\big[1+|X_n|^2 + h^{\frac{1}{2}}|X_n|^{3\varkappa} + h^{\frac{3}{2}}|X_n|^{4\varkappa}\big]\\ &\quad + M\sum_{l=3}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}h^{\frac{l}{2}}\big[1+|X_n|^{l\varkappa}\big]\\ &\le \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh^{\frac{3}{2}}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p+3\varkappa-2}\end{aligned}$$

$$
\begin{aligned}
&+ Mh^{\frac{5}{2}}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p+4\varkappa-2} + M\sum_{l=2}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}h^{\frac{l}{2}} \\
&+ M\sum_{l=3}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p+l(\varkappa-1)}h^{\frac{l}{2}} \\
\le\ & \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + M\sum_{l=2}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}h^{\frac{l}{2}} \\
&+ Mh\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)\left[h^{\frac{1}{2}}|X_n|^{2p+3\varkappa-2} + h^{\frac{3}{2}}|X_n|^{2p+4\varkappa-2} + \sum_{l=3}^{2p}|X_n|^{2p+l(\varkappa-1)}h^{\frac{l}{2}-1}\right]. \qquad (4.26)
\end{aligned}
$$

On the basis of (4.9), we further assume

$$
G(\varkappa) = 6\varkappa - 4. \qquad (4.27)
$$

Notice $\varkappa \ge 1, 6\varkappa - 4 \ge 2\varkappa - 2$. Then for $l = 3, 4, \dots, 2p$, we have

$$
\begin{aligned}
&\chi_{\tilde{\Omega}_{R,n}}(\omega)h^{\frac{1}{2}}|X_n|^{2p+3\varkappa-2} \le \chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p}, \\
&\chi_{\tilde{\Omega}_{R,n}}(\omega)h^{\frac{3}{2}}|X_n|^{2p+4\varkappa-2} \le \chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p}, \\
&\chi_{\tilde{\Omega}_{R,n}}(\omega)h^{\frac{l}{2}-1}|X_n|^{2p+l(\varkappa-1)} \le \chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p}.
\end{aligned}
$$

Thus, (4.4) can be written as

$$
\begin{aligned}
&\mathbb{E}\chi_{\tilde{\Omega}_{R,n+1}}(\omega)|X_{n+1}|^{2p} \\
\le\ & \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + M\sum_{l=2}^{2p}\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p-l}h^{\frac{l}{2}} \\
\le\ & \mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} + Mh, \qquad (4.28)
\end{aligned}
$$

where in the last line we have used Young's inequality. Using (4.28) and Gronwall's inequality, we can show that

$$
\mathbb{E}\chi_{\tilde{\Omega}_{R,n}}(\omega)|X_n|^{2p} \le M\big(1 + \mathbb{E}|X_0|^{2p}\big). \qquad (4.29)
$$

From the inequalities (4.2) and (4.29), we can obtain

$$
\begin{aligned}
\mathbb{E}\chi_{\tilde{\Omega}_{R,n-1}}(\omega)|X_n|^{2p} &\le \mathbb{E}\chi_{\tilde{\Omega}_{R,n-1}}(\omega)(|X_{n-1}| + s + 1)^{2p} \\
&\le M\mathbb{E}\chi_{\tilde{\Omega}_{R,n-1}}(\omega)\big(|X_{n-1}|^{2p} + (s+1)^{2p}\big) \\
&\le M\big(1 + \mathbb{E}|X_0|^{2p}\big). \qquad (4.30)
\end{aligned}
$$

To estimate $\mathbb{E}\chi_{\tilde{\Lambda}_{R,n}}(\omega)|X_n|^{2p}$, we denote

$$
\begin{aligned}
\chi_{\tilde{\Lambda}_{R,n}} &= 1 - \chi_{\tilde{\Omega}_{R,n}} = 1 - \chi_{\tilde{\Omega}_{R,n-1}}\chi_{|X_n|\le R} = \chi_{\tilde{\Lambda}_{R,n-1}} + \chi_{\tilde{\Omega}_{R,n-1}}\chi_{|X_n|>R} \\
&= \cdots = \sum_{l=0}^{n}\chi_{\tilde{\Omega}_{R,l-1}}\chi_{|X_l|>R},
\end{aligned}
$$

where we put $\chi_{\tilde{\Omega}_{R,-1}} = 1$. Combining (2.2), (4.2), (4.29), (4.30), the Cauchy-Bunyakovsky-Schwarz and Markov inequalities, we have

$$
\mathbb{E}\chi_{\tilde{\Lambda}_{R,n}}(\omega)|X_n|^{2p} = \mathbb{E}\sum_{l=0}^{n}\chi_{\tilde{\Omega}_{R,l-1}}\chi_{|X_l|>R}|X_n|^{2p}
$$

$$
\begin{aligned}
&\leq \big(\mathbb{E}|X_0+n(s+1)|^{4p}\big)^{\frac{1}{2}}\sum_{l=0}^{n}\big(\mathbb{E}[\chi_{\tilde{\Omega}_{R,l-1}}\chi_{|X_l|>R}]\big)^{\frac{1}{2}}\\
&=\big(\mathbb{E}|X_0+n(s+1)|^{4p}\big)^{\frac{1}{2}}\sum_{l=0}^{n}\big(P\big((\chi_{\tilde{\Omega}_{R,l-1}}|X_l|)>R\big)\big)^{\frac{1}{2}}\\
&\leq \big(\mathbb{E}|X_0+n(s+1)|^{4p}\big)^{\frac{1}{2}}\sum_{l=0}^{n}\frac{\big(\mathbb{E}\big(\chi_{\tilde{\Omega}_{R,l-1}}|X_l|^{2(2p+1)G(\varkappa)}\big)\big)^{\frac{1}{2}}}{R^{(2p+1)G(\varkappa)}}\\
&\leq M\big(\mathbb{E}|X_0+n(s+1)|^{4p}\big)^{\frac{1}{2}}\big(\mathbb{E}\big(1+|X_0|^{2(2p+1)G(\varkappa)}\big)\big)^{\frac{1}{2}}h^{2p+1}\\
&\leq M\big(1+\mathbb{E}|X_0|^{4p+2(2p+1)G(\varkappa)}\big)^{\frac{1}{2}}.
\end{aligned}
$$

The above estimation results indicate that conclusion (4.1) is correct for integer $p\geq 1$. Then, by Jensen's inequality, (4.1) holds for non-integer p as well. □

5. Strong Convergence of the Tamed Runge-Kutta-Chebyshev Methods

Before introducing the main results, we present the following hypothesis.

Assumption 5.1. Let the functions $f(t,X)$ and $\sum_{r=1}^{m} g_r(t,X)$ have continuous first-order partial derivatives in t and X. And there exist constants $c_3,c_4,c_5,c_6\geq 0$ such that for $t\in[t_0,T]$ and $X\in\mathbb{R}^d$,

$$
\left|\frac{\partial f(t,X)}{\partial t}\right|^2\leq c_3(1+|X|^{2\varkappa}),\quad \sum_{r=1}^{m}\left|\frac{\partial g_r(t,X)}{\partial t}\right|^2\leq c_4(1+|X|^{2\varkappa}),\tag{5.1}
$$

$$
\left|\frac{\partial f(t,X)}{\partial X}\right|^2\leq c_5(1+|X|^{2\varkappa}),\quad \sum_{r=1}^{m}\left|\frac{\partial g_r(t,X)}{\partial X}\right|^2\leq c_6(1+|X|^{2\varkappa}).\tag{5.2}
$$

Lemma 5.1. *Let the functions $f(t,X),\sum_{r=1}^{m} g_r(t,X)$ satisfy Assumption* 2.1 *and their continuous first-order partial derivatives satisfy Assumption* 5.1. *Then for $C\geq 0, l\geq 1,\varkappa\geq 1$ and $\varphi\geq t$, we have*

$$
\mathbb{E}\big|f\big(\varphi,X_{t,x}(\varphi)\big)-f(t,x)\big|^l\leq C(1+|x|^{2\varkappa l-l})\big[(\varphi-t)^{\frac{l}{2}}+(\varphi-t)^l\big],\tag{5.3}
$$

$$
\mathbb{E}\sum_{r=1}^{m}\big|g_r\big(\varphi,X_{t,x}(\varphi)\big)-g_r(t,x)\big|^l\leq C(1+|x|^{2\varkappa l-l})\big[(\varphi-t)^{\frac{l}{2}}+(\varphi-t)^l\big].\tag{5.4}
$$

Proof. For $C\geq 0, l\geq 1,\varkappa\geq 1$ and $\varphi\geq t$, by the mean value theorem, inequality (2.5) and [24, Lemma 2.2], we have

$$
\begin{aligned}
&\mathbb{E}\big|f\big(\varphi,X_{t,x}(\varphi)\big)-f(t,x)\big|^l\\
&\leq \mathbb{E}\big(C\big|f\big(\varphi,X_{t,x}(\varphi)\big)-f(\varphi,x)\big|^l+C|f(\varphi,x)-f(t,x)|^l\big)\\
&\leq \mathbb{E}\bigg(C\big[\big(1+|X_{t,x}(\varphi)|^{\varkappa-1}+|x|^{\varkappa-1}\big)|X_{t,x}(\varphi)-x|\big]^l+C\bigg|\frac{\partial}{\partial\varphi}f\big(\varphi+\xi(\varphi-t),x\big)(\varphi-t)\bigg|^l\bigg)\\
&\leq C\mathbb{E}\big(1+|X_{t,x}(\varphi)|^{\varkappa-1}+|x|^{\varkappa-1}\big)^l\bigg|\int_t^{\varphi}f\big(\theta,X_{t,x}(\theta)\big)d\theta\bigg|^l
\end{aligned}
$$

$$
\begin{aligned}
&+ C\mathbb{E}\big(1+|X_{t,x}(\varphi)|^{\varkappa-1}+|x|^{\varkappa-1}\big)^l\left|\int_t^\varphi \sum_{r=1}^m g_r\big(\theta, X_{t,x}(\theta)\big)dW_r(\theta)\right|^l \\
&+ C(1+|x|^{l\varkappa})(\varphi-t)^l \\
\le\ & C(1+|x|^{l(\varkappa-1)})\left[\mathbb{E}\left|\int_t^\varphi \big(1+|X_{t,x}(\theta)|^{\varkappa}\big)d\theta\right|^{2l}\right]^{\frac{1}{2}} \\
&+ C(1+|x|^{l(\varkappa-1)})\left[\mathbb{E}\left|\int_t^\varphi \sum_{r=1}^m g_r\big(\theta, X_{t,x}(\theta)\big)dW_r(\theta)\right|^{2l}\right]^{\frac{1}{2}} \\
&+ C(1+|x|^{l\varkappa})(\varphi-t)^l.
\end{aligned}
$$

By using the Hölder's integral inequality and the inequality for powers of Itô integrals from [5, p. 26], we obtain that

$$
\begin{aligned}
\mathbb{E}\left[\int_t^\varphi |1+X_{t,x}(\theta)|^{\varkappa}d\theta\right]^{2l} &\le C(\varphi-t)^{2l-1}\int_t^\varphi \mathbb{E}\big|1+X_{t,x}(\theta)\big|^{2l\varkappa}d\theta \\
&\le C(\varphi-t)^{2l}(1+|x|^{2l\varkappa}), \\
\mathbb{E}\left|\int_t^\varphi \sum_{r=1}^m g_r\big(\theta, X_{t,x}(\theta)\big)dW_r(\theta)\right|^{2l} &\le C(\varphi-t)^{l-1}\int_t^\varphi \mathbb{E}\left|\sum_{r=1}^m g_r\big(\theta, X_{t,x}(\theta)\big)\right|^{2l}d\theta \\
&\le C(\varphi-t)^{l}(1+|x|^{2l\varkappa}).
\end{aligned}
$$

Hence,

$$
\mathbb{E}\big|f\big(\varphi, X_{t,x}(\varphi)\big)-f(t,x)\big|^l \le C(1+|x|^{2l\varkappa-l})\big[(\varphi-t)^{\frac{l}{2}}+(\varphi-t)^l\big].
$$

It is not difficult to prove that the conclusion (5.4) is also established by a similar proof process. The proof is complete. □

Theorem 5.1. *Assume that the coefficients* $f(t,X)$ *and* $\sum_{r=1}^m g_r(t,X)$ *have continuous first-order partial derivatives in* t, X. *Additionally, Assumptions* 2.1 *and* 5.1 *hold. Then the method* (3.10) *satisfies the inequalities* (2.12) *and* (2.13) *with* $q_1 = 3/2$ *and* $q_2 = 1$. *This means that the mean square convergence order of the method* (3.10) *is* 1/2.

Proof. By the definition of the method (3.10), we have

$$
|K_1| = \left|X_n + \frac{\mu_1 h f(t_0,K_0)}{1+|\mu_1 h f(t_0,K_0)|}\right| \le |X_n| + Mh\big(1+|X_n|^{\varkappa}\big), \tag{5.5}
$$

$$
\begin{aligned}
|K_2| &= \left|X_n + \sum_{j=1}^2 \frac{A(3,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j h f(t_{j-1},K_{j-1})|}\right| \\
&\le |X_n| + Mh\left(1+\sum_{j=1}^2 |K_{j-1}|^{\varkappa}\right) \\
&\le |X_n| + Mh\big(1+|X_n|^{\varkappa^2}\big).
\end{aligned} \tag{5.6}
$$

Similarly, for the case of $i = 3, 4, \ldots, s-1$, we have

$$|K_i| = \left| X_n + \sum_{j=1}^{i} \frac{A(i+1,j)hf(t_{j-1}, K_{j-1})}{1 + |\mu_j hf(t_{j-1}, K_{j-1})|} \right| \le |X_n| + Mh\left(1 + \sum_{j=1}^{i} |K_{j-1}|^{\varkappa}\right) \le |X_n| + Mh\big(1 + |X_n|^{\varkappa^i}\big). \tag{5.7}$$

In order to make a distinction, let $\tilde{X} = X_{n+1}$ for the method (3.9) and $\hat{X} = X_{n+1}$ for the method (3.10). From (2.9) and (3.9), the one-step error of the SRKC method (3.9) can be defined as

$$\tilde{\rho}(t_n, X_n) := X_{t_n,X_n}(t_n + h) - \tilde{X}.$$

Then, using Lemma 5.1, Taylor formula and (2.7), (3.8), (5.1), (5.2), (5.5)-(5.7), $t_{j-1} = t_n + c_{j-1}h, j = 1, \ldots, s, c_{j-1} \in \mathbb{R}$, we have

$$\begin{aligned}
|\mathbb{E}\tilde{\rho}(t_n, X_n)| &= \left| \mathbb{E} \int_{t_n}^{t_n+h} f\big(\theta, X_{t_n,X_n}(\theta)\big) d\theta - \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1}, K_{j-1}) \right| \\
&= \Bigg| \mathbb{E} \int_{t_n}^{t_n+h} f\big(\theta, X_{t_n,X_n}(\theta)\big) d\theta - hf(t_n, X_n) + hf(t_n, X_n) \\
&\qquad - \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1}, K_{j-1}) \Bigg| \\
&\le M\mathbb{E} \int_{t_n}^{t_n+h} \big| f\big(\theta, X_{t_n,X_n}(\theta)\big) - f(t_n, X_n) \big| d\theta + \sum_{j=1}^{s} A(s+1,j)h, \\
&\mathbb{E}\left| \sum_{j=1}^{s} \frac{\partial f}{\partial t}\big(t_n + \xi(t_{j-1} - t_n), X_n + \xi(K_{j-1} - X_n)\big)c_{j-1}h \right| \\
&\quad + \sum_{j=1}^{s} A(s+1,j)h\,\mathbb{E}\left| \sum_{j=1}^{s} \frac{\partial f}{\partial X}\big(t_n+\xi(t_{j-1}-t_n), X_n+\xi(K_{j-1}-X_n)\big)(K_{j-1}-X_n) \right| \\
&\le Mh^{\frac{3}{2}}\big(1 + |X_n|^{2\varkappa-1}\big) + Mh^2\big(1 + |X_n|^{\varkappa^s}\big) \\
&\quad + \sum_{j=1}^{s} A(s+1,j)h\,\mathbb{E}\Bigg| \sum_{j=1}^{s} \frac{\partial f}{\partial X}\big(t_n + \xi(t_{j-1} - t_n), X_n + \xi(K_{j-1} - X_n)\big) \\
&\qquad \times \left(\sum_{j=1}^{s-1} \frac{hf(t_{j-1}, K_{j-1})}{1 + |\mu_j hf(t_{j-1}, K_{j-1})|} \right) \Bigg| \\
&\le Mh^{\frac{3}{2}}\big(1 + |X_n|^{2\varkappa-1}\big) + Mh^2\big(1 + |X_n|^{\varkappa^s}\big) + Mh^2\big(1 + |X_n|^{\varkappa^{s-1}(2\varkappa+1)}\big) \\
&\le Mh^{\frac{3}{2}}\big(1 + |X_n|^{\varkappa^{s-1}(2\varkappa+1)}\big).
\end{aligned} \tag{5.8}$$

In addition,

$$\big|\mathbb{E}\tilde{\rho}^{2p}(t_n, X_n)\big| \le M\mathbb{E}\left| \int_{t_n}^{t_n+h} f\big(\theta, X_{t_n,X_n}(\theta)\big) d\theta - \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1}, K_{j-1}) \right|^{2p}$$

$$+ M\mathbb{E}\sum_{r=1}^{m}\bigg|\int_{t_n}^{t_n+h} g_r\big(\theta, X_{t_n,X_n}(\theta)\big)dW_r(\theta) - g_r(t_{s-1}, K_{s-1})\Delta W_r\bigg|^{2p}. \quad (5.9)$$

To make it easier to deal with the above inequality, we set

$$\begin{aligned}
L_1 :&= \mathbb{E}\bigg|\int_{t_n}^{t_n+h} f\big(\theta, X_{t_n,X_n}(\theta)\big)d\theta - \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1}, K_{j-1})\bigg|^{2p} \\
&= \mathbb{E}\bigg|\int_{t_n}^{t_n+h} f\big(\theta, X_{t_n,X_n}(\theta)\big)d\theta - \sum_{j=1}^{s} A(s+1,j)hf(t_n, X_n) \\
&\qquad + \sum_{j=1}^{s} A(s+1,j)hf(t_n, X_n) - \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1}, K_{j-1})\bigg|^{2p} \\
&\le Mh^{2p-1}\mathbb{E}\int_{t_n}^{t_n+h}\bigg|f\big(\theta, X_{t_n,X_n}(\theta)\big) - \sum_{j=1}^{s} A(s+1,j)f(t_n, X_n)\bigg|^{2p} d\theta \\
&\quad + M\sum_{j=1}^{s}\big(A(s+1,j)h\big)^{2p}\mathbb{E}\sum_{j=1}^{s}\bigg|\frac{\partial f}{\partial t}\big(t_n + \xi(t_{j-1} - t_n), X_n + \xi(K_{j-1} - X_n)\big)c_{j-1}h\bigg|^{2p} \\
&\quad + M\sum_{j=1}^{s}\big(A(s+1,j)h\big)^{2p} \\
&\qquad \times \mathbb{E}\sum_{j=1}^{s}\bigg|\frac{\partial f}{\partial X}\big(t_n + \xi(t_{j-1} - t_n), X_n + \xi(K_{j-1} - X_n)\big)(K_{j-1} - X_n)\bigg|^{2p} \\
&\le Mh^{3p}\big(1 + |X_n|^{4p\varkappa-2p}\big) + Mh^{4p}\big(1 + |X_n|^{2p\varkappa^s}\big) + Mh^{4p}\big(1 + |X_n|^{\varkappa^{s-1}(4p\varkappa+2p)}\big) \\
&\le Mh^{3p}\big(1 + |X_n|^{\varkappa^{s-1}(4p\varkappa+2p)}\big),
\end{aligned} \quad (5.10)$$

$$\begin{aligned}
L_2 :&= \sum_{r=1}^{m}\mathbb{E}\bigg|\int_{t_n}^{t_n+h} g_r\big(\theta, X_{t_n,X_n}(\theta)\big)dW_r(\theta) - g_r(t_{s-1}, K_{s-1})\Delta W_r\bigg|^{2p} \\
&= \sum_{r=1}^{m}\mathbb{E}\bigg|\int_{t_n}^{t_n+h} g_r\big(\theta, X_{t_n,X_n}(\theta)\big)dW_r(\theta) - \sum_{j=1}^{s} A(s+1,j)g_r(t_n, X_n)\Delta W_r \\
&\qquad + \sum_{j=1}^{s} A(s+1,j)g_r(t_n, X_n)\Delta W_r - g_r(t_{s-1}, K_{s-1})\Delta W_r\bigg|^{2p} \\
&\le Mh^{p-1}\sum_{r=1}^{m}\mathbb{E}\int_{t_n}^{t_n+h}\bigg|g_r\big(\theta, X_{t_n,X_n}(\theta)\big) - \sum_{j=1}^{s} A(s+1,j)g_r(t_n, X_n)\bigg|^{2p} d\theta \\
&\quad + Mh^p\mathbb{E}\sum_{r=1}^{m}\bigg|\frac{\partial g_r}{\partial t}\big(t_n + \xi(t_{s-1} - t_n), X_n + \xi(K_{s-1} - X_n)\big)c_{s-1}h\bigg|^{2p} \\
&\quad + Mh^p\mathbb{E}\sum_{r=1}^{m}\bigg|\frac{\partial g_r}{\partial X}\big(t_n + \xi(t_{s-1} - t_n), X_n + \xi(K_{s-1} - X_n)\big)(K_{s-1} - X_n)\bigg|^{2p} \\
&\le Mh^{2p}\big(1 + |X_n|^{4p\varkappa-2p}\big) + Mh^{3p}\big(1 + |X_n|^{2p\varkappa^s}\big) + Mh^{3p}\big(1 + |X_n|^{\varkappa^{s-1}(4p\varkappa+2p)}\big) \\
&\le Mh^{2p}\big(1 + |X_n|^{\varkappa^{s-1}(4p\varkappa+2p)}\big).
\end{aligned} \quad (5.11)$$

Hence,

$$\left|\mathbb{E}\tilde{\rho}^{2p}(t_n,X_n)\right| \le L_1 + L_2 \le Mh^{2p}\big(1+|X_n|^{\varkappa^{s-1}(4p\varkappa+2p)}\big). \tag{5.12}$$

Let $\hat{\rho}(t_n,X_n)=\tilde{X}-\hat{X}$, then the one-step error of the t-SRKC method (3.10) can be written as

$$\rho(t_n,X_n) = X_{t,x} - \hat{X} = X_{t,x} - \tilde{X} + \tilde{X} - \hat{X} = \tilde{\rho}(t_n,X_n) + \hat{\rho}(t_n,X_n). \tag{5.13}$$

By (3.9) and (3.10), we have

$$\begin{aligned}
\hat{\rho}(t_n,X_n) &= \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1},K_{j-1}) + \sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r \\
&\quad - \sum_{j=1}^{s} \frac{A(s+1,j)hf(t_{j-1},K_{j-1})}{1+|\mu_j hf(t_{j-1},K_{j-1})|} - \frac{\sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r}{1+\big|\sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r\big|} \\
&= \sum_{j=1}^{s} A(s+1,j)hf(t_{j-1},K_{j-1})\frac{|\mu_j hf(t_{j-1},K_{j-1})|}{1+|\mu_j hf(t_{j-1},K_{j-1})|} \\
&\quad + \sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r - \frac{\sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r}{1+\sum_{r=1}^{m} |g_r(t_{s-1},K_{s-1})\Delta W_r|}.
\end{aligned} \tag{5.14}$$

Combing the inequalities (2.7), (5.5)-(5.7) and the equality (4.6) yields

$$\begin{aligned}
|\mathbb{E}\hat{\rho}(t_n,X_n)| &= \left|\mathbb{E}\left(\sum_{j=1}^{s} A(s+1,j)hf(t_{j-1},K_{j-1})\frac{|\mu_j hf(t_{j-1},K_{j-1})|}{1+|\mu_j hf(t_{j-1},K_{j-1})|}\right)\right| \\
&\le Mh^2 \sum_{j=1}^{s} \mathbb{E}\big(1+|K_{j-1}|^{2\varkappa}\big) \\
&\le Mh^2\big(1+|X_n|^{2\varkappa^s}\big).
\end{aligned} \tag{5.15}$$

Based on (5.8) and (5.15), we can show that the method (3.10) satisfies the inequality (2.12) with $q_1 = 3/2$. Using similar proof techniques, we have

$$\begin{aligned}
\mathbb{E}|\hat{\rho}^{2p}(t_n,X_n)| &= \mathbb{E}\left|\sum_{j=1}^{s} A(s+1,j)hf(t_{j-1},K_{j-1})\frac{\sum_{j=1}^{s}|\mu_j hf(t_{j-1},K_{j-1})|}{1+\sum_{j=1}^{s}|\mu_j hf(t_{j-1},K_{j-1})|}\right. \\
&\qquad \left. + \sum_{r=1}^{m} g_r(t_{s-1},K_{s-1})\Delta W_r \frac{\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|}{1+\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})\Delta W_r|}\right|^{2p} \\
&\le \mathbb{E}\left|\sum_{j=1}^{s} |A(s+1,j)hf(t_{j-1},K_{j-1})|\,|\mu_j hf(t_{j-1},K_{j-1})|\right. \\
&\qquad \left. + \sum_{r=1}^{m} |g_r(t_{s-1},K_{s-1})\Delta W_r|\,|g_r(t_{s-1},K_{s-1})\Delta W_r|\right|^{2p} \\
&\le M\mathbb{E}\left(h^{4p}\sum_{j=1}^{s}|f(t_{j-1},K_{j-1})|^{4p} + h^{2p}\sum_{r=1}^{m}|g_r(t_{s-1},K_{s-1})|^{4p}\right) \\
&\le Mh^{2p}\big(1+|X_n|^{4p\varkappa^s}\big).
\end{aligned} \tag{5.16}$$

Inequalities (5.12), (5.16) and equality (5.13) indicate that the method (3.10) satisfies the inequality (2.13) with $q_2 = 1$. According to Theorem 2.1, the t-SRKC method (3.10) has the mean square convergence order 1/2. □

6. Numerical Results

In this section, we will compare t-SRKC methods with the EM method, the balanced-type Euler-Maruyama (BEM) method in [24], the stochastic second kind orthogonal Runge-Kutta-Chebyshev (SK-ROCK) method in [1] and the SRKC method (3.5) through several examples.

We can find from reference [24] that the BEM method can be written as

$$X_{k+1} = X_k + \frac{a(t_k, X_k)h + \sum_{r=1}^{m} \sigma_r(t_k, X_k)\xi_{rk} h^{\frac{1}{2}}}{1 + h|a(t_k, X_k)| + h^{\frac{1}{2}} \sum_{r=1}^{m} |\sigma_r(t_k, X_k)\xi_{rk}|}, \tag{6.1}$$

where ξ_{rk} are Gaussian $\mathcal{N}(0,1)$ i.i.d. random variables. The SK-ROCK method given by reference [1] can be written as

$$\begin{aligned} &K_0 = X_0, \\ &K_1 = X_0 + \mu_1 h f(X_0 + \nu_1 Q) + \kappa_1 Q, \\ &K_i = \mu_i h f(K_{i-1}) + \nu_i K_{i-1} + \kappa_i K_{i-2}, \quad i = 2, \ldots, s, \\ &X_1 = K_s, \end{aligned} \tag{6.2}$$

where

$$Q = \sum_{r=1}^{m} g_r(X_0)\Delta W_r, \quad \mu_1 = \frac{\omega_1}{\omega_0}, \quad \nu_1 = \frac{s\omega_1}{2}, \quad \kappa_1 = \frac{s\omega_1}{\omega_0},$$

and $\mu_i, \nu_i, \kappa_i, i = 2, 3, \ldots, s$ are defined by (3.4).

We take the Monte Carlo simulation and choose 5000 independent trajectories in each simulation. The error is denoted by ϵ,

$$\epsilon = \sqrt{\frac{1}{5000} \sum_{i=1}^{5000} \left| X_N^i - X^i(t_N) \right|^2},$$

where X_N^i and $X^i(t_N)$ indicate respectively numerical solution and exact solution on the i-th sample path of the stochastic differential equation at time $t = t_N$. We treat the numerical solution calculated by the BEM method with a small step size h_1 as the "exact solution" when the analytical expression of the exact solution cannot be found. For SK-ROCK, SRKC and t-SRKC methods, we take constant $\eta = 1$ in this section.

Table 6.1: Calculation cost table.

Method	N_f	N_g	N_r
EM	d	md	m
BEM	d	md	m
SK-ROCK	sd	md	m
SRKC	sd	md	m
t-SRKC	sd	md	m

To better compare the computational efficiency of the methods, inspired by literature [23], we define the computational cost of the methods as follows:

$$\hat{Q} = (N_f + N_g + N_r) \times N,$$

where N_f, N_g denote the number of evaluations for the drift and diffusion functions at each step of the methods, respectively. N_r represents the number of random variables that need to be generated at each step of each method, and N denotes the number of calculated steps.

Example 6.1. Consider a 2-dimensional linear SDE system in [30]

$$dX(t) = UX(t)dt + VX(t)dW(t), \quad X(0) = X_0, \tag{6.3}$$

where U and V are matrices defined by

$$U = \begin{pmatrix} -u & u \\ u & -u \end{pmatrix}, \quad V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}.$$

We set $t \in [0,1], X_0 = (2,1)^T, u = 0.3, v = 0.1$. Taking $h_1 = 2^{-14}$ and $h = 2^{-1}, 2^{-2}, \cdots, 2^{-10}$ in turn, we obtain Table 6.2. It can be seen that all the methods performed well for the non-stiff linear problem. Additionally, the red line in Fig. 6.1 shows a reference line with a slope of 0.5. By comparing with the reference line, it is not difficult to find that all the methods are convergent of order 0.5, which is consistent with the theoretical results.

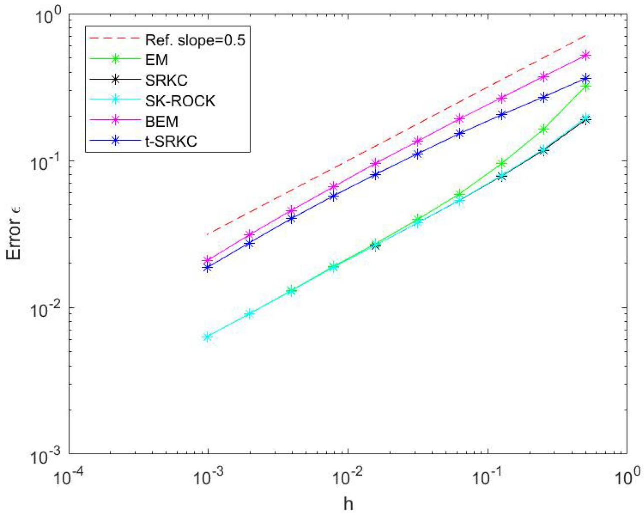

Fig. 6.1. Mean-square convergence orders of the methods EM, SRKC, SK-ROCK, BEM and t-SRKC for the component X_2 of Example 6.1.

Table 6.2: Errors of the methods EM, SRKC, SK-ROCK, BEM and t-SRKC for the component X_2 of Example 6.1.

h	2^{-1}	2^{-2}	2^{-3}	$\cdots$	2^{-9}	2^{-10}
EM	0.0383	0.0195	0.0114	$\cdots$	0.0011	0.0008
SRKC ($s = 15$)	0.0227	0.0141	0.0094	$\cdots$	0.0011	0.0008
SK-ROCK ($s = 15$)	0.0234	0.0143	0.0094	$\cdots$	0.0011	0.0008
BEM	0.0622	0.0447	0.0320	$\cdots$	0.0037	0.0025
t-SRKC ($s = 15$)	0.0434	0.0324	0.0245	$\cdots$	0.0033	0.0022

Example 6.2. Consider the Eq. (2.8). First, for the non-stiff case 1, take $h_1 = 2^{-9}$ and $h = 2^{-1}, 2^{-2}, \cdots, 2^{-5}$ in turn, then we can obtain Table 6.3. We can find from Table 6.3 that BEM method and t-SRKC methods can solve the case 1. However, the methods EM, SRKC and SK-ROCK are unable to solve the case 1. This is because the convergence of these three methods can only be guaranteed under the global Lipschitz condition, however, the case 1 does not satisfy the global Lipschitz condition.

Now let us consider the stiff case 2 of the Eq. (2.8). Take $h_1 = 2^{-12}$ and $h = 2^{-4}, 2^{-5}, \cdots, 2^{-8}$ in turn, then we can obtain Table 6.4. We can find from Table 6.4 that the methods EM, SRKC and SK-ROCK still can not solve the case 2. It is more regrettable that the BEM method is also not suitable for this case, because it suffers from the strict step size restriction due to stability problems. Fortunately, our t-SRKC method can solve case 2 very well.

Table 6.3: Errors of the methods EM, SRKC, SK-ROCK, BEM and t-SRKC for the case 1 of Example 6.2.

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
EM	NaN	NaN	NaN	NaN	NaN
SRKC ($s = 5$)	NaN	NaN	NaN	NaN	NaN
SK-ROCK ($s = 5$)	NaN	NaN	NaN	NaN	NaN
BEM	1.0895	1.0036	0.9049	0.8012	0.6959
t-SRKC ($s = 5$)	0.9901	0.9324	0.8111	0.7854	0.6395

Table 6.4: Errors of the methods EM, SRKC, SK-ROCK, BEM and t-SRKC for the case 2 of Example 6.2.

h	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
EM	NaN	NaN	NaN	NaN	NaN
SRKC ($s = 15$)	NaN	NaN	NaN	NaN	NaN
SK-ROCK ($s = 15$)	NaN	NaN	NaN	NaN	NaN
BEM	983.9664	967.9664	935.9664	871.9664	743.9665
t-SRKC ($s = 15$)	0.5281	0.2151	0.1330	0.0702	0.0245

Example 6.3. Consider the following nonlinear problem in [7]:

$$dX(t) = -X^3(t)dt + dW(t), \quad X(0) = X_0, \tag{6.4}$$

and set $t \in [0, 10]$, $X_0 = 1000$. Take $h_1 = 2^{-11}$ and $h = 2^{-1}, 2^{-2}, \cdots, 2^{-7}$ in turn, then we can obtain Table 6.5.

Example 6.4. Consider the one-dimensional stochastic Ginzburg-Landau SDE in [9]

$$dX(t) = b_1 X(t)\left[b_2 - X^2(t)\right] dt + b_3 X(t) dW(t), \quad X(0) = X_0, \tag{6.5}$$

and set $t \in [0, 1], X_0 = 1000, b_1 = 0.1, b_2 = 1, b_3 = 0.2$. Take $h_1 = 2^{-15}, h = 2^{-5}, 2^{-6}, \cdots, 2^{-11}$ in turn, then we can obtain Table 6.6.

As can be seen from the results in Tables 6.5 and 6.6 that the BEM method is not suitable for the stiff problems due to strict step size restriction. But, our t-SRKC method can still provide good computational results even using a large step size. This means that our t-SRKC method has a wider range of applications than the BEM method.

Table 6.5: Errors of BEM method and t-SRKC method for Example 6.3.

h	2^{-1}	2^{-2}	2^{-3}	$\cdots$	2^{-6}	2^{-7}
BEM	980.0220	960.0220	920.0220	$\cdots$	360.0222	0.2273
t-SRKC ($s = 15$)	0.2312	0.1748	0.1472	$\cdots$	0.0677	0.0490

Table 6.6: Errors of BEM method and t-SRKC method for Example 6.4.

h	2^{-5}	2^{-6}	$\cdots$	2^{-9}	2^{-10}	2^{-11}
BEM	965.6391	933.6397	$\cdots$	485.6847	10.0430	0.8544
t-SRKC ($s = 15$)	0.2484	0.1856	$\cdots$	0.0227	0.0117	0.0062

Example 6.5. Consider the stochastic partial differential equation in [9]

$$\frac{\partial u(t,x)}{\partial t} = \left[e_1 \frac{\partial^2 u(t,x)}{\partial x^2} + e_2 u(t,x) + u^3(t,x) - e_3 u^5(t,x)\right] dt + e_4 u^2(t,x) d\widehat{W}(t,x) \tag{6.6}$$

with $t \geq 0, x \in [0,1]$ and zero Dirichlet boundary conditions. We take the initial data $u_0(0,x) = 2\sin(\pi x), e_1 = 0.1, e_2 = 11, e_3 = 2, e_4 = 0.2$ and the trace class noise $\widehat{W}$

$$\widehat{W}(t,x) = \sum_{j=1}^{m} j^{-\frac{3}{2}} \sin(j\pi x) W_j(t), \quad t \geq 0, \quad x \in [0,1]$$

for some $m \in \mathbb{N}\backslash\{0\}$. We introduce a grid of $d+2$ uniformly spaced points $x_k = k\Delta x$ on $[0,1]$ for $k = 1, \ldots, d+2$.

Then, the finite difference approximation in space leads to a system of d-dimensional SDEs

$$dU(t) = \left[e_1 AU(t) + e_2 U(t) + U^3(t) - e_3 U^5(t)\right] dt + e_4 U^2(t) dW(t), \quad t \geq 0, \tag{6.7}$$

where

$$U(t) := \big(u_1(t), u_2(t), \ldots, u_d(t)\big)^\top, \quad u_k(t) \approx u(t, x_k),$$
$$W(t) := \big(\widehat{W}(t,x_1), \widehat{W}(t,x_2), \ldots, \widehat{W}(t,x_d)\big)^\top.$$

The matrix A is

$$A = -\frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

We take $T = 1$ for this example. Since the results are similar, for convenience, we will only show the results of a certain component of $U(t)$. Take $d = m = 20, h_1 = 2^{-12}$ and $h = 2^{-2}, 2^{-3}, \cdots, 2^{-8}$ in turn, then we can obtain Fig. 6.2. Take $d = m = 50, h_1 = 2^{-14}$ and $h = 2^{-3}, 2^{-4}, \cdots, 2^{-10}$ in turn, then we can obtain Fig. 6.3.

As can be seen from Figs. 6.2 and 6.3, our t-SRKC method has obvious advantages when the step size is large. When the step size is small enough, the advantage of the BEM method is realized, because the step size restriction has disappeared at this point. In addition, the step size restriction of the BEM method will increase with the reduction of spatial subdivision step size.

To better compare the computational efficiency of the BEM method and the t-SRKC method, based on Table 6.1, we have provided Fig. 6.4. We can observe from Fig. 6.4 that the accuracy of the t-SRKC method is higher than that of the BEM method with the equivalent calculation cost. This means that our t-SRKC method is more efficient than the BEM method.

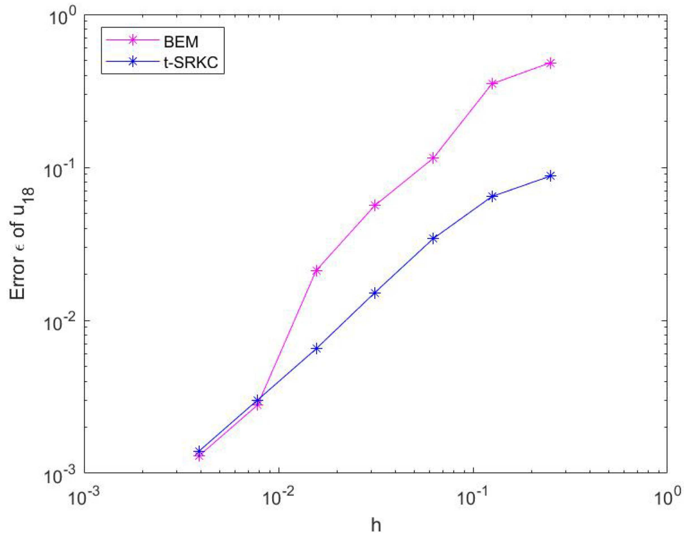

Fig. 6.2. The errors of BEM method and t-SRKC method for Example 6.5 ($d = m = 20, s = 20$).

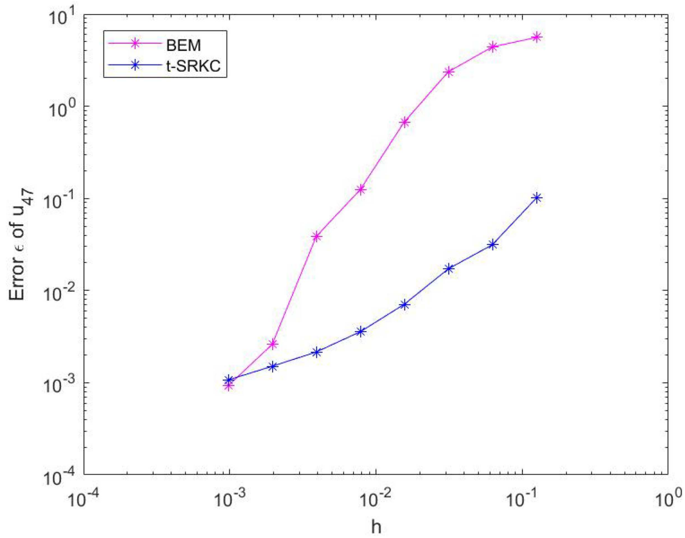

Fig. 6.3. The errors of BEM method and t-SRKC method for Example 6.5 ($d = m = 50, s = 40$).

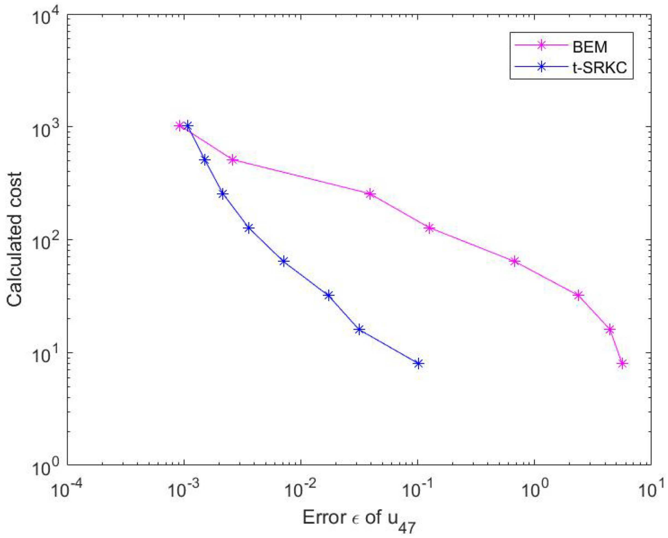

Fig. 6.4. Calculated cost-Error ϵ of the BEM method and t-SRKC method for Example 6.5 (d=m=50, $s = 40$).

7. Conclusion

A new class of explicit methods named the t-SRKC methods have been developed in this paper. Under certain non-globally Lipschitz conditions, we analyzed the strong convergence of our t-SRKC methods and proved that the order of strong convergence is 1/2. The numerical results validated our theoretical conclusions and indicated that our t-SRKC method performs better than the EM method, the BEM method, and the SRKC methods for the stiff SDEs.

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