A Local Inverse Conical Shock Problem for the Steady Supersonic Potential Flow

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th Birthday.

Abstract. This paper studies an inverse problem of reconstructing the shape of the circular symmetric cone for the given leading shock front. Assuming that the attack angle is small and that the incoming flow has a large Mach Number, we show that the shape of the cone can be reconstructed near the vertex from the data of the given shock and establish an asymptotic expansion for the velocity and the slope of cone.

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Key words: Inverse problem, steady supersonic potential flow, conical shock, cone.

1 Introduction

In Courant and Friedrichs famous book [9], for the steady supersonic flow moving at constant speed toward a straight circular cone with attack angle less than critical value, it is shown that there is a straight circular conical shock issuing from the vertex of wedge, ahead of which is the constant supersonic state and behind of which is a selfsimilar solution given by the apple curve. This shock is called the supersonic shock or transonic shock depending on the solution between the shock front and the cone. Such problem is called the direct prob-

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lem, and has been studied by many authors in various cases, see for instance, [1,3–8,10,11,16–18,26,29,37–39] and references therein.

In this paper, we are concerned with its inverse problem, which is to determine the shape of the cone for the given conical shock. Now we describe the problem as follows. The flow and the cone are assumed to be axial-symmetrical. Let $x$ be the abscissa along the axis, $y$ the distance from the axis, $u$ and $v$ the axial and radial components of velocity respectively, see Figs. 1 and 2. Then, the governed equations of the flow can be written as

\[
\begin{align*}
    (\rho u)_x + (\rho v)_y + \frac{\rho v}{y} &= 0, \\
    v_x - u_y &= 0, \\
    \frac{1}{2}(u^2 + v^2) + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} &= \frac{1}{2}(u_\infty^2) + \frac{c_\infty^2}{\gamma - 1}.
\end{align*}
\] (1.1)

Here $c(\rho) = \sqrt{\gamma \rho^{\gamma-1}}$ is the sonic speed, and the adiabatic exponent $\gamma$ is a constant with $\gamma > 1$, $(u_\infty, 0)$ and $\rho_\infty$ are the velocity and the density of incoming flow with $u_\infty > c(\rho_\infty) = c_\infty$.

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**Figure 1:** Supersonic conical flow.

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**Figure 2:** Supersonic conical flow.
Let $\Gamma_0 = \{(x,y) \mid y = f(x), x \geq 0\}$ be the curve that forms the cone and $\Gamma_1 = \{(x,y) \mid y = \phi(x), x \geq 0\}$ be the curve that forms shock front, see Fig. 2. Then on the $\Gamma_0$ that is to be determined, there holds
\[
u(x,f(x)) f'(x) = \nu(x,f(x)),
\] (1.2)
while on the shock front $\Gamma_1$ there holds
\[
\begin{cases}
\rho(u\phi' - v) = \rho_\infty u\phi', \\
u + \nu\phi' = u\infty, \\
\rho_+ > \rho_-
\end{cases}
\] (1.3)
where the last inequality is the entropy condition for the shock, and
\[
\rho_+ = \rho_+(x,\phi(x)) = \lim_{(t,s) \to (x,\phi(x)), s \to \phi(t)} \rho(t,s),
\]
\[
\rho_- = \rho_-(x,\phi(x)) = \lim_{(t,s) \to (x,\phi(x)), s \to \phi(t)} \rho(t,s).
\]
Denote
\[
\Omega = \{(x,y) \mid f(x) < y < \phi(x), x > 0\}.
\] Assuming that the velocity $u_\infty$ of incoming flow and the shock front $\Gamma_1$ are given, our aim is to find smooth functions $(\nu, \nu, \rho)$ and $f$ such that $(\nu, \nu, \rho)$ solves the Eqs. (1.1) in $\Omega$ and satisfy the boundary conditions (1.2) and (1.3).

The main result is stated as follows.

**Theorem 1.1.** Assume that $1 < \gamma < 3$ and $\phi'(0)$ is small. Then for $M_\infty$ sufficiently large and for $\phi$ with $\|\phi' - \phi'(0)\|_{C^2[0,1]} \leq K M_\infty^{2/(\gamma - 1)}$ for some positive constant $K$ independent of $M_\infty$, the inverse conical shock problem for (1.1)-(1.3) has a unique solution $(\nu(x,y), \nu(x,y), \rho(x,y), f(x))$ in domain $\Omega_X = \Omega \cap \{0 < x < X\}$ for some $X > 0$, with $(\nu, \nu) \in C^{1,1}(\Omega_X)$ and $f' \in C^{1,1}(0,X]$.

In what follows, we will prove Theorems 3.1 and 4.1, which lead to the main result Theorem 1.1. The difficulty in the proof comes from the term $1/y$ in the righthand-side of (1.1). Different from the previous works for the corresponding direct problem, we introduce a singular transformation of coordinates to deal with the singular term $1/y$ in the righthand side of (1.1). Then the problem (1.1)-(1.3) is reduced to an equivalent problem (2.8)-(2.10), which is a hyperbolic system provided that $y$ is regarded as the time variable and the incoming flow has high speed. At the line $x = 0$, the problem (2.8)-(2.10) can be reduced to a two-point boundary value problem (3.7)-(3.9). Using the expansions in the high Mach
number for the solutions of the Rankine-Hugoniot equations for the shock, we prove the sequence of the approximate solutions of (3.7)-(3.9) is convergent. The key point in the proof is to get the higher convergent rate of the sequence of the angles $\theta^{(n)}$. Finally, using characteristic method ([25], see also [2, 12, 15, 20, 30–33]) and using the perturbation argument, we can get the solution of (2.8)-(2.10). We also remark that there are two types of inverse planar supersonic shock problems that previously have been considered. One is the problem of reconstructing the shape of wedge and the velocity in the planar steady supersonic shock problems from the given location of leading shock front. This one and the related inverse piston problems have been treated in [19, 21–24, 35, 36], where the leading shock front is assumed to be smooth and the characteristic method is used to find the piecewise smooth solution. The other one is the problem of reconstructed the shape of the wedge with given pressure distribution and the velocity and pressure in the planar steady supersonic flow, see [28]. The second one has relation to the design of the airfoil, see for instance [1, 13, 14, 27, 34] and the references therein. Similar to the first type problem, this paper is devoted to consider the case of the circular symmetric conic flow.

The remaining part is organized as follow. In Section 2, the problem (1.1)-(1.3) is reduced to an equivalent problem (2.8)-(2.10). In Section 3, the problem (2.8)-(2.10) is restricted to $x=0$ and become a two-point boundary value problem (3.7)-(3.9). The sequence of approximate solutions defined by the Picard iterative scheme is shown to be convergent. The key point in the proof is to prove Lemma 3.7. In Section 4, we carry out the perturbation argument to prove the main result on problem (2.8)-(2.10).

2 Reduction of the problem

For smooth solution $(u,v)$ in $\Omega$, the equation becomes

\[
\begin{pmatrix}
1 - \frac{u^2}{c^2} & -\frac{uv}{c^2} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u \\
u
\end{pmatrix}
x
+ \begin{pmatrix}
-\frac{uv}{c^2} & 1 - \frac{v^2}{c^2} \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
u \\
u
\end{pmatrix}
y
= \begin{pmatrix}
-\frac{v}{y} \\
0
\end{pmatrix}.
\] (2.1)

We consider (2.1) in supersonic case, i.e. if $u^2 + v^2 > c^2$. Regarding $x$ as time variable, then (2.1) is strictly hyperbolic with two eigenvalues

\[
\lambda_{\pm} = \frac{uv \pm c \sqrt{q^2 - c^2}}{u^2 - c^2},
\]

where $q = \sqrt{u^2 + v^2}$. 
Now we introduce two new variables $\theta$ and $q$ as
\[
\theta = \arctan \frac{v}{u}, \quad F(q) = \int_{u_{\infty}}^{q} \frac{\sqrt{q^2 - c^2}}{cq} dq,
\]
and let
\[
r_{\pm} = \theta \pm F(q).
\]
Then (1.2) can be reduced to an equivalent form as
\[
\begin{align*}
\frac{\partial r_+}{\partial x} + \lambda - \frac{\partial r_+}{\partial y} &= \frac{vc(-vc + u\sqrt{q^2 - c^2})}{yq^2(u^2 - c^2)}, \\
\frac{\partial r_-}{\partial x} + \lambda + \frac{\partial r_-}{\partial y} &= \frac{vc(-vc - u\sqrt{q^2 - c^2})}{yq^2(u^2 - c^2)},
\end{align*}
\]
which can be written in a compact form as
\[
W_x + \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} W_y = \frac{1}{y} H(U),
\]
where $W = (r_+, r_-)^T$ and
\[
H = \begin{pmatrix} 
\frac{vc(-vc + u\sqrt{q^2 - c^2})}{q^2(u^2 - c^2)} \\
\frac{vc(-vc - u\sqrt{q^2 - c^2})}{q^2(u^2 - c^2)} 
\end{pmatrix}.
\]
Here, the direct computation leads to the following.

**Lemma 2.1.** For $u > c$ and $u^2 + v^2 < c^2$, a pair of functions
\[
\begin{align*}
&\begin{cases}
  r_+ = r_+(u, v), \\
  r_- = r_-(u, v)
\end{cases}
\end{align*}
\]
give an invertible smooth mapping $W = \Phi(U)$ of $U$. Here
\[
c_* = \left( \frac{(\gamma - 1)u_{\infty}^2 + 2c_{\infty}^2}{\gamma + 1} \right)^{\frac{1}{2}}
\]
is a constant determined from the Bernoulli law, and $U = (u, v)^T, W = (r_+, r_-)^T$. 
The right-hand side of (2.3) has singular factor $1/y$, which may bring the difficulty. To deal with this term, we introduce a transformation of the coordinates, namely,

$$
\begin{align*}
\begin{cases}
\tilde{x} = x, \\
\tilde{y} = \frac{y - f(x)}{\phi(x) - f(x)}
\end{cases}
\end{align*}
$$

in the domain $\Omega = \{(x,y) | f(x) < y < \phi(x), x > 0\}$. Then the image of domain $\Omega$ via (2.5) is

$$
\tilde{\Omega}_0 = \{(\tilde{x}, \tilde{y}) | 0 < \tilde{y} < 1, \tilde{x} > 0\},
$$

and the inverse transformation of (2.5) is given by

$$
\begin{align*}
\begin{cases}
x = \tilde{x}, \\
y = f(\tilde{x}) + \tilde{y}(\phi(\tilde{x}) - f(\tilde{x}))
\end{cases}
\end{align*}
$$

In $\tilde{\Omega}_0$, the equations become

$$
(\phi(\tilde{x}) - f(\tilde{x})) W_{\tilde{x}} + \left( \begin{array}{cc}
\lambda_- & 0 \\
0 & \lambda_+
\end{array} \right) - \sigma I_2 \right) W_{\tilde{y}} = \frac{\phi(\tilde{x}) - f(\tilde{x})}{\tilde{\sigma}} H,
$$

(2.7)

where $\tilde{\sigma} = f(\tilde{x}) + y(\phi(\tilde{x}) - f(\tilde{x}))$ and $\sigma = f'(\tilde{x}) + y(\phi'(\tilde{x}) - f'(\tilde{x}))$. In the sequel, for simplification, we omit the $\tilde{\cdot}$, that is, we use $W(x,y), f(x)$ and $\phi(x)$ to take the place of $W(\tilde{x}, \tilde{y}), f(\tilde{x})$ and $\phi(\tilde{x})$ respectively, that is, we consider the equations

$$
\begin{align*}
(\phi(x) - f(x)) & \left( \begin{array}{cc}
1 & 0 \\
\lambda_- - \sigma & 1
\end{array} \right) W_x + W_y \\
& = \frac{\phi(x) - f(x)}{\sigma} \left( \begin{array}{cc}
1 & 0 \\
0 & \lambda_- - \sigma
\end{array} \right) H
\end{align*}
$$

(2.8)

in $\Omega_0 = \{0 < y < 1, x > 0\}$ with

$$
W(x,y = 1+0) = (r_-(u_\infty,0), r_+(u_\infty,0))^\top
$$

(2.9)

and

$$
\begin{align*}
f'(x) = \frac{v(x,y = 0)}{u(x,y = 0)}.
\end{align*}
$$

(2.10)

Here $\tilde{\sigma} = f(\tilde{x}) + y(\phi(\tilde{x}) - f(\tilde{x}))$ and $\sigma = \tilde{\sigma}_x = f'(\tilde{x}) + y(\phi'(\tilde{x}) - f'(\tilde{x}))$.

Assume that $\phi(x)$ is a given smooth function, our problem is reduced to the one of finding smooth functions $W(x,y)$ and $f(x)$, which solve the problem (2.8)-(2.10).
3 Boundary value problem for ODE in $x=0$

3.1 Result on the boundary value problem for ODE in $x=0$

To deal with the problem (2.8) and (2.10), we first recall some results on the hypersonic flow past a wedge, which have been obtained in [37].

Let $(u_\infty,0)$ be the velocity of incoming flow with $u_\infty > c_\infty$, and let $s_0$ and $b_0$ be the slopes of shock front and the upper boundary of the wedge, and $(u_0,v_0)$ be the constant velocity of flow behind the shock. Then, there hold that

\[
\begin{align*}
\rho_0(u_0s_0 - v_0) &= \rho_\infty u_\infty s_0, \\
v_0s_0 + u_0 &= u_\infty, \\
v_0 - b_0u_0 &= 0, \\
\frac{1}{2}(u_0^2 + v_0^2) + \frac{c_0^2}{\gamma - 1} &= \frac{1}{2}u_\infty^2 + \frac{c_\infty^2}{\gamma - 1},
\end{align*}
\]

(3.1)

Let

\[
b_* = \left\{ \frac{1}{2} \left( -1 + \sqrt{\frac{\gamma + 7}{\gamma - 1}} \right) \right\}^{\frac{1}{2}}
\]

and let

\[
\tau = (b_0M_\infty)^{-\frac{2}{\gamma - 1}}.
\]

As in [37], we have the following.

**Lemma 3.1.** For $1 < \gamma < 3$ and $b_0 \in (0,b_*)$, there exist constants $K'_0 > K'_0 > 0$ and $K''''_0 > 0$, independent of $M_\infty$ and $b_0$ such that for $b_0M_\infty > K''''_0$ the Eq. (3.1) has a unique solution $(\rho_0,u_0,v_0,s_0)$ with the estimates

\[
\begin{align*}
s_0 &\in \left( b_0 \{1 + K'_0 \tau\}, b_0 \{1 + K'''_0 \tau\} \right), \\
u_0 \over u_\infty &= \frac{1}{1 + b_0^2} + O(1)b_0 \tau, \\
v_0 \over u_\infty &= \frac{b_0}{1 + b_0^2} + O(1)b_0^2 \tau, \\
c_0 \over u_\infty &= \frac{\sqrt{(\gamma - 1)b_0^2}}{2(1 + b_0^2)} + O(1)b_0 \tau + O(1)b_0^2 \tau^{-1}\gamma - 1.
\end{align*}
\]
Moreover,
\[ \lim_{M_\infty \to \infty} \frac{u_0}{c_0} > 1, \quad \lim_{M_\infty \to \infty} \pm (\lambda_\pm (u_0, v_0) - b_0) > 0, \quad \lim_{M_\infty \to \infty} \cos (\theta_0 \pm \theta^0_{\text{ma}}) > 0. \]

Here the bounds of \( O(1) \) are independent of \( M_\infty \) and \( b_0 \), and \( \theta_0 = \arctan(v_0 / u_0) \), \( \theta^0_{\text{ma}} = \arctan(c_0 / q_0) \).

**Proof.** The equations in (3.1) give
\[
\begin{align*}
u_0 &= \frac{u_\infty}{1 + b_0 s_0}, \quad (3.2) \\
v_0 &= \frac{u_\infty b_0}{1 + b_0 s_0}, \quad (3.3) \\
\rho_0 &= \frac{\rho_\infty (1 + b_0 s_0) s_0}{s_0 - b_0}. \quad (3.4)
\end{align*}
\]

Then, substituting (3.2)-(3.4) into the Bernoulli equation in (3.1), we have
\[
\frac{1}{2} \left\{ \frac{1 + b_0^2}{(1 + b_0 s_0)^2} - 1 \right\} + \frac{1}{\gamma - 1} \left\{ \frac{(1 + b_0 s_0)^{\gamma - 1} |s_0|^{\gamma - 1}}{M_\infty^2 |s_0 - b_0|^{\gamma - 1}} - \frac{1}{M_\infty^2} \right\} = 0. \quad (3.5)
\]

To solve the Eq. (3.5), we set
\[
f(s) = \frac{1}{2} \left\{ \frac{1 + b_0^2}{(1 + b_0 s)^2} - 1 \right\} + \frac{1}{\gamma - 1} \left\{ \frac{(1 + b_0 s)^{\gamma - 1} |s|^{\gamma - 1}}{M_\infty^2 |s - b_0|^{\gamma - 1}} - \frac{1}{M_\infty^2} \right\}
\]
for \( s > b_0 \). Note that \( b_0 > 0 \). It is easy to verify that
\[
\lim_{s \to b_0^+} f(s) = \lim_{s \to -\infty} f(s) = +\infty. \quad (3.6)
\]

In addition, for \( K > 0 \),
\[
f(b_0 + Kb_0 \tau) = f_0(K, \tau),
\]
where \( \tau = (b_0 M_\infty)^{-2/(\gamma - 1)} \) and
\[
f_0(K, t) = \frac{1}{2} \left\{ \frac{1 + b_0^2}{(1 + b_0^2 + b_0^2 K t)^2} - 1 \right\} + \frac{1}{\gamma - 1} \left\{ \frac{(1 + b_0^2 + b_0^2 K t)^{\gamma - 1} |b_0 + Kb_0^2 t|^{\gamma - 1}}{(K)^{\gamma - 1} (b_0)^{\gamma - 3}} - b_0^2 t^{\gamma - 1} \right\}.
\]
Since
\[ f_0(K,0) = \frac{1}{2} \left\{ \frac{1}{1+b_0^2} - 1 \right\} + \frac{1}{\gamma-1} \left\{ \frac{(1+b_0^2)^{\gamma-1}b_0^2}{K^{\gamma-1}} \right\}, \]
we can choose \( K' > 0 \) and \( K'' > 0 \) such that
\[ (K')^{\gamma-1} < \frac{2(1+b_0^2)^{\gamma-1}}{\gamma-1} < (K'')^{\gamma-1}. \]
Then
\[ f_0(K',0) < 0, \quad f_0(K'',0) > 0. \]
These lead to
\[ f(b_0 + K'b_0\tau) < 0, \quad f(b_0 + K''b_0\tau) > 0 \]
for \( \tau = (b_0M_\infty)^{-2/(\gamma-1)} \) sufficiently small, which implies that the equation \( f(s) = 0 \) has two solutions which lie in \((b_0 + K'b_0\tau, b_0 + K''b_0\tau)\) and \((b_0 + K'b_0\tau, +\infty)\) respectively.

On the other hand, the property of the shock polar implies that \( f(s) = 0 \) has at most two solutions in \((b_0, +\infty)\). Therefore, \( f(s) = 0 \) has a unique solution \( s = s_0 \) in \((b_0 + K'b_0\tau, b_0 + K''b_0\tau)\) for \( \tau = (b_0M_\infty)^{-2/(\gamma-1)} \) sufficiently small, which leads to the uniqueness of \((\rho_0,u_0,v_0,s_0)\). Then by (3.2)-(3.4), we can get the desired estimates. The proof is complete.

Moreover, let the shock polar through the state \((u_\infty,0)\) be parameterized by \( \Psi(s) = (u_c(s),v_c(s)) \in C^\infty \), where the parameter \( s \) is the slope of shock and \((u_c(s),v_c(s))\) lies in the supersonic region. Then, we can carry out same argument as in the proof of Lemma 3.1 to prove the following.

**Lemma 3.2.** Assume that \( 1 < \gamma < 3 \) and \( b_0 \in (0,b_*) \) and suppose that \( \tau \) is small. Then, for \( s \in (b_0 + K'b_0\tau, b_0 + K''b_0\tau) \), there holds that

\[ K'b_0\tau < s - b_s < K''b_0\tau \]
for some two positive constants \( K' \) and \( K'' \) independent of \( s \) and \( b_0, M_\infty \). Here the constants \( K'_0 \) and \( K''_0 \) are given in Lemma 3.1, and \( b_s = v_c(s)/u_c(s) \). Therefore,
and while the boundary conditions becomes

Here and in sequel, \( U^* \) and denote \( \Phi^* \) and \( \sigma^* \) are independent of \( M_\infty, u_\infty, b_0 \) and \( s \).

Now we consider the Eqs. (2.8) in the case that \( x = 0 \). For \( x = 0 \), let \( s = \phi'(0) \) and \( b = f'(0) \). Then the Eqs. (2.8) become

\[
W_y = \frac{s-b}{\sigma} \begin{pmatrix}
1 & 0 \\
\lambda_- - \sigma & 1
\end{pmatrix} H,
\]

while the boundary conditions becomes

\[
W(y=1+0) = (r_-(u_\infty,0),r_+(u_\infty,0))',
\]

and

\[
b = \frac{\nu}{u} \quad \text{on} \quad \{y=0\}.
\]

Here \( \sigma = b + y(s-b) \), and \( \sigma_0 = b_0 + y(s-b_0) \).

Let

\[
\hat{\theta}_0(y) = \arctan b_s + (\ln s - \ln(b_0 + y(s-b_0))) \sin \theta_0 \cos \theta_0,
\]

\[
\hat{q}_0 = q_s = \sqrt{(u_c(s))^2 + (v_c(s))^2},
\]

and denote \( \hat{b}_0 = \tan \hat{\theta}_0(y=1) \), and \( \hat{\sigma}_0 = \hat{b}_0 + y(s-\hat{b}_0) \). Here \( b_s = v_c(s)/u_c(s) \) is given in Lemma 3.2.

Then we have the main result of this section as follows.

**Theorem 3.1.** Assume that \( 1 < \gamma < 3 \) and \( b_0 \in (0,\hat{b}_s) \) for some \( \hat{b}_s > 0 \) and suppose that \( \tau \) is small. Then, for \( |s-s_0| < \epsilon_0 b_0 \tau \) with some constant \( \epsilon_0 \in (0,1) \) independent of \( b_0, M_\infty \) and \( u_\infty \), the inverse conical shock problem for (3.7)-(3.9) has a unique solution \((W_*(y), b_*)\). Moreover,

\[
\max_{0 \leq y \leq 1} |\theta_*(y) - \hat{\theta}_0| \leq \mathcal{O}(1)b_0^2 \tau,
\]

\[
\max_{0 \leq y \leq 1} \left| \frac{q_*(y)}{u_\infty} - \hat{q}_0 \right| \leq \mathcal{O}(1)b_0^2 \tau,
\]

\[
|b_* - \hat{b}_0| \leq \mathcal{O}(1)b_0^2 \tau.
\]

Here and in sequel, \( U_* = \Phi^{-1}(W_*) = (u_*, v_*) \), \( \theta_* = \arctan(v_*/u_*) \), and \( q_* = \sqrt{u_*^2 + v_*^2} \), \( \tau = (b_0 M_\infty)^{-2/(\gamma-1)} \); and the bounds of \( \mathcal{O}(1) \) are independent of \( b_0, M_\infty, u_\infty, b_0 \) and \( y \).
3.2 Bounds on the approximate solutions

To solve the problem (3.7)-(3.9) we use the iterative scheme as follows to define the approximate solutions \((W^{(n)}, b^{(n)})\):

\[
W^{(n+1)}_y = \frac{s - b^{(n)}}{\sigma^{(n)}} \begin{pmatrix} 1 & 0 \\ \lambda^{(n)}_+ - \sigma^{(n)} & 1 \end{pmatrix} H\left(\Phi^{-1}(W^{(n)})\right),
\]

\[
b^{(n+1)} = \frac{v^{(n)}}{u^{(n)}} \bigg|_{y=0}.
\]

Here \(\lambda^{(n)}_+ = \lambda_+ (u^{(n)}, v^{(n)})\) and \(\lambda^{(n)}_- = b^{(n)} + y(s - b^{(n)})\), \(W^{(n)} = \Phi(u^{(n)}, v^{(n)})\).

And, we initially set \(\theta^{(1)} = \hat{\theta}_0(y), q^{(1)} = \hat{q}_0\) and let

\[
b^{(1)} = \hat{b}_0 = \tan(\hat{\theta}_0(y = 1)), \quad \sigma^{(1)} = \hat{\sigma}_0 = \hat{b}_0 + y(s - \hat{b}_0).
\]

To show the convergence of the sequence \(\{(W^{(n)}, b^{(n)})\}\), we first establish the following estimates for the right-hand side and the Riemann invariants.

**Lemma 3.3.** Assume that \(1 < \gamma < 3\) and \(b_0 \in (0, b_*)\) and suppose that \(\tau\) is small. Then, there exist positive constants \(\delta_0\) and \(K_0\) such that if

\[
\left| \frac{1}{\sqrt{1 + b_0^2}} - \frac{1}{\sqrt{1 + \hat{b}_0^2}} \right| < b_0 \delta_0, \quad |\theta - \hat{\theta}_0| < \delta_0, \quad |s - \hat{b}_0| < b_0 \delta_0, \quad |b - \hat{b}_0| < b_0 \delta_0,
\]

then

\[
\frac{\sqrt{\hat{q}^2 - \hat{c}^2}}{\hat{c} \hat{q}} > \frac{K_0}{b_0},
\]

\[
|\lambda_+ (u, v) - \sigma| > K_0 b_0.
\]

Here \(\hat{q} = \sqrt{u^2 + v^2}/u_\infty, \hat{c} = c/u_\infty\).

**Proof.** For \(u = u_0, v = v_0\) and \(s = s_0 = b_0 + O(1) b_0 \tau\), the direct computation by Lemma 3.1 shows that

\[
\frac{\sqrt{\hat{q}_0^2 - \hat{c}_0^2}}{\hat{c}_0 \hat{q}_0} = \frac{K_0'''}{b_0} + O(1) \tau + O(1) b_0 \tau^{\gamma - 1},
\]

\[
|\lambda_+ (u_0, v_0) - \sigma(u_0, v_0)| = |\tan(\arctan(b_0) \pm \sigma_{in}) - b_0| + O(1) b_0 \tau + O(1) b_0 \tau^{\gamma - 1},
\]
where \( K''_0 \) is a constant satisfying
\[
K''_0 \geq \lim_{M_0 \to +\infty} \frac{u_0}{c_0} > 0,
\]
and \( \theta^\infty_{ma} = \lim_{M_\infty \to +\infty} c_0/q_0 \).

Therefore, by Lemmas 3.1 and 3.2, the desired result can be deduced from (3.16) and (3.17). The proof is complete.

The above estimates leads to the followings.

**Lemma 3.4.** Assume that \( 1 < \gamma < 3 \) and \( b_0 \in (0, b^*_\gamma) \) and suppose that \( \tau \) is small, and let \( \delta_0 \) be given by Lemma 3.3. Then, there exist positive constants \( K'_j, j=1,2 \), independent of \( b_0, M_\infty \) and \( u_\infty \) such that for
\[
q_j \left\lvert \frac{1}{u_\infty} - \frac{1}{\sqrt{1 + b_0^2}} \right\rvert < b_0 \delta_0, \quad |\theta_j - \theta_0| < b_0 \delta_0, \quad |s - b_0| < b_0 \delta_0, \quad b_0 M_\infty > \frac{1}{\delta_0},
\]
the following inequality holds:
\[
|F(q_2) - F(q_1)| \geq \frac{K'_1}{b_0} \left\lvert \frac{q_2}{u_\infty} - \frac{q_1}{u_\infty} \right\rvert.
\]

**Proof.** Let \( \tilde{q}_j = q_j/u_\infty \), and \( \tilde{c} = c(q)/u_\infty \). Then we have
\[
F(q_2) - F(q_1) = \int_{\tilde{q}_1}^{\tilde{q}_2} \sqrt{\frac{q^2 - \tilde{c}^2}{\tilde{c}\tilde{q}}} d\tilde{q},
\]
which leads to (3.18) by Lemma 3.3. The proof is complete.

Now, we set the inductive hypothesis as follow.

\[
\begin{align*}
H(n;1): \quad &|b^{(n)} - \hat{b}_0| \leq K_1 b_0^2 \tau, \\
H(n;2): \quad &\max_{0 \leq y \leq 1} |\theta^{(n)} - \tilde{\theta}_0| \leq K_1 b_0^2 \tau, \\
H(n;3): \quad &\max_{0 \leq y \leq 1} \left\lvert \frac{q^{(n)}}{u_\infty} - \tilde{q}_0 \right\rvert \leq K_1 b_0^2 \tau,
\end{align*}
\]
where the positive constant \( K_1 = 4K''_0 \) and \( \tau = (b_0 M_\infty)^{-2/(\gamma-1)} \). Here \( K''_0 \) is given by Lemma 3.2.

Before deriving \( H(n+1;1)-H(n+1;3) \) from \( H(k;1)-H(k;3), \) \( 1 \leq k \leq n \), we have the following lemma.
Lemma 3.5. Assume that $1 < \gamma < 3$ and $b_0 \in (0,b_*)$ and suppose that $\tau$ is small. If $H(k;1)$-H(k;3) hold for $1 \leq k \leq n$, then

$$\frac{1}{\lambda_{\pm}^{(n)} - \sigma^{(n)}} = \frac{1}{\lambda_{\pm}(\hat{U}_0) - b_0} \left( 1 + \frac{y(s - \hat{b}_0)}{\lambda_{\pm}(\hat{U}_0) - b_0} + O(1)b_0\tau \right), \quad (3.20)$$

$$\frac{1}{\sigma^{(n)}} = \frac{1}{b_0} \left( 1 - \frac{y(s - \hat{b}_0)}{b_0} + O(1)b_0\tau \right). \quad (3.21)$$

Therefore,

$$\frac{1}{\lambda_-^{(n)} - \sigma^{(n)}} + \frac{1}{\lambda_+^{(n)} - \sigma^{(n)}} = -2\sin\hat{\theta}_0 \cos\hat{\theta}_0 + \frac{y\cos^2\hat{\theta}_0 (s - b_0) (\hat{u}_0^2 - c_0^2)}{c_0^2} + O(1)b_0\tau, \quad (3.22)$$

for small $\tau$. Here and in sequel, the bounds of $O(1)$ is independent of $b_0$, $\tau$, $y$ and $n$.

Proof. Assumptions $H(n;1)$-H(n;3) give

$$\eta_1^{(n)} = \frac{\lambda_{\pm}^{(n)} - \sigma^{(n)} - (\lambda_{\pm}(\hat{U}_0) - \hat{b}_0)}{\lambda_{\pm}(\hat{U}_0) - \hat{b}_0} = \frac{y(s - \hat{b}_0)}{\lambda_{\pm}(\hat{U}_0) - \hat{b}_0} + O(1)\hat{b}_0\tau,$$

$$\eta_2^{(n)} = \frac{\sigma^{(n)} - \hat{b}_0}{\hat{b}_0} = 1 + \frac{y(s - \hat{b})}{\hat{b}} + O(1)b_0\tau.$$

Then, by the Taylor expansions of the following,

$$\frac{1}{\lambda_{\pm}^{(n)} - \sigma^{(n)}} = \frac{1}{(\lambda_{\pm}(\hat{U}_0) - \hat{b}_0) (1 + \eta_1^{(n)})},$$

$$\frac{1}{\sigma^{(n)}} = \frac{1}{\hat{b}_0 (1 + \eta_2^{(n)})},$$

we can get (3.20) and (3.21). The relations (3.22) follow from (3.20) and Lemmas 3.1-3.2 via direct computations. The proof is complete. \qed

Now we can have the following for the inductive steps.
Lemma 3.6. Assume that $1 < \gamma < 3$ and suppose that $b_0$ and $\tau$ are small. Then, there exists a positive constant $\varepsilon_1$, independent of $b_0, M_\infty, u_\infty$ and $n$, such that if $|s-s_0|<\varepsilon_1 b_0 \tau$ then $H(n; 1)-H(n; 3)$ hold for $n \geq 1$.

Proof. We use the inductive argument to prove the result and suppose that $H(k; 1)-H(k; 3)$ hold for $1 \leq k \leq n$, and let

$$
\hat{U}_0 = (\hat{u}_0, \hat{v}_0)^\top = (\hat{q}_0 \cos \hat{\theta}_0, \hat{q}_0 \sin \hat{\theta}_0), \quad \hat{W}_0 = \Phi(\hat{U}_0).
$$

First the direct computation shows that $H(1; 1)-H(1; 3)$ is true. Here and in sequel, for simplification, we use the notation $H(n; 1)-H(n; 3)$ hold for $n \geq 0$ are small so that $
abla \cdot \mathbf{v} < \varepsilon_1$.

For $k \geq 1$, we use the same notations as in Lemmas 3.1-3.4 and denote

$$
D_1^{(n)} = \text{diag} \left\{ \frac{1}{\lambda_-^{(n)} - \sigma^{(n)}}, \frac{1}{\lambda_+^{(n)} - \sigma^{(n)}} \right\} \left( H(U^{(n)}) - H(\hat{U}_0) \right),
$$

$$
D_2^{(n)} = \text{diag} \left\{ \frac{1}{\lambda_-^{(n)} - \sigma^{(n)}} - \frac{1}{\lambda_-^{(\hat{U}_0)} - \hat{\sigma}_0}, \frac{1}{\lambda_+^{(n)} - \sigma^{(n)}} - \frac{1}{\lambda_+^{(\hat{U}_0)} - \hat{\sigma}_0} \right\} H(\hat{U}_0),
$$

$$
D_3^{(n)} = \frac{s-b^{(n)}}{\sigma^{(n)}} \text{diag} \left\{ \frac{1}{\lambda_-^{(\hat{U}_0)} - \hat{\sigma}_0}, \frac{1}{\lambda_+^{(\hat{U}_0)} - \hat{\sigma}_0} \right\} H(\hat{U}_0),
$$

and

$$
D_0 = (\hat{W}_0)_y = \frac{s-b_0}{\sigma_0} \sin \theta_0 \cos \theta_0 (1, 1)^\top.
$$

Here and in sequel, for simplification, we use the notation

$$
\text{diag} \{ a_1, a_2 \} = \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right).
$$

Then (3.12) can be rewritten as

$$
(W^{(n+1)} - \hat{W}_0)_y = \frac{s-b^{(n)}}{\sigma^{(n)}} \left( D_1^{(n)} + D_2^{(n)} \right) + D_3^{(n)} - D_0. \quad (3.23)
$$

To get the bounds of $(\theta^{(n+1)}, q^{(n+1)}, b^{(n+1)})$, we have to deal with the right-hand side of (3.23). First, by the inductive hypotheses $H(n; 1)-H(n; 3)$, and by Lemmas 3.2, 3.4 and 3.5, we have

$$
D_1^{(n)} = \mathcal{O}(1) b_0 (\tau + \tau^{-1}) = \mathcal{O}(1) b_0^3,
$$

$$
D_2^{(n)} = \mathcal{O}(1) b_0 (\tau + \tau^{-1}) = \mathcal{O}(1) b_0^3,
$$

$$
D_3^{(n)} - D_0 = \mathcal{O}(1) b_0 \tau (\tau + \tau^{-1}) = \mathcal{O}(1) b_0^3 \tau
$$

for \( \tau \) sufficiently small, and have

\[
s - b^{(n)} = O(1)b_0\tau,
\]

where to get the estimates for \( D_3^{(n)} - D_0 \), we use the identity

\[
D_0 = \frac{s - b_0}{\sigma_0} \text{diag} \left\{ \frac{1}{\lambda_-(U_0) - \sigma_0^+}, \frac{1}{\lambda_+(U_0) - \sigma_0} \right\} H(U_0).
\]

Then, from the above argument about the terms on right-hand side of (3.23), we can conclude that

\[
(W^{(n+1)} - \hat{W}_0)_y = D_3^{(n)} - D_0 + O(1)b_0^3\tau,
\]

which gives that

\[
(\theta^{(n+1)} - \hat{\theta}_0)_y = O(1)b_0^3\tau,
\]

\[
(F(q^{(n+1)}) - F(\hat{q}_0))_y = O(1)b_0^3\tau.
\]

Moreover,

\[
b^{(n+1)} - b_0 = \tan(\theta^{(n)}(y = 0) - \tan(\hat{\theta}_0) = O(1)|\theta^{(n)} - \hat{\theta}_0|.
\]

Then, by Lemma 3.4, we can get \( H(n+1;1) - H(n+1;3) \) from (3.26)-(3.28). Hence, the proof is complete.

\[\square\]

### 3.3 The convergence of the approximate solutions

Let

\[
\Theta_n = \max_{0 \leq y \leq 1} |\theta^{(n+1)} - \theta^{(n)}|,
\]

\[
Q_n = \max_{0 \leq y \leq 1} \left| \frac{q^{(n+1)}}{u_\infty} - \frac{q^{(n)}}{u_\infty} \right|,
\]

\[
B_n = |b^{(n+1)} - b^{(n)}|.
\]

Then we have the following estimates for the induction.

**Lemma 3.7.** Assume that \( 1 < \gamma < 3 \) and suppose that \( b_0 \) and \( \tau \) are small, and let \( \epsilon_1 \) be the constant given by Lemma 3.6. Then, there exists positive constants \( K_j, j = 2, 3, 4 \) such
that for $|s-s_0|<\epsilon_1 b_0\tau$ and $n \geq 1$, we have

\[
\Theta_{n+1} \leq K_2 (\tau + \tau^{-1})(\Theta_n + Q_n) + K_3 B_n, \tag{3.29}
\]
\[
Q_{n+1} \leq K_2 (\tau + \tau^{-1})(\Theta_n + Q_n + B_n), \tag{3.30}
\]
\[
B_{n+1} \leq K_4 \Theta_n, \tag{3.31}
\]

where $K_3$ and $K_4$ can be chosen to satisfy the relations

\[
0 < K_3 \leq \cos^2 \theta_0 \left\{ 1 - (K_* + \mathcal{O}(1)) b_0 \tau \right\}, \tag{3.32}
\]
\[
0 < K_4 \leq \frac{1}{\cos^2 \theta_0} \left\{ 1 + \mathcal{O}(1) b_0 \tau \right\} \tag{3.33}
\]

for some constant $K_* > 0$. Here the constants $K_2$ and $K_*$ and the bounds of $\mathcal{O}(1)$ are independent of $b_0, \tau, n$ and $y$.

Proof. First we have

\[
(W^{(n+1)} - W^{(n)})_y = \frac{s - b^{(n)}}{\sigma^{(n)}} \left( E_1^n + E_2^n \right) + E_3^n + \frac{b^{(n)} - b^{(n-1)}}{\sigma^{(n-1)}} \left( E_4^n + E_5^n \right), \tag{3.34}
\]

where

\[
E_1^n = \text{diag} \left\{ \frac{1}{\lambda_s^{(n)} - \sigma^{(n)}}, \frac{1}{\lambda_s^{(n)}} \right\} \left( H(U^{(n)}) - H(U^{(n-1)}) \right),
\]
\[
E_2^n = \text{diag} \left\{ \frac{1}{\lambda_s^{(n-1)} - \sigma^{(n-1)}}, \frac{1}{\lambda_{s-}^{(n)} - \sigma^{(n)}}, \frac{1}{\lambda_{s+}^{(n)} - \sigma^{(n-1)}}, \frac{1}{\lambda_{s+}^{(n-1)} - \sigma^{(n-1)}} \right\} H(U^{(n-1)}),
\]
\[
E_3^n = \left( s - b^{(n)} \right) \left( \sigma^{(n)} - \sigma^{(n-1)} \right) \text{diag} \left\{ \frac{1}{\lambda_{s-}^{(n-1)} - \sigma^{(n-1)}}, \frac{1}{\lambda_{s+}^{(n-1)} - \sigma^{(n-1)}} \right\} H(U^{(n-1)}),
\]
\[
E_4^n = D_1^{(n-1)} + D_2^{(n-1)},
\]
\[
E_5^n = \text{diag} \left\{ \frac{1}{\lambda_{s-}^{(0)} - \hat{\sigma}_0}, \frac{1}{\lambda_{s+}^{(0)} - \hat{\sigma}_0} \right\} H(\hat{U}_0),
\]

where $D_1^{(n-1)}$ and $D_2^{(n-1)}$ are given in the proof of Lemma 3.6.

Next to get the estimate on $\theta^{(n+1)} - \theta^{(n)}$, we consider the summations of the components for $E_j^n, j = 1, 2, 3,$ and for $D_3^{(n-1)}$, and let $E_j^n = (E_{j,+}^n, E_{j,-}^n)$ and $D_3^{(n-1)} = (D_{3,+}^{(n-1)}, D_{3,-}^{(n-1)})$. By Lemmas 3.5 and 3.7, we have

\[
\frac{s - b^{(n)}}{\sigma^{(n)}} (E_{1,+}^n + E_{1,-}^n) = \mathcal{O}(1) b_0 \tau (\Theta_n + Q_n), \tag{3.35}
\]
\[
E_{3,+}^n + E_{3,-}^n = \mathcal{O}(1) b_0 \tau B_n, \tag{3.36}
\]
and due to the argument in the proof of Lemma 3.5, we have

$$\frac{1}{\sigma(n-1)}E_n^4 = O(1)b_0^2\tau.$$  \hfill (3.37)

For $E_n^2$, by Lemmas 3.5 and 3.6, we do the direct computation to get the relation

$$E_{n,2,+} + E_{n,2,-} = (1 - y)\Lambda_n (b(n) - b(n-1)) + O(1)(\tau + \tau^{-1})(\Theta_n + Q_n),$$  \hfill (3.38)

where

$$\Lambda_n = \sin(\theta(n-1))\cos(\theta(n-1))\left(\frac{1}{\lambda_-(n) - \sigma(n)} + \frac{1}{\lambda_+(n) - \sigma(n)}\right) = O(1)b_0^2.$$  

And for $E_n^5$, again by Lemmas 3.5 and 3.6, we have

$$\frac{E_{n,5,+} + E_{n,5,-}}{\sigma(n)} = -2\cos^2(\hat{\theta}_0)\left(1 - \frac{y(s - \hat{b}_0)}{b_0} + O(1)b_0\tau\right) = -2\cos^2\theta_0\left(1 - \frac{y(s - \hat{b}_0)}{b_0} + O(1)b_0\tau\right).$$  \hfill (3.39)

Noticing that for the first order term in (3.39) there holds that

$$\lim_{\tau \to 0} \frac{y(s - \hat{b}_0)}{b_0\tau} = 4K_* y > 0$$

for some constant $K_*$ independent of $b_0, M_\infty, y$ and $n$, we can get (3.29) by (3.35)-(3.39). Moreover, in the same way as above we can prove (3.30).

Finally, by Lemma 3.7 we have

$$b^{(n+1)} - b^{(n)} = \tan(\theta(n)(y = 0) - \tan(\theta(n-1)(y = 0)) = \frac{\sin(\theta(n) - \theta(n-1))(y = 0)}{\cos(\theta(n)\cos(\theta(n-1))} = \frac{1}{\cos^2\theta_0} (1 + O(1)b_0\tau)(\theta(n) - \theta(n-1)),$$  \hfill (3.40)

which leads to (3.31). Thus, the proof is complete. \hfill \Box

Next, to deal with the inequalities obtained in Lemma 3.7, we need the following.
Lemma 3.8. Suppose that \( b_0 \) and \( \tau \) are small. Then, for small \( \varepsilon > 0 \), the following matrix:

\[
P(\varepsilon) = \begin{pmatrix} \varepsilon & \varepsilon & K_3 \\ \varepsilon & \varepsilon & \varepsilon \\ K_4 & 0 & 0 \end{pmatrix}
\]

has a positive eigenvalue \( \mu_0 \in (\varepsilon, 1) \) and has a eigenvector \( (a_1, a_2, a_3) \) with \( a_j > 0, j = 1, 2, 3 \), corresponding to \( \mu_0 \), that is, \( \mu_0 (a_1, a_2, a_3) = (a_1, a_2, a_3) P(\varepsilon) \).

Proof. Let \( h(\mu) = \det(\mu I_3 - P(\varepsilon)) \). Then

\[
h(\mu) = \mu^3 - 2\varepsilon \mu^2 - K_3 K_4 (\mu - \varepsilon) - K_4 \varepsilon^2,
\]

where the estimates \( 0 < K_3 K_4 < 1 \) follows from Lemma 3.7 for small \( b_0 \) and \( \tau \). Therefore, the direct computation shows that

\[
h(\mu = 1) = 1 - K_3 K_4 (1 - \varepsilon) - 2\varepsilon - K_4 \varepsilon^2 > 0
\]

for small \( \varepsilon > 0 \) and \( h(\mu = \varepsilon) < 0 \), which imply that the matrix \( P(\varepsilon) \) has an eigenvalue \( \mu_0 \in (\varepsilon, 1) \). Moreover, from the identity that \( \mu_0 (a_1, a_2, a_3) = (a_1, a_2, a_3) P(\varepsilon) \), we have

\[
-\varepsilon a_1 + (\mu_0 - \varepsilon) a_2 = 0, \\
-K_3 a_1 - \varepsilon a_2 + \mu_0 a_3 = 0,
\]

which imply that \( a_1, a_2 \) and \( a_3 \) are of same sign and could be chosen to be positive. Hence, the proof is complete.

Then, applying Lemma 3.8 to the estimates obtained in Lemma 3.7 leads to the following.

Lemma 3.9. Let \( a_j, j = 1, 2, 3 \), and \( \mu_0 \in (0, 1) \) be given by Lemma 3.8, and suppose that \( \tau \) and \( b_0 \) are small. Then, there holds that

\[
a_1 \Theta_{n+1} + a_2 Q_{n+1} + a_3 B_{n+1} \leq \mu_0 (a_1 \Theta_n + a_2 Q_n + a_3 B_n)
\]

for \( n \geq 1 \).

Proof. Let \( \varepsilon \) be the constant given by Lemma 3.8, and suppose that \( b_0 \) and \( \tau \) are small constants such that \( 0 < K_3 K_4 < 1 \) and \( 0 < K_2 (\tau + \tau^{n-1}) < \varepsilon \). Then, by Lemmas 3.7 and 3.8, get the desired result. The proof is complete.

Now we are in the position to prove Theorem 3.1.
Proof of Theorem 3.1. First, Lemma 3.6 gives the bounds of the approximate solutions $\{(W^{(n)}, b^{(n)})\}$, while Lemma 3.9 implies the convergence of the sequence $\{(W^{(n)}, b^{(n)})\}$ uniformly in $[0,1]$.

Then, let $(W_*, b_*)$ be the limit of the sequence $\{(W^{(n)}, b^{(n)})\}$ and let $U_* = \Phi^{-1}(W_*)$. Then $(W_*, b_*)$ is the solution to the inverse shock problem (3.7)-(3.9).

To prove the uniqueness of the solution, let $(W_*, b_*)$ also be a solution to the inverse shock problem (3.7)-(3.9) and let $U_* = \Phi^{-1}(W_*) = (u_*, v_*)$.

Denote $$
\Theta_* = \max_{0 \leq y \leq 1} |\theta_* - \vartheta_*|, \quad Q_* = \max_{0 \leq y \leq 1} \left| \frac{q_*}{u_*} - \frac{q_*}{u_*} \right|, \quad B_* = |b_* - b_*|.
$$

Then, carrying the same argument as in the proof of Lemmas 3.7 and 3.9, we have

$$
a_1 \Theta_* + a_2 Q_* + a_3 B_* \leq \mu_0 (a_1 \Theta_* + a_2 Q_* + a_3 B_*),
$$

(3.44)

which implies that $U_* = U_*$ and $b_* = b_*$. Thus, the proof is complete. \(\square\)

4 Boundary value problem for (2.8)

Now we consider the problem (2.8)-(2.10). It is can be regarded as the perturbation of the problem (3.7) and (3.9).

4.1 The iterative scheme for PDE

In the sequel, we regard $y$ as the time variable, and assume that $b_0 \in (0, b_*)$ and $1 < \gamma < 1$. Denote the domain $D_\delta$ for $\delta > 0$ by

$$
D_\delta = \left\{ (u,v) \left| \theta - \arctan b_0 < b_0 \delta, \left| \frac{q}{u_*} - \frac{1}{\sqrt{b_*^2 + 1}} < b_0 \delta \right| \right. \right\},
$$

and denote the domain $\Omega_T$ for $0 < T \leq 1$ by

$$
\Omega_T = \{(x,y) \mid 0 \leq y \leq 1, 0 \leq x \leq \eta y + T\}.
$$

Here

$$
\eta = \frac{2}{|\lambda_0^+ - b_0|} + \frac{4}{|\lambda_0^+ - b_0|}, \quad \lambda_0^\pm = \lim_{M_* \to +\infty} \lambda_* (U_0).
$$

Then, the direct computation gives the following.
Lemma 4.1. There exist a constant $\delta_1 > 0$ such that if $(u,v) \in D_{\delta_1}$ and
\[ \|f' - b_0\|_{C^1[0,1]} < \delta_1, \quad \|\phi' - s_0\|_{C^1[0,1]} < \delta_1, \]
then (2.8) has two eigenvalues
\[ \mu_\pm = \frac{\phi(x) - f(x)}{\lambda_\pm(u,v) - \delta}, \quad x \geq 0 \]
with $|\mu_\pm| \leq \eta/4$ for $0 \leq x \leq 1$.

To solve the problem (2.8)-(2.10), we inductively define the sequence of approximate solutions $\{(W^{(n+1)}(x,y), f^{(n+1)}(x))\}, n \geq 0$ by the iterative scheme as follows:
\[ L^{(n)}W^{(n+1)} = -\frac{\phi(x) - f^{(n)}(x)}{\bar{o}^{(n)}}\Lambda^{(n)}H^{(n)}, \tag{4.1} \]
\[ \frac{df^{(n+1)}(x)}{dx} = \frac{v^{(n)}(x,y = 0)}{u^{(n)}(x,y = 0)}, \tag{4.2} \]
\[ W^{(n+1)}(x,y = 1) = W_\infty(x), \tag{4.3} \]
where $W^{(1)} = \hat{W}_0$ and $f^{(1)}(x) = \hat{b}_0 = \tan\hat{\theta}_0(y = 1)$, and $W_\infty(x)$ is the state after the shock at $(x,y = 1)$.

Here and in sequel we use the following notations for simplification:
- $U^{(n)} = (u^{(n)}(x,y), v^{(n)}(x,y))^\top$,
- $\theta^{(n)}(x,y) = \arctan \frac{\, v^{(n)}(x,y) \,}{u^{(n)}(x,y)}$, \quad $q^{(n)} = \sqrt{|u^{(n)}(x,y)|^2 + |v^{(n)}(x,y)|^2}$,
- $W^{(n)}(x,y) = (r_+^{(n)}(x,y), r_-^{(n)}(x,y))^\top$, \quad $r_\pm^{(n)} = \theta^{(n)} \pm F(q^{(n)})$,
- $H^{(n)} = H(U^{(n)}) = H(\Phi^{-1}(W^{(n)})) = (h_+(U^{(n)}), h_-(U^{(n)}))^\top$, \quad $h_\pm^{(n)} = h_\pm(U^{(n)})$,
- $\bar{o}^{(n)} = f^{(n)}(x) + y(\phi(x) - f^{(n)}(x))$,
- $\sigma^{(n)} = (\bar{o}^{(n)})_x = (f^{(n)})'(x) + y(\phi'(x) - (f^{(n)})'(x))$,
- $O(1)$ denotes the quantity with the bounds independent of $b_0, \tau, n$ and $(x,y)$,
which are same as the ones used in Section 3 but are the functions of \((x,y)\), and denote the operator \(L^{(n)}\) by

\[
L^{(n)} W = \left( \phi(x) - f^{(n)}(x) \right) \Lambda^{(n)} W_x + W_y,
\]

\[
\Lambda^{(n)} = \begin{pmatrix}
1 & 0 \\
\frac{\lambda^{(n)}_+ - \sigma^{(n)}}{\lambda^{(n)}_- - \sigma^{(n)}} & 1 \\
0 & \frac{\lambda^{(n)}_- - \sigma^{(n)}}{\lambda^{(n)}_+ - \sigma^{(n)}}
\end{pmatrix}
\]

for any smooth function \(W\).

With the above notations, (4.1) can also be written in two components as

\[
\frac{dr^{(n+1)}_\pm}{dy} \left( \psi^{(n)}_\pm(y; x_0, y_0), y \right) = \pm \frac{(\phi - f^{(n)}) h^{(n)}_\pm}{\lambda^{(n)}_\pm - \sigma^{(n)}} \left( x = \psi^{(n)}_\pm(y; x_0, y_0), y \right),
\]

(4.4)

where \(x = \psi^{(n)}_\pm(y; x_0, y_0)\) are characteristic curves satisfying

\[
\frac{d\psi^{(n)}_\pm(y; x_0, y_0)}{dy} = \mu^{(n)}_\pm( x = \psi^{(n)}_\pm(y; x_0, y_0), y),
\]

(4.5)

\[
\psi^{(n)}_\pm(y = y_0; x_0, y_0) = x_0.
\]

(4.6)

Now let \(s = \phi'(0)\), we set the following inductive hypotheses on the sequence \(\{(W^{(n)}(x,y), f^{(n)}(x))\}, n \geq 1\).

**H(n;4):** \(\left\| \frac{d}{dx} f^{(n)} - \hat{b}_0 \right\|_{C[0,2T]} \leq K_1 b_0^2 \tau,\)

**H(n;5):** \(\| \theta^{(n)} - \hat{\theta}_0 \|_{C(\overline{T})} \leq K_1 b_0^2 \tau,\)

**H(n;6):** \(\| \frac{q^{(n)} - \hat{q}_0}{u_\infty} \|_{C(\overline{T})} \leq K_1 b_0^2 \tau,\)

**H(n;7):** \(\left\| \frac{d}{dx} f^{(n)} - b_0 \right\|_{C^2([0,T])} \leq \hat{K}_1 b_0 \tau, \quad \| \theta^{(n)} - \arctan b_0 \|_{C^2(\overline{T})} \leq \hat{K}_1 b_0 \tau, \quad \left\| \frac{q^{(n)} - \hat{q}_0}{u_\infty} \right\|_{C^2(\overline{T})} \leq \hat{K}_1 b_0 \tau.\)
We use the inductive argument to prove the result and suppose that $H(k;4)$ holds for $1 \leq k \leq 7$. Then there exist positive constants $\varepsilon_2$ and $K_1$, independent of $b_0$, $M_\infty$, $u_\infty$ and $n$ such that if $\|\phi'-s_0\|_{C^2[0,2T]} < \varepsilon_2 b_0 \tau$, then for $n \geq 1$ the assumptions $H(n;4)$-$H(n;7)$ hold.

**Lemma 4.2.** Assume that $1 < \gamma < 3$ and $0 < T \leq 1$, and suppose that $b_0$ and $\tau$ are small. Then there exist positive constants $\varepsilon_2$ and $K_1$, independent of $b_0$, $M_\infty$, $u_\infty$ and $n$ such that if $\|\phi'-s_0\|_{C^2[0,2T]} < \varepsilon_2 b_0 \tau$, then for $n \geq 1$ the assumptions $H(n;4)$-$H(n;7)$ hold.

**Proof.** We use the inductive argument to prove the result and suppose that $H(k;4)$-$H(k;7)$ hold for $1 \leq k \leq n$. As in proof of Lemma 3.6, it is obvious that $H(1;4)$-$H(1;7)$ is true. Then, we have to prove $H(n+1;4)$-$H(n+1;7)$ in next.

To this end, we first denote

$$R^{(n)} = \frac{\phi(x) - f^{(n)}(x)}{\sigma^{(n)}(x)} \text{diag} \left\{ \frac{1}{\lambda_-^{(n)} - \sigma^{(n)}}, \frac{1}{\lambda_+^{(n)} - \sigma^{(n)}} \right\} H(U^{(n)}).$$

Then,

$$R^{(n)} = \int_0^1 (\phi' - (f^{(n)})'(x\tau)) \sigma^{(n)}(x\tau) d\tau \text{diag} \left\{ \frac{1}{\lambda_-^{(n)} - \sigma^{(n)}}, \frac{1}{\lambda_+^{(n)} - \sigma^{(n)}} \right\} H(U^{(n)})$$

are smooth. Applying $\partial_j$ and $\partial_i \partial_j$ to (4.4) respectively yield that

$$L^{(n)}(\partial_i W^{(n+1)}) + ((\phi - f^{(n)})\Lambda^{(n)})_{x_i} W_x^{(n+1)} = -\partial_i (R^{(n)}),$$

$$L^{(n)}(\partial_i \partial_j W^{(n+1)}) + ((\phi - f^{(n)})\Lambda^{(n)})_{x_i} \partial_j W_x^{(n+1)} + ((\phi - f^{(n)})\Lambda^{(n)})_{x_j} \partial_i W_x^{(n+1)}$$

$$+ ((\phi - f^{(n)})\Lambda^{(n)})_{x_i x_j} W_x^{(n+1)} = -\partial_i \partial_j (R^{(n)}).$$

Here for simplification, we use the notations $(x_1,x_2) = (x,y)$ and $\partial_i = \partial / \partial x_i$. Moreover, by Lemma 4.1 and by assumptions $H(n;4)$-$H(n;7)$, we deduce that the backward characteristic curves of (4.9) and (4.10) through the point $(x_0,y_0) \in \overline{\Omega_T}$ lie in the $\overline{\Omega_T}$ for $\tau$ sufficiently large and for $\|\phi'-b_0\|_{C^2[0,2T]} < \varepsilon_2 b_0 \tau$, that is,

$$(x = \psi^{(n)}_+ (y,x_0,y_0),y) \subset \overline{\Omega_T}, \quad 0 \leq y \leq y_0.$$

Here $\varepsilon_3 > 0$ is chosen so that

$$\frac{d}{dx} f^{(n)}(x) < b_0 + \varepsilon_2 M_\infty^{-\frac{1}{2}} < s_0 - \varepsilon_2 M_\infty^{-\frac{1}{2}} < \phi'(x), \quad x \geq 0,$$
as in the proof of Lemma 3.6.

Let
\[
\hat{K}_1 = 5 \left( \left\| \frac{d}{dx} f^{(1)} \right\|_{C^2[0,T]} + \left\| \theta^{(1)} \right\|_{C^2(11T)} + \left\| \frac{q^{(1)}}{u_\infty} \right\|_{C^2(11T)} + K_1 \right).
\]

Then we can apply the method of characteristic to (4.4), (4.9) and (4.10) (see [20–25] for instance) and carry out the same arguments as in the proof of Lemma 3.6 to prove \( H(n+1;4) - H(n+1;7) \). Hence, the proof is complete.

\[ \square \]

### 4.2 The convergence of the approximate solutions

Let
\[
\Theta_n = \max_{\Omega_T} |\theta^{(n+1)} - \theta^{(n)}|,
\]
\[
Q_n = \max_{\Omega_T} \left| \frac{\phi^{(n+1)} - \phi^{(n)}}{u_\infty} \right|,
\]
\[
A_n = \max_{[0,T+\eta]} \left| \frac{d}{dx} f^{(n+1)} - \frac{d}{dx} f^{(n)} \right|.
\]

Then, by Lemma 4.2, we carry out the same argument as in the proof of Lemma 3.7 to deduce the following.

**Lemma 4.3.** Assume that \( 1 < \gamma < 3 \) and \( 1 < T \leq 1 \), and suppose that \( b_0 \) and \( \tau \) are small, and let \( \epsilon_2 \) be the constant given in Lemma 4.2. Then, there exist positive constants \( K_j, j = 5,6,7 \) such that for \( \phi \) with \( \| \phi' - s_0 \|_{C^2[0,2T]} < \epsilon_2 b_0 \tau \) and \( n \geq 1 \), we have
\[
\Theta_{n+1} \leq K_5 (\tau + \tau^{\gamma-1}) (\Theta_n + Q_n) + K_6 A_n,
\]
\[
Q_{n+1} \leq K_5 (\tau + \tau^{\gamma-1}) (\Theta_n + Q_n + A_n),
\]
\[
A_{n+1} \leq K_7 \Theta_n,
\]
where \( K_6 \) and \( K_7 \) can be chosen to satisfy the followings:
\[
0 < K_6 \leq \cos^2 \theta_0 \left\{ 1 - (K^* + O(1)b_0) \tau \right\}, \quad (4.11)
\]
\[
0 < K_7 \leq \frac{1}{\cos^2 \theta_0} \left\{ 1 + O(1)b_0 \tau \right\} \quad (4.12)
\]

for some constant \( K^* > 0 \). Here the constants \( K_5 \) and \( K^* \) and the bounds of \( O(1) \) are independent of \( b_0, \tau \) and \( n \).
Proof. Due to the iterated scheme, we have
\[ L^{(n)}(W^{(n+1)} - W^{(n)}) = E_1^{(n)} + E_2^{(n)}, \]
where

\[
E_1^{(n)} = \left\{ \begin{array}{c}
(\phi(x) - f^{(n)}(x)) \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda_-(n)} - \sigma(n) & 1 \end{pmatrix} \\
- (\phi(x) - f^{(n-1)}(x)) \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda_-(n-1)} - \sigma(n-1) & 1 \end{pmatrix}
\end{array} \right\} W_x^{(n)},
\]

\[
E_2^{(n)} = \frac{\phi(x) - f^{(n)}(x)}{\sigma(n)} \text{diag} \left\{ \begin{array}{c}
\frac{1}{\lambda_-(n)} - \sigma(n) \\
\frac{1}{\lambda_+(n)} - \sigma(n)
\end{array} \right\} H(U^{(n)})
+ \frac{\phi(x) - f^{(n-1)}(x)}{\sigma(n-1)} \text{diag} \left\{ \begin{array}{c}
\frac{1}{\lambda_-(n-1)} - \sigma(n-1) \\
\frac{1}{\lambda_+(n-1)} - \sigma(n-1)
\end{array} \right\} H(U^{(n-1)}).
\]

Again, by Lemma 4.2 we can carry out the same argument as in proof of Lemma 3.7 to prove the desired result. The proof is complete.

Now we can state our main result for problem (2.8)-(2.10).

**Theorem 4.1.** Assume that \( 1 < \gamma < 3 \) and that \( 0 < T \leq 1 \), and suppose that \( b_0 \) and \( \tau \) are small. Then, for \( \phi \) with \( \| \phi' - s_0 \|_{C^2([0,2T])} < \epsilon_2 b_0 \tau \), the inverse conical shock problem for (2.8)-(2.10) has a unique solution \((W(x,y), f(x))\) in domain \( \Omega_T \) with \( W \in C^{1,1}(\Omega_T) \) and \( f' \in C^{1,1}([0,T]) \). Moreover, there hold relations

1. \( \| f' - b_0 \|_{C^{1,1}(0,T]} \leq \tilde{K} b_0 \tau, \)
2. \( \| \theta - \arctan b_0 \|_{C^{1,1}(T)} \leq \tilde{K} b_0 \tau, \)
3. \( \left\| \frac{q}{u_{\infty}} - \frac{1}{\sqrt{1 + b_0^2}} \right\|_{C^{1,1}(T)} \leq \tilde{K} b_0 \tau \)

for some constant \( \tilde{K} > 0 \). Here \( b_0 \) and \( s_0 \) are the slopes given in Section 3, \( \epsilon_2 > 0 \) is a small constant given by Lemma 4.2.
Proof. Lemmas 4.2 and 4.3 imply that the sequence \( \{(W^{(n)}, f^{(n)})\} \) is convergent in \( C^1(\Omega_T) \times C^1[0,T] \) and is uniformly bounded in \( C^2(\Omega_T) \times C^2[0,T] \).

Let \((W, f)\) be the limit of the sequence \( \{(W^{(n)}, b^{(n)})\} \). Then, in the same way as in the proof of 3.1, we can show that \((W, b)\) is the unique solution to the inverse shock problem (2.8)-(2.10) with \( W \in C^{1,1}(\Omega_\eta) \) and \( f' \in C^{1,1}[0,2\eta] \) and with the desired estimates. Thus, the proof is complete.

Finally, we remark that the main result, Theorem 1.1, can be derived from Theorem 4.1 via the inverse coordinate transformation of (2.5) introduced in Section 2.

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References