East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-168.180923

Vol. **xx**, No. x, pp. 1-21
xxx 2024

Pointwise Error Estimates of $L1$ Method for Multi-Singularity Problems Arising in Delay Fractional Equations

Dakang Cen[1], Hui Liang[2] and Seakweng Vong[1,*]

[1] *Department of Mathematics, University of Macau, Macao, SAR, China.*
[2] *School of Science, Harbin Institute of Technology, Shenzhen 518055, China.*

Received 17 July 2023; Accepted (in revised version) 18 September 2023.

Abstract. Error estimates of $L1$ scheme for delay fractional equations are derived by discrete Laplace transform method. Theoretical result shows that the convergence order is $\min\{(k+1)\alpha, 1\}$ at $(k\tau)^+$, where $k \in \mathbb{N}$, τ is delay factor, $\alpha \in (0,1)$ is the order of Caputo fractional derivative. At the points without derivative discontinuities, first order convergence is achieved. The uniqueness of the inverse problem, the reaction coefficient, and the delay factor are established by employing asymptotic expansions and the monotonicity of the Mittag-Leffler function. An inversion algorithm based on the Tikhonov regularization method is given.

AMS subject classifications: 34A08

Key words: Delay fractional equation, multi-singularity problem, $L1$ method, pointwise error estimate, simultaneous inversion of multi-parameters.

1. Introduction

We start with a simple popular delay fractional model — viz.

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(t) + \lambda u(t) + b(t)u(t-\tau) = f(t), \quad && t \in (0, T], && (1.1)\\
&u(t) = \phi(t), && t \in [-\tau, 0], && (1.2)
\end{aligned}
$$

where λ is a positive constant, $b(t)$ a smooth time-dependent function, $\tau > 0$ a constant time delay, $T = K\tau$, $K \in \mathbb{Z}^+$, and

$$
{}_a^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^\alpha} ds, \quad a \geq 0, \quad \alpha \in (0,1).
$$

This problem attracted considerable interest in recent years. If $\lambda = 0$ and $f(t) = 0$, the stability regions of the model are given in [5]. Furthermore, the finite time stability of the

*Corresponding author. *Email addresses:* `cendakang@163.com` (D. Cen), `wise2peak@126.com` (H. Liang), `swvong@um.edu.mo` (S. Vong)

 ©2024 Global-Science Press

problem has been proven by using a delayed Mittag-Leffler type matrix function — cf. [16]. If $b(t)$ is a time-independent function and $f(t)=0$, the stability and the asymptotics of the solutions are derived in [4]. For generalised nonlinear systems, the existence, uniqueness, exponential boundedness, and convergence of solutions have been studied in [31, 32].

As theoretical foundations gradually improve, numerical methods have begun to receive attention. The existing numerical analysis is usually based on the assumption that the solutions are smooth [8, 34–36]. However, an initial singularity may propagate at integer multiples of the fixed delay even in the case of smooth source functions $f(t)$ [2,23,29]. Note that global error estimates are recently established under singularity conditions [1, 2]. There are also studies concerning fractional equations and weak singularities [3, 6, 7, 12–15, 17–19, 21, 22, 26, 27, 30, 37]. Hence, it is natural to pay attention to pointwise error estimates in the following stage. It is motivation behind the current work. Then, the proposed $L1$ scheme is applied in numerical algorithm for inverse problem.

The main contributions of this work are as follows:

1. Assuming the regularity of the initial function and the source term, we propose pointwise error estimates of the $L1$ scheme for delay fractional equations.

2. We show the unique solvability of the inverse problem involving a fractional order, the reaction coefficient and the delay factor, and introduce an inversion algorithm based on the Tikhonov regularization method.

The structure of the paper is as follows. In Section 2, an $L1$ scheme in the form of discrete Laplace transform is proposed. In Section 3, pointwise error estimates are derived under the multi-singularity conditions. In Section 4, simultaneous inversion of multi-parameters is considered, and analysis of uniqueness and inversion algorithm are obtained. Numerical tests are carried out in Section 5.

2. L1 Scheme in Form of Discrete Laplace Transform

Let

$$v(t)=u(t)-u(0), \quad g(t)=f(t)-b(t)u(t-\tau)-f(0)+b(0)u(-\tau).$$

Then the Eqs. (1.1)-(1.2) take the form

$$\begin{aligned}
&{}_0^C D_t^{\alpha} v(t)+\lambda v(t)=g(t)-\lambda u(0)+f(0)-b(0)u(-\tau), && t\in(0,T], && (2.1)\\
&v(t)=\phi(t)-u(0), && t\in[-\tau,0]. && (2.2)
\end{aligned}$$

Applying the Laplace transform to (2.1)-(2.2) and letting

$$\varphi=-\lambda u(0)+f(0)-b(0)u(-\tau),$$

one has

$$z^{\alpha}\hat{v}(z)+\lambda\hat{v}(z)=z^{-1}\varphi+\hat{g}(z),$$

i.e.

$$\hat{v}(z) = (z^\alpha + \lambda)^{-1} z^{-1} \varphi + (z^\alpha + \lambda)^{-1} \hat{g}(z).$$

Then the exact solution is given by the inverse Laplace transform

$$v(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} (z^\alpha + \lambda)^{-1} z^{-1} \varphi dz + \int_0^t \varpi(t-s) g(s) ds, \tag{2.3}$$

where

$$\Gamma_{\theta,\delta} = \left\{ z \in \mathbb{C} : |z| = \delta, |\arg z| \le \theta \right\} \cup \left\{ z \in \mathbb{C} : z = \rho e^{\pm i\theta}, \rho \ge \delta \right\},$$
$$\varpi(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} (z^\alpha + \lambda)^{-1} dz,$$

and the angle $\theta \in (\pi/2, \pi)$.

Consider the uniform mesh with the time step $h := K\tau/N$, where $N/K \in \mathbb{Z}^+$ and N is the number of grids points. In order to solve the Eqs. (2.1)-(2.2), we apply the following numerical scheme — cf. [28]:

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} a_{n-k}^{(\alpha)} \left[V(t_k) - V(t_{k-1}) \right] + \lambda V(t_n) = \varphi + g(t_n) + G(t_n),$$

where

$$a_l^{(\alpha)} = (l+1)^{1-\alpha} - l^{1-\alpha}, \quad l \ge 0,$$
$$G(t_n) = b(t_n) \left[u(t_{n-N/K}) - U(t_{n-N/K}) \right],$$
$$V(t_n) = U(t_n) - u(0),$$
$$t_n = nh, \qquad -N/K \le n \le N,$$

and $U(t_n)$, $V(t_n)$ are numerical solutions of $u(t_n)$ and $v(t_n)$.

Rearranging the sequence of $a_l^{(\alpha)}$, one has

$$h^{-\alpha} \sum_{k=1}^{n} w_{n-k} V(t_k) + \lambda V(t_n) = \varphi + g(t_n) + G(t_n), \tag{2.4}$$

where

$$w_0 = \frac{a_0^{(\alpha)}}{\Gamma(2-\alpha)}, \quad w_k = \frac{a_k^{(\alpha)} - a_{k-1}^{(\alpha)}}{\Gamma(2-\alpha)}, \quad k = 1, 2, \ldots.$$

We use the notation $\tilde{\chi}(\xi) = \sum_{n=0}^\infty \chi_n \xi^n$ to denote the generating function of a sequence $\{\chi_n\}_{n=0}^\infty$. Multiplying the Eq. (2.4) by ξ^n, $|\xi| < 1$ and summing in n from 1 to ∞ gives

$$h^{-\alpha} \sum_{n=1}^\infty \sum_{k=1}^n w_{n-k} V(t_k) \xi^n + \lambda \sum_{n=1}^\infty V(t_n) \xi^n = \frac{\xi}{1-\xi} \varphi + \sum_{n=1}^\infty g(t_n) \xi^n + \sum_{n=1}^\infty G(t_n) \xi^n.$$

Since $V(0)=0$, we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} w_{n-k}V(t_k)\xi^n=\sum_{n=1}^{\infty}\sum_{k=0}^{n} w_{n-k}V(t_k)\xi^n=\sum_{n=0}^{\infty} w_n\xi^n\sum_{n=0}^{\infty}V(t_n)\xi^n=\tilde{w}(\xi)\tilde{V}(\xi).$$

Furthermore, from $g(0)=0$ and $G(0)=0$, we have

$$\tilde{V}(\xi)=\frac{\xi}{1-\xi}\left[h^{-\alpha}\tilde{w}(\xi)+\lambda\right]^{-1}\varphi+\left[h^{-\alpha}\tilde{w}(\xi)+\lambda\right]^{-1}\left[\tilde{g}(\xi)+\tilde{G}(\xi)\right].$$

Let $z_h^\alpha=\tilde{w}(\xi)/h^\alpha$, $\xi=e^{-zh}$, $V(t_n)$ has been derived by the Cauchy theorem in [10, 11]

$$\begin{aligned} V(t_n)=&\frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\frac{\xi h}{1-\xi}\left(z_h^\alpha+\lambda\right)^{-1}\varphi dz+\int_0^{t_n}\varpi_h(t_n-s)g(s)ds\\ &+\sum_{k=0}^{n}\varpi_h(t_{n-k})G(t_k), \end{aligned} \tag{2.5}$$

where

$$\Gamma_{\theta,\delta}^h=\left\{z\in\Gamma_{\theta,\delta}:|\Im z|\le\pi/h\right\},\quad \varpi_h(t)=\sum_{n=0}^{\infty}\varpi_h(t_n)\delta_{t_n}(t),$$

δ_{t_n} is the Dirac-delta function at t_n (from the left side), and

$$\varpi_h(t_n)=\frac{h}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\left(z_h^\alpha+\lambda\right)^{-1}dz.$$

3. Pointwise Error Estimates

Let $u_{k\tau}(t)$ be the solution in the domain $[(k-1)\tau,k\tau]$, $k=0,1,2,\ldots,K$. From [2], one knows the following regularity result which is critical in error estimates.

Theorem 3.1. *Assume that $\phi\in C^1[-\tau,0]$, $b\in C^1[0,K\tau]$, $f\in C^1[0,K\tau]$, then for the solution $u(t)$ of model* (1.1)-(1.2)*, there exists a positive constant C such that*

$$\left|\frac{du_{k\tau}(t)}{dt}\right|\le C\Big(1+\big(t-(k-1)\tau\big)^{k\alpha-1}\Big),\quad t\in((k-1)\tau,k\tau],\quad k=1,2,\ldots,K.$$

Furthermore, some important lemmas in [9] are needed in the following analysis.

Lemma 3.1. *For all $\pi/2<\theta<\pi$, there exists $\theta_0\in(\pi/2,\pi)$ such that $z_h^\alpha\in\Sigma_{\theta_0}$ for all $z\in\Sigma_{\theta_0}$, where*

$$\Sigma_{\theta_0}=\left\{z\in\mathbb{C}:|\arg z|<\theta_0,z\neq 0\right\}.$$

Lemma 3.2. *Let θ be close to $\pi/2$, $\delta < \pi/2h$, then for any $z \in \Sigma_{\delta,\theta}$, it holds that $|\kappa| \leq c|z|^{-\alpha}h$, where*

$$\kappa = (z^\alpha + \lambda)^{-1}z^{-1} - \frac{\xi h}{1-\xi}\left(z_h^\alpha + \lambda\right)^{-1},$$

and δ is the sector radius.

By inductive arguments, one can conclude the pointwise error estimates as following.

Theorem 3.2. *Assume that $\phi \in C^1[-\tau,0]$, $b \in C^1[0,K\tau]$, $f \in C^1[0,K\tau]$, and $v(t_n)$ and $V(t_n)$ are the solutions of the* (2.3) *and* (2.5)*, respectively. Then:*

1. *If $0 < t_n \leq \tau$, then*

$$|v(t_n) - V(t_n)| \leq Ch\left(1 + t_n^{\alpha-1}\right), \quad 1 \leq n \leq N/K.$$

2. *If $\tau < t_n \leq 2\tau$, then*

$$|v(t_n) - V(t_n)| \leq Ch\left(1 + (t_n - \tau)^{2\alpha-1}\right), \quad N/K + 1 \leq n \leq 2N/K.$$

3. *If $t_n > 2\tau$, then*

$$|v(t_n) - V(t_n)| \leq Ch\left(1 + (t_n - M\tau)^{(M+1)\alpha-1}\right), \quad MN/K + 1 \leq n \leq (M+1)N/K,$$

where $2 \leq M \leq K-1$, C is a positive constant.

Proof. The main result is deduced by method of steps in [23]. Hence, we consider the case $1 \leq n \leq N/K$ at first. From (2.3) and (2.5), one has

$$\begin{aligned}
v(t_n) - V(t_n) &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}} e^{zt_n}(z^\alpha + \lambda)^{-1}z^{-1}\varphi dz - \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\frac{\xi h}{1-\xi}\left(z_h^\alpha + \lambda\right)^{-1}\varphi dz \\
&\quad + \int_0^{t_n}\varpi(t_n - s)g(s)ds - \int_0^{t_n}\varpi_h(t_n - s)g(s)ds - \sum_{k=1}^{n}\varpi_h(t_{n-k})G(t_k) \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}} e^{zt_n}(z^\alpha + \lambda)^{-1}z^{-1}\varphi dz - \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\frac{\xi h}{1-\xi}\left(z_h^\alpha + \lambda\right)^{-1}\varphi dz \\
&\quad + \int_0^{t_n}\left[\varpi(t_n - s) - \varpi_h(t_n - s)\right]g(s)ds, \qquad (3.1)
\end{aligned}$$

the last equation holds for

$$G(t_n) = b(t_n)\left[u(t_{n-N/K}) - U(t_{n-N/K})\right] = 0, \quad 1 \leq n \leq N/K.$$

Hence, from the (3.1), we have

$$v(t_n) - V(t_n) = I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}\backslash\Gamma_{\theta,\delta}^h} e^{zt_n}(z^\alpha+\lambda)^{-1}z^{-1}\varphi dz,\\
I_2 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\left[(z^\alpha+\lambda)^{-1}z^{-1}-\frac{\xi h}{1-\xi}\left(z_h^\alpha+\lambda\right)^{-1}\right]\varphi dz,\\
I_3 &= \int_0^{t_n}\left[\varpi(t_n-s)-\varpi_h(t_n-s)\right]g(s)ds.
\end{aligned}
$$

For I_1, noting that $|(z^\alpha+\lambda)^{-1}|\le C_\theta|z|^{-\alpha}$, $\theta\in(\pi/2,\pi)$ (see [20,33]) and $\int_0^\infty e^{-s}s^{-\alpha}ds=\Gamma(1-\alpha)$, we have

$$
\begin{aligned}
|I_1| &\le \frac{1}{2\pi}\int_{\Gamma_{\theta,\delta}\backslash\Gamma_{\theta,\delta}^h} |e^{zt_n}|\cdot|(z^\alpha+\lambda)^{-1}|\cdot|z^{-1}|\cdot|\varphi||dz|\\
&\le C\int_{\pi/(h\sin\theta)}^\infty e^{rt_n\cos\theta}r^{-\alpha-1}dr|\varphi|\\
&\le Ch\int_0^\infty e^{rt_n\cos\theta}r^{-\alpha}dr|\varphi|\\
&\le Cht_n^{\alpha-1}\int_0^\infty e^{-s}s^{-\alpha}ds|\varphi|,\\
&\le C\Gamma(1-\alpha)ht_n^{\alpha-1}|\varphi|. \qquad (3.2)
\end{aligned}
$$

However, the estimate (3.2) is not robust since $\Gamma(1-\alpha)\to\infty$ as $\alpha\to 1^-$. Actually, we can get a sharp bound for $|I_1|$ with similar arguments

$$
\begin{aligned}
|I_1| \le& Cht_n^{\alpha-1}\int_{-\pi\cot\theta h^{-1}t_n}^\infty e^{-s}s^{-\alpha}ds|\varphi|\\
\le& Cht_n^{\alpha-1}\int_{-\pi\cot\theta}^\infty e^{-s}s^{-\alpha}ds|\varphi|\\
\le& Cht_n^{\alpha-1}|\varphi|. \qquad (3.3)
\end{aligned}
$$

For the estimate of I_2, the main work is to propose a bound for the kernel

$$
\kappa=(z^\alpha+\lambda)^{-1}z^{-1}-\frac{\xi h}{1-\xi}\left(z_h^\alpha+\lambda\right)^{-1}.
$$

Using Lemma 3.2 with $\delta=1/t_n$ gives

$$
|I_2|\le Ch\left(\int_{1/t_n}^{\pi/(h\sin\theta)} e^{rt_n\cos\theta}r^{-\alpha}dr+\int_{-\theta}^{\theta} e^{\cos\psi}t_n^{\alpha-1}d\psi\right)|\varphi|
$$

$$\le Ch\left(t_n^{\alpha-1}\int_{-\cos\theta}^{-n\pi\cot\theta} e^{-s}s^{-\alpha}ds + t_n^{\alpha-1}\int_{-\theta}^{\theta} e^{\cos\psi}d\psi\right)|\varphi|$$
$$\le Ct_n^{\alpha-1}h|\varphi|. \tag{3.4}$$

The term I_3 is estimated in [10], so that

$$|I_3| \le Ch\left(t_n^{\alpha-1}|g(0)| + \int_0^{t_n}(t_n-s)^{\alpha-1}|g'(s)|ds\right). \tag{3.5}$$

Combining the estimates (3.3)-(3.5) gives

$$|v(t_n)-V(t_n)| \le Ch\left(t_n^{\alpha-1}|\varphi| + t_n^{\alpha-1}|g(0)| + \int_0^{t_n}(t_n-s)^{\alpha-1}|g'(s)|ds\right), \quad 1 \le n \le N/K.$$

Assuming that $\phi \in C^1[-\tau,0]$, $b \in C^1[0,K\tau]$ and $f \in C^1[0,K\tau]$, we obtain

$$|v(t_n)-V(t_n)| \le Ch\left(1+t_n^{\alpha-1}\right), \quad 1 \le n \le N/K.$$

Next, we consider the case $N/K+1 \le n \le 2N/K$. Using (2.3) and (2.5), we write

$$v(t_n)-V(t_n) = I_1+I_2+I_3+I_4,$$

where

$$I_1 = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}\setminus\Gamma_{\theta,\delta}^h} e^{zt_n}(z^\alpha+\lambda)^{-1}z^{-1}\varphi dz,$$
$$I_2 = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\delta}^h} e^{zt_n}\left[(z^\alpha+\lambda)^{-1}z^{-1} - \frac{\xi h}{1-\xi}\left(z_h^\alpha+\lambda\right)^{-1}\right]\varphi dz,$$
$$I_3 = \int_0^{t_n}\left[\varpi(t_n-s)-\varpi_h(t_n-s)\right]g(s)ds,$$
$$I_4 = -\sum_{k=0}^{n}\varpi_h(t_{n-k})G(t_k).$$

The estimates (3.3)-(3.5) yields

$$|I_1|+|I_2|+|I_3|$$
$$\le Ch\left(t_n^{\alpha-1}|\varphi| + t_n^{\alpha-1}|g(0)| + \int_0^{t_n}(t_n-s)^{\alpha-1}|g'(s)|ds\right), \quad N/K+1 \le n \le 2N/K, \tag{3.6}$$

where

$$g'(s) = f'(s) - b'(s)u(s-\tau) - b(s)u'(s-\tau).$$

Theorem 3.1 implies that

$$\left|u'_{1\tau}(t)\right| \le C(1+t^{\alpha-1}), \quad t \in (0,\tau].$$

Hence, let

$$g_1'(s) = f'(s) - b'(s)u(s-\tau),$$

we get that

$$\begin{aligned}
&\int_0^{t_n} (t_n - s)^{\alpha-1} |g'(s)| ds \\
\le &\int_0^{t_n} (t_n - s)^{\alpha-1} |g_1'(s)| ds + C \int_0^{\tau} (t_n - s)^{\alpha-1} |u'(s-\tau)| ds \\
&\quad + C \int_{\tau}^{t_n} (t_n - s)^{\alpha-1} |u'(s-\tau)| ds \\
\le &\, C + C \int_{\tau}^{t_n} (t_n - s)^{\alpha-1} (s-\tau)^{\alpha-1} ds \\
\le &\, C + C(t_n - \tau)^{2\alpha-1}.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), one has

$$|I_1| + |I_2| + |I_3| \le Ch\left(1 + (t_n - \tau)^{2\alpha-1}\right), \quad N/K + 1 \le n \le 2N/K.$$

Evaluating the term I_4, we first write

$$|I_4| \le \sum_{k=0}^{n} |\varpi_h(t_{n-k})||G(t_k)|. \tag{3.8}$$

Simple calculation, Lemma 3.1 and [33, Lemma 2.6] give

$$\begin{aligned}
|\varpi_h(t_n)| &\le \left| \frac{h}{2\pi i} \int_{\Gamma_{\theta,\delta}^h} e^{zt_n} \left(z_h^\alpha + \lambda\right)^{-1} dz \right| \\
&\le Ch \int_{\Gamma_{\theta,\delta}^h} e^{|z|t_n} |z|^{-\alpha} d|z| \\
&\le Ch \left(\int_{1/t_n}^{\pi/(h\sin\theta)} e^{rt_n\cos\theta} r^{-\alpha} dr + \int_{-\theta}^{\theta} e^{\cos\psi} t_n^{\alpha-1} d\psi \right) \\
&\le Ch t_n^{\alpha-1}, \quad \delta = 1/t_n, \quad n \ge 1.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
|\varpi_h(t_0)| &\le \left| \frac{h}{2\pi i} \int_{\Gamma_{\theta,\delta}^h} e^{zt_0} \left(z_h^\alpha + \lambda\right)^{-1} dz \right| \\
&\le Ch \int_{\Gamma_{\theta,\delta}^h} e^{|z|t_0} |z|^{-\alpha} d|z|
\end{aligned}$$

$$
\begin{aligned}
&\leq Ch\left(\int_{1/h}^{\pi/(h\sin\theta)} e^{rt_0\cos\theta} r^{-\alpha} dr + \int_{-\theta}^{\theta} e^{h^{-1}t_0\cos\psi} h^{\alpha-1} d\psi\right)\\
&\leq Ch^{\alpha}, \quad \delta = 1/h.
\end{aligned}
$$

Let $\tau = Nh/K$ be the delay constant. Since

$$
G(t_n) = b(t_n)\big[u(t_{n-N/K}) - U(t_{n-N/K})\big],
$$

we obtain from (3.8) that

$$
\begin{aligned}
|I_4| &\leq \sum_{k=0}^{n} |\varpi_h(t_{n-k})||G(t_k)|\\
&\leq \sum_{k=N/K+1}^{n-1} |\varpi_h(t_{n-k})||G(t_k)| + |\varpi_h(t_0)||G(t_n)|\\
&\leq Ch^2 \sum_{k=N/K+1}^{n-1} t_{n-k}^{\alpha-1}\left(1 + t_{k-N/K}^{\alpha-1}\right) + Ch^{1+\alpha}\left(1 + t_{n-N/K}^{\alpha-1}\right)\\
&\leq Ch \sum_{k=N/K+1}^{n-1} t_{n-k}^{\alpha-1} h + Ch \sum_{k=N/K+1}^{n-1} t_{n-k}^{\alpha-1} t_{k-N/K}^{\alpha-1} h + Ch\big(1 + (t_n - \tau)^{2\alpha-1}\big)\\
&\leq Ch\big(1 + (t_n - \tau)^{2\alpha-1}\big).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
|v(t_n) - V(t_n)| &\leq |I_1| + |I_2| + |I_3| + |I_4|\\
&\leq Ch\left(1 + (t_n - \tau)^{2\alpha-1}\right), \quad N/K + 1 \leq n \leq 2N/K.
\end{aligned}
$$

Assume that

$$
\begin{aligned}
&|v(t_n) - V(t_n)| \leq Ch\left(1 + (t_n - m\tau)^{(m+1)\alpha-1}\right),\\
&mN/K + 1 \leq n \leq (m+1)N/K, \quad 0 \leq m \leq M-1, \quad M \leq K-1.
\end{aligned}
$$

Consider the case $MN/K + 1 \leq n \leq (M+1)N/K$. Recalling the estimates (3.3)-(3.5), we have

$$
|I_1| + |I_2| + |I_3| \leq Ch\left(t_n^{\alpha-1}|\varphi| + t_n^{\alpha-1}|g(0)| + \int_0^{t_n} (t_n - s)^{\alpha-1}|g'(s)|ds\right),
$$

where

$$
g'(s) = f'(s) - b'(s)u(s-\tau) - b(s)u'(s-\tau).
$$

Theorem 3.1 implies that

$$
\left|\frac{du_{k\tau}(t)}{dt}\right| \leq C\left(1 + \big(t - (k-1)\tau\big)^{k\alpha-1}\right), \quad t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \ldots, M.
$$

Setting

$$g_1'(s) = f'(s) - b'(s)u(s-\tau),$$

we get

$$\begin{aligned}
&\int_0^{t_n} (t_n - s)^{\alpha-1} |g'(s)| ds \\
\le\, & C \int_0^{t_n} (t_n - s)^{\alpha-1} |g_1'(s)| ds + C \int_0^{M\tau} (t_n - s)^{\alpha-1} |u'(s-\tau)| ds \\
& + C \int_{M\tau}^{t_n} (t_n - s)^{\alpha-1} |u'(s-\tau)| ds \\
\le\, & C + C \int_{M\tau}^{t_n} (t_n - s)^{\alpha-1} (s - M\tau)^{M\alpha-1} ds \\
\le\, & C + C(t_n - M\tau)^{(M+1)\alpha-1}.
\end{aligned}$$

Therefore, we get

$$|I_1| + |I_2| + |I_3| \le Ch\left(1 + (t_n - M\tau)^{(M+1)\alpha-1}\right), \quad MN/K + 1 \le n \le (M+1)N/K. \tag{3.9}$$

Consequently, the term I_4 can be estimated as follows:

$$\begin{aligned}
|I_4| \le\, & \sum_{k=0}^{n} |\varpi_h(t_{n-k})||G(t_k)| \\
\le\, & \sum_{k=N/K+1}^{n-1} |\varpi_h(t_{n-k})||G(t_k)| + |\varpi_h(t_0)||G(t_n)| \\
\le\, & Ch^2 \sum_{l=1}^{M-1} \sum_{k=lN/K+1}^{(l+1)N/K} t_{n-k}^{\alpha-1} \left(1 + t_{k-lN/K}^{l\alpha-1}\right) \\
& + Ch^2 \sum_{k=MN/K+1}^{n-1} t_{n-k}^{\alpha-1} \left(1 + t_{k-MN/K}^{M\alpha-1}\right) \\
& + Ch^{1+\alpha}\left(1 + (t_n - M\tau)^{M\alpha-1}\right) \\
\le\, & Ch\left(1 + (t_n - M\tau)^{(M+1)\alpha-1}\right).
\end{aligned} \tag{3.10}$$

Combining (3.9)-(3.10) yields

$$|v(t_n) - V(t_n)| \le Ch\left(1 + (t_n - M\tau)^{(M+1)\alpha-1}\right), \quad MN/K + 1 \le n \le (M+1)N/K.$$

The proof is complete. □

4. Simultaneous Inversion Of Multi-Parameters

In this section, we consider identifying the parameters in model (1.1)-(1.2) when the solution is known. The uniqueness of the inverse problem involving the fractional order, the reaction coefficient and the delay factor is deduced by using the asymptotic expansion and the monotonicity of the Mittag-Leffler function. The theoretical result implies only the measured data in the first delay period is needed to obtain the required parameters. An inversion algorithm is then given based on the Tikhonov regularization method.

In [2], the solution of the problem (1.1)-(1.2) have been determined by the method of steps and the Laplace transform. Let $u_{k\tau}(t)$ be the solution in the domain $[(k-1)\tau, k\tau]$, $k = 0, 1, 2, \ldots, K$. Then

$$u_{k\tau}(t) = \phi(0)E_{\alpha,1}(-\lambda t^{\alpha}) + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^{\alpha})g_{k\tau}(t-s)ds,$$

where

$$g_{k\tau}(t) = f(t) - b(t)u_{(n-1)\tau}(t-\tau), \quad (n-1)\tau < t \le n\tau, \quad n = 1, 2, \ldots, k,$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta, z \in \mathbb{C}$$

is the Mittag-Leffler function.

4.1. Uniqueness of the inverse problem

The main result is presented by Theorem 4.1, where $\tau_{\min}$ and $\tau_{\max}$ are the minimum and maximum values of the delay factor in the model (1.1)-(1.2).

Theorem 4.1. *Assume that* $b \neq 0$, $t \in [0, \tau_{\max}]$, $f \in C[0, +\infty)$, $\phi \in C(-\infty, 0]$, $\phi = 0$, $t \le -\tau_{\min}$, ϕ *is a monotone function for* $t \in (-\tau_{\min}, 0]$ *and* $f(0) - \phi(0)\lambda \neq 0$. *Suppose that* u_i *is the solution of the Eqs.* (1.1)-(1.2) *with respect to* $\alpha = \alpha_i$, $\lambda = \lambda_i$ *and* $\tau = \tau_i$, $\tau_{\min} \le \tau_i \le \tau_{\max}$, $i = 1, 2$. *If* $u_1(t) = u_2(t)$, $0 < t \le \min\{\tau_1, \tau_2\}$, *then* $\alpha_1 = \alpha_2$, $\lambda_1 = \lambda_2$, $\tau_1 = \tau_2$ *and* $u_1(t) = u_2(t)$, $t > \min\{\tau_1, \tau_2\}$.

Proof. We start with the case $0 < t \le \min\{\tau_1, \tau_2\}$. Let

$$g_{1\tau_i}(t) = f(t) - b(t)\phi(t-\tau_i), \quad i = 1, 2.$$

The condition $u_1(t) = u_2(t)$ implies

$$\begin{aligned} &\phi(0)E_{\alpha_1,1}(-\lambda_1 t^{\alpha_1}) + \int_0^t s^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_1 s^{\alpha_1})\, g_{1\tau_1}(t-s)ds \\ =\ &\phi(0)E_{\alpha_2,1}(-\lambda_2 t^{\alpha_2}) + \int_0^t s^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-\lambda_2 s^{\alpha_2})\, g_{1\tau_2}(t-s)ds. \end{aligned} \tag{4.1}$$

Since

$$E_{\alpha,1}(-\lambda t^{\alpha}) = 1 - \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + t^{2\alpha}\lambda^{2}E_{\alpha,2\alpha+1}(-\lambda t^{\alpha}),$$

Eq. (4.1) can be written as

$$\begin{aligned}
&\phi(0)\Big[1 - \frac{\lambda_1 t^{\alpha_1}}{\Gamma(\alpha_1+1)} + t^{2\alpha_1}\lambda_1^{2}E_{\alpha_1,2\alpha_1+1}(-\lambda_1 t^{\alpha_1})\Big] \\
&\quad + \int_0^t s^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_1 s^{\alpha_1})\, g_{1\tau_1}(t-s)ds \\
&= \phi(0)\Big[1 - \frac{\lambda_2 t^{\alpha_2}}{\Gamma(\alpha_2+1)} + t^{2\alpha_2}\lambda_2^{2}E_{\alpha_2,2\alpha_2+1}(-\lambda_2 t^{\alpha_2})\Big] \\
&\quad + \int_0^t s^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-\lambda_2 s^{\alpha_2})\, g_{1\tau_2}(t-s)ds,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&-\phi(0)\frac{\lambda_1 t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \phi(0)t^{2\alpha_1}\lambda_1^{2}E_{\alpha_1,2\alpha_1+1}(-\lambda_1 t^{\alpha_1}) \\
&\quad + \int_0^t s^{\alpha_1-1}E_{\alpha_1,\alpha_1}(-\lambda_1 s^{\alpha_1})\, g_{1\tau_1}(t-s)ds \\
&= -\phi(0)\frac{\lambda_2 t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \phi(0)t^{2\alpha_2}\lambda_2^{2}E_{\alpha_2,2\alpha_2+1}(-\lambda_2 t^{\alpha_2}) \\
&\quad + \int_0^t s^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-\lambda_2 s^{\alpha_2})\, g_{1\tau_2}(t-s)ds. \qquad (4.2)
\end{aligned}$$

The relation

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) \le 0,$$

cf. [25] and the mean value theorem give

$$\begin{aligned}
&\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\lambda s^{\alpha})g_{1\tau}(t-s)ds \\
&= \frac{g_{1\tau}(\xi(t))}{\lambda}\big[1 - E_{\alpha,1}(-\lambda t^{\alpha})\big] \\
&= g_{1\tau}\big(\xi(t)\big)\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} - t^{2\alpha}\lambda E_{\alpha,2\alpha+1}(-\lambda t^{\alpha})\Big], \quad \xi(t)\in(0,t).
\end{aligned}$$

Therefore, (4.2) yields

$$\begin{aligned}
&\frac{[f(\xi_1(t)) - \phi(0)\lambda_1]t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \big[\phi(0)\lambda_1 - f(\xi_1(t))\big]t^{2\alpha_1}\lambda_1 E_{\alpha_1,2\alpha_1+1}(-\lambda_1 t^{\alpha_1}) \\
&- b\big(\xi_1(t)\big)\phi\big(\xi_1(t)-\tau_1\big)\Big[\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} - t^{2\alpha_1}\lambda_1 E_{\alpha_1,2\alpha_1+1}(-\lambda_1 t^{\alpha_1})\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[f(\xi_2(t)) - \phi(0)\lambda_2]t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \big[\phi(0)\lambda_2 - f(\xi_2(t))\big]t^{2\alpha_2}\lambda_2 E_{\alpha_2,2\alpha_2+1}(-\lambda_2 t^{\alpha_2}) \\
&\qquad - b\big(\xi_2(t)\big)\phi\big(\xi_2(t)-\tau_2\big)\bigg[\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} - t^{2\alpha_2}\lambda_2 E_{\alpha_2,2\alpha_2+1}(-\lambda_2 t^{\alpha_2})\bigg], \\
&\xi_i(t) \in (0,t), \quad i = 1,2. \qquad (4.3)
\end{aligned}
$$

Recall that $|E_{\alpha,\beta}(z)| \le C$ if $\alpha \in (0,1)$, $\beta, z \in \mathbb{R}$, $z < 0$ [24]. When $t \to 0$, it implies that

$$
\begin{aligned}
&\big[\phi(0)\lambda_i - f(\xi_i(t))\big]t^{2\alpha_i}\lambda_i E_{\alpha_i,2\alpha_i+1}(-\lambda_i t^{\alpha_i}) = \mathscr{O}\big(t^{2\alpha_i}\big), && i = 1,2, \\
&-b\big(\xi_i(t)\big)\phi\big(\xi_i(t)-\tau_i\big)\bigg[\frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} - t^{2\alpha_i}\lambda_i E_{\alpha_i,2\alpha_i+1}(-\lambda_i t^{\alpha_i})\bigg] = o(t^{\alpha_i}), && i = 1,2,
\end{aligned}
$$

where $\xi_i(t) \in (0,t)$, $i = 1,2$, the second equality follows from $\phi(t) = 0$, $t \le -\tau_{\min}$ and $\tau_i \ge \tau_{\min}$, $i = 1,2$. As a result, (4.3) can be rewritten as follows:

$$
\begin{aligned}
&\frac{[f(\xi_1(t)) - \phi(0)\lambda_1]t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \mathscr{O}(t^{2\alpha_1}) + o(t^{\alpha_1}) \\
&= \frac{[f(\xi_2(t)) - \phi(0)\lambda_2]t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \mathscr{O}(t^{2\alpha_2}) + o(t^{\alpha_2}), \quad t \to 0. \qquad (4.4)
\end{aligned}
$$

Assuming that $\alpha_1 > \alpha_2$ and multiplying the above equation by $t^{-\alpha_2}$ gives

$$
\begin{aligned}
&\frac{[f(\xi_1(t)) - \phi(0)\lambda_1]t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1+1)} + \mathscr{O}(t^{2\alpha_1-\alpha_2}) + o(t^{\alpha_1-\alpha_2}) \\
&= \frac{[f(\xi_2(t)) - \phi(0)\lambda_2]}{\Gamma(\alpha_2+1)} + \mathscr{O}(t^{\alpha_2}) + o(1).
\end{aligned}
$$

Therefore,

$$
\frac{[f(\xi_2(t)) - \phi(0)\lambda_2]}{\Gamma(\alpha_2+1)} \to 0 \quad \text{as} \quad t \to 0,
$$

which contradicts the condition given. In the case $\alpha_1 < \alpha_2$, we get similar results. Hence, $\alpha_1 = \alpha_2$.

Eq. (4.4) can be now written as

$$
\begin{aligned}
&\frac{[f(\xi_1(t)) - \phi(0)\lambda_1]t^{\alpha}}{\Gamma(\alpha+1)} + \mathscr{O}(t^{2\alpha}) + o(t^{\alpha}) \\
&= \frac{[f(\xi_2(t)) - \phi(0)\lambda_2]t^{\alpha}}{\Gamma(\alpha+1)} + \mathscr{O}(t^{2\alpha}) + o(t^{\alpha}) \quad \text{as} \quad t \to 0. \qquad (4.5)
\end{aligned}
$$

Multiplying (4.5) by $t^{-\alpha}$, we obtain

$$
\frac{f(\xi_1(t)) - \phi(0)\lambda_1}{\Gamma(\alpha+1)} + \mathscr{O}(t^{\alpha}) + o(1) = \frac{f(\xi_2(t)) - \phi(0)\lambda_2}{\Gamma(\alpha+1)} + \mathscr{O}(t^{\alpha}) + o(1),
$$

so that

$$
\left|\frac{f(\xi_1(t)) - \phi(0)\lambda_1}{\Gamma(\alpha+1)} - \frac{f(\xi_2(t)) - \phi(0)\lambda_2}{\Gamma(\alpha+1)}\right| \to 0 \quad \text{as} \quad t \to 0.
$$

We also note that

$$\left|\frac{f(\xi_1(t))}{\Gamma(\alpha+1)} - \frac{f(\xi_2(t))}{\Gamma(\alpha+1)}\right| \to 0 \quad \text{as} \quad t \to 0,$$

because the condition $\phi(0) \neq 0$ implies that $\lambda_1 = \lambda_2$, $f(t)$ is continuous at 0, and $\xi_i(t) \in (0,t)$, $i = 1,2$.

Since $\alpha_1 = \alpha_2$, $\lambda_1 = \lambda_2$, (4.1) yields

$$\begin{aligned}
0 &= \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^\alpha) b(t-s)[\phi(t-s-\tau_1) - \phi(t-s-\tau_2)]ds \\
&= \int_0^{t-c} s^{\alpha-1} E_{\alpha,\alpha}(-\lambda s^\alpha) b(t-s)[\phi(t-s-\tau_1) - \phi(t-s-\tau_2)]ds \\
&= \frac{1}{\lambda}\big(b(\xi_3(t))[\phi(\xi_3(t)-\tau_1) - \phi(\xi_3(t)-\tau_2)]\big)\big[1 - E_{\alpha,1}(-\lambda(t-c)^\alpha)\big], \\
c &= \min\{\tau_1 - \tau_{\min}, \tau_2 - \tau_{\min}\}, \quad \xi_3(t) \in (c,t).
\end{aligned}$$

In the above consideration we used the condition $\phi(t) = 0$, $t \leq -\tau_{\min}$ and the mean value theorem. Since $E_{\alpha,1}(-\lambda t^\alpha) \in (0,1)$, $t > 0$, one has $1 - E_{\alpha,1}(-\lambda t^\alpha) > 0$ [25]. We get that $\tau_1 = \tau_2$ from the condition that $\phi(t)$ is a monotone function. $u_1(t) = u_2(t)$, $t > \min\{\tau_1, \tau_2\}$ is deduced by the method of steps. The proof is complete. □

4.2. Multi-parameter inversion algorithm

Let $\theta(t)$ be measured data. Define a forward operator $\mathscr{F} : (\alpha,\lambda,\tau) \to u(t;\alpha,\lambda,\tau)$, where $u(t;\alpha,\lambda,\tau)$ is the solution of Eqs. (1.1)-(1.2). Hence, the multi-parameter inversion is equivalent to solving the equation $\mathscr{F}(\alpha,\lambda,\tau) = \theta(t)$. Let $a^k = (\alpha^k, \lambda^k, \tau^k)$. Set $a^{k+1} = a^k + \delta a^k$, where δa^k is the perturbation of a^k at the k-th approximation. Taking the Taylor expansion for nonlinear $\mathscr{F}(a)$, one has the linearized form — i.e. $\mathscr{F}(a) \approx \mathscr{F}(a^k) + \nabla\mathscr{F}(a^k)\cdot\delta a^k$. Therefore, the required δa^k is obtained by solving the regularized minimization problem

$$\min_{\delta a^k \in \mathbb{R}^3} \mathscr{J}(\delta a^k) = \min_{\delta a^k \in \mathbb{R}^3} \left\{ \left\| \nabla\mathscr{F}(a^k)\cdot\delta a^k - [\theta(t) - \mathscr{F}(a^k)] \right\|^2 + \mu_k \|\delta a^k\|^2 \right\},$$

where k is the iteration step, k_0 an a priori chosen number, $\beta > 0$, and

$$\mu_k = \frac{1}{1 + \exp(\beta(k-k_0))}$$

a sigmoid-type regularization parameter.

To solve the problem by the finite difference method, we consider its discrete form on temporal grids $0 = t_0 < t_1 < \cdots < t_N = T$. If h is the numerical differential step, then we have

$$\nabla\mathscr{F}\left(a^k; t_i\right)\cdot\delta a^k \approx \frac{\mathscr{F}(\alpha^k + h, \lambda^k, \tau^k; t_i) - \mathscr{F}(\alpha^k, \lambda^k, \tau^k; t_i)}{h}\delta\alpha^k$$

$$+\frac{\mathscr{F}(\alpha^k,\lambda^k+h,\tau^k;t_i)-\mathscr{F}(\alpha^k,\lambda^k,\tau^k;t_i)}{h}\delta\lambda^k$$
$$+\frac{\mathscr{F}(\alpha^k,\lambda^k,\tau^k+h;t_i)-\mathscr{F}(\alpha^k,\lambda^k,\tau^k;t_i)}{h}\delta\tau^k,\quad i=1,2,\ldots,N.$$

Let A be the matrix defined by the operator $\nabla\mathscr{F}(a^k;t_i)$. The minimization problem is reduced to following normal equation:

$$\left(\mu_k I + AA^T\right)\cdot\left(\delta a^k\right)^T = Ab^k,$$

where

$$b^k=\left(\theta(t_1)-\mathscr{F}(a^k;t_1),\ldots,\theta(t_N)-\mathscr{F}(a^k;t_N)\right)^T.$$

5. Numerical Test

In this section, numerical examples are carried out. To determine the convergence rate in Examples 5.1 and 5.2, we choose numerical solutions obtained for $N = 20480$ as the reference solutions. We consider the absolute error between the numerical solutions and the corresponding reference solutions. The abbreviations MAE and EO in the tables are respectively used for the maximum absolute error and expected order. Besides, for the time step h, numerical solutions are determined at the points h, $\tau+h$ and $2\tau+h$ if $t\to 0^+$, $t\to\tau^+$ and $t\to(2\tau)^+$, respectively. The temporal convergence order is defined as

$$\text{Rate}=\log_2\left(\frac{\text{Error}(N/2)}{\text{Error}(N)}\right)$$

or by

$$\text{Rate}=\log_2\left(\frac{\text{MAE}(N/2)}{\text{MAE}(N)}\right).$$

Example 5.1 ([2]). Consider $\phi(t)=1$, $b(t)=1$, $f(t)=t$, $\lambda=1$, $\tau=1$ and $T=5$ in Eqs. (1.1)-(1.2).

Example 5.2. Let $\phi(t)=\exp(5t)$, $b(t)=1+t$, $f(t)=\sin(t)$, $\lambda=1$, $\tau=1$ and $T=5$ in Eqs. (1.1)-(1.2).

Tables 1 and 2 show that the convergence order agrees with the theoretical results except the case $\alpha=0.99$, $t\to 0^+$. Specifically, the convergence order is $\min\{(k+1)\alpha,1\}$ at $(k\tau)^+$, $k=0,1,2$. At the derivative continuous points from the left side $t=1,2,3$, the first order convergence is reached. Furthermore, the results presented in Tables 3 and 4 also confirm the theoretical conclusion.

Example 5.3. Take $\phi(t)=0$, $t<-0.2$, $\phi(t)=5(t+0.2)$, $t\in[-0.2,0]$, $b(t)=1+t$, $f(t)=\sin(2t)$ and $T=3$ in Eqs. (1.1)-(1.2) and 300 grid points on $[0,T]$. Let the numerical solution solved by $L1$ scheme be measured data $u(t)$. The initial guess is taken as $\alpha=0.5$,

Table 1: Example 5.1. Convergence order as t tends to a singularity point.

α	N	$t \to 0^+$		$t \to 1^+$		$t \to 2^+$	
		Error(N)	Rate	Error(N)	Rate	Error(N)	Rate
0.25	320	7.9206e-02	$*$	3.3660e-02	$*$	1.0471e-02	$*$
	640	7.0669e-02	0.1645	2.6274e-02	0.3574	7.0824e-03	0.5641
	1280	6.1706e-02	0.1957	1.9833e-02	0.4057	4.5923e-03	0.6250
	2560	5.1953e-02	0.2482	1.4232e-02	0.4787	2.7946e-03	0.7166
0.6	320	3.5983e-02	$*$	1.2246e-03	$*$	2.0510e-03	$*$
	640	2.3553e-02	0.6114	7.2893e-04	0.7485	9.9157e-04	1.0486
	1280	1.5003e-02	0.6507	4.0953e-04	0.8318	4.7127e-04	1.0732
	2560	9.0841e-03	0.7238	2.1751e-04	0.9129	2.1664e-04	1.1213
0.99	320	6.7027e-04	$*$	8.5236e-03	$*$	2.6168e-03	$*$
	640	2.4046e-04	1.4789	4.1459e-03	1.0398	1.3138e-03	0.9940
	1280	9.2531e-05	1.3778	1.9915e-03	1.0578	6.3739e-04	1.0436
	2560	3.6395e-05	1.3462	9.2504e-04	1.1063	2.9591e-04	1.1070
EO		α		$\min\{2\alpha, 1\}$		$\min\{3\alpha, 1\}$	

Table 2: Example 5.1. Convergence order at the derivative continuous points from the left side.

α	N	$t = 1$		$t = 2$		$t = 3$	
		Error(N)	Rate	Error(N)	Rate	Error(N)	Rate
0.25	320	1.0022e-03	$*$	5.8956e-04	$*$	5.9564e-04	$*$
	640	4.8905e-04	1.0352	2.8642e-04	1.0415	2.9009e-04	1.0379
	1280	2.3548e-04	1.0543	1.3753e-04	1.0584	1.3949e-04	1.0563
	2560	1.0958e-04	1.1036	6.3888e-05	1.1061	6.4852e-05	1.1050
0.6	320	3.0196e-03	$*$	2.0166e-03	$*$	1.1556e-03	$*$
	640	1.4405e-03	1.0678	9.7403e-04	1.0499	5.7799e-04	0.9995
	1280	6.8085e-04	1.0811	4.6431e-04	1.0689	2.8259e-04	1.0324
	2560	3.1223e-04	1.1247	2.1419e-04	1.1162	1.3278e-04	1.0896
0.99	320	8.2985e-03	$*$	2.7942e-03	$*$	5.8441e-03	$*$
	640	4.0877e-03	1.0216	1.3568e-03	1.0422	2.8833e-03	1.0193
	1280	1.9770e-03	1.0480	6.4767e-04	1.0669	1.3932e-03	1.0493
	2560	9.2158e-04	1.1012	2.9829e-04	1.1185	6.4807e-04	1.1042
EO		1		1		1	

$\lambda = 2.5$, $\tau = 0.5$. Let $u^{(k)}(t; \alpha^{inv}, \lambda^{inv}, \tau^{inv})$ be the numerical solution at the k-th iteration. The inversion algorithm stops when

$$E_k = \frac{1}{301}\left(\sum_{i=0}^{300}\left|u^{(k)}(t_i; \alpha^{inv}, \lambda^{inv}, \tau^{inv}) - u(t_i)\right|^2\right)^{1/2} < 5 \times 10^{-4}$$

or $k = 25$. We choose $k_0 = 3$, $\beta = 0.05$ and the numerical step size $h = 0.01$.

Table 3: Example 5.2. Convergence order when $\alpha = 0.25$.

α	N	$t \to 0^+$		$t \to 1^+$		$t \in [1.5, 2]$	
		Error(N)	Rate	Error(N)	Rate	MAE(N)	Rate
0.25	320	3.9911e-02	*	3.4139e-02	*	1.8857e-03	*
	640	3.5592e-02	0.1652	2.6502e-02	0.3653	9.1934e-04	1.0364
	1280	3.1070e-02	0.1960	1.9968e-02	0.4084	4.4311e-04	1.0529
	2560	2.6155e-02	0.2484	1.4322e-02	0.4795	2.0646e-04	1.1018
EO		0.25		0.5		1	

Table 4: Example 5.2. Convergence order when $\alpha = 0.6$.

α	N	$t \in [0, 0.5]$		$t \in [1, 1.5]$		$t \in [4.5, 5]$	
		MAE(N)	Rate	MAE(N)	Rate	MAE(N)	Rate
0.6	320	1.8206e-02	*	4.6594e-03	*	3.2697e-02	*
	640	1.1888e-02	0.6149	2.3277e-03	1.0013	1.6042e-02	1.0273
	1280	7.5627e-03	0.6525	1.1484e-03	1.0193	7.8783e-03	1.0259
	2560	4.5761e-03	0.7248	5.4686e-04	1.0704	3.7714e-03	1.0628
EO		0.6		1		1	

Table 5: Example 5.3(I). Noise-free data.

(α, λ, τ)	α^{inv}	λ^{inv}	τ^{inv}	k
(0.3,1.5,0.3)	0.3008	1.5127	0.2999	15
(0.3,1.5,0.7)	0.3001	1.5008	0.7000	7
(0.3,3.5,0.3)	0.2984	3.5370	0.3053	22
(0.3,3.5,0.7)	0.3127	3.4757	0.6900	8

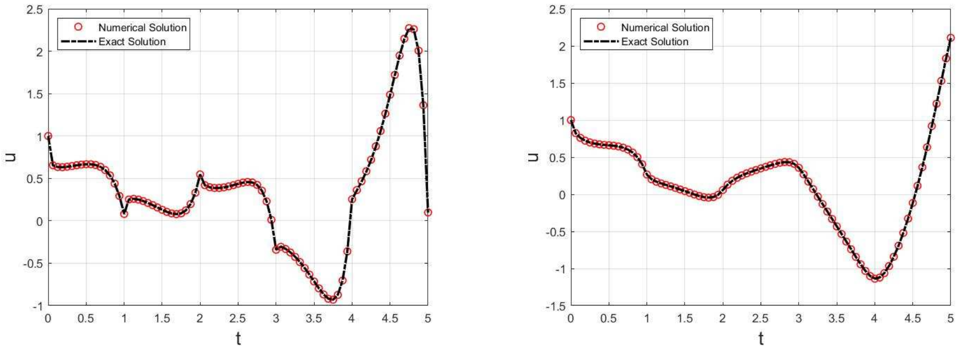
Table 6: Example 5.3(II). Noise-free data.

(α, λ, τ)	α^{inv}	λ^{inv}	τ^{inv}	k
(0.7,1.5,0.3)	0.7003	1.5511	0.3087	4
(0.7,1.5,0.7)	0.6943	1.4836	0.6978	6
(0.7,3.5,0.3)	0.6960	3.4805	0.3042	5
(0.7,3.5,0.7)	0.7006	3.4558	0.6911	5

The results of simultaneous inversion are presented in Tables 5 and 6. The diagrams of the numerical solutions of two cases in Tables 5 and 6 are given — cf. Fig. 2. Furthermore, we consider the case of the noise data $u(t)^\epsilon$, where $u(t)^\epsilon = u(t) + \epsilon u(t) \cdot \text{rand}(\text{size}(u(t)))$, where ϵ is the noise parameter. The details of numerical results are displayed in Table 7.

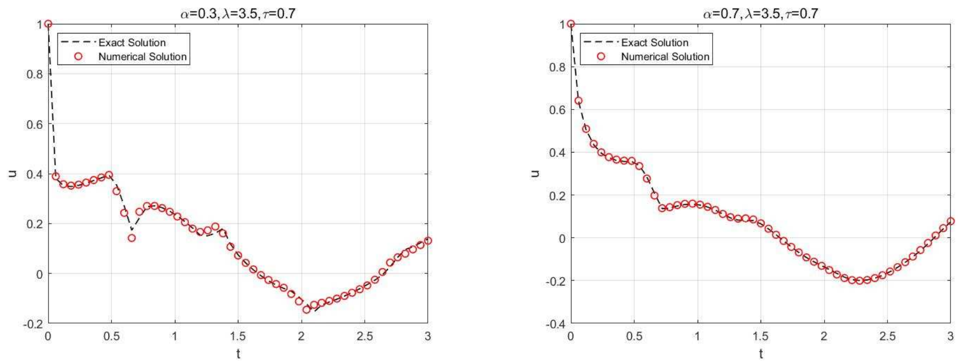
Table 7: Example 5.3. Noise data.

$(\alpha,\lambda,\tau,\epsilon)$	α^{inv}	λ^{inv}	τ^{inv}	k
(0.3,3.5,0.3,1%)	0.2985	3.5068	0.3051	22
(0.7,3.5,0.3,3%)	0.7039	3.3997	0.2984	5
(0.3,1.5,0.7,1%)	0.3004	1.4907	0.6997	7
(0.7,1.5,0.7,3%)	0.7009	1.4397	0.6922	6



$\alpha = 0.25$ $\alpha = 0.6$

Figure 1: Numerical result in Example 5.2.



The fourth case in Table 5. The forth case in Table 6.

Figure 2: Distribution diagrams in Example 5.2.

Example 5.4. Furthermore, we change the initial function in Example 5.3 to the following two cases:

(i) $\phi(t) = 5(t + 0.2), \quad t \leq 0,$

(ii) $\phi(t) = \sin\left(\frac{25\pi}{2}(t + 0.2)\right), \quad t \leq 0.$

Some bad cases are given in Table 8. Specifically, the coefficient α is out of the region $(0, 1)$ in case (i). And, for case (ii), the delay factor is trapped in a deadlock. These two cases illustrate the necessity of the assumption condition of initial function.

Table 8: Example 5.4. Noise-free data.

(α,λ,τ)		α^{inv}	λ^{inv}	τ^{inv}	k
case(i)	(0.3,1.5,0.7)	1.3048	1.9547	0.4998	1
case(i)	(0.3,1.5,0.7)	NaN	NaN	NaN	2
case(ii)	(0.7,3.5,0.3)	0.6943	3.8525	0.4544	10
case(ii)	(0.7,3.5,0.3)	0.6943	3.8541	0.4544	20

6. Conclusion

$L1$ method for multi-singularity problems arising from delay fractional equations is investigated. Pointwise error estimates are derived by discrete Laplace transform technique. Then, we consider identifying the parameters including the fractional order, the reaction coefficient and the delay factor. The uniqueness and $L1$ simultaneous inversion algorithm of inverse problem are proposed. All the theoretical results are confirmed by numerical experiments.

Acknowledgments

D. Cen thanks the Harbin Institute of Technology, Shenzhen for the hospitality during his visit there.

This research was partly supported by the Science and Technology Development Fund, Macau SAR (File No. 0151/2022/A), by the University of Macau (File Nos. MYRG2020-00035-FST and MYRG2022-00076-FST), by the NNSF of China (Grant 12171122), by the Guangdong Provincial Natural Science Foundation of China (Grant 2023A1515010818), by the Fundamental Research Project of Shenzhen (Grant JCYJ20190806143201649), and by the Shenzhen Science and Technology Program (Grant RCJC20210609103755110).

References

[1] D. Cen, C. Ou and S. Vong, *Corrected L-type method for multi-singularity problems arising from delay fractional equations*, J. Sci. Comput. **97**, 15 (2023).

[2] D. Cen and S. Vong, *The tracking of derivative discontinuities for delay fractional equations based on fitted L1 method*, Comput. Methods Appl. Math. **23**, 591–601 (2023).

[3] D. Cen, Z. Wang and Y. Mo, *Second order difference schemes for time-fractional KdV-Burgers equation with initial singularity*, Appl Math Lett. **112**, 106829 (2021).

[4] J. Cermak, Z. Dosla and T. Kisela, *Fractional differential equations with a constant delay: Stability and asymptotics of solutions*, Appl. Math. Comput. **298**, 336–350 (2017).

[5] J. Cermak, J. Hornicek and T. Kisela, *Stability regions for fractional differential systems with a time delay*, Commun. Nonlinear Sci. Numer. Simul. **31**, 108–123 (2016).

[6] H. Chen, Y. Shi, J. Zhang and Y. Zhao, *Sharp error estimate of a Grünwalds-Letnikov scheme for reaction-subdiffusion equations*, Numer. Algor. **89**, 1465–1477 (2022).

[7] J. Gracia, E. O'Riordan and M. Stynes, *A fitted scheme for a Caputo initial-boundary value problem*, J. Sci. Comput. **76**, 583–609 (2018).

[8] A. Hendy, M. Zaky and R. Staelen, *A general framework for the numerical analysis of high-order finite difference solvers for nonlinear multi-term time-space fractional partial differential equations with time delay*, Appl. Numer. Math. **169**, 108–121 (2021).

[9] B. Jin, R. Lazarov and Z. Zhou, *An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data*, IMA J. Numer. Anal. **36**, 197–221 (2016).

[10] B. Jin, B. Li and Z. Zhou, *Correction of high-order BDF convolution quadrature for fractional evolution equations*, SIAM J. Sci. Comput. **39**, A3129–A3152 (2017).

[11] B. Jin, B. Li and Z. Zhou, *An analysis of the Crank-Nicolson method for subdiffusion*, IMA J. Numer. Anal. **38**, 518–541 (2018).

[12] N. Kopteva, *Error analysis of the L1 method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions*, Math. Comput. **88**, 2135–2155 (2019).

[13] N. Kopteva, *Error analysis of an L2-type method on graded meshes for a fractional-order parabolic problem*, Math. Comp. **90**, 19–40 (2021).

[14] D. Li, H. Qin and J. Zhang, *Sharp pointwise-in-time error estimate of L1 scheme for nonlinear subdiffusion equations*, J. Comput. Math. doi:10.4208/jcm.2205-m2021-0316.

[15] D. Li, M. She, H. Sun and X. Yan, *A novel discrete fractional Grönwall-type inequality and its application in pointwise-in-time error estimates*, J. Sci. Comput. **91**, 27 (2022).

[16] M. Li and J. Wang, *Finite time stability of fractional delay differential equations*, Appl. Math. Lett. **64**, 170–176 (2017).

[17] H. Liang and H. Brunner, *The convergence of collocation solutions in continuous piecewise polynomial spaces for weakly singular Volterra integral equations*, SIAM J. Numer. Anal. **57**, 1875–1896 (2019).

[18] H. Liao, W. Mclean and J. Zhang, *A discrete Grönwall inequality with applications to numerical schemes for subdiffusion problems*, SIAM J. Numer. Anal. **57**, 218–237 (2019).

[19] H. Liao, T. Tang and T. Zhou, *An energy stable and maximum bound preserving scheme with variable time steps for time fractional Allen-Cahn equation*, SIAM J. Sci. Comput. **43**, A3503–A3526 (2021).

[20] C. Lubich, I. Sloan and V. Thomee, *Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term*, Math. Comp. **65**, 1–17 (1996).

[21] P. Lyu and S. Vong, *A high-order method with a temporal nonuniform mesh for a time-fractional Benjamin-Bona-Mahony equation*, J. Sci. Comput. **80**, 1607–1628 (2019).

[22] W. McLean, K. Mustapha, R. Ali and O. Knio, *Regularity theory for time-fractional advection diffusion-reaction equations*, Comput. Math. Appl. **79**, 947–961 (2020).

[23] M. Morgado, N. Ford and P. Lima, *Analysis and numerical methods for fractional differential equations with delay*, J. Comput. Appl. Math. **252**, 159–168 (2013).

[24] I. Podlubny, *Fractional Differential Equations*, Academic Press (1999).

[25] H. Pollard, *The completely monotonic character of the Mittag-Leffler function $e_\alpha(-x)$*, Bull. Am. Math. Soc. **54**, 1115–1116 (1948).

[26] J. Shen, Z. Sun and R. Du, *Fast finite difference schemes for the time-fractional diffusion equations with a weak singularity at the initial time*, East Asian J. Appl. Math. **8**, 834–858 (2018).

[27] M. Stynes, *Too much regularity may force too much uniqueness*, Fract. Calc. Appl. Anal. **19**, 1554–1562 (2016).

[28] Z. Sun and X. Wu, *A fully discrete scheme for a diffusion wave system*, Appl. Numer. Math. **56**, 193–209 (2006).

[29] T. Tan, W. Bu and A. Xiao, *L1 method on nonuniform meshes for linear time-fractional diffusion equations with constant time delay*, J. Sci. Comput. **92**, 98 (2022).

[30] T. Tang, H. Yu and T. Zhou, *On energy dissipation theory and numerical stability for time-fractional phase field equations*, SIAM J. Sci. Comput. **41**, A3757–A3778 (2019).

[31] H. Tuan, *A qualitative theory of time delay nonlinear fractional-order systems*, SIAM J. Control Optim. **58**, 1491–1518 (2020).

[32] D. Wang and J. Zou, *Dissipativity and contractivity analysis for fractional functional differential equations and thier numerical approximations*, SIAM J. Numer. Anal. **57**, 1445–1470 (2019).

[33] Y. Yan, M. Khan and N. Ford, *An analysis of the modified L1 scheme for time-fractional partial differential equations with nonsmooth data*, SIAM J. Numer. Anal. **56**, 210–227 (2018).

[34] M. Zayernouri, W. Cao, Z. Zhang and G. Karniadakis, *Spectral and discontinuous spectral element methods for fractional delay equation*, SIAM J. Sci. Comput. **36**, B904–B929 (2014).

[35] Q. Zhang, M. Ran and D. Xu, *Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay*, Appl. Anal. **96**, 1867–1884 (2017).

[36] Y. Zhang and Z. Wang, *Numerical simulation for time-fractional diffusion-wave equations with time delay*, J. Appl. Math. Comput. **69**, 137–157 (2023).

[37] Y. Zhou and M. Stynes, *Optimal convergence rates in time-fractional discretisations: The L1, $\bar{L}1$ and Alikhanov schemes*, East Asian J. Appl. Math. **12**, 503–520 (2022).