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The Characteristics of Solutions to Semilinear Wave Equation with Logarithmic Plus Polynomial Nonlinearities

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Abstract. The semilinear wave equation with logarithmic and polynomial nonlinearities is considered in this paper. By adjusting and using potential well method, we attain the global-in-time existence and infinite time blowup solutions at subcritical initial energy level $E(0)<d$. Then using additional conditions on initial data, these results are enlarged at critical case $E(0)=d$ and arbitrarily positive case $E(0)>0$.

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1 Introduction

In this article, we are exploring the initial boundary value problem of the semilinear wave equation containing logarithmic plus polynomial source terms

$$\begin{cases} u_{tt}-\Delta u=u\ln|u|^k+|u|^{p-1}u, & x\in\Omega, \quad t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t)=0, & x\in\partial\Omega, \quad t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain and $k>1$, $2<p<\frac{n+2}{n-2}$ $(n\geq 3)$ are constants. The semilinear hyperbolic equations are very important nonlinear evolution equations in the field of mathematical physics. The polynomial nonlinearities model the external force that enhances energy and drives the system toward possible instability [12]. The evolution equations with logarithmic nonlinearities come naturally in inflation cosmology and supersymmetric field theory (see [4, 18]). Moreover, we can see its implementation in different area of physics for example nuclear physics, optics, and geophysics (see [5, 16]). According to existing literature, some special analytical solutions of the problem with logarithmic nonlinearities can be found in the logarithmic quantum mechanics (see [3, 25]).

In order to recall the related work of problem (1.1), we give the following hyperbolic model with general nonlinearity $f(u)$ to discuss different cases of the nonlinearity

$$u_{tt}-\Delta u=f(u). \tag{1.2}$$

The background survey will be started with very important work of Sattinger [29], which was revolutionary for investigating nonlinear wave equations. The author [29] first introduced the concept of potential well W to study the above semi-linear wave equation with polynomial source term when initial data u_0 lie in the so-called potential well to get the solution still belongs to the potential well as described above. For precisely specified class of initial data finite blowup result was also studied. In [27], Payane and Sattinger proved finite blowup of solution of the problem (1.2) while u_0 lies outside the potential well W. Besides, they discussed potential energy J with availability of saddle point and gave explanatory description of potential well W. The way to search blow up results for the abstract problem (1.2) was first developed in [21]. In [2] point-wise blow up in finite time was shown for (1.2). Using assumption $(u_0,u_1)\geq 0$, the proof of blow up (global nonexistence) was treated in [17] and [28] for definitely positive initial energy case. The so-called family of potential wells was proposed by Liu Yacheng [20] which incorporates single potential well W as a special case, and the previous results were developed in $E(0)<d$ for special

nonlinear term

$$f(u)=|u|^{p-1}u.$$

For the case $I(u_0)\geq 0$, $E(0)=d$, threshold results were obtained in [19]. Considering damping term, the author [10] successfully proved finite time blowup solution for $E(0)>0$. Later, many investigation carried out for the arbitrarily high initial energy (see [9, 26, 30, 32, 33]). All these works were about polynomial source term, and for more recent work on polynomial source terms, we can refer to [8, 24, 34]. However, there were many investigation already done with logarithmic nonlinearity (see [1, 6, 7, 11, 13–15]). Inspiring by their study, we went into polynomial and logarithmic nonlinearity together, i.e., polynomial term multiplying logarithmic term, and got some interesting result. The nonlinear term of a semilinear wave equation with logarithmic nonlinearity in [22] is $u\ln|u|^k$, the nonlinear term of a semilinear wave equation with polynomial nonlinearity in [23] is $|u|^p\ln|u|$. Here in this paper, the nonlinear term is $u\ln|u|^k+|u|^{p-1}u$, for the first time we are considering the problem (1.1) with logarithmic plus polynomial source term to see the nature of the solution.

In the present paper, we consider another case of the combination of the logarithmic and polynomial nonlinearities, i.e., polynomial term plus logarithmic term, to investigate the effect of such combined nonlinearity on the dynamical properties of the solution to problem (1.1).

We are organizing the article as following. Some fundamental concepts of potential wells and essential lemmas are discussing in Section 2. In Section 3, we conclude the main result for subcritical initial energy level. The results of the critical case are given in Section 4. Finally, the proof under arbitrarily positive initial energy case is studied in Section 5.

2 Preliminaries

Definition 2.1. *A function $u(x,t)$ is called a weak solution of* (1.1) *on $\Omega\times[0,T)$ if*

$$u\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right)$$

and maintains

$$\begin{aligned}&(u_t,v)+\int_0^t(\nabla u,\nabla v)d\tau\\ =&\int_0^t(u\ln|u|^k,v)d\tau+\int_0^t(|u|^{p-1}u,v)d\tau+(u_1,v)\end{aligned}$$

for all $v\in H_0^1(\Omega)$, $t\in[0,T)$ and $u(x,0)=u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0)=u_1(x)$ in $L^2(\Omega)$.

2.1 Potential wells

In this part, potential wells and their essential characteristics are discussed. Later these will be important ingredients for the successive sections.

To begin with, we propose two C^1 functions on $H_0^1(\Omega)\to\mathbb{R}$, which are potential energy function and Nehari function respectively as below

$$J(u):=\frac{1}{2}\|\nabla u\|^2-\frac{k}{2}\int_\Omega u^2\ln|u|dx+\frac{k}{4}\|u\|^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}$$

and

$$I(u):=\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx-\|u\|_{p+1}^{p+1}. \tag{2.1}$$

After that we can write

$$J(u):=\frac{1}{2}I(u)+\frac{k}{4}\|u\|^2+\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} \tag{2.2}$$

and

$$\begin{aligned}E(t):=&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{k}{2}\int_\Omega u^2\ln|u|dx+\frac{k}{4}\|u\|^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\\=&\frac{1}{2}\|u_t\|^2+J(u)\\=&\frac{1}{2}\|u_t\|^2+\frac{1}{2}I(u)+\frac{k}{4}\|u\|^2+\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}.\end{aligned} \tag{2.3}$$

Now, we introduce the Nehari manifold

$$\mathcal{N}(u):=\{u\in H_0^1(\Omega)|\ I(u)=0,\ \|\nabla u\|^2\neq 0\}$$

and the mountain pass level or depth of potential well

$$d:=\inf_{u\in\mathcal{N}}J(u).$$

Then we define the stable set

$$W:=\{u\in H_0^1(\Omega)|\ I(u)>0,\ J(u)<d\}\cup\{0\}$$

and the unstable set

$$V:=\{u\in H_0^1(\Omega)|\ I(u)<0,\ J(u)<d\}.$$

Then family potential wells for $\delta>0$ expanding above sets as following

$$J_\delta(u):=\frac{\delta}{2}\|\nabla u\|^2+\frac{k}{4}\|u\|^2-\frac{k}{2}\int_\Omega u^2\ln|u|dx-\frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$I_\delta(u):=\delta\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx-\|u\|_{p+1}^{p+1},$$
$$\mathcal{N}_\delta(u):=\{u\in H_0^1(\Omega)|\ I_\delta(u)=0,\ \|\nabla u\|^2\neq 0\},$$

and

$$d(\delta):=\inf_{u\in\mathcal{N}_\delta} J(u), \tag{2.4}$$

and

$$W_\delta:=\{u\in H_0^1(\Omega)|\ I_\delta(u)>0,\ J(u)<d(\delta)\}\cup\{0\},$$

and the outside of W_δ as

$$V_\delta:=\{u\in H_0^1(\Omega)|\ I_\delta(u)<0,\ J(u)<d(\delta)\}.$$

To investigate the problem (1.1) at $E(0)=d$ we introduce

$$V':=\{u\in H_0^1(\Omega)|\ I(u)<0\}.$$

The below lemma apprises us about the critical point of potential energy function.

Lemma 2.1. *For $u\in H_0^1(\Omega)$ with $\|u\|\neq 0$, set $m(\lambda):=J(\lambda u)$. Then*

(i). $\lim_{\lambda\to 0} m(\lambda)=0$, $\lim_{\lambda\to+\infty} m(\lambda)=-\infty$;

(ii). for $\lambda\in(0,+\infty)$, there exists a unique $\lambda^*=\lambda^*(u)$ at which

$$\frac{d}{d\lambda}m(\lambda)=0;$$

(iii). $m(\lambda)$ increases on $(0,\lambda^*)$, decreases on $(\lambda^*,+\infty)$ and reaches its maximum at $\lambda=\lambda^*$.

(iv). $I(\lambda u)=\lambda\frac{d}{d\lambda}m(\lambda)>0$ for $0<\lambda<\lambda^*$; $I(\lambda u)<0$ for $\lambda^*<\lambda<+\infty$ and $I(\lambda^* u)=0$.

Proof. Proof of (i). We have

$$\begin{aligned}m(\lambda):=&J(\lambda u)\\=&\frac{1}{2}\lambda^2\|\nabla u\|^2-\frac{k}{2}\lambda^2\int_\Omega u^2\ln|u|dx+\frac{k}{4}\lambda^2\|u\|^2-\frac{k}{2}\lambda^2\ln\lambda\|u\|^2-\frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1}.\end{aligned}$$

From $\|u\|\neq 0$, it is obvious $m(0)=0$, $m(+\infty)=-\infty$.

Proof of (ii). Differentiating $m(\lambda)$, then having equal zero, we get

$$\begin{aligned}m'(\lambda)=&\frac{d}{d\lambda}J(\lambda u)\\=&\lambda\|\nabla u\|^2-k\lambda\ln\lambda\|u\|^2-k\lambda\int_\Omega u^2\ln|u|dx-\lambda^p\|u\|_{p+1}^{p+1}=0,\end{aligned}$$

that implies

$$\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx=k\ln\lambda\|u\|^2+\lambda^{p-1}\|u\|_{p+1}^{p+1}. \tag{2.5}$$

Let

$$l(\lambda):=k\ln\lambda\|u\|^2+\lambda^{p-1}\|u\|_{p+1}^{p+1}.$$

Differentiating this we get

$$\begin{aligned}l'(\lambda)=&\frac{k}{\lambda}\|u\|^2+(p-1)\lambda^{p-2}\|u\|_{p+1}^{p+1}\\=&\frac{1}{\lambda}\left(k\|u\|^2+(p-1)\lambda^{p-1}\|u\|_{p+1}^{p+1}\right).\end{aligned}$$

Clearly $l'(\lambda)>0$ for every $\lambda>0$. So $l(\lambda)$ is increasing on $(0,+\infty)$. Thus, we can write

$$\begin{aligned}&\lim_{\lambda\to 0^+}l(\lambda)=-\infty,\\&\lim_{\lambda\to\infty}l(\lambda)=\infty.\end{aligned}$$

Hence, there is a unique λ_0 available for which $l(\lambda_0)=0$,

$$\begin{aligned}&l(\lambda)<0 \quad \text{for} \quad 0<\lambda<\lambda_0,\\&l(\lambda)>0 \quad \text{for} \quad \lambda_0<\lambda<\infty.\end{aligned}$$

As a result, we can obtain a unique $\lambda^*>\lambda_0$ so that (2.5) holds for some u for which left side of (2.5) is positive.

Proof of (iii). We know

$$\frac{d}{d\lambda}J(\lambda u)=\lambda\left(\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx-l(\lambda)\right).$$

From the proof of (ii) it implies that

1. if $0<\lambda\leq\lambda_0$, then $l(\lambda)\leq 0$;

2. if $\lambda_0<\lambda<\lambda^*$, then
$$0<l(\lambda)<\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx;$$

3. if $\lambda^*<\lambda<\infty$, then
$$l(\lambda)>\|\nabla u\|^2-k\int_\Omega u^2\ln|u|dx.$$

Hence, we arrive at

1. If $0<\lambda<\lambda^*$,
$$\frac{d}{d\lambda}J(\lambda u)=m'(\lambda)>0;$$

2. If $\lambda=\lambda^*$,
$$\frac{d}{d\lambda}J(\lambda u)=m'(\lambda)=0;$$

3. If $\lambda^*<\lambda<+\infty$,
$$\frac{d}{d\lambda}J(\lambda u)=m'(\lambda)<0.$$

Hence, (iii) follows.

(iv). we can have the conclusion using the proof of (iii) and

$$\begin{aligned}I(\lambda u)=&\lambda^2\|\nabla u\|^2-k\lambda^2\int_\Omega u^2\ln|u|dx-k\lambda^2\ln\lambda\|u\|^2-\lambda^{p+1}\|u\|_{p+1}^{p+1}\\=&\lambda\frac{d}{d\lambda}J(\lambda u).\end{aligned}$$

Thus, we complete the proof. □

The below lemma gives some key characteristics of $I_\delta(u)$ with respect to H_0^1.

Lemma 2.2. *Consider a ball in H_0^1 with radius $r:=\|\nabla u\|$, then for any $\delta>0$*

(i). $I_\delta(u)>0$, when $0<\|\nabla u\|\leq r(\delta)$;

(ii). $r>r(\delta)$, when $I_\delta(u)<0$;

(iii). $r > r(\delta)$ or $r=0$, when $I_\delta(u)=0$, where the real value $r(\delta)$ solves equation $\phi(r)=\delta$ uniquely

$$\phi(r):=(kC^*+2)C^{p+1}r^{p-1},$$
$$C=\sup_{u\in H_0^1(\Omega)}\frac{\|u\|_{p+1}}{\|\nabla u\|},$$

and C^* is a large enough constant.

Proof. First, we prove

$$k\int_\Omega u^2\ln|u|dx<(kC^*+1)\|u\|_{p+1}^{p+1}.$$

Since $\ln|u|<|u|$, we can write

$$k\int_\Omega u^2\ln|u|dx<k\int_\Omega u^3dx<k\|u\|_3^3+\|u\|_{p+1}^{p+1}.$$

Using the embedding $L^{p+1}(\Omega)\hookrightarrow L^3(\Omega)$ for $p>2$, where C_* is the constant of that embedding, we can obtain

$$k\int_\Omega u^2\ln|u|dx<kC_*^3\|u\|_{p+1}^3+\|u\|_{p+1}^{p+1}.$$

Now, we can write

$$\begin{aligned}
&kC_*^3\|u\|_{p+1}^3=kC_*^3\left(\int_{\Omega_1}|u|^{p+1}dx\right)^{\frac{3}{p+1}}\\
&=kC_*^3\left(\int_{\Omega_1}|u|^{p+1}dx+\int_{\Omega_2}|u|^{p+1}dx\right)^{\frac{3}{p+1}}\\
&<kC_*^3\left(\int_{\Omega_1}|u|^{p+1}dx+\int_{\Omega_2}1dx\right)^{\frac{3}{p+1}}\\
&=kC_*^3\left(\int_{\Omega_1}|u|^{p+1}dx+|\Omega_2|\right)^{\frac{3}{p+1}}\\
&<kC_*^3\left(\int_{\Omega_1}|u|^{p+1}dx+|\Omega_2|\right)\\
&<kC^*\int_{\Omega_1}|u|^{p+1}dx\\
&<kC^*\int_\Omega|u|^{p+1}dx=kC^*\|u\|_{p+1}^{p+1},
\end{aligned}$$

where

$$\Omega_1=\left\{x\in\Omega\,\middle|\,\left(\int_{\Omega_1}|u|^{p+1}dx\right)^{\frac{1}{p+1}}\geq 1\right\},$$
$$\Omega_2=\left\{x\in\Omega\,\middle|\,\left(\int_{\Omega_2}|u|^{p+1}dx\right)^{\frac{1}{p+1}}< 1\right\}.$$

Therefore, we have

$$\begin{aligned}k\int_\Omega u^2\ln|u|dx&<kC^*\|u\|_{p+1}^{p+1}+\|u\|_{p+1}^{p+1}\\&=(kC^*+1)\|u\|_{p+1}^{p+1}.\end{aligned}$$

Proof of (i). Using $0<\|\nabla u\|\leq r(\delta)$ we obtain $\|u\|_{p+1}>0$ and $0<\phi(\|\nabla u\|)\leq\delta$. By

$$\begin{aligned}k\int_\Omega u^2\ln|u|dx+\|u\|_{p+1}^{p+1}&<(kC^*+2)\|u\|_{p+1}^{p+1}\\&\leq(kC^*+2)C^{p+1}\|\nabla u\|^{p+1}\\&=\phi(\|\nabla u\|)\|\nabla u\|^2\leq\delta\|\nabla u\|^2,\end{aligned}\tag{2.6}$$

we have $I_\delta(u)>0$.

Proof of (ii). If $I_\delta(u)<0$, then we can write

$$\begin{aligned}\delta\|\nabla u\|^2&<k\int_\Omega u^2\ln|u|dx+\|u\|_{p+1}^{p+1}\\&<(kC^*+2)\|u\|_{p+1}^{p+1}\leq\phi(\|\nabla u\|)\|\nabla u\|^2,\end{aligned}$$

which implies $\|\nabla u\|>r(\delta)$.

Proof of (iii). $\|\nabla u\|=0$ implies $I_\delta(u)=0$. Again, If $I_\delta(u)=0$ and $\|\nabla u\|\neq 0$, then

$$\begin{aligned}\delta\|\nabla u\|^2&=k\int_\Omega u^2\ln|u|dx+\|u\|_{p+1}^{p+1}\\&<(kC^*+2)\|u\|_{p+1}^{p+1}\leq\phi(\|\nabla u\|)\|\nabla u\|^2,\end{aligned}\tag{2.7}$$

which gives $\|\nabla u\|>r(\delta)$. □

The more about $d(\delta)$ will be discussed in the below lemma.

Lemma 2.3. *As function of δ, $d(\delta)$ shows the following behavior*

(i). $d(\delta)>a(\delta)r^2(\delta)$, $\forall\delta\in(0,1)$, where $a(\delta)=\frac{1}{2}-\frac{\delta}{2}$;

(ii). there is a unique $\delta_0 > 1$*, for which* $d(\delta_0) = 0$*, and* $d(\delta) > 0$ *for* $0 < \delta < \delta_0$*;*

(iii). $d(\delta)$ *increases on* $(0,1)$*, decreases on* $(1,\delta_0)$ *and takes its maximum* $d = d(1)$ *at* $\delta = 1$*.*

Proof. Proof of (i). We notice that $I_\delta(u) = 0$ and $\|\nabla u\| \neq 0$ implies $\|\nabla u\| > r(\delta)$ by Lemma 2.2(iii). Applying this we can obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|^2 - \frac{k}{2}\int_\Omega u^2 \ln|u| dx + \frac{k}{4}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &> \frac{1}{2}\|\nabla u\|^2 - \frac{k}{2}\int_\Omega u^2 \ln|u| dx + \frac{k}{4}\|u\|^2 - \frac{1}{2}\|u\|_{p+1}^{p+1} \\ &= \left(\frac{1}{2} - \frac{\delta}{2}\right)\|\nabla u\|^2 + \frac{1}{2} I_\delta(u) + \frac{k}{4}\|u\|^2 \\ &= \frac{1}{2}(1-\delta)\|\nabla u\|^2 + \frac{k}{4}\|u\|^2 > a(\delta) r^2(\delta). \end{aligned}$$

Proof of (ii). For every $u \in H_0^1(\Omega)$, $\|\nabla u\| \neq 0$, we introduce $\lambda = \lambda(\delta)$ such that

$$\delta\lambda^2\|\nabla u\|^2 - k\lambda^2 \int_\Omega u^2 \ln|u| dx = k\lambda^2 \ln\lambda \|u\|^2 + \lambda^{p+1}\|u\|_{p+1}^{p+1}, \tag{2.8}$$

i.e.,

$$\delta\|\nabla u\|^2 - k\int_\Omega u^2 \ln|u| dx = k\ln\lambda\|u\|^2 + \lambda^{p-1}\|u\|_{p+1}^{p+1}. \tag{2.9}$$

So, $I_\delta(\lambda u) = 0$ means $\lambda u \in \mathcal{N}_\delta$. Now, from (2.9) we see that $\lambda(\delta)$ is increasing as δ increases, so we can obtain

$$\lim_{\delta\to+\infty} \lambda(\delta) = +\infty.$$

Thus by Lemma 2.1, we achieve

$$\lim_{\delta\to+\infty} J(\lambda u) = \lim_{\lambda\to+\infty} J(\lambda u) = -\infty.$$

Therefore,

$$\lim_{\delta\to+\infty} d(\delta) = -\infty.$$

Again, by (i), we have $d(\delta) > 0$ for $0 < \delta < 1$ and becomes maximum at a ($a = 1$ will be proved in the next part), and hence it certainly hits the δ-axis at some point $\delta_0 > 1$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 < \delta < \delta_0$.

Proof of (iii). We prove that $d(\delta')<d(\delta'')$ for any $0<\delta'<\delta''<1$ or $1<\delta''<\delta'<\delta_0$. Clearly it is sufficient to prove that for any $0<\delta'<\delta''<1$ or $1<\delta''<\delta'<\delta_0$. We prove this part considering following two cases.

Case i. $(0<\delta'<\delta''<1)$. For any $u\in H_0^1(\Omega)$, $I_{\delta''}(u)=0$ and $\|\nabla u\|\neq 0$ there exist a $v\in H_0^1(\Omega)$ and a fix value $\varepsilon(\delta',\delta'')>0$ for which $I_{\delta'}(v)=0$, $\|\nabla v\|\neq 0$ and $J(v)<J(u)-\varepsilon(\delta',\delta'')$. In fact, for these u, $\lambda(\delta)$ can be defined by (2.8) so that $I_\delta(\lambda(\delta)u)=0$, $\lambda(\delta'')=1$ and (2.9) is true. Suppose $m(\lambda)=J(\lambda u)$. Then

$$\begin{aligned}\frac{d}{d\lambda}m(\lambda)=&\frac{1}{\lambda}\big((1-\delta)\|\nabla(\lambda u)\|^2+I_\delta(\lambda u)\big)\\=&(1-\delta)\lambda\|\nabla u\|^2.\end{aligned}$$

Take $v=\lambda(\delta')u$, then $I_{\delta'}(v)=0$ and $\|\nabla v\|\neq 0$. If $0<\delta'<\delta''<1$, then

$$\begin{aligned}J(u)-J(v)=&m(1)-m(\lambda(\delta'))\\>&(1-\delta'')\lambda(\delta')r^2(\delta'')(1-\lambda(\delta'))\\=&\varepsilon(\delta',\delta'').\end{aligned}$$

Case ii. $(1<\delta''<\delta'<\delta_0)$. From the argument in Case i, it is obvious that

$$\begin{aligned}J(u)-J(v)=&m(1)-m(\lambda(\delta'))\\>&(\delta''-1)\lambda(\delta'')r^2(\delta'')(\lambda(\delta')-1)\\=&\varepsilon(\delta',\delta'').\end{aligned}$$

This completes the proof. □

The below lemma will be required to discuss details about the invariant set.

Lemma 2.4. *Suppose* $0<J(u)<d$ *for some* $u\in H_0^1(\Omega)$ *and* $0<\delta_1<1<\delta_2$ *is two solutions of* $d(\delta)=J(u)$, *then* $I_\delta(u)$ *remains same in sign on* $0<\delta_1<\delta<\delta_2$.

Proof. To begin with, $J(u)>0$ leads $\|\nabla u\|\neq 0$. If the sign of $I_\delta(u)$ are changeable for $0<\delta_1<\delta<\delta_2$, then there exists a $\bar{\delta}\in(\delta_1,\delta_2)$ such that $I_{\bar{\delta}}(u)=0$. Hence, by (2.4), we have

$$J(u)\geq d(\bar{\delta}),$$

which contradicts

$$J(u)=d(\delta_1)=d(\delta_2)<d(\bar{\delta}).$$

This completes the proof. □

2.2 Invariant sets

Here, in the below lemma the invariant set will be explored.

Lemma 2.5 (Invariant sets)**.** *Let* $u_0 \in H_0^1(\Omega)$ *and* $u_1(x) \in L^2(\Omega)$. *Assume that* $0<e<d$, $\delta_1<\delta_2$ *are the two roots of equation* $d(\delta)=e$. *Then*

(i). all solutions of problem (1.1) *with* $0<E(0)\leq e$ *belong to* W_δ *for* $\delta_1<\delta<\delta_2$, *provided* $I(u_0)>0$ *or* $\|\nabla u_0\|=0$;

(ii). all solutions of problem (1.1) *with* $0<E(0)\leq e$ *belong to* V_δ *for* $\delta_1<\delta<\delta_2$, *provided* $I(u_0)<0$.

Proof. (i). Let $u(t)$ be any solution of problem (1.1) with $E(0)=e$ and $I(u_0)>0$ or $\|\nabla u_0\|=0$, T be the existence time of $u(t)$. If $\|\nabla u_0\|=0$, then obviously $u_0(x)\in W_\delta$ on $0<\delta<\delta_0$. Since $I(u_0)>0$ and Lemma 2.4 implies the sign of $I_\delta(u)$ is not changeable on $\delta_1<\delta<\delta_2$, we have $I_\delta(u_0)>0$ for $\delta\in(\delta_1,\delta_2)$. From the energy equality

$$\frac{1}{2}\|u_1\|^2+J(u_0)=E(0)\leq d(\delta_1)=d(\delta_2)<d(\delta), \quad \delta\in(\delta_1,\delta_2),$$

we have $J(u_0)<d(\delta)$, i.e., $u_0(x)\in W_\delta$ for $\delta_1<\delta<\delta_2$. Next, we prove $u(t)\in W_\delta$ for $\delta_1<\delta<\delta_2$ and $0<t<T$, where T is the maximal existence time of $u(t)$. Arguing by contradiction, we suppose that there must exist a first $t_0\in(0,T)$ such that $u(t_0)\in\partial W_\delta$ for some $\delta\in(\delta_1,\delta_2)$, i.e.,

$$I_\delta(u(t_0))=0, \quad \|\nabla u(t_0)\|\neq 0 \quad \text{or} \quad J(u(t_0))=d(\delta).$$

From the energy inequality

$$\frac{1}{2}\|u_t\|^2+J(u)\leq E(0)<d(\delta), \quad t\in(0,T), \quad \delta\in(\delta_1,\delta_2), \tag{2.10}$$

$J(u(t_0))=d(\delta)$ is not possible. But, if $I_\delta(u(t_0))=0$ and $\|\nabla u(t_0)\|\neq 0$, then (2.4) implies $J(u(t_0))\geq d(\delta)$ that violates (2.10).

(ii). The proof follows from (i). □

Indeed, we still have the following lemmas to discuss about solutions.

Lemma 2.6. *For* $E(0)=0$, *all the solutions of* (1.1), *which are not trivial, lie in*

$$B_{r_0}^c=\left\{u\in H_0^1(\Omega) \;\middle|\; \|\nabla u\|\geq r_0:=\left(\frac{1}{(kC^*+2)C^{p+1}}\right)^{\frac{1}{p-1}}\right\}.$$

Proof. Let $u(t)$ be any solution of (1.1) with initial energy $E(0)=0$, and T be the maximum existence time of $u(t)$. Then by

$$\frac{1}{2}\|u_t\|^2+J(u)\equiv E(0)=0,$$

we get $J(u)\leq 0$ on $0\leq t<T$. Thus by (2.2) we obtain

$$\frac{1}{2}I(u)+\frac{k}{4}\|u\|^2+\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}\leq 0,$$

which means that $I(u)\leq 0$. Therefore, using (2.1), we get

$$\begin{aligned}\|\nabla u\|^2 &\leq k\int_\Omega u^2\ln|u|dx+\|u\|_{p+1}^{p+1}\\ &\leq (kC^*+2)\|u\|_{p+1}^{p+1}\\ &\leq (kC^*+2)C^{p+1}\|\nabla u\|^{p-1}\|\nabla u\|^2, \quad 0\leq t<T.\end{aligned}$$

By which we have either

$$\|\nabla u\|=0$$

or

$$\|\nabla u\|\geq r_0=\left(\frac{1}{(kC^*+2)C^{p+1}}\right)^{\frac{1}{p-1}}.$$

If $\|\nabla u_0\|=0$, then $\|\nabla u\|\equiv 0$ for $0\leq t<T$. Otherwise there exists a $t_0\in(0,T)$ such that $0<\|\nabla u(t_0)\|<r_0$. By similar logics we can show that if $\|\nabla u_0\|\geq r_0$, then $\|\nabla u\|\geq r_0$ for $0<t<T$. □

Lemma 2.7. *Suppose $u_0(x)\in H_0^1(\Omega)$ and $u_1(x)\in L^2(\Omega)$. Assume that $E(0)<0$ or $E(0)=0$, $\|\nabla u_0\|\neq 0$. Then V_δ on $0<\delta<1$ contains all the solutions of* (1.1).

Proof. Let $u(t)$ be the any solution of (1.1) with initial energy $E(0)=0$, and T be the maximum existence time of it. The following gives

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2+a(\delta)\|\nabla u\|^2+\frac{1}{p+1}I_\delta(u)\\ \leq&\frac{1}{2}\|u_t\|^2+J(u)=E(0), \quad 0<\delta<1.\end{aligned} \tag{2.11}$$

Combining $E(0)<0$ and (2.11) implies $I_\delta(u)<0$, $J(u)<0<d(\delta)$, where $d(\delta)>0$ for $0<\delta<1$ by Lemma 2.3; if $E(0)=0$, $\|\nabla u_0\|\neq 0$, then by Lemma 2.6, we have $\|\nabla u_0\|\geq r_0$ for $0\leq t<T$. Again by (2.11) we get $I_\delta(u)<0$, $J(u)<0<d(\delta)$ for $0<\delta<1$. Thus for above two cases we always have $u(t)\in V_\delta$ for $0<\delta<1$, $0\leq t<T$. □

3 Global existence and exponential growth at subcritical initial energy level ($E(0)<d$)

Herein, the proof of the global existence and blowup characteristics of the solutions for (1.1) will be given.

Theorem 3.1 (Global existence for $E(0)<d$**.** *Suppose* $u_0(x)\in H_0^1(\Omega)$ *and* $u_1(x)\in L^2(\Omega)$*. Then, on the assumption* $0<E(0)<d$ *and* $I(u_0)>0$ *or* $\|\nabla u_0\|=0$*, the problem* (1.1) *possess a global weak solution*

$$u(t)\in L^\infty\big(0,\infty;H_0^1(\Omega)\big)$$

with

$$u_t(t)\in L^\infty\big(0,\infty;L^2(\Omega)\big) \quad \text{and} \quad u(t)\in W$$

for $0\le t<\infty$*.*

Proof. Similar to [20], we consider approximate solutions $u_m(x,t)$ of problem (1.1). Then by similar arguments used in the proof of Theorem 3.2 in [20] we obtain

$$\frac{1}{2}\|u_{mt}\|^2+J(u_m)=E_m(0)<d, \quad 0\le t<\infty, \tag{3.1}$$

and $u_m(t)\in W$ for sufficiently large m and $0\le t<\infty$. From (3.1) we can write

$$J(u_m)<d.$$

From this and by (2.2) we can write

$$\frac{1}{2}I(u_m)+\frac{k}{4}\|u_m\|^2+\frac{p-1}{2(p+1)}\|u_m\|_{p+1}^{p+1}<d,$$

which implies that

$$\|u_m\|^2<\frac{4d}{k}, \quad \|u_m\|_{p+1}^{p+1}<\frac{2(p+1)d}{p-1}, \tag{3.2}$$

and

$$I(u_m)<2d.$$

From this and (2.1) we get

$$\begin{aligned}\|\nabla u_m\|^2 &<2d+k\int_\Omega u_m^2\ln|u_m|dx+\|u_m\|_{p+1}^{p+1}\\ &<2d+(kC^*+2)\|u_m\|_{p+1}^{p+1}\\ &<2d+\frac{2(p+1)(kC^*+2)d}{p-1}.\end{aligned}$$

Again from (3.1) we can write

$$\|u_{mt}\|^2 < 2d, \quad 0 \le t < \infty.$$

Now, using calculation and the fact $(\ln|y|)^2 \le y^2$ for $|y|>1$, we have

$$\begin{aligned}\int_\Omega \left(u_m \ln|u_m|^k\right)^2 dx &= k^2 \int_{\Omega_1} (u_m \ln|u_m|)^2 dx + k^2 \int_{\Omega_2} (u_m \ln|u_m|)^2 dx \\ &\le |\Omega| k^2 e^{-2} + |\Omega| k^2 \|u_m\|_4^4 \\ &\le |\Omega| k^2 e^{-2} + |\Omega| k^2 S^* \|\nabla u_m\|^4, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Omega_1 &= \{x \in \Omega |\ u_m(x) \le 1\}, \\ \Omega_2 &= \{x \in \Omega |\ u_m(x) > 1\},\end{aligned}$$

and S^* is the constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. By (3.2)-(3.3) and compactness method, the problem (1.1) admits a global weak solution

$$u(t) \in L^\infty\left(0,\infty; H_0^1(\Omega)\right)$$

with

$$u_t(t) \in L^\infty\left(0,\infty; L^2(\Omega)\right).$$

Finally, $u(t) \in W$ on time $0 \le t < \infty$ by Lemma 2.5. □

Theorem 3.2 (Exponential growth for $E(0)<d$)**.** *Suppose $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Then, on the assumption $E(0)<d$, $I(u_0)<0$ and $(u_0,u_1) \ge 0$, the L^2-norm of the solution of problem* (1.1) *will grow up as an exponential function as time goes to infinity.*

Proof. For $E(0)<d$ and $I(u_0)<0$, we suppose $u(x,t)$ to be any solution of the problem (1.1). Now, we consider the function $L(t):[0,+\infty) \to R^+$ defined as following

$$L(t) = \|u\|^2.$$

Differentiating this we have

$$L'(t) = 2(u,u_t)$$

and

$$\begin{aligned} L''(t) &= 2\|u_t\|^2 + 2(u,u_{tt}) \\ &= 2\|u_t\|^2 - 2\left(\|\nabla u\|^2 - k\int_\Omega u^2 \ln|u| dx - \|u\|_{p+1}^{p+1}\right) \\ &= 2\|u_t\|^2 - 2I(u). \end{aligned} \tag{3.4}$$

Using energy inequality and (2.3), we have

$$\begin{aligned}E(0)\geq&\frac{1}{2}\|u_t\|^2+\frac{1}{2}I(u)+\frac{k}{4}\|u\|^2+\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}\\ \geq&\frac{1}{2}\|u_t\|^2+\frac{1}{2}I(u)+\frac{k}{4}\|u\|^2,\end{aligned}$$

which implies

$$4E(0)\geq 2\|u_t\|^2+2I(u)+k\|u\|^2.$$

By this we can have

$$2I(u)\leq 4E(0)-k\|u\|^2-2\|u_t\|^2. \tag{3.5}$$

Using (3.4) and (3.5), we get

$$\begin{aligned}L''(t)\geq&2\|u_t\|^2+2k\|u\|^2+2\|u_t\|^2-4E(0)\\ =&4\|u_t\|^2+2k\|u\|^2-4E(0)\\ =&4\|u_t\|^2+2kL(t)-4E(0).\end{aligned} \tag{3.6}$$

(i). For $E(0)\leq 0$, (3.6) leads to

$$L''(t)\geq 4\|u_t\|^2. \tag{3.7}$$

(ii). If $0<E(0)<d$, then applying Lemma 2.5, we can write $u(t)\in V_\delta$ on $1<\delta<\delta_2$ and $t>0$. Hence $I_\delta(u)<0$, and by Lemma 2.2(ii), $\|\nabla u\|>r(\delta)$ for $1<\delta<\delta_2$ and $t>0$. So, we obtain $I_{\delta_2}(u)\leq 0$ and $\|\nabla u\|\geq r(\delta_2)$ for $t>0$. And by (3.4), we get

$$L''(t)\geq 2(\delta_2-1)\|\nabla u\|^2-2I_{\delta_2}(u)\geq 2(\delta_2-1)r^2(\delta_2)>0.$$

Therefore, we obtain

$$\begin{aligned}L'(t)\geq&2(\delta_2-1)r^2(\delta_2)t+L'(0)\\ \geq&2(\delta_2-1)r^2(\delta_2)t,\end{aligned}$$

and

$$\begin{aligned}L(t)\geq&(\delta_2-1)r^2(\delta_2)t^2+L(0)\\ \geq&(\delta_2-1)r^2(\delta_2)t^2.\end{aligned}$$

So, for sufficiently large t we have $2kL(t)>4E(0)$ and (3.7) holds. Finally, (3.7) gives

$$L(t)L''(t)-(L'(t))^2\geq 4\left(\|u\|^2\|u_t\|^2-(u,u_t)^2\right)\geq 0.$$

Now, by direct calculation we can have

$$(\ln L(t))' = \frac{L'(t)}{L(t)} \tag{3.8}$$

and

$$(\ln L(t))'' = \left(\frac{L'(t)}{L(t)}\right)' = \frac{L(t)L''(t) - (L'(t))^2}{L^2(t)} \geq 0.$$

So, $(\ln L(t))' = \frac{L'(t)}{L(t)}$ is increasing on t. Using this fact and integrating (3.8) from t_0 to t, we have

$$\begin{aligned} \ln L(t) - \ln L(t_0) &= \int_{t_0}^{t} (\ln L(t))' d\tau \\ &= \int_{t_0}^{t} \frac{L'(t)}{L(t)} d\tau \geq \frac{L'(t_0)}{L(t_0)}(t - t_0), \end{aligned}$$

where $0 \leq t_0 < t$. From this we obtain

$$L(t) \geq L(t_0) \exp\left(\frac{L'(t_0)}{L(t_0)}(t - t_0)\right). \tag{3.9}$$

If we can take a t_0 sufficiently small such that $L'(t_0) > 0$ and $L(t_0) > 0$, then from (3.9), we have

$$\lim_{t \to +\infty} L(t) = +\infty,$$

which means that the L^2-norm of the solution of problem (1.1) grows up as an exponential function as time goes to infinity. □

4 Global existence and exponential growth at critical initial energy level($E(0) = d$)

Here, the results will be proved for critical case.

Theorem 4.1 (Global existence for $E(0) = d$)**.** *Suppose $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Then, on the assumption $E(0) = d$ and $I(u_0) \geq 0$, the problem* (1.1) *possess a global weak solution*

$$u(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$$

with

$$u_t(t) \in L^\infty\left(0, \infty; L^2(\Omega)\right) \quad \text{and} \quad u(t) \in W \cup \partial W$$

for $0 \leq t < \infty$.

Proof. It will be proved in two steps.

(i). $\|\nabla u_0\| \neq 0$. Suppose $\lambda_m = 1-\frac{1}{m}$ and $u_{0m} = \lambda_m u_0$, $m = 2,3,\cdots$. Consider the initial data like [22] for the problem (1.1) as following

$$u(x,0) = u_{0m}(x), \quad u_t(x,0) = u_1(x).$$

Now, from $I(u_0) \geq 0$ and Lemma 2.1, we have $\lambda^* = \lambda^*(u_0) \geq 1$. Thus, $I(u_{0m}) > 0$ and $J(u_{0m}) = J(\lambda_m u_0) < J(u_0)$. Furthermore,

$$\begin{aligned} 0 < E_m(0) &\equiv \frac{1}{2}\|u_1\|^2 + J(u_{0m}) \\ &< \frac{1}{2}\|u_1\|^2 + J(u_0) \\ &= E(0) = d. \end{aligned}$$

Hence, by Theorem 3.1, for every m the problem (1.1) with above initial data possess a global solution

$$u_m(t) \in L^\infty\left(0,\infty; H_0^1(\Omega)\right)$$

and $u_m(t) \in W$ on $0 \leq t < \infty$ meeting

$$(u_{mt}, v) + \int_0^t (\nabla u_m, \nabla v) d\tau = \int_0^t (f(u_m), v) d\tau + (u_1, v)$$

for every $v \in H_0^1(\Omega)$, $0 \leq t < \infty$ and

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) < d, \quad 0 \leq t < \infty.$$

The remainder part of the proof follows Theorem 3.1. □

To prove the blowup result at $E(0) = d$ the Lemma 4.1 is required and for the proof see Lemma 2.7 in [31].

Lemma 4.1. *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Then, on the assumption $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$, the set V' is invariant under the flow of* (1.1).

Theorem 4.2 (Exponential growth for $E(0) = d$)**.** *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Then, on the assumption $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$, the L^2-norm of the solution of problem* (1.1) *will grow up exponentially as time goes to infinity.*

Proof. From (3.6), we have

$$\begin{aligned} L''(t) &\geq 4\|u_t\|^2 + 2kL(t) - 4E(0) \\ &= 4\|u_t\|^2 + 2kL(t) - 4d, \end{aligned} \tag{4.1}$$

then $L''(t)>0$ on $0\leq t<\infty$ by (3.4) and Lemma 4.1. Hence, $L'(t)$ is increasing. For $L'(0)=2(u_0,u_1)\geq 0$, $\forall t_0>0$ we can obtain

$$\begin{aligned} &L'(t)\geq L'(t_0)>0, && t\geq t_0,\\ &L(t)\geq L'(t_0)(t-t_0)+L(t_0)>L'(t_0)(t-t_0), && t\geq t_0. \end{aligned}$$

Therefore, for large enough t, we can get

$$2kL(t)>4d.$$

From this and (4.1) we get

$$L''(t)\geq 4\|u_t\|^2.$$

Hence

$$L(t)L''(t)-(L'(t))^2\geq 4\left(\|u\|^2\|u_t\|-(u,u_t)^2\right)\geq 0.$$

The remainder proof is likewise to Theorem 3.2. □

5 Exponential growth at arbitrarily positive initial energy level $E(0)>0$

The blowup results in arbitrarily positive initial energy will be discussed in below theorem.

Theorem 5.1 (Exponential growth for $E(0)>0$)**.** *Suppose* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *meet below conditions*

(i). $E(0)>0$*;*

(ii). $(u_0,u_1)>0$*;*

(iii). $\|u_0\|^2>\frac{4}{k}E(0)$*;*

(iv). $I(u_0)<0$*,*

then the solution of the problem (1.1) *will grow up exponentially as time goes to infinity.*

Proof. The proof will be shown in two steps:

Step 1. First, we show $I(u)<0$ and $\|u(t)\|^2>\frac{4}{k}E(0)$ for all $t\in(0,T)$. For $I(u)<0$, arguing by contradiction we suppose that there exists available a first time $t_0\in(0,T)$

for which $I(u(t_0))=0$ and $I(u)<0$ on $t\in[0,t_0)$. Again, for the function $L(t)$ in Theorem 5.1(iv), there should be

$$L'(t)=2(u,u_t)$$

and

$$L''(t)=2\|u_t\|-2I(u).$$

Using these and assumption of this theorem, we can prove $L(t)$ is strictly increasing. Hence, we can write

$$L(t)>\|u_0\|^2>\frac{4}{k}E(0),\quad \forall t\in(0,t_0).$$

So, we get

$$L(t_0)>\frac{4}{k}E(0). \tag{5.1}$$

Meanwhile, we know

$$J(u((t_0))\leq E(t_0)\leq E(0)$$

that is

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t_0)\|^2-\frac{k}{2}\int_\Omega u^2(t_0)\ln|u(t_0)|dx+\frac{k}{4}\|u(t_0)\|^2-\frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1}\\ \leq& E(t_0)\leq E(0).\end{aligned} \tag{5.2}$$

Moreover, $I(u(t_0))=0$ leads

$$\begin{aligned}\frac{1}{2}\|\nabla u(t_0)\|^2=&\frac{k}{2}\int_\Omega u^2(t_0)\ln|u(t_0)|dx+\frac{1}{2}\|u(t_0)\|_{p+1}^{p+1}\\ >&\frac{k}{2}\int_\Omega u^2(t_0)\ln|u(t_0)|dx+\frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1}.\end{aligned}$$

Now, we can write left side of (5.2) as below

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t_0)\|^2-\frac{k}{2}\int_\Omega u^2(t_0)\ln|u(t_0)|dx+\frac{k}{4}\|u(t_0)\|^2-\frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1}\\ =&\frac{1}{2}\|\nabla u(t_0)\|^2+\frac{k}{4}\|u(t_0)\|^2-\left(\frac{k}{2}\int_\Omega u^2(t_0)\ln|u(t_0)|dx+\frac{1}{p+1}\|u(t_0)\|_{p+1}^{p+1}\right)\\ >&\frac{1}{2}\|\nabla u(t_0)\|^2+\frac{k}{4}\|u(t_0)\|^2-\frac{1}{2}\|\nabla u(t_0)\|^2=\frac{k}{4}\|u(t_0)\|^2.\end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3), it is obvious that

$$\frac{k}{4}\|u(t_0)\|^2 \le E(0)$$

is equivalent to

$$L(t_0) \le \frac{4}{k}E(0),$$

that violates (5.1).

Step 2. In this step, infinite time blow up result will be proved. By (3.6) and $L(t) = \|u(t)\|^2 > \frac{4}{k}E(0)$ for any $t \in (0,T)$ we can reach

$$L''(t) \ge 4\|u_t\|^2.$$

Thus, we get

$$L(t)L''(t) - (L'(t))^2 \ge 4\left(\|u\|^2\|u_t\|^2 - (u,u_t)^2\right) \ge 0.$$

The rest of the proof follows Theorem 3.2. □

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