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SUPERCONVERGENCE ERROR ESTIMATES OF THE LOWEST-ORDER RAVIART-THOMAS GALERKIN MIXED FINITE ELEMENT METHOD FOR NONLINEAR THERMISTOR EQUATIONS*

Huaijun Yang and Dongyang Shi[1)]
School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, China;
School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China
Email: huaijunyang@zua.edu.cn, shi_dy@zzu.edu.cn

Abstract

This paper is concerned with the superconvergence error estimates of a classical mixed finite element method for a nonlinear parabolic/elliptic coupled thermistor equations. The method is based on a popular combination of the lowest-order rectangular Raviart-Thomas mixed approximation for the electric potential/field $(\phi, \boldsymbol{\theta})$ and the bilinear Lagrange approximation for temperature u. In terms of the special properties of these elements above, the superclose error estimates with order $\mathcal{O}(h^2)$ are obtained firstly for all three components in such a strongly coupled system. Subsequently, the global superconvergence error estimates with order $\mathcal{O}(h^2)$ are derived through a simple and effective interpolation post-processing technique. As by a product, optimal error estimates are acquired for potential/field and temperature in the order of $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively. Finally, some numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65M15, 65M30, 65N15, 65N30.
Key words: Nonlinear thermistor equations, Galerkin mixed finite element method, Interpolation post-processing technique, Superclose and superconvergence error estimates.

1. Introduction

In this paper, we focus on superconvergence error analysis of the lowest-order Raviart-Thomas mixed finite element method for nonlinear and coupled thermistor equations, which are modeled as a coupled system of nonlinear partial differential equations with a quadratic growth on the gradient of one of the unknowns, defined by

$$u_t - \Delta u = \sigma(u)|\nabla\phi|^2, \qquad (\boldsymbol{x},t) \in \Omega \times J, \tag{1.1}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0, \qquad (\boldsymbol{x},t) \in \Omega \times J, \tag{1.2}$$

$$u(\boldsymbol{x},t) = 0, \quad \phi(\boldsymbol{x},t) = g(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \partial\Omega \times J, \tag{1.3}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangular domain with boundary $\partial\Omega, \boldsymbol{x} = (x,y), J = (0,T]$. The system (1.1)-(1.4) models the electric heating of a conducting body, which plays an important role in many micro-electromechanical systems. The unknowns $\phi = \phi(\boldsymbol{x},t)$ and $u = u(\boldsymbol{x},t)$ are the

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1) Corresponding author

distributions of the electrical potential and the temperature in Ω, respectively. $\sigma(u)$ is the temperature dependent electrical conductivity, $\sigma(u)|\nabla\phi|^2$ is the Joule heating. Moreover, u_0 and g are given smooth functions.

A lot of theoretical and numerical analysis have been devoted to system (1.1)-(1.4) by many authors due to its wide applications (see [2–5, 12, 16, 19–22, 32, 49–52, 56, 59–62, 64] and the references therein). More precisely, for theoretical analysis, the existence of time-dependent thermistor equations was shown by means of supersolutions and subsolutions, the maximum principle and fixed point argument in [4]. The existence of weak solutions was studied with Faedo-Galerkin method for an arbitrarily large interval of time in [12]. The existence and uniqueness of C^α solution for thermistor problem with mixed boundary conditions was established in [60]. For numerical analysis, a linearized Euler Galerkin scheme with linear finite element approximation applied in spatial direction was presented and analyzed in [61]. Due to some pollution arising from the approximation used the nonlinear term $\sigma(u)|\nabla\phi|^2$, only a sub-optimal error estimate was obtained. In [16], optimal error estimate was established based on the duality argument under the time-step condition $\tau = \mathcal{O}(h^{d/6})$, where d is the dimension, for completely discrete scheme with minimal regularity assumptions. Based on a standard finite element method used in spatial direction and the combinations of rational implicit and explicit multistep schemes used in temporal direction, some higher-order linearly implicit finite element schemes were developed in [2] and optimal error estimates were proved under the time-step condition $\tau = \mathcal{O}(h^{3/(2p)})$ and $r \geq 2$, where p is the order of discretization in time and r is the degree of piecewise polynomial approximations used in space, respectively. In terms of an error splitting technique proposed in [33,34], the unconditionally optimal error estimates for Lagrange finite element methods with different time approximation schemes were established in [2, 19, 20, 32]. Subsequently, the superconvergence error estimates were derived in [49–52] with the help of the high precision integral identity technique under the appropriate restriction between time step size and space step size.

Since mixed finite element methods allow simultaneous computation of the original variable and its gradient, both of them being equally accurate [18], and these methods have been applied to many problems [6,7,10,13,23,25–27] and the references therein. As pointed out in [22], $\boldsymbol{\theta} = \sigma(u)\nabla\phi$ denotes the electric field, which is more important in physics, it is natural to solve the system with a mixed finite element method to approximate potential/field and temperature $(\phi, \boldsymbol{\theta}, u)$. Mixed finite element methods may produce a better approximation to the electric field $\boldsymbol{\theta}$ and the nonlinear source term $\sigma(u)|\nabla\phi|^2$. In [62], based on the Raviart-Thomas mixed finite element approximation used for the electric potential/field $(\phi, \boldsymbol{\theta})$ and classical Lagrange finite element approximation applied for the temperature u, a mixed finite element scheme was proposed and investigated. It should be pointed out that a higher-order mixed finite element space was required in [62] to obtain optimal error estimate for temperature, in which the lowest-order Raviart-Thomas mixed finite element space is excluded. As we known, the lowest-order Raviart-Thomas mixed finite element space is the most popular and widely used in practical applications [8,43,53] due to the ease of implementation and less computational costs. In terms of an H^{-1}-norm estimate of a classical mixed finite element method, which the lowest Raviart-Thomas mixed used to approximate the electric potential/field $(\phi, \boldsymbol{\theta})$ and the linear Lagrange element used to approximate the temperature u, and a nonclassical elliptic map, optimal error estimates were derived in [22]. Meanwhile, a simple one-step recovery technique with one-order Raviart-Thomas mixed finite element space was developed to obtain a new numerical electric potential/field of second-order accuracy.

On the other hand, the superconvergence technique is a simple and effective way to improve the accuracy of numerical solutions for linear or nonlinear boundary value problems [37,40] and has become a hot topic for the numerical methods of partial differential equations, such as the second-order elliptic equation [39, 45], linear elasticity problem [44], the Schrödinger equation [47, 48], Maxwell's equation [30, 38, 58], Stokes/Navier-Stokes equations [42, 57] and so on.

As point out in [22], the approximation in the lowest-order Raviart-Thomas mixed element is in the order $\mathcal{O}(h)$, while the linear Lagrange element approximation is in the order $\mathcal{O}(h^2)$. Due to the strong coupling and nonlinearity of the thermistor system (we refer to [11, 24, 35, 36, 55] for other problems with strong nonlinearity), the inconsistency makes the theoretical analysis on optimal and superconvergence error estimates of the temperature more difficult and more challenging in the traditional sense. In this work, as an attempt, different from optimal estimates obtained in [2, 16, 19, 20, 32, 61], the main aim herein is to investigate the superclose and superconvergent error estimates of the lowest-order Raviart-Thomas mixed finite element space and bilinear Lagrange finite element space approximating potential/field and temperature $(\phi, \boldsymbol{\theta}, u)$ with a linearized backward Euler scheme for system (1.1)-(1.4). The analysis relies on the refined estimates with the aid of high accuracy analysis of the lowest-order Raviart-Thomas mixed element and bilinear element on the rectangular mesh, as well as the mean value technique and interpolated postprocessing approach.

The rest of this paper is organized as follows. In Section 2, some notations, preliminaries and a widely used the lowest-order Raviart-Thomas mixed finite element space are introduced. In Section 3, the detailed superclose error estimates (see Theorem 3.1) are derived firstly. Then, based on the interpolation postprocessing technique, the global superconvergence results (see Theorem 3.2) are obtained efficiently. Finally, in Section 4, some numerical results are provided to confirm the theoretical analysis and show the effectiveness of the interpolation postprocessing technique.

Throughout this paper, we assume that the temperature-dependent electric conductivity $\sigma(\cdot) \in W^{1,\infty}(\mathbb{R}) \cap C^2(\mathbb{R})$ satisfies

$$0 < a_* < \sigma(s) < a^*, \quad \forall\, s \in \mathbb{R} \tag{1.5}$$

for some positive constants a_* and a^*. It should be pointed out that the above assumption on σ is satisfied in many engineering applications [29]. Moreover, we denote C a generic positive constant, which may be different in different places, and independent of n (time level), h (spatial parameter) and τ (time step).

2. Some Preliminaries

Let $W^{m,p}(\Omega)$ be the Sobolev space with the norm $\|\cdot\|_{m,p}$ and seminorm $|\cdot|_{m,p}$ as the definition in [1], and $(\cdot\,,\cdot)$ denote the inner product. When $p = 2$, we also denote $H^m(\Omega) = W^{m,2}(\Omega)$. Moreover, in the investigation of nonstationary problem we shall work with functions which depend on time and have values in a Banach space. Such functions are elements of the so-called Bochner spaces (see [14] for more details). More precisely, for any Banach space Y and function $f : [0, T] \to Y$, define the norm

$$\|f\|_{L^p(Y)} = \begin{cases} \left(\displaystyle\int_0^T \|f(t)\|_Y^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \text{ess sup}_{t\in(0,T)} \|f(t)\|_Y, & p = \infty. \end{cases}$$

Let $\mathcal{T}_h$ be a finite element partition of Ω into uniform rectangles and $h = \max_{K\in\mathcal{T}_h}\{h_K\}$, and h_K is the diameter of element K. The bilinear finite element space is defined by

$$V_h = \{v_h \in C(\overline{\Omega});\ v_h|_K \in \mathrm{span}\{1, x, y, xy\},\ v_h|_{\partial\Omega} = 0,\ \forall\, K \in \mathcal{T}_h\}$$

with $I_h : u \in H^2(\Omega) \to I_h u \in V_h$ be the nodal Lagrangian interpolation operator. Moreover, define $R_h : H_0^1(\Omega) \to V_h$ to be a Ritz projection operator by

$$\big(\nabla(u - R_h u), \nabla v_h\big) = 0, \quad \forall\, v_h \in V_h. \tag{2.1}$$

Then, by the classical finite element theory [9,54], there holds for $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u - R_h u\|_0 + h\|\nabla(u - R_h u)\|_0 \le Ch^2|u|_2, \tag{2.2}$$

$$\|u - I_h u\|_0 + h\|\nabla(u - I_h u)\|_0 \le Ch^2|u|_2. \tag{2.3}$$

In addition, from [46,48], we have the following superclose error estimate between $I_h u$ and $R_h u$ in H^1-seminorm for $u \in H^3(\Omega)$,

$$\|\nabla(I_h u - R_h u)\|_0 \le Ch^2\|u\|_3. \tag{2.4}$$

To present the mixed formulation, the lowest-order Raviart-Thomas space, which satisfies the Babuska-Brezzi condition, is defined as follows:

$$\boldsymbol{H}_h = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}|_K = (v_1, v_2)|_K \in Q_{10}(K) \times Q_{01}(K),\ \forall\, K \in \mathcal{T}_h\}, \tag{2.5}$$

$$M_h = \big\{w \in L^2(\Omega) : w|_K \in Q_{00}(K),\ \forall\, K \in \mathcal{T}_h\big\}, \tag{2.6}$$

where $Q_{mn}(K)$ is the space of polynomials of degree no more than m and n in x and y on K, respectively and

$$\boldsymbol{H}(\mathrm{div};\Omega) = \big\{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \big(L^2(\Omega)\big)^2,\ \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)\big\}$$

with

$$\|\boldsymbol{\sigma}\|_{\mathrm{div}}^2 = \|\boldsymbol{\sigma}\|_0^2 + \|\nabla\cdot\boldsymbol{\sigma}\|_0^2.$$

From [17,31], we recall the lowest-order Raviart-Thomas projection as follows:

$$\boldsymbol{\Pi}_h \times P_h : \boldsymbol{H}(\mathrm{div};\Omega) \times L^2(\Omega) \ \to\ \boldsymbol{H}_h \times M_h$$

is defined by the following conditions:

$$\int_{l_i} (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}) \cdot \boldsymbol{n}_i ds = 0, \quad i = 1, 2, 3, 4, \tag{2.7}$$

$$\int_K (w - P_h w)\, d\boldsymbol{x} = 0, \tag{2.8}$$

where $l_i, i = 1, 2, 3, 4$, are the four edges of the rectangle $K \in \mathcal{T}_h$ and $\boldsymbol{n}_i$ is the outward normal vector along to the edge l_i. Moreover, we have the following results from [15,17,31]:

(1) P_h is the local $L^2(\Omega)$ projection.

(2) $\boldsymbol{\Pi}_h$ and P_h satisfy

$$\big(\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}), \omega_h\big) = 0, \quad \forall\, \omega_h \in M_h, \tag{2.9}$$

$$(w - P_h w, \nabla\cdot\boldsymbol{\chi}_h) = 0, \qquad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{H}_h. \tag{2.10}$$

(3) There hold the approximation properties

$$\|\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{0,p}\le Ch\|\boldsymbol{\sigma}\|_{1,p},\quad 1\le p\le\infty, \tag{2.11}$$

$$\|w-P_hw\|_{0,p}\le Ch\|w\|_{1,p},\quad 1\le p\le\infty. \tag{2.12}$$

Here, we present some lemmas, which play a key role, in the error analysis.

Lemma 2.1. *Suppose that $\boldsymbol{\sigma}\in(H^2(\Omega))^2$, we have*

$$(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma},\boldsymbol{\chi})_K=\int_K(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma})\cdot\boldsymbol{\chi}\,d\boldsymbol{x}\le Ch_K^2\|\boldsymbol{\sigma}\|_{2,K}\|\boldsymbol{\chi}\|_{0,K},\quad\forall\,\boldsymbol{\chi}\in\boldsymbol{H}_h. \tag{2.13}$$

Proof. For any element $K\in\mathcal{T}_h$ (see Fig. 2.1), let (x_K,y_K) stand for the center of K and let $2h_x$ and $2h_y$ stand for the side lengths of K in the x- and y-direction, respectively. Define two error functions for x and y as follows [37, 40, 41]:

$$E(x):=\frac{1}{2}\big[(x-x_K)^2-h_x^2\big],\quad F(y):=\frac{1}{2}\big[(y-y_K)^2-h_y^2\big].$$

Then, one can check that

$$\begin{aligned}
&E(x)|_{l_i}=0,\quad i=1,3, && F(y)|_{l_i}=0,\quad i=2,4,\\
&E''(x)=\big(E(x)\big)_{xx}=1, && F''(y)=\big(F(y)\big)_{yy}=1,\\
&(x-x_K)=\frac{1}{6}\big(E^2(x)\big)'''=\frac{1}{6}\big(E^2(x)\big)_{xxx}, && (y-y_K)=\frac{1}{6}\big(F^2(y)\big)'''=\frac{1}{6}\big(F^2(y)\big)_{yyy}.
\end{aligned}$$

Note that

$$\begin{aligned}
(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma},\boldsymbol{\chi})_K&=\int_K(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h\boldsymbol{\sigma})\cdot\boldsymbol{\chi}\,d\boldsymbol{x}\\
&=\int_K(\sigma_1-\Pi_h\sigma_1)\chi_1\,d\boldsymbol{x}+\int_K(\sigma_2-\Pi_h\sigma_2)\chi_2\,d\boldsymbol{x}=:I+II.
\end{aligned}$$

Since the treatment for II is the same as that for I, we deal only with I in the following.

Noting that $\chi_1|_K\in Q_{10}(K)=\text{span}\{1,x\}$, we have

$$\chi_1(x,y)|_K=\chi_1(x_K,y)+(x-x_K)\chi_{1x}(x,y), \tag{2.14}$$

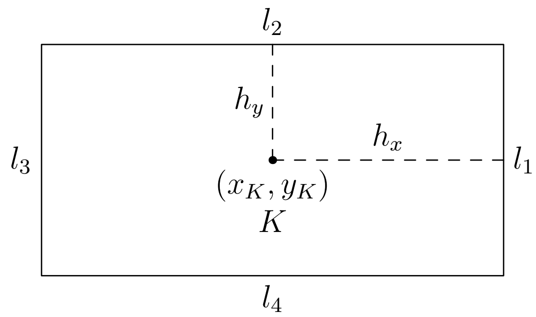


Fig. 2.1. The illustration of element K.

where $\chi_1(x_K, y)$ and $\chi_{1x}(x, y)$ are constants on element K. Hence, I can be rewritten as

$$\begin{aligned} I &= \int_K (\sigma_1 - \Pi_h\sigma_1)\chi_1 \, d\boldsymbol{x} = \int_K (\sigma_1 - \Pi_h\sigma_1)\big(\chi_1(x_K, y) + (x - x_K)\chi_{1x}(x, y)\big) \, d\boldsymbol{x} \\ &= \chi_1(x_K, y) \int_K (\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} + \chi_{1x}(x, y) \int_K (x - x_K)(\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} \\ &=: I_1 + I_2. \end{aligned} \tag{2.15}$$

In terms of $E^{''}(x) = 1$, one can obtain by integration by parts

$$\begin{aligned} \int_K (\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} &= \int_K E^{''}(x)(\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} \\ &= \left(\int_{l_1} - \int_{l_3}\right) E^{'}(x)(\sigma_1 - \Pi_h\sigma_1) \, dy - \int_K E^{'}(x)(\sigma_1 - \Pi_h\sigma_1)_x \, d\boldsymbol{x} \\ &= -\left(\int_{l_1} - \int_{l_3}\right) E(x)(\sigma_1 - \Pi_h\sigma_1)_x \, dy + \int_K E(x)(\sigma_1 - \Pi_h\sigma_1)_{xx} \, d\boldsymbol{x} \\ &= \int_K E(x)\sigma_{1xx} \, d\boldsymbol{x}, \end{aligned} \tag{2.16}$$

where we have used $E^{'}(x)|_{l_1}$ and $E^{'}(x)|_{l_3}$ are constants and interpolation condition (2.7), which shows

$$\left(\int_{l_1} - \int_{l_3}\right) E^{'}(x)(\sigma_1 - \Pi_h\sigma_1) \, dy = 0,$$

and $E(x)|_{l_1} = 0$ and $E(x)|_{l_3} = 0$ in the above estimate. Combining (2.14) and (2.16), I_1 becomes that

$$\begin{aligned} I_1 &= \int_K E(x)\sigma_{1xx}\big(\chi_1(x, y) - (x - x_K)\chi_{1x}(x, y)\big) \, d\boldsymbol{x} \\ &= \int_K E(x)\sigma_{1xx}\chi_1 \, d\boldsymbol{x} - \int_K E(x)(x - x_K)\sigma_{1xx}\chi_{1x} \, d\boldsymbol{x} \\ &\leq Ch_x^2|\sigma_1|_{2,K}|\chi_1|_{0,K} \leq Ch_K^2|\sigma_1|_{2,K}|\chi_1|_{0,K}, \end{aligned} \tag{2.17}$$

where we have used

$$\|\chi_{1x}\|_{0,K} \leq Ch_x^{-1}|\chi_1|_{0,K}.$$

Moreover, according to $(x - x_K) = (E^2(x))^{'''}/6$, one can obtain by integration by parts

$$\begin{aligned} &\int_K (x - x_K)(\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} \\ &= \frac{1}{6}\int_K \big(E^2(x)\big)^{'''}(\sigma_1 - \Pi_h\sigma_1) \, d\boldsymbol{x} \\ &= \frac{1}{6}\left[\left(\int_{l_1} - \int_{l_3}\right)\big(E^2(x)\big)^{''}(x)(\sigma_1 - \Pi_h\sigma_1) \, dy - \int_K \big(E^2(x)\big)^{''}(\sigma_1 - \Pi_h\sigma_1)_x \, d\boldsymbol{x}\right] \\ &= -\frac{1}{6}\left[\left(\int_{l_1} - \int_{l_3}\right)\big(E^2(x)\big)^{'}(x)(\sigma_1 - \Pi_h\sigma_1)_x \, dy - \int_K \big(E^2(x)\big)^{'}(\sigma_1 - \Pi_h\sigma_1)_{xx} \, d\boldsymbol{x}\right] \\ &= \frac{1}{3}\int_K (x - x_K)E(x)\sigma_{1xx} \, d\boldsymbol{x}, \end{aligned} \tag{2.18}$$

where we have used $E''(x)|_{l_1}$ and $E''(x)|_{l_3}$ are constants and interpolation condition (2.7), which shows

$$\left(\int_{l_1} - \int_{l_3}\right) E''(x)(\sigma_1 - \Pi_h\sigma_1)\, dy = 0,$$

and $E(x)|_{l_1} = 0$ and $E(x)|_{l_3} = 0$ in the above estimate. Combining (2.14) and (2.18), I_2 becomes

$$I_2 = \frac{1}{3}\int_K (x - x_K)E(x)\sigma_{1xx}\chi_{1x}\, d\boldsymbol{x} \le Ch_K^2|\sigma_1|_{2,K}|\chi_1|_{0,K}, \tag{2.19}$$

where we have used

$$\|\chi_{1x}\|_{0,K} \le Ch_x^{-1}|\chi_1|_{0,K}.$$

Substituting (2.17) and (2.19) into (2.15) yields that

$$I = \int_K (\sigma_1 - \Pi_h\sigma_1)\chi_1\, d\boldsymbol{x} \le Ch_K^2|\sigma_1|_{2,K}|\chi_1|_{0,K}. \tag{2.20}$$

In a similar way, we also have

$$II = \int_K (\sigma_2 - \Pi_h\sigma_2)\chi_2\, d\boldsymbol{x} \le Ch_K^2|\sigma_2|_{2,K}|\chi_2|_{0,K}. \tag{2.21}$$

Combining (2.20) and (2.21), we derive

$$(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}, \boldsymbol{\chi})_K = \int_K (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}) \cdot \boldsymbol{\chi}\, d\boldsymbol{x} \le Ch_K^2\|\boldsymbol{\sigma}\|_{2,K}\|\boldsymbol{\chi}\|_{0,K}, \tag{2.22}$$

which is the desired result and the proof is complete. □

Lemma 2.2. *Let $\boldsymbol{m} = (m_1, m_2)$ be a constant vector and suppose that $\boldsymbol{\sigma} \in (H^2(\Omega))^2$, we have*

$$\big(\boldsymbol{m} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}), w\big)_K = \int_K \boldsymbol{m} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma})w\, d\boldsymbol{x} \le Ch_K^2\|\boldsymbol{\sigma}\|_{2,K}\|w\|_{0,K}, \quad \forall\, w \in M_h. \tag{2.23}$$

Proof. Noting that

$$\begin{aligned}\int_K \boldsymbol{m} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma})w\, d\boldsymbol{x} &= \int_K m_1(\sigma_1 - \Pi_h\sigma_1)w\, d\boldsymbol{x} + \int_K m_2(\sigma_2 - \Pi_h\sigma_2)w\, d\boldsymbol{x} \\ &= m_1 w|_K \int_K (\sigma_1 - \Pi_h\sigma_1)\, d\boldsymbol{x} + m_2 w|_K \int_K (\sigma_2 - \Pi_h\sigma_2)\, d\boldsymbol{x},\end{aligned}$$

and using the same process as estimate (2.16), we have

$$\begin{aligned}\int_K \boldsymbol{m} \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma})w\, d\boldsymbol{x} &= m_1 \int_K E(x)\sigma_{1xx}w\, d\boldsymbol{x} + m_2 \int_K F(y)\sigma_{2yy}w\, d\boldsymbol{x} \\ &\le Ch_K^2\|\boldsymbol{\sigma}\|_{2,K}\|w\|_{0,K},\end{aligned}$$

which is the desired result and the proof is complete. □

Lemma 2.3 ([28], Discrete Gronwall's Inequality). *Let τ, H and a_n, b_n, c_n, d_n be non-negative numbers for integers $n \ge 0$ such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \tau\sum_{k=0}^{n} d_k a_k + \tau\sum_{k=0}^{n} c_k + H.$$

Suppose that $\tau d_k < 1$ for all k and set $\sigma_k = (1 - \tau d_k)^{-1}$, then we have

$$a_n + \tau\sum_{k=0}^{n} b_k \le \exp\left(\tau\sum_{k=1}^{n} \sigma_k d_k\right)\left(\tau\sum_{k=0}^{n} c_k + H\right).$$

For the fully discrete analysis, we introduce the following notations. Let

$$0 = t_0 < t_1 < \cdots < t_N = T$$

be a uniform partition of the time interval $[0, T]$ with time step size $\tau = T/N$. For a smooth function ω on $[0, T]$, we denote $\omega^n = \omega(\cdot, t_n)$ for $1 \le n \le N$ and the backward Euler discretization operator by

$$D_\tau \omega^n = \frac{\omega^n - \omega^{n-1}}{\tau}.$$

3. Superclose and Superconvergence Error Analysis

To introduce the mixed form, we shall define an extra variable $\boldsymbol{\theta} = \sigma(u)\nabla\phi$ and denote $\beta(u) = 1/\sigma(u)$. Then, the original thermistor equation (1.1)-(1.2) can be rewritten as

$$u_t - \Delta u = \beta(u)|\boldsymbol{\theta}|^2, \tag{3.1}$$

$$\beta(u)\boldsymbol{\theta} = \nabla\phi, \tag{3.2}$$

$$-\nabla \cdot \boldsymbol{\theta} = 0. \tag{3.3}$$

Here, we present the weak formulation of (3.1)-(3.3) is to seek $u \in H_0^1(\Omega)$, $(\boldsymbol{\theta}, \phi) \in \boldsymbol{H}(\mathrm{div}; \Omega) \times L^2(\Omega)$ such that

$$(u_t, v) + (\nabla u, \nabla v) = \big(\beta(u)|\boldsymbol{\theta}|^2, v\big), \quad \forall\, v \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.4}$$

$$\big(\beta(u)\boldsymbol{\theta}, \boldsymbol{\chi}\big) + (\phi, \nabla \cdot \boldsymbol{\chi}) = \langle g, \boldsymbol{\chi} \cdot \boldsymbol{n}\rangle, \quad \forall\, \boldsymbol{\chi} \in \boldsymbol{H}(\mathrm{div}; \Omega), \tag{3.5}$$

$$-(\nabla \cdot \boldsymbol{\theta}, \omega) = 0, \quad \forall\, \omega \in L^2(\Omega), \tag{3.6}$$

where g is the boundary data and $\boldsymbol{n}$ denotes the normal vector along to $\partial\Omega$.

Based on the above weak formulation and notations, a semi-implicit backward Euler mixed finite element scheme with the lowest-order Raviart-Thomas pair is to find $(u_h^{n+1}, \boldsymbol{\theta}^{n+1}, \phi_h^{n+1}) \in (V_h, \boldsymbol{H}_h, M_h)$, such that

$$\big(D_\tau u_h^{n+1}, v_h\big) + \big(\nabla u_h^{n+1}, \nabla v_h\big) = \big(\beta\big(u_h^n\big)|\boldsymbol{\theta}_h^{n+1}|, v_h\big), \quad \forall\, v_h \in V_h, \tag{3.7}$$

$$\big(\beta\big(u_h^n\big)\boldsymbol{\theta}_h^{n+1}, \boldsymbol{\chi}_h\big) + \big(\phi_h^{n+1}, \nabla \cdot \boldsymbol{\chi}_h\big) = \langle g^{n+1}, \boldsymbol{\chi}_h \cdot \boldsymbol{n}\rangle, \quad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{H}_h, \tag{3.8}$$

$$-\big(\nabla \cdot \boldsymbol{\theta}_h^{n+1}, \omega_h\big) = 0, \quad \forall\, \omega_h \in M_h, \tag{3.9}$$

where g^{n+1} is the boundary data of ϕ at $t = t_{n+1}$. For the initial step approximations, we choose $u_h^0 = R_h u^0$ and $(\boldsymbol{\theta}_h^0, \phi_h^0)$ is the finite element solution of

$$\big(\beta(u^0)\boldsymbol{\theta}_h^0, \boldsymbol{\chi}_h\big) + \big(\phi_h^0, \nabla \cdot \boldsymbol{\chi}_h\big) = \big\langle g^0, \boldsymbol{\chi}_h \cdot \boldsymbol{n}\big\rangle, \quad \forall\, \boldsymbol{\chi}_h \in \boldsymbol{H}_h, \tag{3.10}$$

$$-\big(\nabla \cdot \boldsymbol{\theta}_h^0, \omega_h\big) = 0, \quad \forall\, \omega_h \in M_h. \tag{3.11}$$

The mixed finite element numerical scheme (3.7)-(3.9) is semi-implicit and decoupled, one needs to solve tow linear systems at each time step. One can check that the coefficient matrix in (3.7) is symmetric positive definite and $(\boldsymbol{\theta}_h^n, \phi_h^n)$ is the mixed finite element solution to a linear elliptic equation, the numerical scheme (3.7)-(3.9) is uniquely solvable at each time step [22]. Now, we present the optimal and superclose error estimates in the following theorem.

Theorem 3.1. *Let* $(\boldsymbol{\theta}^n, \phi^n, u^n)$ *and* $(\boldsymbol{\theta}_h^n, \phi_h^n, u_h^n)$ *be the solutions of* (3.4)-(3.6) *and* (3.7)-(3.9)*, respectively. Suppose that* $\boldsymbol{\theta} \in L^\infty((H^2(\Omega))^2 \cap (W^{1,\infty}(\Omega))^2), \boldsymbol{\theta}_t \in L^\infty((H^2(\Omega))^2 \cap (W^{1,\infty}(\Omega))^2), \phi \in L^\infty(H^1(\Omega)), u \in L^\infty(H^3(\Omega)), u_t \in L^\infty(H^2(\Omega))$ *and* $u_{tt} \in L^\infty(L^2(\Omega))$*. Under a temporal stepsize restriction* $\tau = \mathcal{O}(h^{1+\alpha}), \alpha > 0$*, we have*

$$\|\nabla(I_h u^n - u_h^n)\|_0 + \|\boldsymbol{\Pi}_h \boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0 + \|P_h \phi^n - \phi_h^n\|_0 \le C(h^2 + \tau), \quad n = 0, 1, 2, \ldots, N, \tag{3.12}$$
$$\|\nabla(u^n - u_h^n)\|_0 + \|\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0 + \|\phi^n - \phi_h^n\|_0 \le C(h + \tau), \quad n = 0, 1, 2, \ldots, N. \tag{3.13}$$

Proof. For simplicity, we split the errors as follows:

$$\begin{aligned} u^n - u_h^n &= u^n - R_h u^n + R_h u^n - u_h^n := \xi_u^n + e_u^n, \\ \boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n &= \boldsymbol{\theta}^n - \boldsymbol{\Pi}_h \boldsymbol{\theta}^n + \boldsymbol{\Pi}_h \boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n := \xi_{\boldsymbol{\theta}}^n + e_{\boldsymbol{\theta}}^n, \\ \phi^n - \phi_h^n &= \phi^n - P_h \phi^n + P_h \phi^n - \phi_h^n := \xi_\phi^n + e_\phi^n. \end{aligned}$$

From (3.4)-(3.6), we have at $t = t_{n+1}$

$$\begin{aligned} (D_\tau u^{n+1}, v_h) + (\nabla u^{n+1}, \nabla v_h) = {} & (\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2, v_h) + (D_\tau u^{n+1} - u_t^{n+1}, v_h) \\ & + ((\beta(u^{n+1}) - \beta(u^n))|\boldsymbol{\theta}^{n+1}|^2, v_h), \end{aligned} \tag{3.14}$$
$$(\beta(u^n)\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h) + (\phi^{n+1}, \nabla \cdot \boldsymbol{\chi}_h) = \langle g^{n+1}, \boldsymbol{\chi}_h \cdot \boldsymbol{n} \rangle + ((\beta(u^n) - \beta(u^{n+1}))\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h), \tag{3.15}$$
$$-(\nabla \cdot \boldsymbol{\theta}^{n+1}, \omega_h) = 0. \tag{3.16}$$

Thus, from (3.14)-(3.16) and (3.7)-(3.9), we obtain the following error equations:

$$\begin{aligned} (D_\tau e_u^{n+1}, v_h) + (\nabla e_u^{n+1}, \nabla v_h) = {} & -(D_\tau \xi_u^{n+1}, v_h) - (\nabla \xi_u^{n+1}, \nabla v_h) \\ & + (\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta(u_h^n)|\boldsymbol{\theta}_h^{n+1}|^2, v_h) \\ & + (D_\tau u^{n+1} - u_t^{n+1}, v_h) \\ & + ((\beta(u^{n+1}) - \beta(u^n))|\boldsymbol{\theta}^{n+1}|^2, v_h), \quad \forall v_h \in V_h, \end{aligned} \tag{3.17}$$
$$\begin{aligned} (\beta(u_h^n) e_{\boldsymbol{\theta}}^{n+1}, \boldsymbol{\chi}_h) = {} & -((\beta(u^n) - \beta(u_h^n))\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h) \\ & + (\beta(u_h^n)(\boldsymbol{\Pi}_h \boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}), \boldsymbol{\chi}_h) \\ & - (\xi_\phi^{n+1}, \nabla \cdot \boldsymbol{\chi}_h) - (e_\phi^{n+1}, \nabla \cdot \boldsymbol{\chi}_h) \\ & + ((\beta(u^n) - \beta(u^{n+1}))\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in \boldsymbol{H}_h, \end{aligned} \tag{3.18}$$
$$(\nabla \cdot \xi_{\boldsymbol{\theta}}^{n+1}, \omega_h) + (\nabla \cdot e_{\boldsymbol{\theta}}^{n+1}, \omega_h) = 0, \quad \forall \omega_h \in M_h. \tag{3.19}$$

Taking $\boldsymbol{\chi}_h = e_{\boldsymbol{\theta}}^{n+1}$ in (3.18), we obtain by (1.5) and the definition of $\beta(\cdot)$ that

$$\begin{aligned} \frac{1}{a^*}\|e_{\boldsymbol{\theta}}^{n+1}\|_0^2 \le {} & -((\beta(u^n) - \beta(u_h^n))\boldsymbol{\theta}^{n+1}, e_{\boldsymbol{\theta}}^{n+1}) + (\beta(u_h^n)(\boldsymbol{\Pi}_h \boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}), e_{\boldsymbol{\theta}}^{n+1}) \\ & - (\xi_\phi^{n+1}, \nabla \cdot e_{\boldsymbol{\theta}}^{n+1}) - (e_\phi^{n+1}, \nabla \cdot e_{\boldsymbol{\theta}}^{n+1}) + ((\beta(u^n) - \beta(u^{n+1}))\boldsymbol{\theta}^{n+1}, e_{\boldsymbol{\theta}}^{n+1}) \\ =: {} & A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned} \tag{3.20}$$

Now, we estimate $A_i, i = 1, 2, 3, 4, 5$, term by term. By Cauchy-Schwarz inequality and (2.2), there holds

$$\begin{aligned} |A_1| &\le \|\beta(u^n) - \beta(u_h^n)\|_0 \|\boldsymbol{\theta}^{n+1}\|_{0,\infty} \|e_{\boldsymbol{\theta}}^{n+1}\|_0 \le C\|u^n - u_h^n\|_0 \|e_{\boldsymbol{\theta}}^{n+1}\|_0 \\ &\le C(h^2 + \|e_u^n\|_0)\|e_{\boldsymbol{\theta}}^{n+1}\|_0. \end{aligned}$$

Note that

$$\begin{aligned} A_2 &= \big(\beta\big(u_h^n\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big) \\ &= \big(\big(\beta\big(u_h^n\big) - \beta(u^n)\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big) \\ &\quad + \big(\beta(u^n)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big) = A_{21} + A_{22}. \end{aligned}$$

By Cauchy-Schwarz inequality again, we have

$$\begin{aligned} |A_{21}| &\le \big\|\beta(u^n) - \beta\big(u_h^n\big)\big\|_0 \big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big\|_0 \big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_{0,\infty} \\ &\le C\big\|u^n - u_h^n\big\|_0 (Ch)\big(Ch^{-1}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\big) \\ &\le C\big(h^2 + \big\|e_u^n\big\|_0\big)\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0. \end{aligned}$$

In order to estimate the term A_{22}, we define

$$\overline{\omega} = \frac{1}{|K|}\int_K \omega \, d\boldsymbol{x}, \quad \omega \in W^{1,\infty}(K), \tag{3.21}$$

there holds

$$\|\omega - \overline{\omega}\|_{0,\infty,K} \le Ch_K|\omega|_{1,\infty,K}, \tag{3.22}$$

thus, we have

$$\begin{aligned} A_{22} &= \big(\beta(u^n)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big) \\ &= \sum_{K\in\mathcal{T}_h} \big(\beta(u^n)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big)_K \\ &= \sum_{K\in\mathcal{T}_h} \big(\big(\beta(u^n) - \overline{\beta(u^n)}\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), e_{\boldsymbol{\theta}}^{n+1}\big)_K \\ &\quad + \sum_{K\in\mathcal{T}_h} \overline{\beta(u^n)}\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}, e_{\boldsymbol{\theta}}^{n+1}\big)_K \\ &\le \sum_{K\in\mathcal{T}_h} \|\beta(u^n) - \overline{\beta(u^n)}\|_{0,\infty,K}\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_{0,K} \\ &\quad + \sum_{K\in\mathcal{T}_h} \overline{\beta(u^n)}\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}, e_{\boldsymbol{\theta}}^{n+1}\big)_K \\ &\le \sum_{K\in\mathcal{T}_h} C_K h_K^2 \|\boldsymbol{\theta}^{n+1}\|_{1,K}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_{0,K} + \sum_{K\in\mathcal{T}_h} C_K h_K^2 |\boldsymbol{\theta}^{n+1}|_{2,K}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_{0,K} \\ &\le Ch^2 \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_{0,K} \le Ch^2\|\boldsymbol{\theta}^{n+1}\|_2\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0 \le Ch^2\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0, \end{aligned}$$

where Lemma 2.1 have been used in the above estimate. Therefore, one can check that

$$|A_2| \le C\left(h^2 + \big\|e_u^n\big\|_0\right)\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0.$$

According to (2.10), it follows that

$$A_3 = 0.$$

Moreover, taking $\omega_h = e_\phi^{n+1}$ in (3.19) yields

$$\big(e_\phi^{n+1}, \nabla\cdot e_{\boldsymbol{\theta}}^{n+1}\big) = -\big(\nabla\cdot \xi_{\boldsymbol{\theta}}^{n+1}, e_\phi^{n+1}\big),$$

together with (2.9) shows that

$$A_4 = 0.$$

With the help of Taylor's expansion, one can check that

$$\begin{aligned}
|A_5| &= \big(\big(\beta(u^n) - \beta(u^{n+1})\big)\boldsymbol{\theta}^{n+1}, e_{\boldsymbol{\theta}}^{n+1}\big) \\
&\leq \|\beta(u^n) - \beta(u^{n+1})\|_0 \|\boldsymbol{\theta}^{n+1}\|_{0,\infty} \big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0 \\
&\leq C\|u^n - u^{n+1}\|_0 \big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0 \\
&\leq C\tau \big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0.
\end{aligned}$$

Substituting the estimates $A_1 \sim A_5$ into (3.20) gives that

$$\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0^2 \leq C\big(h^2 + \big\|e_u^n\big\|_0\big)\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0 + C\tau\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0,$$

which implies

$$\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0 \leq C\big(h^2 + \tau + \big\|e_u^n\big\|_0\big). \tag{3.23}$$

From (3.18), we have

$$\begin{aligned}
\big(e_\phi^{n+1}, \nabla\cdot\boldsymbol{\chi}_h\big) = &-\big(\big(\beta(u^n) - \beta\big(u_h^n\big)\big)\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h\big) + \big(\beta\big(u_h^n\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), \boldsymbol{\chi}_h\big) - \big(\xi_\phi^{n+1}, \nabla\cdot\boldsymbol{\chi}_h\big) \\
&+ \big(\big(\beta(u^n) - \beta(u^{n+1})\big)\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h\big) - \big(\beta\big(u_h^n\big)e_{\boldsymbol{\theta}}^{n+1}, \boldsymbol{\chi}_h\big), \quad \forall \boldsymbol{\chi}_h \in \boldsymbol{H}_h.
\end{aligned}$$

Then, by Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
-\big(\big(\beta(u^n) - \beta\big(u_h^n\big)\big)\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h\big) &\leq \big\|\beta(u^n) - \beta\big(u_h^n\big)\big\|_0 \|\boldsymbol{\theta}^{n+1}\|_{0,\infty}\|\boldsymbol{\chi}_h\|_0 \\
&\leq C\big(h^2 + \big\|e_u^n\big\|_0\big)\|\boldsymbol{\chi}_h\|_0.
\end{aligned} \tag{3.24}$$

In a similar way as A_2, one can check that

$$\begin{aligned}
\big(\beta\big(u_h^n\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), \boldsymbol{\chi}_h\big) &= \Big(\big(\beta\big(u_h^n\big) - \beta(u^n)\big)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), \boldsymbol{\chi}_h\Big) \\
&\quad + \big(\beta(u^n)\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n+1}\big), \boldsymbol{\chi}_h\big) \\
&\leq C\big(h^2 + \big\|e_u^n\big\|_0\big)\|\boldsymbol{\chi}_h\|_0.
\end{aligned} \tag{3.25}$$

According to (2.10), there holds

$$\big(\xi_\phi^{n+1}, \nabla\cdot\boldsymbol{\chi}_h\big) = 0. \tag{3.26}$$

An application of Taylor's expansion, we have

$$\begin{aligned}
\big(\big(\beta(u^n) - \beta(u^{n+1})\big)\boldsymbol{\theta}^{n+1}, \boldsymbol{\chi}_h\big) &\leq \|\beta(u^n) - \beta(u^{n+1})\|_0\|\boldsymbol{\theta}^{n+1}\|_{0,\infty}\|\boldsymbol{\chi}_h\|_0 \\
&\leq C\tau\|\boldsymbol{\chi}_h\|_0.
\end{aligned} \tag{3.27}$$

Using (1.5), (3.23) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
-\big(\beta\big(u_h^n\big)e_{\boldsymbol{\theta}}^{n+1}, \boldsymbol{\chi}_h\big) &\leq \big|\beta\big(u_h^n\big)\big|\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\|\boldsymbol{\chi}_h\|_0 \\
&\leq C\big(h^2 + \tau + \big\|e_u^n\big\|_0\big)\|\boldsymbol{\chi}_h\|_0.
\end{aligned} \tag{3.28}$$

Based on the above estimates (3.24)-(3.28) and in terms of the inf-sup condition of Raviart-Thomas mixed finite element method, e_ϕ^{n+1} can be bounded by

$$\big\|e_\phi^{n+1}\big\|_0 \leq C \sup_{\boldsymbol{\chi}_h \in \boldsymbol{H}_h} \frac{\big(e_\phi^{n+1}, \nabla\cdot\boldsymbol{\chi}_h\big)}{\|\boldsymbol{\chi}_h\|_{\mathrm{div}}} \leq C\big(h^2 + \tau + \big\|e_u^n\big\|_0\big). \tag{3.29}$$

Next, we pay our attention to estimate e_u^n. To do this, letting $v_h = e_u^{n+1}$ in (3.17) results in

$$\begin{aligned}
&\frac{1}{2\tau}\big(\|e_u^{n+1}\|_0^2 - \|e_u^n\|_0^2 + \|e_u^{n+1} - e_u^n\|_0^2\big) + \|\nabla e_u^{n+1}\|_0^2 \\
=\ & -\big(D_\tau \xi_u^{n+1}, e_u^{n+1}\big) - \big(\nabla \xi_u^{n+1}, \nabla e_u^{n+1}\big) \\
&+ \big(\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2, e_u^{n+1}\big) + \big(D_\tau u^{n+1} - u_t^{n+1}, e_u^{n+1}\big) \\
&+ \big(\big(\beta(u^{n+1}) - \beta(u^n)\big)|\boldsymbol{\theta}^{n+1}|^2, e_u^{n+1}\big) \\
=:\ & B_1 + B_2 + B_3 + B_4 + B_5. \qquad (3.30)
\end{aligned}$$

In what follows, we prove

$$\|e_u^n\|_0 \le C^*(h^2 + \tau), \quad n = 0, 1, 2, \ldots, N, \qquad (3.31)$$

by mathematical induction, where C^* is a positive constant independent of h, τ and n. In fact, since $u_h^0 = R_h u_0$, we have $\|e_u^0\|_0 = 0$, which shows that (3.31) holds for $n = 0$. Now, we assume that (3.31) holds for $n \le k - 1$ for some positive integer k. We should find C^* independent of h, τ and n such that (3.31) also holds for $n \le k$.

By Cauchy-Schwarz inequality and (2.2), it follows that

$$|B_1| \le \|D_\tau \xi_u^{n+1}\|_0 \|e_u^{n+1}\|_0 \le Ch^2 \|e_u^{n+1}\|_0. \qquad (3.32)$$

According to Ritz projection, there holds

$$B_2 = 0. \qquad (3.33)$$

Note that

$$\begin{aligned}
&\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2 \\
=\ & \beta(u^n)\big(|\boldsymbol{\theta}^{n+1}|^2 - \big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2\big) + \beta(u^n)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2, \\
&\beta(u^n)\big(|\boldsymbol{\theta}^{n+1}|^2 - \big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2\big) \\
=\ & 2\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big) - \beta(u^n)\big|\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2, \\
&\beta(u^n)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2 \\
=\ & \big(\beta(u^n) - \beta(u_h^n)\big)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2 + 2\beta(u_h^n)\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big) - \beta(u_h^n)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big|^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
B_3 &= \big(\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2, e_u^{n+1}\big) \\
&= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big) \\
&\quad - \big(\beta(u^n)\big|\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2, e_u^{n+1}\big) \\
&\quad + \big(\big(\beta(u^n) - \beta(u_h^n)\big)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2, e_u^{n+1}\big) \\
&\quad + 2\big(\beta(u_h^n)\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big), e_u^{n+1}\big) \\
&\quad - \big(\beta(u_h^n)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big|^2, e_u^{n+1}\big) =: \sum_{k=1}^{5} B_{3k}.
\end{aligned}$$

One can check that

$$
\begin{aligned}
B_{31} &= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big) \\
&= 2\sum_{K\in\mathcal{T}_h}\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big)_K \\
&= 2\sum_{K\in\mathcal{T}_h}\big(\big(\beta(u^n)-\overline{\beta(u^n)}\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big)_K \\
&\quad + 2\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big(\big(\boldsymbol{\theta}^{n+1}-\overline{\boldsymbol{\theta}^{n+1}}\big)\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big)_K \\
&\quad + 2\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big(\overline{\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), \overline{e_u^{n+1}}\big)_K \\
&\quad + 2\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big(\overline{\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}-\overline{e_u^{n+1}}\big)_K \\
&\le 2\sum_{K\in\mathcal{T}_h}\|\beta(u^n)-\overline{\beta(u^n)}\|_{0,\infty,K}\|\boldsymbol{\theta}^{n+1}\|_{0,\infty,K}\big\|\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_u^{n+1}\big\|_{0,K} \\
&\quad + 2\sum_{K\in\mathcal{T}_h}C_K\|\boldsymbol{\theta}^{n+1}-\overline{\boldsymbol{\theta}^{n+1}}\|_{0,\infty,K}\big\|\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_u^{n+1}\big\|_{0,K} \\
&\quad + 2\sum_{K\in\mathcal{T}_h}C_K h_K^2\|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|\overline{e_u^{n+1}}\big\|_{0,K} + 2\sum_{K\in\mathcal{T}_h}C_K\|\boldsymbol{\theta}^{n+1}-\overline{\boldsymbol{\theta}^{n+1}}\|_{0,K}\big\|e_u^{n+1}-\overline{e_u^{n+1}}\big\|_{0,K} \\
&\le Ch^2\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|e_u^{n+1}\big\|_{0,K} + Ch^2\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\theta}^{n+1}\|_{1,K}\big\|e_u^{n+1}\big\|_{1,K} \\
&\le Ch^2\big\|e_u^{n+1}\big\|_0 + Ch^2\big\|\nabla e_u^{n+1}\big\|_0,
\end{aligned}
$$

where we have used Lemma 2.2, and

$$
\|\overline{w}\|_{0,K}\le\|w\|_{0,K},\quad \|w-\overline{w}\|_{0,K}\le C_K h_K\|w\|_{1,K}.
$$

By Cauchy-Schwarz inequality, there holds

$$
|B_{32}|\le\|\beta(u^n)\|_{0,\infty}\big\|\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,4}^2\big\|e_u^{n+1}\big\|_0\le Ch^2\big\|e_u^{n+1}\big\|_0.
$$

Using Cauchy-Schwarz inequality again, we have

$$
\begin{aligned}
|B_{33}| &\le \big\|\beta(u^n)-\beta\big(u_h^n\big)\big\|_0\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,\infty}^2\big\|e_u^{n+1}\big\|_0 \\
&\le C\big(h^2+\big\|e_u^n\big\|_0\big)\big\|e_u^{n+1}\big\|_0.
\end{aligned}
$$

Moreover, it follows that

$$
\begin{aligned}
B_{34} &\le 2\big\|\beta\big(u_h^n\big)\big\|_{0,\infty}\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{\boldsymbol{n+1}}\big\|_{0,\infty}\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}_h^{n+1}\big\|_0\big\|e_u^{n+1}\big\|_0 \\
&\le C\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\big\|e_u^{n+1}\big\|_0\le C\big(h^2+\tau+\big\|e_u^n\big\|_0\big)\big\|e_u^{n+1}\big\|_0,
\end{aligned}
$$

where we have used (3.23).

By Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
B_{35} &\le \big\|\beta\big(u_h^n\big)\big\|_{0,\infty}\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}_h^{n+1}\big\|_0\big\|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}_h^{n+1}\big\|_{0,\infty}\big\|e_u^{n+1}\big\|_0 \\
&\le C\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\big(Ch^{-1}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\big)\big\|e_u^{n+1}\big\|_0 \\
&\le C\big(h^2+\tau+\big\|e_u^n\big\|_0\big)\big\|e_u^{n+1}\big\|_0,
\end{aligned}
$$

where we have used (3.23) and (3.31), which shows that for $n \le k-1$,

$$\begin{aligned}\|e_{\boldsymbol{\theta}}^{n+1}\|_{0,\infty} &\le Ch^{-1}\|e_{\boldsymbol{\theta}}^{n+1}\|_0 \le Ch^{-1}\big(h^2+\tau+\|e_u^n\|_0\big)\\ &\le C(1+C^*)(h+h^{-1}\tau) \le 1, \end{aligned}\tag{3.34}$$

provided that $\tau = \mathcal{O}(h^{1+\alpha}), \alpha > 0$ and $2C(1+C^*)\max\{h, h^\alpha\} \le 1$ for sufficiently small h. With the above estimates $B_{3i}, i = 1,2,3,4,5$, we have

$$\begin{aligned}B_3 &= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big)\\ &\le C\big(h^2+\tau+\|e_u^n\|_0\big)\|e_u^{n+1}\|_0 + Ch^2\|\nabla e_u^{n+1}\|_0.\end{aligned}\tag{3.35}$$

An application of Taylor's expansion, one can check that

$$|B_4| \le \big\|D_\tau u^{n+1} - u_t^{n+1}\big\|_0\|e_u^{n+1}\|_0 \le C\tau\|e_u^{n+1}\|_0, \tag{3.36}$$

$$|B_5| \le \big\|\beta(u^{n+1}) - \beta(u^n)\big\|_0\|\boldsymbol{\theta}^{n+1}\|_{0,\infty}^2\|e_u^{n+1}\|_0 \le C\tau\|e_u^{n+1}\|_0. \tag{3.37}$$

Substituting the estimates $B_1 \sim B_5$ into (3.30) yields that

$$\frac{1}{2\tau}\big(\|e_u^{n+1}\|_0^2 - \|e_u^n\|_0^2\big) + \|\nabla e_u^{n+1}\|_0^2 \le C\big(h^2+\tau+\|e_u^n\|_0\big)\|e_u^{n+1}\|_0 + Ch^2\|\nabla e_u^{n+1}\|_0,$$

which shows that

$$\frac{1}{2\tau}\big(\|e_u^{n+1}\|_0^2 - \|e_u^n\|_0^2\big) \le C(h^4+\tau^2) + C\big(\|e_u^{n+1}\|_0^2 + \|e_u^n\|_0^2\big). \tag{3.38}$$

Summing up the above inequality and using $e_u^0 = 0$, we have

$$\|e_u^{n+1}\|_0^2 \le C(h^4+\tau^2) + C\tau\sum_{k=1}^{n+1}\|e_u^k\|_0^2. \tag{3.39}$$

Thanks to Gronwall's inequality (see Lemma 2.3), there holds for $C\tau \le 1/2$,

$$\begin{aligned}\|e_u^{n+1}\|_0^2 &\le \exp\left(\frac{CT}{1-C\tau}\right)C(h^4+\tau^2)\\ &\le C\exp(2CT)(h^4+\tau^2)\\ &\le C^*(h^2+\tau)^2,\end{aligned}\tag{3.40}$$

where we take $C^* \ge \sqrt{C\exp(2CT)}$. Thus, the estimate (3.31) is also valid for $n \le k$ and we complete the induction.

Finally, substituting (3.40) into (3.23) and (3.29), we have

$$\|e_{\boldsymbol{\theta}}^{n+1}\|_0 + \|e_\phi^{n+1}\|_0 \le C(h^2+\tau). \tag{3.41}$$

On the other hand, we pay attention to estimate the H^1-norm of e_u^n. To do this, letting $v_h = D_\tau e_u^{n+1}$ in (3.17) results in

$$\begin{aligned}&\|D_\tau e_u^{n+1}\|_0^2 + \frac{1}{2\tau}\big(\|\nabla e_u^{n+1}\|_0^2 - \|\nabla e_u^n\|_0^2 + \|\nabla(e_u^{n+1}-e_u^n)\|_0^2\big)\\ =& -\big(D_\tau\xi_u^{n+1}, D_\tau e_u^{n+1}\big) - \big(\nabla\xi_u^{n+1}, \nabla D_\tau e_u^{n+1}\big)\\ &+ \big(\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta(u_h^n)\big|\boldsymbol{\theta}_h^{n+1}\big|^2, D_\tau e_u^{n+1}\big) + \big(D_\tau u^{n+1} - u_t^{n+1}, D_\tau e_u^{n+1}\big)\\ &+ \big(\big(\beta(u^{n+1}) - \beta(u^n)\big)|\boldsymbol{\theta}^{n+1}|^2, D_\tau e_u^{n+1}\big) =: \sum_{k=1}^5 E_k.\end{aligned}\tag{3.42}$$

By Cauchy-Schwarz inequality, there holds

$$E_1 \leq \big\|D_\tau \xi_u^{n+1}\big\|_0 \big\|D_\tau e_u^{n+1}\big\|_0 \leq Ch^2 \big\|D_\tau e_u^{n+1}\big\|_0. \tag{3.43}$$

An application of Ritz projection, it follows that

$$E_2 = 0. \tag{3.44}$$

Note that

$$\begin{aligned}
E_3 &= \big(\beta(u^n)|\boldsymbol{\theta}^{n+1}|^2 - \beta\big(u_h^n\big)\big|\boldsymbol{\theta}_h^{n+1}\big|^2, D_\tau e_u^{n+1}\big) \\
&= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), D_\tau e_u^{n+1}\big) \\
&\quad - \big(\beta(u^n)\big|\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2, D_\tau e_u^{n+1}\big) \\
&\quad + \big(\big(\beta(u^n) - \beta\big(u_h^n\big)\big)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big|^2, D_\tau e_u^{n+1}\big) \\
&\quad + 2\big(\beta\big(u_h^n\big)\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big), D_\tau e_u^{n+1}\big) \\
&\quad - \big(\beta\big(u_h^n\big)\big|\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}_h^{n+1}\big|^2, D_\tau e_u^{n+1}\big) =: \sum_{k=1}^{5} E_{3k}.
\end{aligned}$$

In order to estimate E_{31}, using summation by parts yields

$$\begin{aligned}
E_{31} &= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), D_\tau e_u^{n+1}\big) \\
&= \frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big) - \big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n - \boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big), e_u^n\big)\big] \\
&\quad - \frac{2}{\tau}\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big) - \beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n - \boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big), e_u^n\big),
\end{aligned}$$

and noting that

$$\begin{aligned}
&\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big) - \beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n - \boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big) \\
&= \big(\beta(u^n) - \beta(u^{n-1})\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big) \\
&\quad + \beta(u^{n-1})(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n)\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big) \\
&\quad + \beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n - \boldsymbol{\Pi}_h(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n)\big).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
E_{31} &= 2\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), D_\tau e_u^{n+1}\big) \\
&= \frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^{n+1}\big) - \big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n - \boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big), e_u^n\big)\big] \\
&\quad - 2\big(\big(D_\tau\beta(u^n)\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\big) \\
&\quad - 2\big(\beta(u^{n-1})\big(D_\tau\boldsymbol{\theta}^{n+1}\big)\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\big) \\
&\quad - 2\big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(D_\tau\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h D_\tau\boldsymbol{\theta}^{n+1}\big), e_u^n\big).
\end{aligned}$$

Moreover, using mean value technique (3.22), we have

$$\begin{aligned}
&\big(\big(D_\tau\beta(u^n)\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\big) \\
&= \sum_{K\in\mathcal{T}_h}\big(\big(D_\tau\beta(u^n)\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\big)_K
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h} \Big(\big(D_\tau\beta(u^n) - \overline{D_\tau\beta(u^n)}\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{D_\tau\beta(u^n)}\Big(\big(\boldsymbol{\theta}^{n+1} - \overline{\boldsymbol{\theta}^{n+1}}\big)\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{D_\tau\beta(u^n)\boldsymbol{\theta}^{n+1}}\Big(\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n - \overline{e_u^n}\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{D_\tau\beta(u^n)\boldsymbol{\theta}^{n+1}}\Big(\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), \overline{e_u^n}\Big)_K \\
&\le \sum_{K\in\mathcal{T}_h} \big\|D_\tau\beta(u^n) - \overline{D_\tau\beta(u^n)}\big\|_{0,\infty,K}\|\boldsymbol{\theta}^{n+1}\|_{0,\infty,K}\big\|\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_u^n\big\|_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} C_K\|\boldsymbol{\theta}^{n+1} - \overline{\boldsymbol{\theta}^{n+1}}\|_{0,\infty,K}\big\|\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_u^n\big\|_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} C_K\big\|\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,K}\big\|e_u^n - \overline{e_u^n}\big\|_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} C_K h_K^2\|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|\overline{e_u^n}\big\|_{0,K} \\
&\le \sum_{K\in\mathcal{T}_h} C_K h_K^2|\boldsymbol{\theta}^{n+1}|_{1,K}\big\|e_u^n\big\|_{0,K} + \sum_{K\in\mathcal{T}_h} C_K h_K^2|\boldsymbol{\theta}^{n+1}|_{1,K}\big\|e_u^n\big\|_{1,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} C_K h_K^2\|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|e_u^n\big\|_{0,K} \\
&\le Ch^2\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\theta}^{n+1}\|_{2,K}\big\|e_u^n\big\|_{0,K} + Ch^2\sum_{K\in\mathcal{T}_h}|\boldsymbol{\theta}^{n+1}|_{1,K}\big\|e_u^n\big\|_{1,K} \\
&\le Ch^2\big\|e_u^n\big\|_0 + Ch^2\big\|\nabla e_u^n\big\|_0 \le Ch^2\big\|\nabla e_u^n\big\|_0.
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
&\Big(\beta(u^{n-1})\big(D_\tau\boldsymbol{\theta}^{n+1}\big)\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big) \\
&= \sum_{K\in\mathcal{T}_h} \Big(\beta(u^{n-1})\big(D_\tau\boldsymbol{\theta}^{n+1}\big)\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&= \sum_{K\in\mathcal{T}_h} \Big(\big(\beta(u^{n-1}) - \overline{\beta(u^{n-1})}\big)\big(D_\tau\boldsymbol{\theta}^{n+1}\big)\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{\beta(u^{n-1})}\Big(\big(D_\tau\boldsymbol{\theta}^{n+1} - \overline{D_\tau\boldsymbol{\theta}^{n+1}}\big)\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{\beta(u^{n-1})}\Big(\overline{D_\tau\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), e_u^n - \overline{e_u^n}\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \overline{\beta(u^{n-1})}\Big(\overline{D_\tau\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big), \overline{e_u^n}\Big)_K \\
&\le Ch^2\big\|e_u^n\big\|_0 + Ch^2\big\|\nabla e_u^n\big\|_0 \le Ch^2\big\|\nabla e_u^n\big\|_0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(D_\tau\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h D_\tau\boldsymbol{\theta}^{n+1}\big), e_u^n\Big) \\
&= \sum_{K\in\mathcal{T}_h} \Big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(D_\tau\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h D_\tau\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K \\
&= \sum_{K\in\mathcal{T}_h} \Big(\big(\beta(u^{n-1}) - \overline{\beta(u^{n-1})}\big)\boldsymbol{\theta}^n\big(D_\tau\boldsymbol{\theta}^{n+1} - \mathbf{\Pi}_h D_\tau\boldsymbol{\theta}^{n+1}\big), e_u^n\Big)_K
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^{n-1})}\big((\boldsymbol{\theta}^n-\overline{\boldsymbol{\theta}^n})\big(D_\tau\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_hD_\tau\boldsymbol{\theta}^{n+1}\big),e_u^n\big)_K\\
&+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^{n-1})}\big(\overline{\boldsymbol{\theta}^n}\big(D_\tau\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_hD_\tau\boldsymbol{\theta}^{n+1}\big),e_u^n-\overline{e_u^n}\big)_K\\
&+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^{n-1})}\big(\overline{\boldsymbol{\theta}^n}\big(D_\tau\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_hD_\tau\boldsymbol{\theta}^{n+1}\big),\overline{e_u^n}\big)_K\\
\le{}& Ch^2\big\|e_u^n\big\|_0+Ch^2\big\|\nabla e_u^n\big\|_0\le Ch^2\big\|\nabla e_u^n\big\|_0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
E_{31}\le{}&\frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}(\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}),e_u^{n+1}\big)-\big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n-\mathbf{\Pi}_h\boldsymbol{\theta}^n\big),e_u^n\big)\big]\\
&+Ch^2\big\|\nabla e_u^n\big\|_0. \qquad (3.45)
\end{aligned}
$$

In terms of Cauchy-Schwarz inequality, it follows that

$$
E_{32}\le 2\|\beta(u^n)\|_{0,\infty}\big\|\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,4}^2\big\|D_\tau e_u^{n+1}\big\|_0\le Ch^2\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.46)
$$

Thanks to (3.40), there holds

$$
\begin{aligned}
E_{33}&\le\big\|\beta(u^n)-\beta\big(u_h^n\big)\big\|_0\big\|\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,\infty}^2\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C\big(h^2+\big\|e_u^n\big\|_0\big)\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C(h^2+\tau)\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.47)
\end{aligned}
$$

According to (3.40) and (3.41), we have

$$
\begin{aligned}
E_{34}&\le\big\|\beta\big(u_h^n\big)\big\|_{0,\infty}\big\|\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big\|_{0,\infty}\big\|e_{\boldsymbol{\theta}}^{n+1}\big\|_0\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C\big(h^2+\tau+\big\|e_u^n\big\|_0\big)\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C(h^2+\tau)\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.48)
\end{aligned}
$$

With the help of (3.34), one can check that

$$
\begin{aligned}
E_{35}&\le\big\|\beta\big(u_h^n\big)\big\|_{0,\infty}\big\|\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}_h^{n+1}\big\|_0\big\|\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}_h^{n+1}\big\|_{0,\infty}\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C\big(h^2+\tau+\big\|e_u^n\big\|_0\big)\big\|D_\tau e_u^{n+1}\big\|_0\\
&\le C(h^2+\tau)\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.49)
\end{aligned}
$$

Based on the above estimates $E_{31}\sim E_{35}$, there holds

$$
\begin{aligned}
E_3\le{}&\frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\mathbf{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)-\big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n-\mathbf{\Pi}_h\boldsymbol{\theta}^n\big),e_u^n\big)\big]\\
&+Ch^2\big\|\nabla e_u^n\big\|_0+C(h^2+\tau)\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.50)
\end{aligned}
$$

With the aid of Taylor's expansion, it follows that

$$
E_4\le\big\|D_\tau u^{n+1}-u_t^{n+1}\big\|_0\big\|D_\tau e_u^{n+1}\big\|_0\le C\tau\big\|D_\tau e_u^{n+1}\big\|_0, \qquad (3.51)
$$

$$
E_5\le\|\beta(u^{n+1})-\beta(u^n)\|_0\|\boldsymbol{\theta}^{n+1}\|_{0,\infty}^2\big\|D_\tau e_u^{n+1}\big\|_0\le C\tau\big\|D_\tau e_u^{n+1}\big\|_0. \qquad (3.52)
$$

Substituting the estimates $E_1\sim E_5$ into (3.42) results in

$$
\big\|D_\tau e_u^{n+1}\big\|_0^2+\frac{1}{2\tau}\big(\big\|\nabla e_u^{n+1}\big\|_0^2-\big\|\nabla e_u^n\big\|_0^2\big)
$$

$$\leq \frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)-\big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n-\boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big),e_u^n\big)\big]$$
$$+Ch^2\|\nabla e_u^n\|_0+C(h^2+\tau)\|D_\tau e_u^{n+1}\|_0,$$

which implies that

$$\frac{1}{2\tau}\big(\|\nabla e_u^{n+1}\|_0^2-\|\nabla e_u^n\|_0^2\big)$$
$$\leq \frac{2}{\tau}\big[\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)-\big(\beta(u^{n-1})\boldsymbol{\theta}^n\big(\boldsymbol{\theta}^n-\boldsymbol{\Pi}_h\boldsymbol{\theta}^n\big),e_u^n\big)\big]$$
$$+C(h^4+\tau^2)+C\|\nabla e_u^n\|_0^2.$$

Summing up the above inequality and using $e_u^0=0$, we have

$$\frac{1}{2\tau}\|\nabla e_u^{n+1}\|_0^2\leq\frac{2}{\tau}\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)$$
$$+C\sum_{k=1}^n(h^4+\tau^2)+C\sum_{k=1}^n\|\nabla e_u^k\|_0^2, \tag{3.53}$$

which shows that

$$\|\nabla e_u^{n+1}\|_0^2\leq 4\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)$$
$$+C\tau\sum_{k=1}^n(h^4+\tau^2)+C\tau\sum_{k=1}^n\|\nabla e_u^k\|_0^2. \tag{3.54}$$

An application of a similar way as E_{31}, one can check that

$$\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)$$
$$=\sum_{K\in\mathcal{T}_h}\big(\beta(u^n)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)_K$$
$$=\sum_{K\in\mathcal{T}_h}\big(\big(\beta(u^n)-\overline{\beta(u^n)}\big)\boldsymbol{\theta}^{n+1}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)_K$$
$$+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big((\boldsymbol{\theta}^{n+1}-\overline{\boldsymbol{\theta}^{n+1}}\big)\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}\big)_K$$
$$+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big(\overline{\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),e_u^{n+1}-\overline{e_u^{n+1}}\big)_K$$
$$+\sum_{K\in\mathcal{T}_h}\overline{\beta(u^n)}\big(\overline{\boldsymbol{\theta}^{n+1}}\big(\boldsymbol{\theta}^{n+1}-\boldsymbol{\Pi}_h\boldsymbol{\theta}^{n+1}\big),\overline{e_u^{n+1}}\big)_K$$
$$\leq Ch^2\|\nabla e_u^{n+1}\|_0.$$

Substituting the above inequality into (3.54) gives that

$$\|\nabla e_u^{n+1}\|_0^2\leq Ch^2\|\nabla e_u^{n+1}\|_0+C\tau\sum_{k=1}^n(h^4+\tau^2)+C\tau\sum_{k=1}^n\|\nabla e_u^k\|_0^2$$
$$\leq\frac{1}{2}\|\nabla e_u^{n+1}\|_0^2+C(h^4+\tau^2)+C\tau\sum_{k=1}^n\|\nabla e_u^k\|_0^2, \tag{3.55}$$

which implies that

$$\|\nabla e_u^{n+1}\|_0^2\leq C(h^4+\tau^2)+C\tau\sum_{k=1}^n\|\nabla e_u^k\|_0^2. \tag{3.56}$$

Thanks to Gronwall's inequality (see Lemma 2.3), we have

$$\left\|\nabla e_u^{n+1}\right\|_0 \le C(h^2+\tau), \tag{3.57}$$

which together with (2.4) and triangle inequality yields that

$$\begin{aligned}\left\|\nabla\left(I_h u^{n+1}-u_h^{n+1}\right)\right\|_0 &\le \left\|\nabla\left(I_h u^{n+1}-R_h u^{n+1}\right)\right\|_0+\left\|\nabla\left(R_h u^{n+1}-u_h^{n+1}\right)\right\|_0\\ &\le Ch^2+C(h^2+\tau)\le C(h^2+\tau). \end{aligned}\tag{3.58}$$

Thus, the desired result (3.12) is obtained with (3.41). Moreover, the desired result (3.13) is derived by triangle inequality, (2.11), (2.12), (3.41) and (3.58). The proof is complete. □

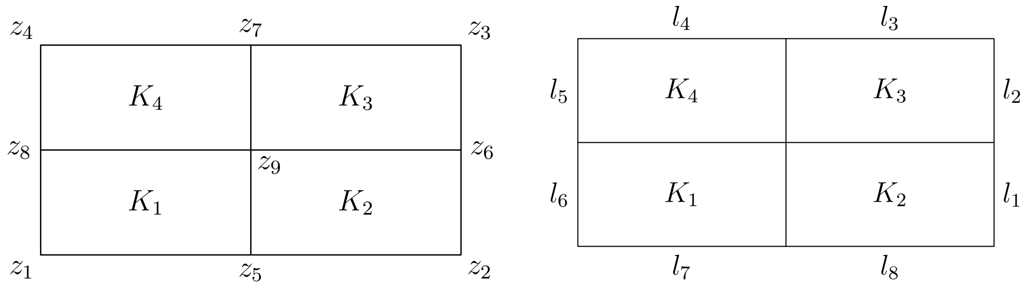
In order to obtain the global superconvergence results, we adopt the interpolation post-processing approach. To do this, we build a macroelement $\widetilde{K}$ consisting 4 elements K_j, $j=1,2,3,4$ (see Fig. 3.1). For numerical solution u_h^n, we adopt the local interpolation operator $I_{2h}: C(\widetilde{K})\to Q_{22}(\widetilde{K})$ as interpolation post-processing operator with the following interpolation conditions (see Fig. 3.1(a)):

$$I_{2h}u(z_i)=u(z_i),\quad i=1,2,\dots,9, \tag{3.59}$$

where $z_i, i=1,2,\dots,9$ are the nine vertices of $\widetilde{K}$.

Moreover, we adopt the following two local interpolation operators $\mathbf{\Pi}_{2h}:\boldsymbol{H}(\mathrm{div};\widetilde{K})\to Q_{11}(\widetilde{K})\times Q_{11}(\widetilde{K})$ and $P_{2h}: L^2(\widetilde{K})\to Q_{11}(\widetilde{K})$ as interpolation post-processing operator with the following interpolation conditions (see Fig. 3.1(b)):

$$\begin{aligned}&\mathbf{\Pi}_{2h}\boldsymbol{u}\in Q_{11}(\widetilde{K})\times Q_{11}(\widetilde{K}),\\ &\int_{l_i}(u_1-\Pi_{2h}u_1)dy=0,\quad i=1,2,5,6,\\ &\int_{l_i}(u_2-\Pi_{2h}u_2)dx=0,\quad i=3,4,7,8,\\ &P_{2h}w\in Q_{11}(\widetilde{K}),\\ &\int_{K_i}(w-P_{2h}w)\,d\boldsymbol{x}=0,\quad i=1,2,3,4.\end{aligned}\tag{3.60}$$



(a) The illustration for I_{2h} (b) The illustration for $\mathbf{\Pi}_{2h}$ and P_{2h}

Fig. 3.1. The macroelement $\widetilde{K}$.

Moreover, from [37] and [63], the following properties for I_{2h}, $\mathbf{\Pi}_{2h}$ and P_{2h} hold, i.e.

$$\begin{aligned} &I_{2h}I_h = I_{2h}, \\ &\|u - I_{2h}u\|_1 \le Ch^2\|u\|_3, \quad \forall\, u \in H^3(\Omega), \\ &\|I_{2h}v_h\|_1 \le C\|v_h\|_1, \qquad \forall\, v_h \in V_h, \end{aligned} \tag{3.61}$$

and

$$\begin{aligned} &\mathbf{\Pi}_{2h}\mathbf{\Pi}_h = \mathbf{\Pi}_{2h}, \\ &\|\boldsymbol{p} - \mathbf{\Pi}_{2h}\boldsymbol{p}\|_0 \le Ch^2\|\boldsymbol{p}\|_2, \quad \forall\, \boldsymbol{p} \in \big(H^2(\Omega)\big)^2, \\ &\|\mathbf{\Pi}_{2h}\boldsymbol{q}_h\|_0 \le C\|\boldsymbol{q}_h\|_0, \qquad \forall\, \boldsymbol{q}_h \in \boldsymbol{H}_h, \\ &P_{2h}P_h = P_{2h}, \\ &\|w - P_{2h}w\|_0 \le Ch^2\|w\|_2, \quad \forall\, w \in H^2(\Omega), \\ &\|P_{2h}\chi_h\|_0 \le C\|\chi_h\|_0, \qquad \forall\, \chi_h \in M_h. \end{aligned} \tag{3.62}$$

With the above properties, we have the following global superconvergence result.

Theorem 3.2. *Suppose that the conditions of Theorem* 3.1 *hold. Moreover, suppose that* $\phi \in L^\infty(H^2(\Omega))$ *, for* $1 \le n \le N$*, there holds*

$$\big\|u^n - I_{2h}u_h^n\big\|_1 + \big\|\boldsymbol{\theta}^n - \mathbf{\Pi}_{2h}\boldsymbol{\theta}_h^n\big\|_0 + \big\|\phi^n - P_{2h}\phi_h^n\big\|_0 \le C(h^2 + \tau). \tag{3.63}$$

Proof. In fact, it is easy to see that by (3.61)

$$\begin{aligned} \big\|u^n - I_{2h}u_h^n\big\|_1 &\le \big\|u^n - I_{2h}I_hu^n\big\|_1 + \big\|I_{2h}I_hu^n - I_{2h}u_h^n\big\|_1 \\ &\le \big\|u^n - I_{2h}u^n\big\|_1 + \big\|I_{2h}\big(I_hu^n - u_h^n\big)\big\|_1 \\ &\le Ch^2 + C\big\|I_hu^n - u_h^n\big\|_1 \\ &\le Ch^2 + C(h^2 + \tau) \le C(h^2 + \tau). \end{aligned}$$

The error estimate of $\boldsymbol{\theta}$ and ϕ follows an analogous approach for u. The proof is complete. □

4. Numerical Results

In this section, we present some numerical results to verify the theoretical analysis. Here, we consider a general system as in [22]

$$\begin{aligned} u_t - \Delta u &= \sigma(u)|\nabla\phi|^2 + f_1, \\ -\nabla\cdot\big(\sigma(u)\nabla\phi\big) &= f_2, \end{aligned}$$

where $\sigma(u)$ takes the form

$$\sigma(u) = \frac{1}{1+u^2} + 1.$$

We set the domain $\Omega = (0,1)\times(0,1)$ and the final time $T = 1.0$ in the computation. Moreover, we set the Dirichlet boundary conditions for u and ϕ. The initial conditions $u_0(\boldsymbol{x})$ and the right-hand side terms f_1, f_2 are chosen such that the exact solution is given by

$$\begin{aligned} u(t,x,y) &= \exp(-t)\sin(\pi x)\sin(\pi y), \\ \phi(t,x,y) &= \exp(-t)\sin(x+y). \end{aligned}$$

To confirm the conclusion presented in Theorems 3.1-3.2, we present the numerical errors with $\tau = h^2$ at $t = 0.1, 0.6, 1.0$ in Tables 4.1-4.9, respectively. Clearly, from Tables 4.1-4.9, we can see that the numerical results agree well with the theoretical analysis. In addition, we also present the graphics of the exact solution and numerical solution at $t = 1.0$ on mesh 32×32 in Figs. 4.1-4.4, which also illustrate that the numerical solution approximates the exact solution very well.

Table 4.1: The numerical errors and convergence rates of u at $t = 0.1$.

h	1/4	1/8	1/16	1/32	1/64
$\|u^n - u_h^n\|_1$	4.5489e-01	2.2774e-01	1.1392e-01	5.6964e-02	2.8483e-02
Order	/	0.99816	0.99940	0.99984	0.99996
$\|I_h u^n - u_h^n\|_1$	8.4898e-02	2.2328e-02	5.6511e-03	1.4171e-03	3.5454e-04
Order	/	1.9269	1.9823	1.9956	1.9989
$\|u^n - I_{2h} u_h^n\|_1$	2.0166e-01	5.1228e-02	1.2855e-02	3.2166e-03	8.0433e-04
Order	/	1.9769	1.9946	1.9987	1.9997

Table 4.2: The numerical errors and convergence rates of $\boldsymbol{\theta}$ at $t = 0.1$.

h	1/4	1/8	1/16	1/32	1/64
$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	1.4946e-01	7.4273e-02	3.7072e-02	1.8528e-02	9.2627e-03
Order	/	1.0089	1.0025	1.0007	1.0002
$\|\boldsymbol{\Pi}_h \boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	1.4332e-02	3.6642e-03	9.2201e-04	2.3093e-04	5.7748e-05
Order	/	1.9677	1.9907	1.9973	1.9996
$\|\boldsymbol{\theta}^n - \boldsymbol{\Pi}_{2h} \boldsymbol{\theta}_h^n\|_0$	3.6943e-02	1.8422e-02	5.8765e-03	1.5072e-03	3.7895e-04
Order	/	1.0039	1.6484	1.9631	1.9918

Table 4.3: The numerical errors and convergence rates of ϕ at $t = 0.1$.

h	1/4	1/8	1/16	1/32	1/64
$\|\phi^n - \phi_h^n\|_0$	5.4561e-02	2.7387e-02	1.3706e-02	6.8548e-03	3.4276e-03
Order	/	0.99440	0.99863	0.99966	0.99992
$\|P_h \phi^n - \phi_h^n\|_0$	6.2129e-04	2.2419e-04	6.0247e-05	1.5337e-05	3.8553e-06
Order	/	1.4705	1.8958	1.9739	1.9920
$\|\phi^n - P_{2h} \phi_h^n\|_0$	9.6224e-03	2.4096e-03	6.0263e-04	1.5067e-04	3.7668e-05
Order	/	1.9976	1.9994	1.9999	2.0000

Table 4.4: The numerical errors and convergence rates of u at $t = 0.6$.

h	1/4	1/8	1/16	1/32	1/64
$\|u^n - u_h^n\|_1$	2.7568e-01	1.3810e-01	6.9089e-02	3.4550e-02	1.7276e-02
Order	/	0.99727	0.99917	0.99978	0.99995
$\|I_h u^n - u_h^n\|_1$	5.9412e-02	1.5540e-02	3.9278e-03	9.8463e-04	2.4632e-04
Order	/	1.9348	1.9842	1.9961	1.9990
$\|u^n - I_{2h} u_h^n\|_1$	1.2608e-01	3.2007e-02	8.0302e-03	2.0093e-03	5.0243e-04
Order	/	1.9779	1.9949	1.9987	1.9997

Table 4.5: The numerical errors and convergence rates of $\boldsymbol{\theta}$ at $t = 0.6$.

h	1/4	1/8	1/16	1/32	1/64
$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	8.9561e-02	4.4710e-02	2.2345e-02	1.1171e-02	5.5855e-03
Order	/	1.0023	1.0006	1.0002	1.0000
$\|\mathbf{\Pi}_h\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	5.0923e-03	1.2978e-03	3.2650e-04	8.1756e-05	2.0439e-05
Order	/	1.9722	1.9909	1.9977	2.0000
$\|\boldsymbol{\theta}^n - \mathbf{\Pi}_{2h}\boldsymbol{\theta}_h^n\|_0$	1.7793e-02	7.1161e-03	1.9246e-03	4.8721e-04	1.2217e-04
Order	/	1.3222	1.8865	1.9819	1.9956

Table 4.6: The numerical errors and convergence rates of ϕ at $t = 0.6$.

h	1/4	1/8	1/16	1/32	1/64
$\|\phi^n - \phi_h^n\|_0$	3.3092e-02	1.6611e-02	8.3133e-03	4.1576e-03	2.0789e-03
Order	/	0.99439	0.99861	0.99965	0.99991
$\|P_h\phi^n - \phi_h^n\|_0$	2.6009e-04	8.0563e-05	2.1171e-05	5.3616e-06	1.3477e-06
Order	/	1.6908	1.9280	1.9814	1.9921
$\|\phi^n - P_{2h}\phi_h^n\|_0$	5.8294e-03	1.4572e-03	3.6427e-04	9.1066e-05	2.2767e-05
Order	/	2.0002	2.0001	2.0000	2.0000

Table 4.7: The numerical errors and convergence rates of u at $t = 1.0$.

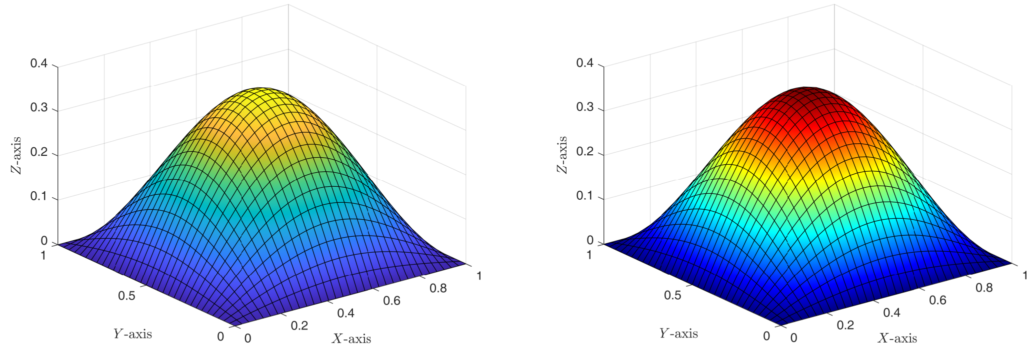
h	1/4	1/8	1/16	1/32	1/64
$\|u^n - u_h^n\|_1$	1.8479e-01	9.2570e-02	4.6312e-02	2.3159e-02	1.1580e-02
Order	/	0.99727	0.99917	0.99978	0.99994
$\|I_hu^n - u_h^n\|_1$	3.9913e-02	1.0434e-02	2.6371e-03	6.6106e-04	1.6538e-04
Order	/	1.9355	1.9843	1.9961	1.9990
$\|u^n - I_{2h}u_h^n\|_1$	8.4579e-02	2.1464e-02	5.3849e-03	1.3474e-03	3.3692e-04
Order	/	1.9784	1.9949	1.9988	1.9997

Table 4.8: The numerical errors and convergence rates of $\boldsymbol{\theta}$ at $t = 1.0$.

h	1/4	1/8	1/16	1/32	1/64
$\|\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	6.0120e-02	3.0039e-02	1.5017e-02	7.5079e-03	3.7539e-03
Order	/	1.0010	1.0003	1.0001	1.0000
$\|\mathbf{\Pi}_h\boldsymbol{\theta}^n - \boldsymbol{\theta}_h^n\|_0$	2.8156e-03	7.1442e-04	1.7938e-04	4.4890e-05	1.1219e-05
Order	/	1.9786	1.9937	1.9986	2.0004
$\|\boldsymbol{\theta}^n - \mathbf{\Pi}_{2h}\boldsymbol{\theta}_h^n\|_0$	1.2300e-02	3.7065e-03	9.5376e-04	2.3985e-04	6.0052e-05
Order	/	1.7305	1.9584	1.9915	1.9979

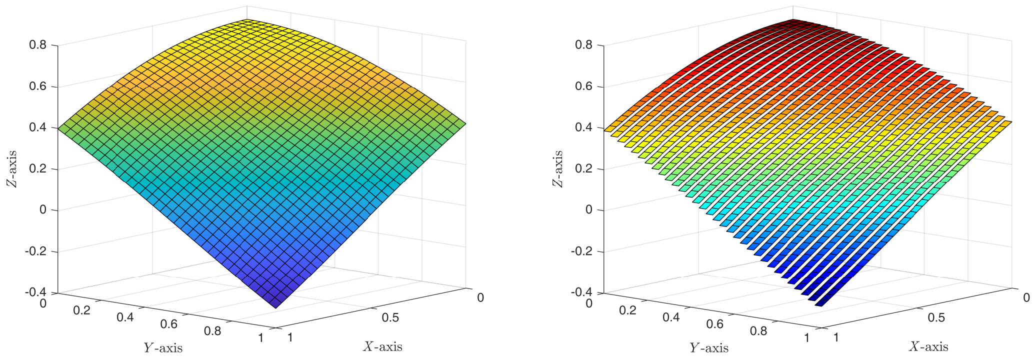
Table 4.9: The numerical errors and convergence rates of ϕ at $t = 1.0$.

h	1/4	1/8	1/16	1/32	1/64
$\|\phi^n - \phi_h^n\|_0$	2.2182e-02	1.1134e-02	5.5725e-03	2.7869e-03	1.3936e-03
Order	/	0.99438	0.99860	0.99965	0.99991
$\|P_h\phi^n - \phi_h^n\|_0$	1.3306e-04	3.8058e-05	9.8436e-06	2.4847e-06	6.2550e-07
Order	/	1.8058	1.9509	1.9861	1.9900
$\|\phi^n - P_{2h}\phi_h^n\|_0$	3.9058e-03	9.7595e-04	2.4396e-04	6.0988e-05	1.5247e-05
Order	/	2.0007	2.0002	2.0000	2.0000



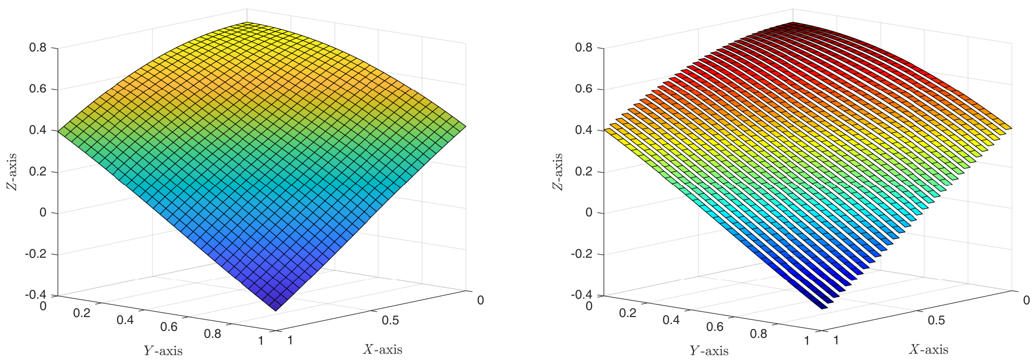
(a) Exact solution u^n (b) Numerical solution u_h^n

Fig. 4.1. The graphics of u^n and u_h^n at $t = 1.0$ on mesh 32×32.



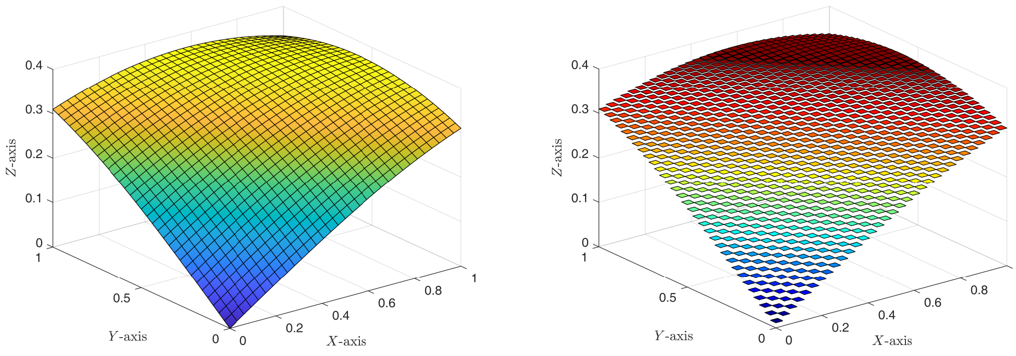
(a) Exact solution θ_1^n (b) Numerical solution θ_{1h}^n

Fig. 4.2. The graphics of θ_1^n and θ_{1h}^n at $t = 1.0$ on mesh 32×32.



(a) Exact solution θ_2^n (b) Numerical solution θ_{2h}^n

Fig. 4.3. The graphics of θ_2^n and θ_{2h}^n at $t = 1.0$ on mesh 32×32.



(a) Exact solution ϕ^n (b) Numerical solution ϕ_h^n

Fig. 4.4. The graphics of ϕ^n and ϕ_h^n at $t = 1.0$ on mesh 32×32.

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References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*, Academic Press, 2003.

[2] G. Akrivis and S. Larsson, Linearly implicit finite element methods for the time-dependent Joule heating problem, *BIT*, **45** (2005), 429–442.

[3] W. Allegretto, Y. Lin, and S. Ma, Existence and long time behaviour of solutions to obstacle thermistor equations, *Discrete Contin. Dyn. Syst.*, **8** (2002), 757–780.

[4] W. Allegretto and H. Xie, Existence of solutions for the time-dependent thermistor equations, *IMA J. Appl. Math.*, **48** (1992), 271–281.

[5] S. Antontsev and M. Chipot, The thermistor problem: Existence, smoothness uniqueness, blowup, *SIAM J. Math. Anal.*, **25**:4 (1994), 1128–1156.

[6] D.N. Arnold, Mixed finite element methods for elliptic problems, *Comput. Meth. Appl. Mech. Eng.*, **82**:1-3 (1990), 281–300.

[7] D.N. Arnold and R.S. Falk, A new mixed formulation for elasticity, *Numer. Math.*, **53** (1988), 13–30.

[8] C. Bahriawati and C. Carstensen, Three Matlab implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control, *Comput. Methods Appl. Math.*, **5** (2005), 333–361.

[9] S. Brenner and L. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2002.

[10] F. Brezzi, J. Douglas, and L.D. Marini, Two families of mixed nite elements for second order elliptic problems, *Numer. Math.*, **47** (1985), 217–235.

[11] W.T. Cai, J.L. Wang, and K. Wang, Convergence analysis of Crank-Nicolson Galerkin-Galerkin FEMs for miscible displacement in porous media, *J. Sci. Comput.*, **83** (2020), 25.

[12] G. Cimatti, Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, *Ann. Mat. Pura Appl.*, **162** (1992), 33–42.

[13] M. Crouzeix and P.A. Raviart, Conforming and nonconforming nite element methods for solving the stationary Stokes equations, *RAIRO Anal. Numer.*, **3** (1973), 33–76.

[14] V. Dolejsi and M. Feistauer, *Discontinouos Galerkin Method: Analysis and Application to Compressible Flow*, Springer, 2015.

[15] J. Douglas, Jr. and J.E. Roberts, Global estimates for mixed methods for second order elliptic equations, *Math. Comput.*, **44** (1985), 39–52.

[16] C.M. Elliott and S. Larsson, A finite element model for the time-dependent Joule heating problem, *Math. Comput.*, **64** (1995), 1433–1453.

[17] G. Fairweather, Q. Lin, Y.P. Lin, J.P. Wang, and S.H. Zhang, Asymptotic expansions and richardson extrapolation of approximate solutions for second order elliptic problems on rectangular domains by mixed finite element methods, *SIAM J. Numer. Anal.*, **44**:3 (2006), 1122–1149.

[18] M. Farhloul and M. Fortin, Review and complements on mixed-hybrid finite element methods for fluid flows, *J. Comput. Appl. Math.*, **140** (2002), 301–313.

[19] H.D. Gao, Optimal error analysis of Galerkin FEMs for nonlinear Joule heating equations, *J. Sci. Comput.*, **58** (2014), 627–647.

[20] H.D. Gao, Unconditional optimal error estimates of BDF-Galerkin FEMs for nonlinear thermistor equations, *J. Sci. Comput.*, **66** (2016), 504–527.

[21] H.D. Gao, B.Y. Li, and W.W. Sun, Stability and error estimates of fully discrete Galerkin FEMs for nonlinear thermistor equations in non-convex polygons, *Numer. Math.*, **136** (2017), 383–409.

[22] H.D. Gao, W.W. Sun, and C.D. Wu, Optimal error estimates and recovery technique of a mixed finite element method for nonlinear thermistor equations, *IMA J. Numer. Anal.*, **41** (2021), 3175–3200.

[23] V. Girault and P.A. Raviart, *Finite Element Methods for the Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag, 1986.

[24] X.P. Gui, B.Y. Li, and J.L. Wang, Convergence of renormalized finite element methods for heat flow of harmonic maps, *SIAM J. Numer. Anal.*, **60** (2022), 312–338.

[25] Y.N. He, A fully discrete stabilized finite element method for the tim-dependent Navier-Stokes problem, *IMA J. Numer. Anal.*, **23**:4 (2003), 665–691.

[26] Y.N. He, The Euler implicit/explicit scheme for the 2D time-dependent Navier-Stokes equations with smooth or non-smooth initial data, *Math. Comput.*, **77**:264 (2008), 2097–2124.

[27] Y.N. He and W.W. Sun, Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations, *SIAM J. Numer. Anal.*, **45**:2 (2007), 837–869.

[28] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. Part IV: Error analysis for the second order time disctetization, *SIAM J. Numer. Anal.*, **2** (1990), 353–384.

[29] M. Holst, M. Larson, A. Malqvist, and R. Soderlund, Convergence analysis of finite element approximations of the Joule heating problem in three spatial dimensions, *BIT*, **50** (2010), 781–795.

[30] Y.Q. Huang, J.C. Li, and Q. Lin, Superconvergence analysis for time-dependent Maxwell's equations in metamaterials, *Numer. Methods Partial Differential Equations*, **6** (2012), 1794–1816.

[31] S.H. Jia, D.L. Li, T. Bay, T. Liu, and S.H. Zhang, Richardson extrapolation and defect correction of mixed finite element methods for integro-differential equations in porous media, *Appl. Math.*, **53**:1 (2018), 13–39.

[32] B.Y. Li, H.D. Gao, and W.W. Sun, Unconditionally optimal error estimates of a Crank-Nicolson Galerkin method for the nonlinear thermistor equations, *SIAM J. Numer. Anal.*, **52** (2014), 933–954.

[33] B.Y. Li and W.W. Sun, Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations, *Int. J. Numer. Anal. Model.*, **10** (2013), 622–633.

[34] B.Y. Li and W.W. Sun, Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media, *SIAM J. Numer. Anal.*, **51** (2013), 1959–1977.

[35] B.Y. Li and W.W. Sun, Linearized FE approximations to a nonlinear gradient flow, *SIAM J. Numer. Anal.*, **52** (2014), 2623–2646.

[36] B.Y. Li, Y. Ueda, and G.Y. Zhou, A second-order stabilization method for linearizing and decoupling nonlinear parabolic systems, *SIAM J. Numer. Anal.*, **58** (2020), 2736–2763.

[37] Q. Lin and J.F. Lin, *Finite Element Methods: Accuracy and Improvement*, Beijing Science Press, 2006.

[38] Q. Lin and J.F. Lin, Superconvergence analysis for Maxwell's equations in dispersive media, *Math. Comput.*, **262** (2013), 757–771.

[39] Q. Lin, L. Tobiska, and A.H. Zhou, Superconvergence and extrapolation of non-conforming low order finite elements applied to the Poisson equation, *IMA J. Numer. Anal.*, 25 (2005), 160–181.

[40] Q. Lin and N. Yan, *The construction and analysis of high efficiency finite element methods*, Hebei University Press, 1996.

[41] Q. Lin, S. Zhang, and N. Yan, Asymptotic error expansion and defect correction for Sobolev and viscoelasticity type equations, *J. Comput. Math.*, **16** (1998), 57–62.

[42] H.P. Liu and N.N. Yan, Superconvergence analysis of the nonconforming quadrilateral linear-constant scheme for Stokes equations, *Adv. Comput. Math.*, **29** (2008), 375–392.

[43] P.A. Raviart and J.M. Thomas, A mixed finite element method for second order elliptic problems, in: *Mathematical Aspects of the Finite Element Method. Lecture Notes in Mathematics*, Springer, **606** (1977), 292–315.

[44] D.Y. Shi and M.H. Li, Superconvergence analysis of the stable conforming rectangular mixed finite elements for the linear elasticity problem, *J. Comput. Math.*, **32** (2014), 205–214.

[45] D.Y. Shi, S.P. Mao, and S.C. Chen, An anisotropic nonconforming finite element with some superconvergence results, *J. Comput. Math.*, **23** (2005) , 261–274.

[46] D.Y. Shi, F.L. Wang, M.Z. Fan, and Y.M. Zhao, A new approach of the lowest order anistropic mixed finite element high accuracy analysis for nonlinear sine-Gordon equations, *Math. Numer. Sin.*, **37**:2 (2015), 148–161.

[47] D.Y. Shi and J.J. Wang, Unconditional superconvergence analysis of a Crank–Nicolson Galerkin FEM for nonlinear Schrödinger equation, *J. Sci. Comput.*, **72** (2017), 1093–1118.

[48] D.Y. Shi, P.L. Wang, and Y.M. Zhao, Superconvergence analysis of anisotropic linear triangular finite element for nonlinear Schrödinger equation, *Appl. Math. Lett.*, **38** (2014), 129–134.

[49] D.Y. Shi and H.J. Yang, Superconvergent estimates of conforming finite element method for nonlinear time-dependent Joule heating equations, *Numer. Methods Partial Differential Equations*, **34**:1 (2017), 336–356.

[50] D.Y. Shi and H.J. Yang, Superconvergence analysis of finite element method for time-fractional thermistor problem, *Appl. Math. Comput.*, **323** (2018), 31–42.

[51] D.Y. Shi and H.J. Yang, Superconvergence analysis of nonconforming FEM for nonlinear time-dependent thermistor problem, *Appl. Math. Comput.*, **347** (2019), 210–224.

[52] X.Y. Shi, L.Z. Lu, and H.J. Wang, New superconvergence estimates of FEM for time-dependent Joule heating problem, *Comput. Math. Appl.*, **111** (2022), 91–97.

[53] W.W. Sun and C.D. Wu, New analysis of Galerkin-mixed FEMs for incompressible miscible flow in porous media, *Math. Comp.*, **90** (2021), 81–102.

[54] V. Thomee, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, 2006.

[55] J.L. Wang, Z.Y. Si, and W.W. Sun, A new error analysis of characteristics-mixed FEMs for miscible displacement in porous media, *SIAM J. Numer. Anal.*, **52** (2014), 3000–3020.

[56] X. Wu and X. Xu, Existence for the thermoelastic thermistor problem, *J. Math. Anal. Appl.*, **319** (2006), 124–138.

[57] H.J. Yang, D.Y. Shi, and Q. Liu, Superconvergence analysis of low order nonconforming mixed finite element methods for time-dependent Navier-Stokes equations, *J. Comput. Math.*, **39**:1 (2021), 63–80.

[58] C. Yao, J. Li, and D. Shi, Superconvergence analysis of nonconforming mixed finite element methods for time-dependent Maxwell's equations in isotropic cold plasma media, *Appl. Math. Comput.*, **12** (2013), 6466–6472.

[59] G. Yuan, Regularity of solutions of the thermistor problem, *Appl. Anal.*, **53** (1994), 149–156.

[60] G. Yuan and Z. Liu, Existence and uniqueness of the C^{α} solution for the thermistor problem with

mixed boundary value, *SIAM J. Math. Anal.*, **25** (1994), 1157–1166.

[61] X. Yue, Numerical analysis of nonstationary thermistor problem, *J. Comput. Math.*, **12** (1994), 213–223.

[62] W.D. Zhao, Convergence analysis of finite element method for the nonstationary thermistor problem, *Shandong Daxue Xuebao*, **29** (1994), 361–367.

[63] Y.M. Zhao, P. Chen, W.P. Bu, X.T. Liu, and Y.F. Tang, Two mixed finite element methods for time-fractional diffusion equations, *J. Sci. Comput.*, **70** (2017), 407–428.

[64] S. Zhou and D.R. Westbroo, Numerical solutions of the thermistor equations, *J. Comput. Appl. Math.*, **79**:1 (1997), 101–118.