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Robust Globally Divergence-Free Weak Galerkin Methods for Stationary Incompressible Convective Brinkman-Forchheimer Equations

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Abstract. This paper develops a class of robust weak Galerkin methods for stationary incompressible convective Brinkman-Forchheimer equations. The methods adopt piecewise polynomials of degrees m $(m \geq 1)$ and $m-1$ respectively for the approximations of velocity and pressure variables inside the elements and piecewise polynomials of degrees k $(k = m-1, m)$, and m respectively for their numerical traces on the interfaces of elements, and are shown to yield globally divergence-free velocity approximation. Existence and uniqueness results for the discrete schemes, as well as optimal a priori error estimates, are established. A convergent linearized iterative algorithm is also presented. Numerical experiments are provided to verify the performance of the proposed methods.

AMS subject classifications: 65M60, 65N30

Key words: Brinkman-Forchheimer equations, weak Galerkin method, divergence-free, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a Lipschitz polygonal/polyhedral domain. We consider the following stationary incompressible convective Brinkman-Forchheimer model:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + \alpha|\boldsymbol{u}|^{r-2}\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u} = (u_1, \cdots, u_n)^\top$ is the velocity vector, p the pressure, $\boldsymbol{f}$ a given forcing function, ν the Brinkman coefficient, $\alpha > 0$ the Forchheimer coefficient, and $2 \leq r < \infty$ when

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$n=2$ and $2\leq r\leq 6$ when $n=3$. The operator $\otimes$ is defined by $\boldsymbol{u}\otimes\boldsymbol{v}=(u_iv_j)_{n\times n}$ for $\boldsymbol{v}=(v_1,\cdots,v_n)^\top$.

The Brinkman-Forchheimer model, which can be viewed as the Navier-Stokes equations with a nonlinear damping term, is used to modelling fast flows in highly porous media [20, 43]. In recent years there have developed many numerical algorithms for Brinkman-Forchheimer equations, such as conforming mixed finite element methods [5, 6, 24, 31, 51], nonconforming mixed finite element methods [33], stabilized mixed methods [27,35], multi-level mixed methods [25,41,58,59], parallel finite element algorithms [48,49]. We refer to [4,7,13,18,23,26,32,39,52,57,61,62] for the study of the properties of weak/strong solutions to the Brinkman-Forchheimer equations.

It is well-known that the divergence constraint $\nabla\cdot\boldsymbol{u}=0$ corresponds to the conservation of mass for incompressible fluid flows, and that numerical methods with poor conservation usually suffer from instabilities [1,19,28,29,38]. Besides, the numerical schemes with exactly divergence-free velocity approximation may automatically lead to pressure-robustness in the sense that the velocity approximation error is independent of the pressure approximation [19,30,36]. We refer to [8,9,11,15,16,21,37,50,55,60] for some divergence-free finite element methods for the incompressible fluid flows.

In this paper we consider a robust globally divergence-free weak Galerkin (WG) finite element discretization of the Brinkman-Forchheimer model (1.1). The WG framework was first proposed in [44, 45] for second-order elliptic problems. It allows the use of totally discontinuous functions on meshes with arbitrary shape of polygons/polyhedra due to the introduction of weakly defined gradient/divergence operators over functions with discontinuity, and has the local elimination property, i.e. the unknowns defined in the interior of elements can be locally eliminated by using the numerical traces defined on the interfaces of elements. We refer to [8, 9, 12, 15–17, 22, 34, 36, 37, 40, 46, 47, 50, 53, 54, 56, 60] for developments and applications of WG methods for fluid flow problems and some other problems. Particularly, a class of robust globally divergence-free weak Galerkin methods were developed in [8] for Stokes equations, and later were extended to solve incompressible quasi-Newtonian Stokes equations [60], natural convection equations [15, 16] and incompressible Magnetohydrodynamics flow equations [56].

The goal of this contribution is to extend the WG methods of [8] to the discretization of the Brinkman-Forchheimer model. The main features of our WG discretization for the model (1.1) are as follows:

- The discretization scheme is arbitrary order, which adopts piecewise polynomials of degrees m $(m\geq 1)$ and $m-1$ to approximate the velocity and pressure inside the elements, respectively, and piecewise polynomials of degrees k $(k=m-1,m)$ and m to approximate the traces of velocity and pressure on the interfaces of elements, respectively.

- The scheme yields globally divergence-free velocity approximation, which automatically leads to pressure-robustness.

- The scheme is parameter-friendly, i.e. the stabilization parameter in the scheme does not require to be sufficiently large.
- The unknowns of the velocity and pressure in the interior of elements can be locally eliminated so as to obtain a reduced discrete system of smaller size.
- The well-posedness and optimal error estimates of the scheme are established.

The rest of this paper is organized as follows. Section 2 gives notations, weak formulations, the WG scheme and some preliminary results. Section 3 establishes the well-posedness of the discrete scheme. Section 4 is devoted to the a priori error analysis. Section 5 derives L^2 error estimate for the velocity. Section 6 shows the local elimination property and proposes an iteration algorithm for the nonlinear WG scheme. Section 7 provides several numerical experiments. Finally, Section 8 gives some concluding remarks.

2. Weak Galerkin finite element scheme

2.1. Notation and weak problem

For any bounded domain $\Lambda \subset \mathbb{R}^l$ $(l = n, n-1)$, nonnegative integer s and real number $1 \leq q < \infty$, let $W^{s,q}(\Lambda)$ and $W_0^{s,q}(\Lambda)$ be the usual Sobolev spaces defined on Λ with norm $\|\cdot\|_{s,q,\Lambda}$ and semi-norm $|\cdot|_{s,q,\Lambda}$. In particular, $H^s(\Lambda) := W^{s,2}(\Lambda)$ and $H_0^s(\Lambda) := W_0^{m,2}(\Lambda)$, with $\|\cdot\|_{s,\Lambda} := \|\cdot\|_{s,2,\Lambda}$ and $|\cdot|_{s,\Lambda} := |\cdot|_{s,2,\Lambda}$. We use $(\cdot,\cdot)_{s,\Lambda}$ to denote the inner product of $H^s(\Lambda)$, with $(\cdot,\cdot)_\Lambda := (\cdot,\cdot)_{0,\Lambda}$. When $\Lambda = \Omega$, we set $\|\cdot\|_s := \|\cdot\|_{s,\Omega}, |\cdot|_s := |\cdot|_{s,\Omega}$, and $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$. Especially, when $\Lambda \subset \mathbb{R}^{n-1}$ we use $\langle\cdot,\cdot\rangle_\Lambda$ to replace $(\cdot,\cdot)_\Lambda$. For a nonnegative integer m, let $P_m(\Lambda)$ be the set of all polynomials defined on Λ with degree no more than m. We also need the following Sobolev spaces:

$$
\begin{aligned}
&L_0^2(\Omega) := \left\{q \in L^2(\Omega) : (q,1) = 0\right\}, \\
&\boldsymbol{H}(\mathrm{div};\Lambda) := \left\{\boldsymbol{v} \in [L^2(\Lambda)]^n : \nabla\cdot\boldsymbol{v} \in L^2(\Lambda)\right\}.
\end{aligned}
$$

Let $\mathcal{T}_h$ be a shape regular partition of Ω into closed simplexes, and let $\mathcal{E}_h$ be the set of all edges (faces) of all the elements in Ω. For any $K \in \mathcal{T}_h$, $e \in \mathcal{E}_h$, we denote by h_K the diameter of K and by h_e the diameter of e, and set $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\boldsymbol{n}_K$ and $\boldsymbol{n}_e$ denote the outward unit normal vectors along the boundary ∂K and e, respectively. We may abbreviate $\boldsymbol{n}_K$ as $\boldsymbol{n}$ when there is no ambiguity. We use ∇_h and $\nabla_h\cdot$ to denote respectively the operators of piecewise-defined gradient and divergence with respect to the decomposition $\mathcal{T}_h$. For convenience, throughout the paper we use $x \lesssim y$ $(x \gtrsim y)$ to denote $x \leq Cy$ $(x \geq Cy)$, where C is a positive constant independent of the mesh size h.

We introduce the spaces

$$
\boldsymbol{V} := \left[H_0^1(\Omega)\right]^n, \quad Q := L_0^2(\Omega), \quad \boldsymbol{V}_0 := \{\boldsymbol{v} \in \boldsymbol{V} : \nabla\cdot\boldsymbol{v} = 0\},
$$

and define the following bilinear and trilinear forms: For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa} \in \boldsymbol{V}$ and $q \in Q$,

$$
\begin{aligned}
& a(\boldsymbol{u}, \boldsymbol{v}) := \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \quad b(\boldsymbol{v}, q) := -(q, \nabla \cdot \boldsymbol{v}), \\
& c(\boldsymbol{\kappa}; \boldsymbol{u}, \boldsymbol{v}) := \alpha(|\boldsymbol{\kappa}|^{r-2} \boldsymbol{u}, \boldsymbol{v}), \\
& d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}\big(\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), \boldsymbol{v}\big) - \frac{1}{2}\big(\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{u}), \boldsymbol{u}\big).
\end{aligned}
$$

We note that the trilinear form $d(\cdot;\cdot,\cdot)$ is the same as that in [15, 16].

Then the weak form of (1.1) is given as follows: Seek $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ such that

$$
\begin{aligned}
& a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = (f, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \\
& b(\boldsymbol{u}, q) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall q \in Q.
\end{aligned} \tag{2.1}
$$

Remark 2.1. As shown in [31, Theorem 1], the weak problem (2.1) admits at least one solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ when Ω is a bounded Lipschitz domain and $\boldsymbol{f} \in [H^{-1}(\Omega)]^n$, and there holds

$$
\|\nabla \boldsymbol{u}\|_0 \leq \frac{\|\boldsymbol{f}\|_*}{\nu}. \tag{2.2}
$$

In addition, if the smallness condition

$$
\frac{\mathcal{N}\|\boldsymbol{f}\|_*}{\nu^2} < 1 \tag{2.3}
$$

holds, then the solution of (2.1) is unique. Here

$$
\|\boldsymbol{f}\|_* := \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}_0} \frac{(\boldsymbol{f}, \boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_0}, \quad \mathcal{N} := \sup_{\boldsymbol{0} \neq \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\kappa} \in \boldsymbol{V}_0} \frac{d(\boldsymbol{\kappa}; \boldsymbol{u}, \boldsymbol{v})}{\|\nabla \boldsymbol{\kappa}\|_0 \|\nabla \boldsymbol{u}\|_0 \|\nabla \boldsymbol{v}\|_0}.
$$

2.2. WG scheme

In order to give the WG scheme to the system (1.1) we introduce, for integer $\gamma \geq 0$, the discrete gradient operator $\nabla_{w,\gamma}$ and the discrete weak divergence operator $\nabla_{w,\gamma}\cdot$ as follows.

Definition 2.1. *For all $K \in \mathcal{T}_h$ and $v \in \mathcal{V}(K) := \{v = \{v_i, v_b\} : v_i \in L^2(K), v_b \in H^{1/2}(\partial K)\}$, the discrete weak gradient $\nabla_{w,\gamma,K} v \in [P_\gamma(K)]^n$ of v on K is defined by*

$$
(\nabla_{w,\gamma,K} v, \boldsymbol{\varsigma})_K = -(v_i, \nabla \cdot \boldsymbol{\varsigma})_K + \langle v_b, \boldsymbol{\varsigma} \cdot \boldsymbol{n}_K \rangle_{\partial K}, \quad \forall \boldsymbol{\varsigma} \in [P_\gamma(K)]^n. \tag{2.4}
$$

Then the global discrete weak gradient operator $\nabla_{w,\gamma}$ is defined as

$$
\nabla_{w,\gamma}|_K := \nabla_{w,\gamma,K}, \quad \forall K \in \mathcal{T}_h.
$$

Moreover, for a vector $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^\top$ with $v_j|_K \in \mathcal{V}(K)$ for $j = 1, \ldots, n$, the discrete weak gradient $\nabla_{w,\gamma} \boldsymbol{v}$ is defined as

$$
\nabla_{w,\gamma} \boldsymbol{v} := (\nabla_{w,\gamma} v_1, \nabla_{w,\gamma} v_2, \cdots, \nabla_{w,\gamma} v_n)^\top.
$$

Definition 2.2. *For all $K \in \mathcal{T}_h$ and $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}(K) := \{\boldsymbol{w} = \{\boldsymbol{w}_i, \boldsymbol{w}_b\} : \boldsymbol{w}_i \in [L^2(K)]^n, \boldsymbol{w}_b \cdot \boldsymbol{n}_K \in H^{-1/2}(\partial K)\}$, the discrete weak divergence $\nabla_{w,\gamma,K} \cdot \boldsymbol{w} \in P_\gamma(K)$ of $\boldsymbol{w}$ on K is defined by*

$$(\nabla_{w,\gamma,K} \cdot \boldsymbol{w}, \varsigma)_K = -(\boldsymbol{w}_i, \nabla\varsigma)_K + \langle \boldsymbol{w}_b \cdot \boldsymbol{n}, \varsigma\rangle_{\partial K}, \quad \forall \varsigma \in P_\gamma(K). \tag{2.5}$$

Then the global discrete weak divergence operator $\nabla_{w,\gamma}\cdot$ is defined as

$$\nabla_{w,\gamma} \cdot |_K := \nabla_{w,\gamma,K} \cdot, \quad \forall K \in \mathcal{T}_h.$$

Moreover, for a tensor $\widetilde{\boldsymbol{w}} = (\boldsymbol{w}_1, \cdots, \boldsymbol{w}_n)^\top$ with $\boldsymbol{w}_j|_K \in \boldsymbol{\mathcal{W}}(K)$ for $j = 1, \ldots, n$, the discrete weak divergence $\nabla_{w,\gamma} \cdot \widetilde{\boldsymbol{w}}$ is defined as

$$\nabla_{w,\gamma} \cdot \widetilde{\boldsymbol{w}} := (\nabla_{w,\gamma} \cdot \boldsymbol{w}_1, \cdots, \nabla_{w,\gamma} \cdot \boldsymbol{w}_n)^\top.$$

For any $K \in \mathcal{T}_h, e \in \mathcal{E}_h$ and nonnegative integer j, let $\Pi_j^* : L^2(K) \to P_j(K)$ and $\Pi_j^B : L^2(e) \to P_j(e)$ be the usual L^2-projection operators. We shall adopt $\boldsymbol{\Pi}_j^*$ to denote Π_j^* for the vector form.

For any integer $m \geq 1$, and integer $k = m-1, m$, we introduce the following finite dimensional spaces:

$$\begin{aligned}
&\boldsymbol{V}_h := \big\{\boldsymbol{v}_h = \{\boldsymbol{v}_{hi}, \boldsymbol{v}_{hb}\} : \boldsymbol{v}_{hi}|_K \in [P_m(K)]^n, \boldsymbol{v}_{hb}|_e \in [P_k(e)]^n, \forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}_h\big\},\\
&\boldsymbol{V}_h^0 := \big\{\boldsymbol{v}_h = \{\boldsymbol{v}_{hi}, \boldsymbol{v}_{hb}\} \in \boldsymbol{V}_h : \boldsymbol{v}_{hb}|_{\partial\Omega} = \boldsymbol{0}\big\},\\
&Q_h := \big\{q_h = \{q_{hi}, q_{hb}\} : q_{hi}|_K \in P_{m-1}(K), q_{hb}|_e \in P_m(e), \forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}_h\big\},\\
&Q_h^0 := \big\{q_h = \{q_{hi}, q_{hb}\} \in Q_h, q_{hi} \in L_0^2(\Omega)\big\}.
\end{aligned}$$

For any $\boldsymbol{u}_h = \{\boldsymbol{u}_{hi}, \boldsymbol{u}_{hb}\}, \boldsymbol{v}_h = \{\boldsymbol{v}_{hi}, \boldsymbol{v}_{hb}\}, \boldsymbol{\kappa}_h = \{\boldsymbol{\kappa}_{hi}, \boldsymbol{\kappa}_{hb}\} \in \boldsymbol{V}_h^0$, and $p_h = \{p_{hi}, p_{hb}\} \in Q_h^0$, we shall define bilinear and trilinear terms as follows:

$$\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \nu(\nabla_{w,m-1}\boldsymbol{u}_h, \nabla_{w,m-1}\boldsymbol{v}_h) + s_h(\boldsymbol{u}_h, \boldsymbol{v}_h),\\
s_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \nu\left\langle \eta(\boldsymbol{\Pi}_k^B\boldsymbol{u}_{hi} - \boldsymbol{u}_{hb}), \boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi} - \boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h},\\
b_h(\boldsymbol{v}_h, q_h) &:= (\nabla_{w,m}q_h, \boldsymbol{v}_{hi}),\\
c_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) &:= \alpha(|\boldsymbol{\kappa}_{hi}|^{r-2}\boldsymbol{u}_{hi}, \boldsymbol{v}_{hi}),\\
d_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) &:= \frac{1}{2}\big(\nabla_{w,m} \cdot \{\boldsymbol{u}_{hi} \otimes \boldsymbol{\kappa}_{hi}, \boldsymbol{u}_{hb} \otimes \boldsymbol{\kappa}_{hb}\}, \boldsymbol{v}_{hi}\big)\\
&\quad - \frac{1}{2}\big(\nabla_{w,m} \cdot \{\boldsymbol{v}_{hi} \otimes \boldsymbol{\kappa}_{hi}, \boldsymbol{v}_{hb} \otimes \boldsymbol{\kappa}_{hb}\}, \boldsymbol{u}_{hi}\big),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial K}$, and the stabilization parameter $\eta|_{\partial K} = h_K^{-1}, \forall K \in \mathcal{T}_h$.

In what follows we assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Based on the above definitions, the WG scheme for (1.1) reads: Seek $\boldsymbol{u}_h = \{\boldsymbol{u}_{hi}, \boldsymbol{u}_{hb}\} \in \boldsymbol{V}_h^0$, $p_h = \{p_{hi}, p_{hb}\} \in Q_h^0$ such that

$$\begin{aligned}
&a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\\
&\quad + d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_{hi}), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^0, \qquad (2.6a)\\
&b_h(\boldsymbol{u}_h, q_h) = 0, \qquad \forall q_h \in Q_h^0. \qquad (2.6b)
\end{aligned}$$

The following theorem shows that the scheme (2.6) yields globally divergence-free velocity approximation.

Theorem 2.1. *Let $\boldsymbol{u}_h = \{\boldsymbol{u}_{hi}, \boldsymbol{u}_{hb}\} \in \boldsymbol{V}_h^0$ be the velocity solution of the WG scheme* (2.6). *Then there hold*

$$\boldsymbol{u}_{hi} \in \boldsymbol{H}(\mathrm{div};\Omega), \quad \nabla \cdot \boldsymbol{u}_{hi} = 0. \tag{2.7}$$

Proof. Define a function $\varphi_{hb} \in L^2(\mathcal{E}_h)$ as follows: For any $e \in \mathcal{E}_h$,

$$\varphi_{hb}|_e := \begin{cases} -\big((\boldsymbol{u}_{hi}\cdot\boldsymbol{n}_e)|_{K_1}\big)|_e - \big((\boldsymbol{u}_{hi}\cdot\boldsymbol{n}_e)|_{K_2}\big)|_e, & \text{if} \quad e = K_1 \cap K_2, \quad K_1, K_2 \in \mathcal{T}_h, \\ 0, \quad \forall e \subset \partial\Omega. \end{cases}$$

Let

$$\varphi_0 := \frac{1}{|\Omega|}\int_\Omega \nabla_h \cdot \boldsymbol{u}_{hi}\, d\boldsymbol{x}.$$

Taking $q_{hi} = \nabla_h \cdot \boldsymbol{u}_{hi} - \varphi_0, q_{hb} = \varphi_{hb} - \varphi_0$ in (2.6b), we obtain

$$\begin{aligned} 0 &= -(\boldsymbol{u}_{hi}, \nabla_{w,m} q_h) \\ &= (\nabla_h \cdot \boldsymbol{u}_{hi}, q_{hi}) - \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{u}_{hi}\cdot\boldsymbol{n}, q_{hb}\rangle_{\partial K} \\ &= (\nabla_h \cdot \boldsymbol{u}_{hi}, \nabla_h \cdot \boldsymbol{u}_{hi} - \varphi_0) - \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{u}_{hi}\cdot\boldsymbol{n}, \varphi_{hb} - \varphi_0\rangle_{\partial K} \\ &= (\nabla_h \cdot \boldsymbol{u}_{hi}, \nabla_h \cdot \boldsymbol{u}_{hi}) - \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{u}_{hi}\cdot\boldsymbol{n}, \varphi_{hb}\rangle_{\partial K} \\ &= \|\nabla_h \cdot \boldsymbol{u}_{hi}\|_0^2 + \sum_{e\in\mathcal{E}_h, e\not\subseteq\partial\Omega} \|(\boldsymbol{u}_{hi}\cdot\boldsymbol{n}_e)|_{K_1} + (\boldsymbol{u}_{hi}\cdot\boldsymbol{n}_e)|_{K_2}\|_{0,e}^2, \end{aligned}$$

which indicates the desired conclusion (2.7). □

2.3. Preliminary results

We first introduce two semi-norms $|||\cdot|||_V$ and $|||\cdot|||_Q$ on the spaces $\boldsymbol{V}_h$ and Q_h, respectively, as follows:

$$\begin{aligned} |||\boldsymbol{v}_h|||_V^2 &:= \|\nabla_{w,m-1}\boldsymbol{v}_h\|_0^2 + \left\|\eta^{\frac{1}{2}}\big(\boldsymbol{\Pi}_k^B \boldsymbol{v}_{hi} - \boldsymbol{v}_{hb}\big)\right\|_{0,\partial\mathcal{T}_h}^2, \quad && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ |||q_h|||_Q^2 &:= \|q_{hi}\|_0^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla_{w,m} q_h\|_{0,K}^2, && \forall q_h \in Q_h, \end{aligned}$$

where

$$\|\cdot\|_{0,\partial\mathcal{T}_h} := \Big(\sum_{K\in\mathcal{T}_h} \|\cdot\|_{0,\partial K}^2\Big)^{\frac{1}{2}},$$

and we recall that $\eta|_{\partial K} = h_K^{-1}$. It is easy to see that $|||\cdot|||_V$ and $|||\cdot|||_Q$ are norms on $\boldsymbol{V}_h^0$ and Q_h^0, respectively (cf. [8, Lemma 3.3]).

The following lemma follows from the trace theorem, the inverse inequality and scaling arguments (cf. [16, Lemma 3.2], [42, Theorem 3.4.1]).

Lemma 2.1. *For all $K \in \mathcal{T}_h, \omega \in H^1(K)$, and $1 \le q \le \infty$, there holds*

$$\|\omega\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}} \|\omega\|_{0,q,K} + h_K^{1-\frac{1}{q}} |\omega|_{1,q,K}.$$

In particular, for all $\omega \in P_m(K)$,

$$\|\omega\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}} \|\omega\|_{0,q,K}.$$

For the projections Π_j^* and Π_j^B with $j \ge 0$, the following approximation and stability results are standard.

Lemma 2.2 (cf. [42, Lemmas 3.5.4-3.5.5]). *For $\forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}_h$ and $1 \le l \le j+1$, there hold*

$$\begin{aligned}
&\|\omega - \Pi_j^* \omega\|_{0,K} + h_K \big|\omega - \Pi_j^* \omega|_{1,K} \lesssim h_K^l |\omega\big|_{l,K}, && \forall \omega \in H^l(K),\\
&\|\omega - \Pi_j^* \omega\|_{0,\partial K} + \big\|\omega - \Pi_j^B \omega\big\|_{0,\partial K} \lesssim h_K^{l-\frac{1}{2}} |\omega|_{l,K}, && \forall \omega \in H^l(K),\\
&\big\|\Pi_j^* \omega\big\|_{0,K} \le \|\omega\|_{0,K}, && \forall v \in L^2(K),\\
&\big\|\Pi_j^B \omega\big\|_{0,e} \le \|\omega\|_{0,e}, && \forall v \in L^2(e).
\end{aligned}$$

In view of the definitions of the discrete weak gradient operator, the Green's formula, the projection operator, the Cauchy-Schwarz inequality, the inverse inequality and the trace inequality, the following lemma holds (cf. [8, Lemma 3.2]).

Lemma 2.3. *For any $K \in \mathcal{T}_h$ and $\boldsymbol{\omega}_h = \{\boldsymbol{\omega}_{hi}, \boldsymbol{\omega}_{hb}\} \in [P_m(K)]^n \times [P_k(\partial K)]^n$ with $0 \le m-1 \le k \le m$, there hold*

$$\begin{aligned}
&\|\nabla \boldsymbol{\omega}_{hi}\|_{0,K} \lesssim \|\nabla_{w,m-1} \boldsymbol{\omega}_h\|_{0,K} + h_K^{-\frac{1}{2}} \big\|\boldsymbol{\Pi}_k^B \boldsymbol{\omega}_{hi} - \boldsymbol{\omega}_{hb}\big\|_{0,\partial K},\\
&\|\nabla_{w,m-1} \boldsymbol{\omega}_h\|_{0,K} \lesssim \|\nabla \boldsymbol{\omega}_{hi}\|_{0,K} + h_K^{-\frac{1}{2}} \big\|\boldsymbol{\Pi}_k^B \boldsymbol{\omega}_{hi} - \boldsymbol{\omega}_{hb}\big\|_{0,\partial K}.
\end{aligned} \tag{2.8}$$

By the definition of the norm $|||\cdot|||_V$, we further have the following conclusion (cf. [16, (3.3)], [56, Lemma 3.3]).

Lemma 2.4. *For any $\boldsymbol{v}_h \in \boldsymbol{V}_h^0$, there hold*

$$\|\nabla_h \boldsymbol{v}_{hi}\|_0 \lesssim |||\boldsymbol{v}_h|||_V \tag{2.9}$$

and

$$\|\boldsymbol{v}_{hi}\|_{0,r} \le C_{\widetilde{r}} |||\boldsymbol{v}_h|||_V \tag{2.10}$$

for r satisfying

$$\begin{cases} 2 \le r < \infty, & \text{if} \quad n = 2, \\ 2 \le r \le 6, & \text{if} \quad n = 3, \end{cases}$$

where $C_{\tilde{r}} > 0$ is a positive constant only depending on r.

For any integer $j \ge 0$, we introduce the local Raviart-Thomas (RT) element space

$$\boldsymbol{RT}_j(K) = [P_j(K)]^n + \boldsymbol{x} P_j(K), \quad \forall K \in \mathcal{T}_h$$

and the RT projection operator $\boldsymbol{P}_j^{RT} : [H^1(K)]^n \to \boldsymbol{RT}_j(K)$ (cf. [3]) defined by

$$\begin{aligned} &\left\langle \boldsymbol{P}_j^{RT} \boldsymbol{\omega} \cdot \boldsymbol{n}_e, \sigma \right\rangle_e = \langle \boldsymbol{\omega} \cdot \boldsymbol{n}_e, \sigma \rangle_e, \quad \forall \sigma \in P_j(e), \ \ e \in \mathcal{E}_h, \ \ e \subset \partial K \ \text{ for } \ j \ge 0, && (2.11) \\ &\left(\boldsymbol{P}_j^{RT} \boldsymbol{\omega}, \boldsymbol{\sigma} \right)_K = (\boldsymbol{\omega}, \boldsymbol{\sigma})_K, \qquad \forall \boldsymbol{\sigma} \in [P_{j-1}(K)]^n \qquad \text{for } \ j \ge 1. && (2.12) \end{aligned}$$

The following lemmas show some properties of $\boldsymbol{P}_j^{RT}$.

Lemma 2.5 (cf. [3, Lemma 3.1]). *For any $\boldsymbol{\omega}_{hi} \in \boldsymbol{RT}_j(K)$, the relation $\nabla \cdot \boldsymbol{\omega}_{hi}|_K = 0$ implies $\boldsymbol{\omega}_{hi} \in [P_j(K)]^n$.*

Lemma 2.6 (cf. [3, Theorem 3.1, Lemma 3.5]). *For any $K \in \mathcal{T}_h$, the following properties hold:*

$$\begin{aligned} &\left(\nabla \cdot \boldsymbol{P}_j^{RT} \boldsymbol{\omega}, q_h \right)_K = (\nabla \cdot \boldsymbol{\omega}, q_h)_K, \quad \forall \boldsymbol{\omega} \in [H^1(K)]^n, \quad q_h \in P_j(K), && (2.13) \\ &\left\| \boldsymbol{\omega} - \boldsymbol{P}_j^{RT} \boldsymbol{\omega} \right\|_{0,K} \lesssim h_K^l |\boldsymbol{\omega}|_{l,K}, \qquad \forall \boldsymbol{\omega} \in [H^l(K)]^n, \quad \forall 1 \le l \le j+1. && (2.14) \end{aligned}$$

Lemma 2.7 (cf. [16, Lemma 4.2]). *For any $K \in \mathcal{T}_h$, $\boldsymbol{\omega} \in [H^l(K)]^n$ and $1 \le l \le j+1$, the following estimates hold:*

$$\begin{aligned} \left| \boldsymbol{\omega} - P_j^{RT} \boldsymbol{\omega} \right|_{1,K} &\lesssim h_K^{l-1} |\boldsymbol{\omega}|_{l,K}, \\ \left| \boldsymbol{\omega} - P_j^{RT} \boldsymbol{\omega} \right|_{0,\partial K} &\lesssim h_K^{l-\frac{1}{2}} |\boldsymbol{\omega}|_{l,K}, \\ \left| \boldsymbol{\omega} - P_j^{RT} \boldsymbol{\omega} \right|_{0,3,K} &\lesssim h_K^{l-\frac{n}{6}} |\boldsymbol{\omega}|_{l,K}, \\ \left| \boldsymbol{\omega} - P_j^{RT} \boldsymbol{\omega} \right|_{0,3,\partial K} &\lesssim h_K^{l-\frac{1}{3}-\frac{n}{6}} |\boldsymbol{\omega}|_{l,K}. \end{aligned}$$

We have the following commutativity properties for the RT projection, the L^2 projections and the discrete weak operators.

Lemma 2.8 (cf. [8, Lemma 3.7]). *For $m \ge 1$, there hold*

$$\begin{aligned} &\nabla_{w,m-1} \left\{ \boldsymbol{P}_m^{RT} \boldsymbol{\omega}, \boldsymbol{\Pi}_k^B \boldsymbol{\omega} \right\} = \boldsymbol{\Pi}_{m-1}^* (\nabla \boldsymbol{\omega}), \quad \forall \boldsymbol{\omega} \in [H^1(\Omega)]^n, \quad k = m, m-1, \\ &\nabla_{w,m} \left\{ \Pi_{m-1}^* q, \Pi_m^B q \right\} = \boldsymbol{\Pi}_m^* (\nabla q), \qquad \forall q \in H^1(\Omega). \end{aligned}$$

Finally, we give several inequalities to be used later (cf. [2, Lemma 4.1], [10, Theorem 5.3.3], [25, Lemma 1]).

Lemma 2.9. *For any $\lambda, \mu \in \mathbb{R}^n$, there hold*

$$\begin{aligned}
&\big||\lambda|^{r-2} - |\mu|^{r-2}\big| \leq (r-2)(|\lambda|^{r-3} + |\mu|^{r-3})|\lambda - \mu|,\\
&\big||\lambda|^{r-2}\lambda - |\mu|^{r-2}\mu\big| \leq C_r(|\lambda| + |\mu|)^{r-2}|\lambda - \mu|,\\
&\big(|\lambda|^{r-2}\lambda - |\mu|^{r-2}\mu\big)\cdot(\lambda-\mu) \gtrsim |\lambda - \mu|^r
\end{aligned}$$

for $r \geq 2$, and

$$\begin{aligned}
&\big||\lambda|^{r-2} - |\mu|^{r-2} - (r-2)|\mu|^{r-4}\mu\cdot(\lambda-\mu)\big|\\
\leq &\frac{(r-1)(r-2)}{2}(|\lambda|^{r-4} + |\mu|^{r-4})|\lambda-\mu|^2
\end{aligned}$$

for $r \geq 4$, where $|\cdot|$ denotes the Euclid norm and C_r is a positive constant, depending only on r, to be used in the latter analysis.

3. Well-posedness of discrete scheme

Lemmas 3.1 and 3.2 give some stability conditions for the discrete scheme (2.6).

Lemma 3.1. *For any $\boldsymbol{\kappa}_h = \{\boldsymbol{\kappa}_{hi}, \boldsymbol{\kappa}_{hb}\}, \boldsymbol{u}_h = \{\boldsymbol{u}_{hi}, \boldsymbol{u}_{hb}\}, \boldsymbol{v}_h = \{\boldsymbol{v}_{hi}, \boldsymbol{v}_{hb}\} \in \boldsymbol{V}_h^0$, there hold*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \lesssim \nu|||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V, \tag{3.1}$$
$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) = \nu|||\boldsymbol{v}_h|||_V^2, \tag{3.2}$$
$$c_h(\boldsymbol{v}_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = \alpha\|\boldsymbol{v}_{hi}\|_{0,r}^r, \tag{3.3}$$
$$c_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \leq \alpha C_{\widetilde{r}}^r|||\boldsymbol{\kappa}_h|||_V^{r-2}|||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V, \tag{3.4}$$
$$d_h(\boldsymbol{\kappa}_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = 0, \tag{3.5}$$
$$d_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \lesssim |||\boldsymbol{\kappa}_h|||_V \cdot |||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V, \tag{3.6}$$

where $C_{\widetilde{r}}$ is the same as in (2.10).

Proof. According to the definition of $a_h(\cdot,\cdot)$, the Cauchy-Schwarz inequality and Lemma 2.4, we easily get (3.1)-(3.2). The results (3.3) and (3.5) follow from the definitions of $c_h(\cdot;\cdot,\cdot)$ and $d_h(\cdot;\cdot,\cdot)$, respectively. From the definition of $c_h(\cdot;\cdot,\cdot)$, the Hölder's inequality and Lemma 2.4 we obtain

$$c_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \leq \alpha\|\boldsymbol{\kappa}_{hi}\|_{0,r}^{r-2}\|\boldsymbol{u}_{hi}\|_{0,r}\|\boldsymbol{v}_{hi}\|_{0,r} \leq C_{\widetilde{r}}^r\alpha|||\boldsymbol{\kappa}_h|||_V^{r-2}|||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V,$$

i.e. (3.4) holds. The inequality (3.6) has been proved in [16, Lemma 3.10]. ☐

We also have the following discrete inf-sup inequality.

Lemma 3.2 ([8, Theorem 3.1])**.** *There holds*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h^0} \frac{b_h(\boldsymbol{v}_h, p_h)}{|||\boldsymbol{v}_h|||_V} \gtrsim |||p_h|||_Q, \quad \forall p_h \in Q_h^0.$$

Denote

$$\boldsymbol{V}_{0h} := \left\{ \boldsymbol{\kappa}_h \in \boldsymbol{V}_h^0 : b_h(\boldsymbol{\kappa}_h, q_h) = 0, \forall q_h \in Q_h^0 \right\}.$$

From the proof of Theorem 2.1 we easily see that

$$\boldsymbol{V}_{0h} = \left\{ \boldsymbol{\kappa}_h \in \boldsymbol{V}_h^0 : \boldsymbol{\kappa}_{hi} \in \boldsymbol{H}(\mathrm{div}; \Omega), \nabla \cdot \boldsymbol{\kappa}_{hi} = 0 \right\}.$$

To prove the existence of solutions to the scheme (2.6), we introduce the following auxiliary system: Find $\boldsymbol{u}_h \in \boldsymbol{V}_{0h}$ such that

$$\mathcal{B}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_{hi}), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}, \tag{3.7}$$

where the trilinear form $\mathcal{B}_h(\cdot; \cdot, \cdot) : \boldsymbol{V}_{0h} \times \boldsymbol{V}_{0h} \times \boldsymbol{V}_{0h} \to \mathbb{R}$ is defined by

$$\mathcal{B}_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + d_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)$$

for any $\boldsymbol{\kappa}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_{0h}$.

We have the following equivalence result.

Lemma 3.3. *The discrete problems* (2.6) *and* (3.7) *are equivalent in the sense that both* (i) *and* (ii) *hold:*

(i) If $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^0 \times Q_h^0$ *solves* (2.6)*, then* $\boldsymbol{u}_h \in \boldsymbol{V}_{0h}$ *solves* (3.7)*.*

(ii) If $\boldsymbol{u}_h \in \boldsymbol{V}_{0h}$ *solves* (3.7)*, then* $(\boldsymbol{u}_h, p_h)$ *solves* (2.6)*, where* $p_h \in Q_h^0$ *is determined by*

$$\begin{aligned} b_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_{hi}) - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\ - d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^0. \end{aligned} \tag{3.8}$$

Define

$$\mathcal{N}_h := \sup_{\boldsymbol{0} \neq \boldsymbol{\kappa}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{d_h(\boldsymbol{\kappa}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)}{|||\boldsymbol{\kappa}_h|||_V \cdot |||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V}, \quad \|\boldsymbol{f}\|_{*,h} := \sup_{\boldsymbol{0} \neq \boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{(\boldsymbol{f}, \boldsymbol{v}_{hi})}{|||\boldsymbol{v}_h|||_V}.$$

From (3.6) and Lemma 2.4 we easily know that

$$\mathcal{N}_h \lesssim 1, \quad \|\boldsymbol{f}\|_{*,h} \lesssim \|\boldsymbol{f}\|_0.$$

Based on Lemma 3.3, we can obtain the following existence and boundedness results for the WG method.

Theorem 3.1. *The WG scheme* (2.6) *admits at least a solution pair* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^0 \times Q_h^0$ *and there hold*

$$|||\boldsymbol{u}_h|||_V \leq \frac{1}{\nu}\|\boldsymbol{f}\|_{*,h}, \tag{3.9}$$

$$|||p_h|||_Q \lesssim \|\boldsymbol{f}\|_{*,h} + \|\boldsymbol{f}\|_{*,h}^2 + \|\boldsymbol{f}\|_{*,h}^{r-1}. \tag{3.10}$$

Proof. We first show the problem (3.7) admits at least one solution $\boldsymbol{u}_h \in \boldsymbol{V}_{0h}$. According to [14, Theorem 1.2], it suffices to show that the following two results hold:

(I) $\mathcal{B}_h(\boldsymbol{v}_h; \boldsymbol{v}_h, \boldsymbol{v}_h) \geq \nu|||\boldsymbol{v}_h|||_V^2, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}$;

(II) $\boldsymbol{V}_{0h}$ is separable, and the relation $\lim_{l\to\infty} \boldsymbol{u}_h^{(l)} = \boldsymbol{u}_h$ (weakly in $\boldsymbol{V}_{0h}$) implies

$$\lim_{l\to\infty} \mathcal{B}_h\left(\boldsymbol{u}_h^{(l)}; \boldsymbol{u}_h^{(l)}, \boldsymbol{v}_h\right) = \mathcal{B}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

In fact, (I) follows from Lemma 3.1 directly. We only need to show (II). Since $\boldsymbol{V}_{0h}$ is a finite dimensional space, we know that $\boldsymbol{V}_{0h}$ is separable and that the weak convergence $\lim_{l\to\infty} \boldsymbol{u}_h^{(l)} = \boldsymbol{u}_h$ on $\boldsymbol{V}_{0h}$ is equivalent to the strong convergence

$$\lim_{l\to\infty} \left|\left|\left|\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h\right|\right|\right|_V = 0. \tag{3.11}$$

On the other hand, by Lemmas 2.4, 2.9, 3.1 and the definition of $\mathcal{N}_h$, we have

$$\begin{aligned}
&\left|\mathcal{B}_h\left(\boldsymbol{u}_h^{(l)}; \boldsymbol{u}_h^{(l)}, \boldsymbol{v}_h\right) - \mathcal{B}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\right| \\
=& \left|a_h\left(\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h, \boldsymbol{v}_h\right) + \left(c_h(\boldsymbol{u}_h^{(l)}; \boldsymbol{u}_h^{(l)}, \boldsymbol{v}_h) - c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\right)\right. \\
&\left. + d_h\left(\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h; \boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h, \boldsymbol{v}_h\right) + d_h\left(\boldsymbol{u}_h; \boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h, \boldsymbol{v}_h\right) + d_h\left(\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h\right)\right| \\
\leq& \nu\left|\left|\left|\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h\right|\right|\right|_V \cdot |||\boldsymbol{v}_h|||_V + C_{\tilde{r}}^r C_r \alpha \left|\left|\left|\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h\right|\right|\right|_V \left(\left|\left|\left|\boldsymbol{u}_h^{(l)}\right|\right|\right|_V + |||\boldsymbol{u}_h|||_V\right)^{r-2} |||\boldsymbol{v}_h|||_V \\
&+ \mathcal{N}_h \left|\left|\left|\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h\right|\right|\right|_V^2 |||\boldsymbol{v}_h|||_V + 2\mathcal{N}_h \left|\left|\left|\boldsymbol{u}_h^{(l)} - \boldsymbol{u}_h\right|\right|\right|_V \cdot |||\boldsymbol{u}_h|||_V \cdot |||\boldsymbol{v}_h|||_V,
\end{aligned}$$

which, together with (3.11), yields

$$\lim_{l\to\infty} \mathcal{B}_h\left(\boldsymbol{u}_h^{(l)}; \boldsymbol{u}_h^{(l)}, \boldsymbol{v}_h\right) = \mathcal{B}_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h},$$

i.e. (II) holds. Hence, (3.7) has at least one solution $\boldsymbol{u}_h \in \boldsymbol{V}_{0h}$.

For a given $\boldsymbol{u}_h \in \boldsymbol{V}_{0h} \subset \boldsymbol{V}_h^0$, by Lemma 3.2 there is a unique $p_h \in Q_h^0$ satisfying (3.8). Thus, in light of Lemma 3.3 we know that $(\boldsymbol{u}_h, p_h)$ is a solution of (2.6). Taking $\boldsymbol{v}_h = \boldsymbol{u}_h, q_h = p_h$ in (2.6) and using Lemma 3.1, we immediately get

$$\nu|||\boldsymbol{u}_h|||_V^2 + \alpha\|\boldsymbol{u}_{hi}\|_{L^r}^r = (\boldsymbol{f}, \boldsymbol{u}_{hi}) \leq \|\boldsymbol{f}\|_{*,h}|||\boldsymbol{u}_h|||_V,$$

which implies (3.9). Finally, the estimate (3.10) follows from Lemmas 2.4, 3.1 and 3.2, the Eq. (3.8) and the estimate (3.9). This finishes the proof. □

Furthermore, we have the following uniqueness result.

Theorem 3.2. *Assume that the smallness condition*

$$\frac{\mathcal{N}_h}{\nu^2}\|\boldsymbol{f}\|_{*,h} < 1 \tag{3.12}$$

holds, then the scheme (2.6) *admits a unique solution* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^0 \times Q_h^0$.

Proof. Let $(\boldsymbol{u}_{h1}, p_{h1})$ and $(\boldsymbol{u}_{h2}, p_{h2})$ be two solutions of (2.6), i.e. for $j = 1, 2$ and $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h^0 \times Q_h^0$ there hold

$$\begin{aligned}
&a_h(\boldsymbol{u}_{hj}, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_{hj}; \boldsymbol{u}_{hj}, \boldsymbol{v}_h) \\
&\quad + d_h(\boldsymbol{u}_{hj}; \boldsymbol{u}_{hj}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_{hj}, p_{hj}) = (\boldsymbol{f}, \boldsymbol{v}_{hi}), \\
&b_h(\boldsymbol{u}_{hj}, q_h) = 0,
\end{aligned}$$

which give

$$\begin{aligned}
&a_h(\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{v}_h) \\
&\quad - c_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h2}, \boldsymbol{v}_h) - b_h(\boldsymbol{v}_h, p_{h1} - p_{h2}) \\
&= d_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h2}, \boldsymbol{v}_h) - d_h(\boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{v}_h), \qquad (3.13\text{a}) \\
&b_h(\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}, q_h) = 0. \qquad (3.13\text{b})
\end{aligned}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}$ and $q_h = p_{h1} - p_{h2}$ in the above two equations and using the relation

$$d_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) = 0$$

due to (3.5), we obtain

$$\begin{aligned}
&a_h(\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) + c_h(\boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) \\
&\quad - c_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h2}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) \\
&= d_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h2}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) - d_h(\boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) \\
&= d_h(\boldsymbol{u}_{h2} - \boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}).
\end{aligned}$$

This relation, together with (3.2), (3.6), (3.9), and the inequality

$$c_h(\boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) - c_h(\boldsymbol{u}_{h2}; \boldsymbol{u}_{h2}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) \gtrsim \|\boldsymbol{u}_{hi1} - \boldsymbol{u}_{hi2}\|_{0,r}^r \geq 0$$

due to Lemma 2.9, implies

$$\begin{aligned}
\nu |||\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}|||_V^2 &\leq d_h(\boldsymbol{u}_{h2} - \boldsymbol{u}_{h1}; \boldsymbol{u}_{h1}, \boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}) \\
&\leq \mathcal{N}_h |||\boldsymbol{u}_{h1}|||_V |||\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}|||_V^2 \\
&\leq \nu^{-1} \mathcal{N}_h \|\boldsymbol{f}\|_{*,h} |||\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}|||_V^2,
\end{aligned}$$

i.e.

$$\left(1 - \frac{\mathcal{N}_h}{\nu^2}\|\boldsymbol{f}\|_{*,h}\right) |||\boldsymbol{u}_{h1} - \boldsymbol{u}_{h2}|||_V^2 \leq 0.$$

This inequality plus the assumption (3.12) yields $\boldsymbol{u}_{h1} = \boldsymbol{u}_{h2}$. Then by (3.13a) we have $b_h(\boldsymbol{v}_h, p_{h1} - p_{h2}) = 0$ which, together with Lemma 3.2, leads to $p_{h1} = p_{h2}$. This finishes the proof. □

4. A priori error estimates

This section is devoted to the error analysis for the WG method (2.6). To this end, we first assume that the weak solution $(\boldsymbol{u}, p)$ of (1.1) satisfies the following regularity conditions for $m \geq 1$:

$$\boldsymbol{u} \in \left[H^{m+1}(\Omega)\right]^n \cap \boldsymbol{V}, \quad p \in H^m(\Omega) \cap L_0^2(\Omega). \tag{4.1}$$

Define

$$\begin{aligned} \boldsymbol{\mathcal{I}}_h \boldsymbol{u}|_K &:= \left\{\boldsymbol{P}_m^{RT}(\boldsymbol{u}|_K), \boldsymbol{\Pi}_k^B(\boldsymbol{u}|_K)\right\}, \\ \mathcal{P}_h p|_K &:= \left\{\Pi_{m-1}^*(p|_K), \Pi_m^B(p|_K)\right\}. \end{aligned} \tag{4.2}$$

Here we recall that $m \geq 1$ and $k = m-1, m$.

Lemma 4.1. *There hold*

$$\boldsymbol{P}_m^{RT}\boldsymbol{u}|_K \in [P_m(K)]^n, \quad \forall K \in \mathcal{T}_h \tag{4.3}$$

and, for any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h^0 \times Q_h^0$*,*

$$\begin{aligned} &a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \mathcal{P}_h p) + c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h) + d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h) \\ &= (\boldsymbol{f}, \boldsymbol{v}_{hi}) + \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) + \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) + \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h), \qquad (4.4a) \\ &b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}, q_h) = 0, \qquad (4.4b) \end{aligned}$$

where

$$\begin{aligned} \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) := &-\frac{1}{2}\left(\boldsymbol{P}_m^{RT}\boldsymbol{u} \otimes \boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u} \otimes \boldsymbol{u}, \nabla_h \boldsymbol{v}_{hi}\right) \\ &+ \frac{1}{2}\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u} \otimes \boldsymbol{\Pi}_k^B\boldsymbol{u} - \boldsymbol{u} \otimes \boldsymbol{u})\boldsymbol{n}, \boldsymbol{v}_{hi}\right\rangle_{\partial\mathcal{T}_h} \\ &- \frac{1}{2}\left((\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{P}_m^{RT}\boldsymbol{u} \cdot \nabla_h)\boldsymbol{P}_m^{RT}\boldsymbol{u}, \boldsymbol{v}_{hi}\right) \\ &- \frac{1}{2}\left\langle(\boldsymbol{v}_{hb} \otimes \boldsymbol{\Pi}_k^B\boldsymbol{u})\boldsymbol{n}, \boldsymbol{P}_m^{RT}\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}, \\ \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) := &\ \nu\left\langle\left(\nabla\boldsymbol{u} - \boldsymbol{\Pi}_{m-1}^*\nabla\boldsymbol{u}\right)\boldsymbol{n}, \boldsymbol{v}_{hi} - \boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h} \\ &+ \nu\left\langle\eta(\boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi} - \boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}, \\ \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) = &\ \alpha\left(|\boldsymbol{P}_m^{RT}\boldsymbol{u}|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{u} - |\boldsymbol{u}|^{r-2}\boldsymbol{u}, \boldsymbol{v}_{hi}\right). \end{aligned}$$

Proof. For all $K \in \mathcal{T}_h$, $\varrho_m \in P_m(K)$, by Lemma 2.6 we have

$$\left(\nabla \cdot \boldsymbol{P}_m^{RT}\boldsymbol{u}, \varrho_m\right)_K = (\nabla \cdot \boldsymbol{u}, \varrho_m)_K = 0,$$

which means that $\nabla \cdot \boldsymbol{P}_m^{RT}\boldsymbol{u} = 0$, i.e. (4.3) holds.

From the definition of discrete weak divergence, the Green's formula and the definition of the trilinear form $d_h(\cdot;\cdot,\cdot)$ we easily have

$$d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h) = \left(\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), \boldsymbol{v}_{hi}\right) + \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h). \tag{4.5}$$

Thus, according to Lemma 2.8, the projection properties and the definition of discrete weak gradient, we get

$$\begin{aligned}
&a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,\mathcal{P}_hp)+c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)+d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)\\
=&\,\nu\left(\nabla_{w,m-1}\{\boldsymbol{P}_m^{RT}\boldsymbol{u},\boldsymbol{\Pi}_k^B\boldsymbol{u}\},\nabla_{w,m-1}\boldsymbol{v}_h\right)+\nu\left\langle\eta\boldsymbol{\Pi}_k^B(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\\
&+\left(\boldsymbol{v}_{hi},\nabla_{w,m}\{\Pi_{m-1}^*p,\Pi_m^Bp\}\right)+\alpha\left(|\boldsymbol{P}_m^{RT}\boldsymbol{u}|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{u},\boldsymbol{v}_{hi}\right)\\
&+\left(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}),\boldsymbol{v}_{hi}\right)+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\\
=&\,\nu\big(\boldsymbol{\Pi}_{m-1}^*(\nabla\boldsymbol{u}),\nabla_{w,m-1}\boldsymbol{v}_h\big)+\nu\left\langle\eta(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\\
&+\big(\boldsymbol{v}_{hi},\boldsymbol{\Pi}_m^*(\nabla p)\big)+\alpha\big(|\boldsymbol{u}|^{r-2}\boldsymbol{u},\boldsymbol{v}_{hi}\big)+\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\\
&+\big(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}),\boldsymbol{v}_{hi}\big)+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\\
=&-\nu\big(\nabla_h\cdot\boldsymbol{\Pi}_{m-1}^*(\nabla\boldsymbol{u}),\boldsymbol{v}_{hi}\big)+\nu\big\langle\boldsymbol{\Pi}_{m-1}^*(\nabla\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\
&+\nu\left\langle\eta(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}+\big(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}),\boldsymbol{v}_{hi}\big)\\
&+(\boldsymbol{v}_{hi},\nabla p)+\alpha\big(|\boldsymbol{u}|^{r-2}\boldsymbol{u},\boldsymbol{v}_{hi}\big)+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)+\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h).
\end{aligned}$$

Consequently, using the relation $\langle(\nabla\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hb}\rangle_{\partial\mathcal{T}_h}=0$, the Green formula, projection properties, and the first equation in (1.1), we obtain

$$\begin{aligned}
&a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,\mathcal{P}_hp)+c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)+d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{v}_h)\\
=&\,\nu\big(\boldsymbol{\Pi}_{m-1}^*(\nabla\boldsymbol{u}),\nabla_h\boldsymbol{v}_{hi}\big)+\nu\big\langle\boldsymbol{\Pi}_{m-1}^*(\nabla\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hb}-\boldsymbol{v}_{hi}\big\rangle_{\partial\mathcal{T}_h}\\
&+\nu\left\langle\eta(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}+\big(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}),\boldsymbol{v}_{hi}\big)\\
&+(\boldsymbol{v}_{hi},\nabla p)+\alpha\big(|\boldsymbol{u}|^{r-2}\boldsymbol{u},\boldsymbol{v}_{hi}\big)+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)+\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\\
=&-\nu(\triangle\boldsymbol{u},\boldsymbol{v}_{hi})+\nu\big\langle(\nabla\boldsymbol{u}-\boldsymbol{\Pi}_{m-1}^*\nabla\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\
&+\nu\left\langle\eta(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}+\big(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}),\boldsymbol{v}_{hi}\big)\\
&+(\boldsymbol{v}_{hi},\nabla p)+\alpha\big(|\boldsymbol{u}|^{r-2}\boldsymbol{u},\boldsymbol{v}_{hi}\big)+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)+\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)\\
=&\,(\boldsymbol{f},\boldsymbol{v}_{hi})+\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)+\xi_{II}(\boldsymbol{u},\boldsymbol{v}_h)+\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h).
\end{aligned}$$

This gives (4.4a).

From the definition of $\nabla_{w,m}$, the fact $\nabla\cdot\boldsymbol{P}_m^{RT}\boldsymbol{u}=0$ and (2.11) it follows

$$b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u},q_h)=-\left(\nabla\cdot\boldsymbol{P}_m^{RT}\boldsymbol{u},q_{hi}\right)+\left\langle\boldsymbol{P}_m^{RT}\boldsymbol{u}\cdot\boldsymbol{n},q_{hb}\right\rangle_{\partial\mathcal{T}_h}=\langle\boldsymbol{u}\cdot\boldsymbol{n},q_{hb}\rangle_{\partial\mathcal{T}_h}=0,$$

i.e. the relation (4.4b) holds. This completes the proof. □

By following a similar line as in the proofs of [16, Lemma 4.3] and [56, Lemma 5.2], we can obtain the estimates of ξ_I, ξ_{II}, and ξ_{III}.

Lemma 4.2. *For any $\boldsymbol{v}_h\in\boldsymbol{V}_h^0$, there hold*

$$|\xi_I(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_h)|\lesssim h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V, \tag{4.6a}$$

$$|\xi_{II}(\boldsymbol{u},\boldsymbol{v}_h)|\lesssim h^m\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V, \tag{4.6b}$$

$$|\xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_h)|\lesssim h^m\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V, \tag{4.6c}$$

for $k=m,m-1$ when $n=2$ and $k=m$ when $n=3$.

Proof. We first estimate the four terms of $\xi_I(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_h)$ one by one. By the triangle inequality, the Hölder's inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\left|-\frac{1}{2}\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\otimes\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\otimes\boldsymbol{u},\nabla_h\boldsymbol{v}_{hi}\right)\right|\\
\lesssim&\left|\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\otimes(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}),\nabla_h\boldsymbol{v}_{hi}\right)\right|+\left|\left((\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u})\otimes\boldsymbol{u},\nabla_h\boldsymbol{v}_{hi}\right)\right|\\
\lesssim&\sum_{K\in\mathcal{T}_h}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right|_{0,3,K}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|_{0,6,K}\|\nabla_h\boldsymbol{v}_{hi}\|_{0,K}\\
&\quad+|\boldsymbol{u}|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right|_{0,K}\|\nabla_h\boldsymbol{v}_{hi}\|_{0,K}\\
=:&\ \mathfrak{B}_1+\mathfrak{B}_2.
\end{aligned}$$

Then, combining Lemmas 2.4, 2.6 and 2.7 we obtain

$$\begin{aligned}
\mathfrak{B}_1\lesssim&\sum_{K\in\mathcal{T}_h}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right|_{0,3,K}\left(|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}|_{0,6,\Omega}+|\boldsymbol{u}|_{0,6,\Omega}\right)\|\nabla_h\boldsymbol{v}_{hi}\|_{0,K}\\
\lesssim&\ h^{m+1-\frac{n}{6}}\|\boldsymbol{u}\|_1|\boldsymbol{u}|_{m+1}\|\nabla_h\boldsymbol{v}_{hi}\|_{0,K}+h^{m+1}|\boldsymbol{u}|_{0,\infty,\Omega}|\boldsymbol{u}|_{m+1}|||\boldsymbol{v}_h|||_V\\
\lesssim&\ h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V,\\
\mathfrak{B}_2\lesssim&\ h^{m+1}|\boldsymbol{u}|_{0,\infty,\Omega}|\boldsymbol{u}|_{m+1}|||\boldsymbol{v}_h|||_V\lesssim h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V.
\end{aligned}$$

Thus, we get the following estimate of the first term of ξ_I:

$$\left|-\frac{1}{2}\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\otimes\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\otimes\boldsymbol{u},\nabla_h\boldsymbol{v}_{hi}\right)\right|\lesssim h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V. \tag{4.7}$$

For the second term of ξ_I, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\otimes\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}\right\rangle_{\partial\mathcal{T}_h}\right|\\
=&\left|\frac{1}{2}\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\otimes\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right|\\
\lesssim&\left|\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u})\otimes(\boldsymbol{\Pi}_m^*\boldsymbol{u}-\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\quad+\left|\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u})\otimes(\boldsymbol{\Pi}_m^*\boldsymbol{u}\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\quad+\left|\left\langle(\boldsymbol{\Pi}_k^*\boldsymbol{u}-\boldsymbol{\Pi}_k^B\boldsymbol{u})\otimes(\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\quad+\left|\left\langle((\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u})\otimes\boldsymbol{\Pi}_k^*\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right|,
\end{aligned}$$

which, together with the Cauchy inequality and the properties of the projections $\boldsymbol{\Pi}_k^B$ and $\boldsymbol{\Pi}_m^*$, further yields

$$\left|\frac{1}{2}\left\langle(\boldsymbol{\Pi}_k^B\boldsymbol{u}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\otimes\boldsymbol{u})\boldsymbol{n},\boldsymbol{v}_{hi}\right\rangle_{\partial\mathcal{T}_h}\right|$$

$$
\begin{aligned}
&\lesssim \sum_{K\in\mathcal{T}_h} \left|\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\right|_{0,\partial K}|\boldsymbol{\Pi}_m^*\boldsymbol{u}-\boldsymbol{u}|_{0,\partial K}\\
&\qquad\times\Big(\left|\boldsymbol{v}_{hi}-\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right|_{0,\infty,\partial K}+\left|\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right|_{0,\infty,\partial K}\Big)\\
&\qquad+\left|\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\right|_{0,\partial K}|\boldsymbol{\Pi}_m^*\boldsymbol{u}|_{0,\infty,\partial K}\\
&\qquad\times\Big(\left|\boldsymbol{v}_{hi}-\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right|_{0,\partial K}+\left|\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right|_{0,\partial K}\Big)\\
&\qquad+\left|\boldsymbol{\Pi}_k^*\boldsymbol{u}-\Pi_k^B\boldsymbol{u}\right|_{0,\partial K}\left|\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\right|_{0,\partial K}\\
&\qquad\times\Big(\left|\boldsymbol{v}_{hi}-\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right|_{0,\infty,\partial K}+\left|\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right|_{0,\infty,\partial K}\Big)\\
&\qquad+\left|\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{u}\right|_{0,\partial K}|\boldsymbol{\Pi}_m^*\boldsymbol{u}|_{0,\infty,\partial K}\\
&\qquad\times\Big(\left|\boldsymbol{v}_{hi}-\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right|_{0,\partial K}+\left|\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right|_{0,\partial K}\Big)\\
&\lesssim h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V.
\end{aligned}
\tag{4.8}
$$

For the third term of ξ_I, from the Hölder's inequality and the properties of the projection $\boldsymbol{P}_m^{RT}$ we obtain

$$
\begin{aligned}
&\left|-\frac{1}{2}\left((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{P}_m^{RT}\boldsymbol{u}\cdot\nabla_h)\boldsymbol{P}_m^{RT}\boldsymbol{u},\boldsymbol{v}_{hi}\right)\right|\\
&\lesssim\left|\left((\boldsymbol{u}-\boldsymbol{P}_m^{RT}\boldsymbol{u})\cdot\nabla\boldsymbol{u},\boldsymbol{v}_{hi}\right)\right|+\left|\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\cdot(\nabla\boldsymbol{u}-\nabla_h\boldsymbol{P}_m^{RT}\boldsymbol{u}),\boldsymbol{v}_{hi}\right)\right|\\
&\lesssim\sum_{K\in\mathcal{T}_h}\left|\boldsymbol{u}-\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|_{0,3,K}|\nabla\boldsymbol{u}|_{0,K}\|\boldsymbol{v}_{hi}\|_{0,6,K}\\
&\qquad+\sum_{K\in\mathcal{T}_h}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|_{0,6,K}\left|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|_{0,K}\|\boldsymbol{v}_{hi}\|_{0,3,K}\\
&\lesssim h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V.
\end{aligned}
\tag{4.9}
$$

For the last term of ξ_I, we apply the triangle inequality to get

$$
\begin{aligned}
&\left|-\frac{1}{2}\left\langle(\boldsymbol{v}_{hb}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u})\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&=\left|\frac{1}{2}\left\langle(\boldsymbol{v}_{hb}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u})\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\Pi}_m^B\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\lesssim\left|\left\langle(\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb})\otimes(\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{\Pi}_k^*\boldsymbol{u})\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\Pi}_m^B\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\qquad+\left|\left\langle\boldsymbol{v}_{hi}\otimes(\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{\Pi}_k^*\boldsymbol{u})\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\Pi}_m^B\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\qquad+\left|\left\langle(\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb})\otimes\boldsymbol{\Pi}_k^*\boldsymbol{u}\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\Pi}_m^B\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|\\
&\qquad+\left|\left\langle\boldsymbol{v}_{hi}\otimes\boldsymbol{\Pi}_k^*\boldsymbol{u}\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\Pi}_m^B\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right|,
\end{aligned}
$$

which plus the Cauchy inequality and the properties of the projection $\boldsymbol{P}_m^{RT}$, $\boldsymbol{\Pi}_k^B$ and

$\mathbf{\Pi}_k^*$ further implies

$$
\begin{aligned}
&\left|-\frac{1}{2}\left\langle(\boldsymbol{v}_{hb}\otimes\mathbf{\Pi}_k^B\boldsymbol{u})\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
\lesssim &\sum_{K\in\mathcal{T}_h}\left(|\boldsymbol{v}_{hi}-\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}|_{0,\infty,\partial K}+|\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}|_{0,\infty,\partial K}\right) \\
&\quad\times\left|\mathbf{\Pi}_k^B\boldsymbol{u}-\mathbf{\Pi}_k^*\boldsymbol{u}\right|_{0,\partial K}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\mathbf{\Pi}_m^B\boldsymbol{u}\right|_{0,\partial K} \\
&\quad+|\boldsymbol{v}_{hi}|_{0,\infty,\partial K}\left|\mathbf{\Pi}_k^B\boldsymbol{u}-\mathbf{\Pi}_k^*\boldsymbol{u}\right|_{0,\partial K}|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\mathbf{\Pi}_m^B\boldsymbol{u}|_{0,\partial K} \\
&\quad+\left(\left|\boldsymbol{v}_{hi}-\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}\right|_{0,\partial K}+\left|\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right|_{0,\partial K}\right) \\
&\quad\times|\mathbf{\Pi}_k^*\boldsymbol{u}|_{0,6,\partial K}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\mathbf{\Pi}_k^B\boldsymbol{u}\right|_{0,3,\partial K} \\
&\quad+|\boldsymbol{v}_{hi}|_{0,3,\partial K}|\mathbf{\Pi}_k^*\boldsymbol{u}|_{0,6,\partial K}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\mathbf{\Pi}_m^B\boldsymbol{u}\right|_{0,\partial K} \\
\lesssim &\, h^m\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V.
\end{aligned}
\tag{4.10}
$$

Combining the above four estimates (4.7)-(4.10) leads to the desired result (4.6a).

Similarly, for the terms $\xi_{II}(\boldsymbol{u},\boldsymbol{v}_h)$ and $\xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_h)$ we can get the following estimates:

$$
\begin{aligned}
|\xi_{II}(\boldsymbol{u},\boldsymbol{v}_h)| \le &\left|\nu\left\langle\left(\nabla\boldsymbol{u}-\mathbf{\Pi}_{m-1}^*\nabla\boldsymbol{u}\right)\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+\left|\nu\left\langle\eta\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right),\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
\lesssim &\sum_{K\in\mathcal{T}_h}\|\nabla\boldsymbol{u}-\mathbf{\Pi}_{m-1}^*\nabla\boldsymbol{u}\|_{0,\partial K} \\
&\times\left(\left\|\boldsymbol{v}_{hi}-\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}\right\|_{0,\partial K}+\left\|\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\|_{0,\partial K}\right) \\
&+\sum_{K\in\mathcal{T}_h}\left\|\eta^{\frac{1}{2}}\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right)\right\|_{0,\partial K}\left\|\eta^{\frac{1}{2}}\left(\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right)\right\|_{0,\partial K} \\
\lesssim &\, h^m\|\boldsymbol{u}\|_{m+1}\left(\|\nabla_h\boldsymbol{v}_{hi}\|_0+\left\|\eta^{\frac{1}{2}}\left(\mathbf{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right)\right\|_{0,\partial K}\right) \\
&+h^m\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V \\
\lesssim &\, h^m\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V
\end{aligned}
$$

and

$$
\begin{aligned}
|\xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}_h)| &\lesssim \alpha\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right\|_{0,3}\left(\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\|_{0,2(r-2)}^{r-2}+\|\boldsymbol{u}\|_{0,2(r-2)}^{r-2}\right)\|\boldsymbol{v}_{hi}\|_{0,6} \\
&\lesssim h^m\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{u}\|_{m+1}|||\boldsymbol{v}_h|||_V,
\end{aligned}
$$

where in the estimate of $|\xi_{III}|$ we have used Lemma 2.9. This finishes the proof. □

Based on Lemmas 4.1 and 4.2, we can obtain the following conclusion.

Theorem 4.1. *Let* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^0 \times Q_h^0$ *be the solution to the WG scheme* (2.6). *Under the regularity assumption* (4.1) *and the discrete smallness condition* (3.12) *with*

$$\vartheta := 1 - \frac{\mathcal{N}_h}{\nu^2}\|\boldsymbol{f}\|_{*,h} > 0, \tag{4.11}$$

there hold the following error estimates:

$$|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V \lesssim \mathcal{M}_1(\boldsymbol{u})h^m, \tag{4.12a}$$
$$|||\mathcal{P}_h p - p_h|||_Q \lesssim \mathcal{M}_2(\boldsymbol{u})h^m + \mathcal{M}_3(\boldsymbol{u})h^{2m}, \tag{4.12b}$$

where

$$\mathcal{M}_1(\boldsymbol{u}) := \vartheta^{-1}\left(1 + \|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^{r-2}\right)\|\boldsymbol{u}\|_{m+1},$$

and $\mathcal{M}_2(\boldsymbol{u})$ *and* $\mathcal{M}_3(\boldsymbol{u})$ *are two positive constants depending only on* $\vartheta, \nu, \|\boldsymbol{f}\|_{*,h}, \|\boldsymbol{u}\|_2$ *and* $\|\boldsymbol{u}\|_{m+1}$.

Proof. Subtracting (2.6a) and (2.6b) from (4.4a) and (4.4b), respectively, we have

$$\begin{aligned}
&a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \mathcal{P}_h p - p_h) + c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h)\\
&\quad - c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{v}_h) - d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\\
&= \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) + \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) + \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^0, &(4.13a)\\
&b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0, \qquad \forall q_h \in Q_h^0. &(4.13b)
\end{aligned}$$

Taking $\boldsymbol{v}_h = \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h$ in Eq. (4.13a) and utilizing Lemmas 2.9 and 3.1, we obtain

$$\begin{aligned}
&\nu|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V^2 + \left\|\boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u}_{hi}\right\|_{0,r}^r\\
&\lesssim \nu|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V^2 + \left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{u} - |\boldsymbol{u}_{hi}|^{r-2}\boldsymbol{u}_{hi}, \boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u}_{hi}\right)\\
&= \xi_I(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) + \xi_{II}(\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) + \xi_{III}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h)\\
&\quad - \left\{d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u}, \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) - d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h)\right\}\\
&= \xi_I(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) + \xi_{II}(\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) + \xi_{III}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h)\\
&\quad - d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h),
\end{aligned}$$

where in the last line we have used the relation $d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u}; \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h) = 0$. In view of Lemma 4.2, the definition of $\mathcal{N}_h$ with $\boldsymbol{u}_h, \boldsymbol{\mathcal{I}}_h\boldsymbol{u} \in \boldsymbol{V}_{0h}$, and the fact that $\|\boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u}_{hi}\|_{0,r}^r \geq 0$, we further obtain

$$\begin{aligned}
\nu|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V^2 \leq\ & Ch^m\left(\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1} + \|\boldsymbol{u}\|_{m+1} + \|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{u}\|_{m+1}\right)|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V\\
&+ \mathcal{N}_h|||\boldsymbol{u}_h|||_V|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V^2,
\end{aligned}$$

which, together with (3.9), yields

$$\nu\left(1 - \frac{\mathcal{N}_h}{\nu^2}\|\boldsymbol{f}\|_{*,h}\right)|||\boldsymbol{\mathcal{I}}_h\boldsymbol{u} - \boldsymbol{u}_h|||_V \leq Ch^m\left(1 + \|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^{r-2}\right)\|\boldsymbol{u}\|_{m+1}.$$

Thus, the desired estimate (4.12a) follows.

Next we estimate the error of pressure. Using the Eq. (4.13a) we get

$$
\begin{aligned}
b_h(\boldsymbol{v}_h, \mathcal{P}_h p - p_h) &= -a_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - c_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h \boldsymbol{u}, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad - d_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h \boldsymbol{u}, \boldsymbol{v}_h) + d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad + \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) + \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) + \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) \\
&= -a_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - \big(c_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u}; \boldsymbol{\mathcal{I}}_h \boldsymbol{u}, \boldsymbol{v}_h) - c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\big) \\
&\quad - d_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) - d_h(\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad - d_h(\boldsymbol{u}_h; \boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad + \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) + \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) + \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^0.
\end{aligned}
$$

In light of Lemmas 2.4, 2.9, 3.1, 3.2, 4.2 and the estimates (3.9) and (4.12a), we have

$$
\begin{aligned}
|||\mathcal{P}_h p - p_h|||_Q &\lesssim \sup_{\mathbf{0} \neq \boldsymbol{v}_h \in \boldsymbol{V}_h^0} \frac{b_h(\boldsymbol{v}_h, \mathcal{P}_h p - p_h)}{|||\boldsymbol{v}_h|||_V} \\
&\lesssim \nu |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V + \alpha C_r C_{\tilde{r}}^r \left(|||\boldsymbol{\mathcal{I}}_h \boldsymbol{u}|||_V + |||\boldsymbol{u}_h|||_V\right)^{r-2} |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V \\
&\quad + |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V^2 + |||\boldsymbol{u}_h|||_V |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V \\
&\quad + \xi_I(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) + \xi_{II}(\boldsymbol{u}, \boldsymbol{v}_h) + \xi_{III}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) \\
&\lesssim \left(\nu + \frac{\|\boldsymbol{f}\|_{*,h}}{\nu} + \frac{\|\boldsymbol{f}\|_{*,h}^{r-2}}{\nu^{r-2}} + \|\boldsymbol{u}\|_2^{r-2}\right) |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V + |||\boldsymbol{\mathcal{I}}_h \boldsymbol{u} - \boldsymbol{u}_h|||_V^2 \\
&\quad + h^m \left(1 + \|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^{r-2}\right) \|\boldsymbol{u}\|_{m+1},
\end{aligned}
$$

which shows (4.12b). □

Finally, based upon Theorem 4.1, Lemmas 2.2, 2.6-2.8, we can obtain the following main conclusion.

Theorem 4.2. *Under the same conditions as in Theorem* 4.1*, there hold*

$$
\|\nabla \boldsymbol{u} - \nabla_h \boldsymbol{u}_{hi}\|_0 + \|\nabla \boldsymbol{u} - \nabla_{w,m-1} \boldsymbol{u}_h\|_0 \lesssim \mathcal{M}_1(\boldsymbol{u}) h^m, \tag{4.14}
$$

$$
\|p - p_{hi}\|_0 \lesssim (\mathcal{M}_2(\boldsymbol{u}) + \|p\|_m) h^m + \mathcal{M}_3(\boldsymbol{u}) h^{2m}. \tag{4.15}
$$

Remark 4.1. The result (4.14) shows that the velocity error estimate is independent of the pressure approximation, which means that the proposed WG scheme is pressure-robust.

5. L^2 error estimation for velocity

We follow standard dual arguments to derive an L^2 error estimate for the velocity solution of the WG scheme with $r = 2, 4$. To this end, we introduce the following dual problem: Seek $(\boldsymbol{\phi}, \psi)$ such that

$$\begin{cases} -\nu\triangle\boldsymbol{\phi} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi} + (\nabla\boldsymbol{u})^\top\boldsymbol{\phi} + \alpha|\boldsymbol{u}|^{r-2}\boldsymbol{\phi} & \\ \quad +\alpha(r-2)|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u} + \nabla\psi = \boldsymbol{e}_{hi}, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\phi} = 0, & \text{in } \Omega, \\ \boldsymbol{\phi} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\boldsymbol{u}$ and $\boldsymbol{u}_h = \{\boldsymbol{u}_{hi}, \boldsymbol{u}_{hb}\}$ are respectively the solutions of (1.1) and (2.6), and $\boldsymbol{e}_{hi} := \boldsymbol{P}_m^{RT}\boldsymbol{u} - \boldsymbol{u}_{hi}$. We assume the following regularity condition holds:

$$\|\boldsymbol{\phi}\|_2 + \|\psi\|_1 \lesssim \|\boldsymbol{e}_{hi}\|_0. \tag{5.2}$$

The corresponding weak form of (5.1) reads: Seek $(\boldsymbol{\phi}, \psi) \in \boldsymbol{V}\times Q$ such that

$$\begin{aligned} &\mathcal{A}_u(\boldsymbol{\phi}, \boldsymbol{v}) + b(\boldsymbol{v}, \psi) = (\boldsymbol{e}_{hi}, \boldsymbol{v}), && \forall \boldsymbol{v}\in\boldsymbol{V}, \\ &b(\boldsymbol{\phi}, q) = 0, && \forall q\in Q, \end{aligned} \tag{5.3}$$

where the bilinear form $\mathcal{A}_u(\cdot,\cdot) : \boldsymbol{V}\times\boldsymbol{V}\to\mathbb{R}$ is defined by

$$\begin{aligned} \mathcal{A}_u(\boldsymbol{\phi}, \boldsymbol{v}) := {}& a(\boldsymbol{\phi}, \boldsymbol{v}) + c(\boldsymbol{u}; \boldsymbol{\phi}, \boldsymbol{v}) + d(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{\phi}) + d(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{\phi}) \\ & + \alpha(r-2)(|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u}, \boldsymbol{v}), \end{aligned} \tag{5.4}$$

and the bilinear forms, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and the trilinear forms, $c(\cdot;\cdot,\cdot)$ and $d(\cdot;\cdot,\cdot)$, are given in Subsection 2.1.

Remark 5.1. According to the Hölder's inequality, the Sobolev inequality and the boundedness result (2.2), we can get the boundedness result

$$\mathcal{A}_u(\boldsymbol{\phi}, \boldsymbol{v}) \lesssim \|\nabla\boldsymbol{\phi}\|_0\|\nabla\boldsymbol{v}\|_0, \quad \forall \boldsymbol{\phi}, \boldsymbol{v}\in\boldsymbol{V}. \tag{5.5}$$

At the same time, under the uniqueness condition (2.3) we can obtain the coercivity result

$$\mathcal{A}_u(\boldsymbol{v}, \boldsymbol{v}) \gtrsim \|\nabla\boldsymbol{v}\|_0^2, \quad \forall \boldsymbol{v}\in\boldsymbol{V}. \tag{5.6}$$

It is standard that the inf-sup inequality

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v}, q)}{\|\nabla\boldsymbol{v}\|_0} \gtrsim \|q\|_0, \quad \forall q\in Q$$

holds (cf. [14]). As a result, the problem (5.3) admits a unique solution.

By taking similar routines as in the proofs of Lemmas 4.1 and 4.2, respectively, we can obtain Lemmas 5.1 and 5.2.

Lemma 5.1. *There hold*

$$
\begin{aligned}
&a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,\mathcal{P}_h\psi)+c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{v}_h)\\
&\quad-d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{v}_h)+d_h(\boldsymbol{v}_h;\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&=(\boldsymbol{e}_{hi},\boldsymbol{v}_{hi})-E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h)+E_{II}(\boldsymbol{\phi},\boldsymbol{v}_h)\\
&\quad+E_{III}(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h)+E_{IV}(\boldsymbol{v}_h;\boldsymbol{u},\boldsymbol{\phi})\\
&\quad-\alpha(r-2)(|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u},\boldsymbol{e}_{hi}),\quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h^0,\\
&b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},q_h)=0,\qquad\qquad\qquad \forall q_h\in Q_h^0,
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h):=&-\frac{1}{2}\left(\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\otimes\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{\phi}\otimes\boldsymbol{u},\nabla_h\boldsymbol{v}_{hi}\right)\\
&+\frac{1}{2}\left\langle\left(\boldsymbol{\Pi}_k^B\boldsymbol{\phi}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u}-\boldsymbol{\phi}\otimes\boldsymbol{u}\right)\boldsymbol{n},\boldsymbol{v}_{hi}\right\rangle_{\partial\mathcal{T}_h}\\
&-\frac{1}{2}\left((\boldsymbol{u}\cdot\nabla)\boldsymbol{\phi}-\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\cdot\nabla_h\right)\boldsymbol{P}_m^{RT}\boldsymbol{\phi},\boldsymbol{v}_{hi}\right)\\
&-\frac{1}{2}\left\langle\left(\boldsymbol{v}_{hb}\otimes\boldsymbol{\Pi}_k^B\boldsymbol{u}\right)\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h},\\
E_{II}(\boldsymbol{\phi},\boldsymbol{v}_h):=&\,\nu\left\langle\left(\nabla\boldsymbol{\phi}-\boldsymbol{\Pi}_{m-1}^*\nabla\boldsymbol{\phi}\right)\boldsymbol{n},\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\\
&+\nu\left\langle\eta\left(\boldsymbol{P}_m^{RT}\boldsymbol{\phi}-\boldsymbol{\phi}\right),\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}-\boldsymbol{v}_{hb}\right\rangle_{\partial\mathcal{T}_h},\\
E_{III}(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h)=&\,\alpha\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{\phi}-|\boldsymbol{u}|^{r-2}\boldsymbol{\phi},\boldsymbol{v}_{hi}\right),\\
E_{IV}(\boldsymbol{v}_h;\boldsymbol{u},\boldsymbol{\phi}):=&\left(\left(\nabla_h\boldsymbol{P}_m^{RT}\boldsymbol{u}\right)^{\top}\boldsymbol{P}_m^{RT}\boldsymbol{\phi}-(\nabla\boldsymbol{u})^{\top}\boldsymbol{\phi},\boldsymbol{v}_{hi}\right)\\
&-\frac{1}{2}\left\langle\left(\boldsymbol{\Pi}_k^B\boldsymbol{\phi}\otimes\boldsymbol{v}_{hb}\right)\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\rangle_{\partial\mathcal{T}_h}\\
&+\frac{1}{2}\left\langle\left(\boldsymbol{\Pi}_k^B\boldsymbol{u}\otimes\boldsymbol{v}_{hb}\right)\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}\\
&-\frac{1}{2}\left\langle\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}\otimes\boldsymbol{v}_{hi}\right)\boldsymbol{n},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Lemma 5.2. *For any $\boldsymbol{v}_h\in\boldsymbol{V}_h^0$, there hold*

$$
\begin{aligned}
&|\xi_I(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})|\lesssim h^{m+1}\|\boldsymbol{u}\|_2\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{\phi}\|_2,\\
&|E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h)|\lesssim h\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{v}_h|||_V,\\
&|\xi_{II}(\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})|+|E_{II}(\boldsymbol{\phi},\boldsymbol{e}_h)|\lesssim h^{m+1}\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{\phi}\|_2,\\
&|\xi_{III}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})|\lesssim h^{m+1}\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{\phi}\|_2,\\
&|E_{III}(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{v}_h)|\lesssim h\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{\phi}\|_2|||\boldsymbol{v}_h|||_V,\\
&|E_{IV}(\boldsymbol{v}_h;\boldsymbol{u},\boldsymbol{\phi})|\lesssim h\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{v}_h|||_V.
\end{aligned}
\tag{5.8}
$$

Based on Lemmas 5.1 and 5.2, we can derive the following L^2 error estimate in the cases $r=2,4$.

Theorem 5.1. *Under the regularity assumption* (5.2) *and the same conditions as in Theorem* 4.1*, there holds*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \lesssim \mathcal{M}_4(\boldsymbol{u})h^{m+1}, \tag{5.9}$$

where $\mathcal{M}_4(\boldsymbol{u})$ *is a positive constant depending on* $\|\boldsymbol{u}\|_2$ *and* $\|\boldsymbol{u}\|_{m+1}$.

Proof. Denote $\boldsymbol{e}_h := \boldsymbol{\mathcal{I}}_h\boldsymbol{u}-\boldsymbol{u}_h$ and $\varepsilon_h := \mathcal{P}_h p - p_h$. Taking $\boldsymbol{v}_h=\boldsymbol{e}_h$ and $q_h=\varepsilon_h$ in (5.7), we derive

$$\begin{aligned}
\|\boldsymbol{e}_{hi}\|_0^2 &= a_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{e}_h)+b_h(\boldsymbol{e}_h,\mathcal{P}_h\psi)+c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{e}_h)\\
&\quad+\alpha(r-2)(|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u},\boldsymbol{e}_{hi})\\
&\quad+d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{e}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+d_h(\boldsymbol{e}_h;\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&\quad+E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{e}_h)-E_{II}(\boldsymbol{\phi},\boldsymbol{e}_h)\\
&\quad-E_{III}(\boldsymbol{u};\boldsymbol{\phi},e_h)-E_{IV}(\boldsymbol{e}_h;\boldsymbol{u},\boldsymbol{\phi}),\\
b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\varepsilon_h)&=0.
\end{aligned}\tag{5.10}$$

Taking $\boldsymbol{v}_h=\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}$ and $q_h=\mathcal{P}_h\psi$ in (4.13), respectively, we have

$$\begin{aligned}
&a_h(\boldsymbol{e}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+b_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\varepsilon_h)+c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})-c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&\quad+d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})-d_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&=\xi_I(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+\xi_{II}(\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+\xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}),\\
&b_h(\boldsymbol{e}_h,\mathcal{P}_h\psi)=0,
\end{aligned}$$

which plus (5.10) give

$$\begin{aligned}
\|\boldsymbol{e}_{hi}\|_0^2 &= \big\{c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi},\boldsymbol{e}_h)-c_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&\qquad+\alpha(r-2)(|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u},\boldsymbol{e}_{hi})\big\}\\
&\quad+\big\{-d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+d_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\\
&\qquad+d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{e}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+d_h(\boldsymbol{e}_h;\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\big\}\\
&\quad+\big\{\xi_I(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})+E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{e}_h)+\xi_{II}(\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})-E_{II}(\boldsymbol{\phi},\boldsymbol{e}_h)\\
&\qquad+\xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})-E_{III}(\boldsymbol{u};\boldsymbol{\phi},e_h)-E_{IV}(\boldsymbol{e}_h;\boldsymbol{u},\boldsymbol{\phi})\big\}\\
&=:\sum_{j=1}^{3}\mathfrak{R}_j.
\end{aligned}\tag{5.11}$$

Then let us estimate $\mathfrak{R}_j$, $j=1,\ldots,3$ one by one.

For the term $\mathfrak{R}_1$, we have

$$\begin{aligned}
\mathfrak{R}_1 &= \alpha\Big\{\Big(\big|\boldsymbol{P}_m^{RT}\boldsymbol{u}\big|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{\phi},\boldsymbol{e}_{hi}\Big)-\Big(\big|\boldsymbol{P}_m^{RT}\boldsymbol{u}\big|^{r-2}\boldsymbol{P}_m^{RT}\boldsymbol{u},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\Big)\\
&\qquad+\big(|\boldsymbol{u}_{hi}|^{r-2}\boldsymbol{u}_{hi},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\big)+(r-2)\big(|\boldsymbol{u}|^{r-4}(\boldsymbol{u}\cdot\boldsymbol{\phi})\boldsymbol{u},\boldsymbol{e}_{hi}\big)\Big\}\\
&=\alpha\Big\{\Big(\Big(\big|\boldsymbol{P}_m^{RT}\boldsymbol{u}\big|^{r-2}-|\boldsymbol{u}_{hi}|^{r-2}\Big)\boldsymbol{e}_{hi},\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-2}-|\boldsymbol{u}_{hi}|^{r-2}\right)\boldsymbol{P}_m^{RT}\boldsymbol{u},\boldsymbol{\phi}-\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right)\\
&-\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-2}-|\boldsymbol{u}_{hi}|^{r-2}-(r-2)|\boldsymbol{u}_{hi}|^{r-4}\boldsymbol{u}_{hi}\cdot\boldsymbol{e}_{hi},\boldsymbol{P}_m^{RT}\boldsymbol{u}\cdot\boldsymbol{\phi}\right)\\
&-(r-2)\Big(|\boldsymbol{u}_{hi}|^{r-4}\boldsymbol{u}_{hi}\left(\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right)\cdot\boldsymbol{\phi},\boldsymbol{e}_{hi}\Big)\\
&+(r-2)\left(|\boldsymbol{u}_{hi}|^{r-4}\boldsymbol{u}_{hi}-|\boldsymbol{u}|^{r-4}\boldsymbol{u}\right)(\boldsymbol{u}\cdot\boldsymbol{\phi}),\boldsymbol{e}_{hi}\Big\}. \qquad (5.12)
\end{aligned}
$$

We easily see that $\mathfrak{R}_1=0$ when $r=2$. When $r=4$, combining (5.12) with Lemma 2.9 further gives

$$
\begin{aligned}
|\mathfrak{R}_1|\le\ &(r-2)\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-3}+|\boldsymbol{u}_{hi}|^{r-3}\right)|\boldsymbol{e}_{hi}|^2,\left|\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right|\right)\\
&+(r-2)\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-3}+|\boldsymbol{u}_{hi}|^{r-3}\right)|\boldsymbol{e}_{hi}|\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|,\left|\boldsymbol{\phi}-\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right|\right)\\
&+\frac{(r-1)(r-2)}{2}\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|^{r-4}+|\boldsymbol{u}_{hi}|^{r-4}\right)|\boldsymbol{e}_{hi}|^2,\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right||\boldsymbol{\phi}|\right)\\
&+(r-2)\left(|\boldsymbol{u}_{hi}|^{r-3}\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right||\boldsymbol{\phi}|,|\boldsymbol{e}_{hi}|\right)\\
&+(r-2)C_r\left((|\boldsymbol{u}_{hi}|+|\boldsymbol{u}|)^{r-4}|\boldsymbol{u}_{hi}-\boldsymbol{u}||\boldsymbol{u}||\boldsymbol{\phi}|,|\boldsymbol{e}_{hi}|\right)\\
\lesssim\ &\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|+|\boldsymbol{u}_{hi}|\right)|\boldsymbol{e}_{hi}|^2,\left|\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right|\right)\\
&+\left(\left(\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|+|\boldsymbol{u}_{hi}|\right)|\boldsymbol{e}_{hi}|\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right|,\left|\boldsymbol{\phi}-\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right|\right)\\
&+\left(|\boldsymbol{e}_{hi}|^2,\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right||\boldsymbol{\phi}|\right)+\left(|\boldsymbol{u}_{hi}|\left|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right||\boldsymbol{\phi}|,|\boldsymbol{e}_{hi}|\right)\\
&+\left(|\boldsymbol{u}_{hi}-\boldsymbol{u}||\boldsymbol{u}||\boldsymbol{\phi}|,|\boldsymbol{e}_{hi}|\right).
\end{aligned}
$$

This estimate, together with the Hölder's inequality, the triangle inequality, Lemmas 2.4, 2.7 and (3.9), yields

$$
\begin{aligned}
|\mathfrak{R}_1|\lesssim\ &\left(\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\|_{0,3}+\|\boldsymbol{u}_{hi}\|_{0,3}\right)\|\boldsymbol{e}_{hi}\|_{0,6}^2\left\|\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right\|_{0,3}\\
&+\left(\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\|_{0,3}+\|\boldsymbol{u}_{hi}\|_{0,3}\right)\|\boldsymbol{e}_{hi}\|_{0,6}\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\|_{0,6}\left\|\boldsymbol{\phi}-\boldsymbol{P}_m^{RT}\boldsymbol{\phi}\right\|_{0,3}\\
&+\|\boldsymbol{e}_{hi}\|_{0,6}^2\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}\right\|_{0,3}\|\boldsymbol{\phi}\|_{0,3}+\|\boldsymbol{u}_{hi}\|_{0,3}\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right\|_{0,3}\|\boldsymbol{\phi}\|_{0,6}\|\boldsymbol{e}_{hi}\|_{0,6}\\
&+\left(\|\boldsymbol{e}_{hi}\|_{0,3}+\left\|\boldsymbol{P}_m^{RT}\boldsymbol{u}-\boldsymbol{u}\right\|_{0,3}\right)\|\boldsymbol{u}\|_{0,3}\|\boldsymbol{\phi}\|_{0,6}\|\boldsymbol{e}_{hi}\|_{0,6}\\
\lesssim\ &\left(\|\boldsymbol{u}\|_2+|||\boldsymbol{u}_h|||_V\right)|||\boldsymbol{e}_h|||_V^2\|\boldsymbol{\phi}\|_2\\
&+\left(\|\boldsymbol{u}\|_2+|||\boldsymbol{u}_h|||_V\right)|||\boldsymbol{e}_h|||_V\|\boldsymbol{u}\|_2h^{2-\frac{n}{6}}\|\boldsymbol{\phi}\|_2\\
&+|||\boldsymbol{e}_h|||_V^2\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2+|||\boldsymbol{u}_h|||_Vh^{2-\frac{n}{6}}\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{e}_h|||_V\\
&+\left(|||\boldsymbol{e}_h|||_V+h^{2-\frac{n}{6}}\|\boldsymbol{u}\|_2\right)\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{e}_h|||_V\\
\lesssim\ &\left(\|\boldsymbol{u}\|_2+\frac{\|\boldsymbol{f}\|_{*,h}}{\nu}\right)\left(|||\boldsymbol{e}_h|||_V+h^{2-\frac{n}{6}}\|\boldsymbol{u}\|_2\right)|||\boldsymbol{e}_h|||_V\|\boldsymbol{\phi}\|_2.
\end{aligned}
$$

From the definition of $d_h(\cdot;\cdot,\cdot)$, the Hölder's inequality, the Sobolev inequality and

Lemmas 2.1, 2.4 and 2.7 it follows

$$\begin{aligned}|\mathfrak{R}_2| &= \big| - d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) + d_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) \\ &\qquad + d_h(\boldsymbol{\mathcal{I}}_h\boldsymbol{u};\boldsymbol{e}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) + d_h(\boldsymbol{e}_h;\boldsymbol{\mathcal{I}}_h\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})\big| \\ &= |d_h(\boldsymbol{e}_h;\boldsymbol{e}_h,\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi})| \\ &\lesssim |||\boldsymbol{e}_h|||_V^2\|\boldsymbol{\phi}\|_2.\end{aligned}$$

By Lemma 5.2 we have

$$\begin{aligned}|\mathfrak{R}_3| &= |\xi_I(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) + E_I(\boldsymbol{u};\boldsymbol{\phi},\boldsymbol{e}_h) + \xi_{II}(\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) - E_{II}(\boldsymbol{\phi},\boldsymbol{e}_h) \\ &\qquad + \xi_{III}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{\mathcal{I}}_h\boldsymbol{\phi}) - E_{III}(\boldsymbol{u};\boldsymbol{\phi},e_h) - E_{IV}(\boldsymbol{e}_h;\boldsymbol{u},\boldsymbol{\phi})| \\ &\lesssim h^{m+1}\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2 + h\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{e}_h|||_V + h^{m+1}\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{\phi}\|_2 \\ &\quad + h^{m+1}\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{\phi}\|_2 + h\|\boldsymbol{u}\|_2^{r-2}\|\boldsymbol{\phi}\|_2|||\boldsymbol{e}_h|||_V + h\|\boldsymbol{u}\|_2\|\boldsymbol{\phi}\|_2|||\boldsymbol{e}_h|||_V.\end{aligned}$$

All the above estimates of $\mathfrak{R}_j$, together with (5.11), the regularity assumption (5.2), Theorem 4.2 and the triangle inequality, lead to the desired conclusion (5.9). ☐

Remark 5.2. For some other cases of r, one may follow a similar line as in the proof of Theorem 5.1 to derive the L^2 estimate, but with additional conditions on r and $\boldsymbol{u}$.

6. Local elimination property and iteration scheme

6.1. Local elimination property

In the subsection, we shall demonstrate that in the WG scheme (2.6) the velocity and pressure approximations, $(\boldsymbol{u}_{hi}, p_{hi})$, defined in the interior of elements can be locally eliminated by the using the numerical traces $(\boldsymbol{u}_{hb}, p_{hb})$ defined on the element interfaces. After the local elimination the resulting system only includes the degrees of freedom of $(\boldsymbol{u}_{hb}, p_{hb})$ as unknowns.

For any $K \in \mathcal{T}_h$, taking $\boldsymbol{v}_{hi}|_{\mathcal{T}_h\setminus K} = \boldsymbol{0}$, $\boldsymbol{v}_{hb} = \boldsymbol{0}$, $q_{hi}|_{\mathcal{T}_h\setminus K} = 0$ and $q_{hb} = 0$ in (2.6), we obtain the following local problem: Seek $(\boldsymbol{u}_{hi}, p_{hi}) \in [P_m(K)]^n \times P_{m-1}(K)$ such that

$$\begin{aligned}&a_{h,K}(\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) + b_{h,K}(\boldsymbol{v}_{hi},p_{hi}) + c_{h,K}(\boldsymbol{u}_{hi};\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) + d_{h,K}(\boldsymbol{u}_{hi};\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) \\ &= F_{h,K}(\boldsymbol{v}_{hi}), \qquad \forall \boldsymbol{v}_{hi} \in [P_m(K)]^n, \\ &b_{h,K}(\boldsymbol{u}_{hi},q_{hi}) = 0, \quad \forall q_{hi} \in P_{m-1}(K),\end{aligned} \tag{6.1}$$

where

$$\begin{aligned}a_{h,K}(\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) &:= \nu\big(\nabla_{w,m-1}\{\boldsymbol{u}_{hi},\boldsymbol{0}\},\nabla_{w,m-1}\{\boldsymbol{v}_{hi},\boldsymbol{0}\}\big)_K + s_{h,K}(\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}), \\ s_{h,K}(\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) &:= \nu\left\langle\eta\boldsymbol{\Pi}_k^B\boldsymbol{u}_{hi},\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right\rangle_{\partial K}, \\ b_{h,K}(\boldsymbol{v}_{hi},p_{hi}) &:= \big(\nabla_{w,m}\{p_{hi},0\},\boldsymbol{v}_{hi}\big)_K, \\ c_{h,K}(\boldsymbol{u}_{hi};\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) &:= \big(\alpha|\boldsymbol{u}_{hi}|^{r-2}\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}\big)_K,\end{aligned}$$

$$
\begin{aligned}
d_{h,K}(\boldsymbol{u}_{hi};\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}) :=& \frac{1}{2}\big(\nabla_{w,m}\cdot\{\boldsymbol{u}_{hi}\otimes\boldsymbol{u}_{hi},\boldsymbol{0}\otimes\boldsymbol{0}\},\boldsymbol{v}_{hi}\big)_K \\
&-\frac{1}{2}\big(\nabla_{w,m}\cdot\{\boldsymbol{v}_{hi}\otimes\boldsymbol{u}_{hi},\boldsymbol{0}\otimes\boldsymbol{0}\},\boldsymbol{u}_{hi}\big)_K, \\
F_{h,K}(\boldsymbol{v}_{hi}) :=& (\boldsymbol{f},\boldsymbol{v}_{hi})_K - \nu\big(\nabla_{w,m-1}\{\boldsymbol{0},\boldsymbol{u}_{hb}\},\nabla_{w,m-1}\{\boldsymbol{v}_{hi},\boldsymbol{0}\}\big)_K \\
&+\nu\left\langle \eta\boldsymbol{u}_{hb},\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right\rangle_{\partial K} - \frac{1}{2}\big(\nabla_{w,m}\cdot\{\boldsymbol{0}\otimes\boldsymbol{0},\boldsymbol{u}_{hb}\otimes\boldsymbol{u}_{hb}\},\boldsymbol{v}_{hi}\big)_K \\
&-\big(\nabla_{w,m}\{0,p_{hb}\},\boldsymbol{v}_{hi}\big)_K.
\end{aligned}
$$

By following the same routines as in the proofs of Theorems 3.1 and 3.2, we can get existence and uniqueness results of (6.1).

Theorem 6.1. *For all $K\in\mathcal{T}_h$ and given numerical traces $\boldsymbol{u}_{hb}|_{\partial K}$ and $p_{hb}|_{\partial K}$, the local problem* (6.1) *admits at least one solution. In addition, under the smallness condition*

$$
\frac{\mathcal{N}_{h,K}\|F_{h,K}\|_{*,h}}{\nu^2} < 1, \tag{6.2}
$$

the problem (6.1) *admits a unique solution. Here*

$$
\begin{aligned}
\mathcal{N}_{h,K} &:= \sup_{\boldsymbol{0}\neq\boldsymbol{\kappa}_{hi},\boldsymbol{u}_{hi},\boldsymbol{v}_{hi}\in\boldsymbol{V}_{0h,K}} \frac{d_{h,K}(\boldsymbol{\kappa}_{hi};\boldsymbol{u}_{hi},\boldsymbol{v}_{hi})}{|||\boldsymbol{\kappa}_{hi}|||_{V,K}\cdot|||\boldsymbol{u}_{hi}|||_{V,K}\cdot|||\boldsymbol{v}_{hi}|||_{V,K}}, \\
\|F_{h,K}\|_{*,h} &:= \sup_{\boldsymbol{0}\neq\boldsymbol{v}_{hi}\in\boldsymbol{V}_{0h,K}} \frac{F_{h,K}(\boldsymbol{v}_{hi})}{|||\boldsymbol{v}_h|||_{V,K}}, \\
\boldsymbol{V}_{0h,K} &:= \Big\{\boldsymbol{\kappa}_{hi}\in[P_m(K)]^n : b_{h,K}(\boldsymbol{\kappa}_{hi},q_{hi})=0,\ \forall q_{hi}\in P_{m-1}(K)\Big\}, \\
|||\boldsymbol{v}_{hi}|||_{V,K} &:= \left(\|\nabla_{w,m-1}\{\boldsymbol{v}_{hi},\boldsymbol{0}\}\|_{0,K}^2 + \left\|\eta^{\frac{1}{2}}\boldsymbol{\Pi}_k^B\boldsymbol{v}_{hi}\right\|_{0,\partial K}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

6.2. Iteration scheme

Due to the nonlinearity of the WG scheme (2.6), we shall employ the following Oseen type iteration algorithm:

Given $\boldsymbol{u}_h^0$, seek $(\boldsymbol{u}_h^l,p_h^l)$ with $l=1,2,\ldots$ such that

$$
\begin{aligned}
&a_h\big(\boldsymbol{u}_h^l,\boldsymbol{v}_h\big)+b_h\big(\boldsymbol{v}_h,p_h^l\big)+c_h\big(\boldsymbol{u}_h^{l-1};\boldsymbol{u}_h^l,\boldsymbol{v}_h\big) \\
&\quad + d_h\big(\boldsymbol{u}_h^{l-1};\boldsymbol{u}_h^l,\boldsymbol{v}_h\big) = (\boldsymbol{f},\boldsymbol{v}_{hi}), \\
&b_h\big(\boldsymbol{u}_h^l,q_h\big)=0
\end{aligned}
\tag{6.3}
$$

for $\forall(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h^0\times Q_h^0$.

It is not difficult to know that the linear system (6.3) is uni-solvent for given $(\boldsymbol{u}_h^{l-1},p_h^{l-1})$ and that it holds

$$
\big|\big|\big|\boldsymbol{u}_h^l\big|\big|\big|_V \le \frac{1}{\nu}\|\boldsymbol{f}\|_{*,h}, \quad l=1,2,\ldots. \tag{6.4}
$$

We have the following convergence result.

Theorem 6.2. *Assume that* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^0 \times Q_h^0$ *is the solution of the WG scheme* (2.6). *Under the condition*

$$2C_{\widetilde{r}}^r C_r \alpha \frac{\|\boldsymbol{f}\|_{*,h}^{r-2}}{\nu^{r-1}} + \frac{\mathcal{N}_h \|\boldsymbol{f}\|_{*,h}}{\nu^2} < 1 \tag{6.5}$$

the Oseen type iteration scheme (6.3) *is convergent in the following sense:*

$$\lim_{l\to\infty} |||\boldsymbol{u}_h^l - \boldsymbol{u}_h|||_V = 0, \quad \lim_{l\to\infty} |||p_h^l - p_h|||_Q = 0. \tag{6.6}$$

Proof. Denote $e_u^l := \boldsymbol{u}_h^l - \boldsymbol{u}_h$ and $e_p^l := p_h^l - p_h$. Subtracting (2.6) from (6.3) gives

$$\begin{aligned} &a_h(e_u^l, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, e_p^l) - c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h^l, \boldsymbol{v}_h) \\ &\quad - d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + d_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h^l, \boldsymbol{v}_h) = 0, \qquad (6.7a) \\ &b_h(e_u^l, q_h) = 0 \qquad (6.7b) \end{aligned}$$

for any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h^0 \times Q_h^0$. Taking $\boldsymbol{v}_h = e_u^l$, $q_h = e_p^l$ in (6.7) and using the definition of $c_h(\cdot;\cdot,\cdot)$ and the relation $d_h(e_u^{l-1}; e_u^l, e_u^l) = 0$, we have

$$\begin{aligned} \nu|||e_u^l|||_V^2 &= c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, e_u^l) - c_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h^l, e_u^l) + d_h(\boldsymbol{u}_h; \boldsymbol{u}_h, e_u^l) - d_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h^l, e_u^l) \\ &= c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, e_u^l) - c_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h, e_u^l) - c_h(\boldsymbol{u}_h^{l-1}; e_u^l, e_u^l) \\ &\quad + d_h(\boldsymbol{u}_h - \boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h, e_u^l) + \Big(d_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h, e_u^l) - d_h(\boldsymbol{u}_h^{l-1}; \boldsymbol{u}_h^l, e_u^l) \Big) \\ &= \alpha \Big((|\boldsymbol{u}_h|^{r-2} - |\boldsymbol{u}_h^{l-1}|^{r-2}) \boldsymbol{u}_h, e_u^l \Big) - c_h(\boldsymbol{u}_h^{l-1}; e_u^l, e_u^l) - d_h(e_u^{l-1}; \boldsymbol{u}_h, e_u^l). \end{aligned}$$

This relation, together with Lemmas 2.4, 2.9 and 3.1, the estimates (3.9) and (6.4), and the fact that $-c_h(\boldsymbol{u}_h^{l-1}; e_u^l, e_u^l) \le 0$, implies

$$\begin{aligned} \nu|||e_u^l|||_V^2 &\le \alpha \Big((|\boldsymbol{u}_h|^{r-2} - |\boldsymbol{u}_h^{l-1}|^{r-2}) \boldsymbol{u}_h, e_u^l \Big) - d_h(e_u^{l-1}; \boldsymbol{u}_h, e_u^l) \\ &\le \Big(C_{\widetilde{r}}^r C_r \alpha (|||\boldsymbol{u}_h|||_V^{r-3} + |||\boldsymbol{u}_h^{l-1}|||_V^{r-3}) + \mathcal{N}_h \Big) |||e_u^{l-1}|||_V \cdot |||\boldsymbol{u}_h|||_V \cdot |||e_u^l|||_V \\ &\le \left(2C_{\widetilde{r}}^r C_r \alpha \frac{\|\boldsymbol{f}\|_{*,h}^{r-2}}{\nu^{r-2}} + \frac{\mathcal{N}_h \|\boldsymbol{f}\|_{*,h}}{\nu} \right) |||e_u^{l-1}|||_V |||e_u^l|||_V, \end{aligned}$$

which gives

$$|||e_u^l|||_V \le \mathcal{M} |||e_u^{l-1}|||_V$$

with

$$\mathcal{M} := 2C_{\widetilde{r}}^r C_r \alpha \frac{\|\boldsymbol{f}\|_{*,h}^{r-2}}{\nu^{r-1}} + \frac{\mathcal{N}_h \|\boldsymbol{f}\|_{*,h}}{\nu^2}.$$

Thus, we get

$$|||e_u^l|||_V \le \mathcal{M} |||e_u^{l-1}|||_V \le \cdots \le \mathcal{M}^l |||e_u^0|||_V. \tag{6.8}$$

In view of (6.5), we know that $0 < \mathcal{M} < 1$. Hence, we obtain

$$\lim_{l\to\infty} |||e_u^l|||_V = \lim_{l\to\infty} |||\boldsymbol{u}_h^l - \boldsymbol{u}_h|||_V = 0. \tag{6.9}$$

The thing left is to prove the second convergence relation of (6.6). From (6.7a) it follows

$$\begin{aligned} b_h\big(\boldsymbol{v}_h, e_p^l\big) &= -a_h\big(e_u^l, \boldsymbol{v}_h\big) + c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) - c_h\big(\boldsymbol{u}_h^{l-1};\boldsymbol{u}_h^l,\boldsymbol{v}_h\big) \\ &\quad + d_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) - d_h\big(\boldsymbol{u}_h^{l-1};\boldsymbol{u}_h^l,\boldsymbol{v}_h\big) \\ &= -a_h\big(e_u^l,\boldsymbol{v}_h\big) + \Big(c_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h) - c_h\big(\boldsymbol{u}_h^{l-1};\boldsymbol{u}_h,\boldsymbol{v}_h\big)\Big) - c_h\big(\boldsymbol{u}_h^{l-1};e_u^l,\boldsymbol{v}_h\big) \\ &\quad - \Big(d_h\big(\boldsymbol{u}_h;e_u^l,\boldsymbol{v}_h\big) + d_h\big(e_u^{l-1};e_u^l,\boldsymbol{v}_h\big) + d_h\big(e_u^{l-1};\boldsymbol{u}_h,\boldsymbol{v}_h\big)\Big) \end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h^0$. By Lemma 3.2 we have

$$|||e_p^l|||_Q \lesssim \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h^0} \frac{b_h(\boldsymbol{v}_h,e_p^l)}{|||\boldsymbol{v}_h|||_V}.$$

The above two results, together with (6.9) and Lemmas 2.4, 2.9 and 3.1, yield the desired conclusion

$$\lim_{l\to\infty} |||e_p^l|||_Q = \lim_{l\to\infty} |||p_h^l - p_h|||_Q = 0.$$

This completes the proof. □

7. Numerical experiments

In this section, we provide some numerical tests to verify the performance of the WG scheme (2.6) for the Brinkman-Forchheimer model (1.1) in two dimensions. We adopt the Oseen type iterative algorithm (6.3) with the initial guess $\boldsymbol{u}_{hi}^0 = \boldsymbol{0}$ and the stop criterion

$$\big\|\boldsymbol{u}_h^l - \boldsymbol{u}_h^{l-1}\big\|_0 < 1e-8 \tag{7.1}$$

in all the numerical examples.

Example 7.1. Set $\Omega = [0,1]\times[0,1]$, $\nu = 1$, $\alpha = 5$ and $r = 10$ in the model (1.1). The exact solution $(\boldsymbol{u}, p)$ is given as follows:

$$\begin{cases} u_1 = 10x^2(x-1)^2y(y-1)(2y-1), \\ u_2 = -10x(x-1)(2x-1)y^2(y-1)^2, \\ p = 10(2x-1)^2(2y-1). \end{cases} \tag{7.2}$$

We compute the scheme (2.6) on uniform triangular meshes (cf. Fig. 1), with $m = 1,2$, $k = m-1, m$. Numerical results of $\|\boldsymbol{u}-\boldsymbol{u}_{hi}\|_0$, $\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_{hi}\|_0$, $\|p-p_{hi}\|_0$ and $\|\nabla\cdot\boldsymbol{u}_{hi}\|_{0,\infty}$ are listed in Tables 1 and 2.

Table 1: History of convergence results for Example 7.1: $m = 1$.

k	$mesh$	$\frac{\lVert\boldsymbol{u}-\boldsymbol{u}_{hi}\rVert_0}{\lVert\boldsymbol{u}\rVert_0}$		$\frac{\lVert\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_{hi}\rVert_0}{\lVert\nabla\boldsymbol{u}\rVert_0}$		$\frac{\lVert p-p_{hi}\rVert_0}{\lVert p\rVert_0}$		$\lVert\nabla_h\cdot\boldsymbol{u}_{hi}\rVert_{0,\infty}$
		Error	Rate	Error	Rate	Error	Rate	Error
0	4×4	5.9583e-01	-	5.1516e-01	-	2.8667e-01	-	4.0593e-16
	8×8	1.5876e-01	1.91	2.7301e-01	0.92	1.4424e-01	0.99	2.3028e-16
	16×16	4.1525e-02	1.93	1.3851e-01	0.98	7.2201e-02	1.00	3.1127e-16
	32×32	1.0641e-02	1.96	6.9420e-02	1.00	3.6100e-02	1.00	8.4459e-17
	64×64	2.6985e-03	1.98	3.4723e-02	1.00	1.8048e-02	1.00	2.7905e-17
	128×128	6.9479e-04	1.96	1.7364e-02	1.00	9.0235e-03	1.00	5.6257e-17
1	4×4	5.6714e-01	-	5.1165e-01	-	2.8667e-01	-	3.4694e-18
	8×8	1.5224e-01	1.90	2.7237e-01	0.91	1.4425e-01	0.99	3.2092e-17
	16×16	3.9918e-02	1.93	1.3841e-01	0.98	7.2213e-02	1.00	1.3010e-18
	32×32	1.0236e-02	1.96	6.9404e-02	1.00	3.6110e-02	1.00	2.9328e-17
	64×64	2.5908e-03	1.98	3.4720e-02	1.00	1.8054e-02	1.00	8.2115e-17
	128×128	6.5160e-04	1.99	1.7363e-02	1.00	9.0267e-03	1.00	1.4732e-17

Table 2: History of convergence results for Example 7.1: $m = 2$.

k	$mesh$	$\frac{\lVert\boldsymbol{u}-\boldsymbol{u}_{hi}\rVert_0}{\lVert\boldsymbol{u}\rVert_0}$		$\frac{\lVert\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_{hi}\rVert_0}{\lVert\nabla\boldsymbol{u}\rVert_0}$		$\frac{\lVert p-p_{hi}\rVert_0}{\lVert p\rVert_0}$		$\lVert\nabla_h\cdot\boldsymbol{u}_{hi}\rVert_{0,\infty}$
		Error	Rate	Error	Rate	Error	Rate	Error
1	4×4	5.9915e-02	-	1.3040e-01	-	3.3330e-02	-	2.2560e-14
	8×8	7.6055e-03	2.98	3.4961e-02	1.90	8.3117e-03	2.00	7.0083e-16
	16×16	9.4731e-04	3.01	8.9617e-03	1.96	2.0761e-03	2.00	1.9606e-15
	32×32	1.1827e-04	3.00	2.2616e-03	1.99	5.1888e-04	2.00	7.4921e-16
	64×64	1.4789e-05	3.00	5.6761e-04	1.99	1.2970e-04	2.00	2.5543e-17
	128×128	1.8494e-06	3.00	1.4215e-04	2.00	3.2423e-05	2.00	3.5312e-16
2	4×4	5.6852e-02	-	1.3016e-01	-	3.3281e-02	-	1.1310e-15
	8×8	7.3762e-03	2.95	3.4894e-02	1.90	8.2971e-03	2.00	1.7295e-15
	16×16	9.3175e-04	2.98	8.9496e-03	1.96	2.0724e-03	2.00	2.5093e-15
	32×32	1.1713e-04	2.99	2.2591e-03	1.99	5.1795e-04	2.00	4.0441e-17
	64×64	1.4693e-05	3.00	5.6704e-04	1.99	1.2947e-04	2.00	1.0278e-15
	128×128	1.8403e-06	3.00	1.4201e-04	2.00	3.2366e-05	2.00	1.1529e-16

Example 7.2. Set $\Omega = [0,1]\times[0,1]$, $\nu = 1, \alpha = 0.01$ and $r = 10$ in the model (1.1). The exact solution $(\boldsymbol{u}, p)$ is given as follows:

$$\begin{cases} u_1 = \sin(\pi x)\cos(\pi y), \\ u_2 = -\cos(\pi x)\sin(\pi y), \\ p = x^6 - y^6. \end{cases} \tag{7.3}$$

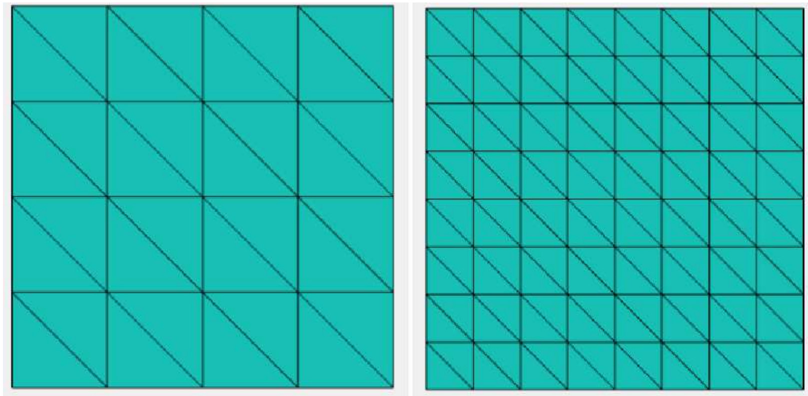


Figure 1: Uniform triangular meshes: 4×4 mesh (left) and 8×8 mesh (right).

We compute the scheme (2.6) on uniform triangular meshes (cf. Fig. 1), with $m = 1, 2$ and $k = m-1, m$. Numerical results of $\|\boldsymbol{u} - \boldsymbol{u}_{hi}\|_0$, $\|\nabla \boldsymbol{u} - \nabla_h \boldsymbol{u}_{hi}\|_0$, $\|p - p_{hi}\|_0$ and $\|\nabla \cdot \boldsymbol{u}_{hi}\|_{0,\infty}$ are listed in Tables 3 and 4.

Table 3: History of convergence results for Example 7.2: $m = 1$.

k	$mesh$	$\frac{\|\boldsymbol{u}-\boldsymbol{u}_{hi}\|_0}{\|\boldsymbol{u}\|_0}$		$\frac{\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_{hi}\|_0}{\|\nabla\boldsymbol{u}\|_0}$		$\frac{\|p-p_{hi}\|_0}{\|p\|_0}$		$\|\nabla_h \cdot \boldsymbol{u}_{hi}\|_{0,\infty}$
		Error	Rate	Error	Rate	Error	Rate	Error
0	4×4	1.6005e-01	-	2.6991e-01	-	5.2020e-01	-	2.5121e-15
	8×8	3.7542e-02	2.13	1.3803e-01	0.98	2.5615e-01	1.01	3.7682e-15
	12×12	1.5153e-02	2.28	9.2644e-02	0.98	1.7035e-01	1.00	1.1723e-14
	16×16	7.7371e-03	2.33	6.9833e-02	0.98	1.2785e-01	1.00	1.0049e-14
1	4×4	1.5501e-01	-	2.6999e-01	-	8.2265e-01	-	7.5364e-15
	8×8	3.9277e-02	1.99	1.3795e-01	0.98	4.0158e-01	1.02	1.2561e-15
	12×12	1.7508e-02	1.99	9.2386e-02	0.99	2.6654e-01	1.01	6.6991e-15
	16×16	9.8669e-03	1.99	6.9411e-02	0.99	1.9970e-01	1.00	2.5121e-15

Table 4: History of convergence results for Example 7.2: $m = 2$.

k	$mesh$	$\frac{\|\boldsymbol{u}-\boldsymbol{u}_{hi}\|_0}{\|\boldsymbol{u}\|_0}$		$\frac{\|\nabla\boldsymbol{u}-\nabla_h\boldsymbol{u}_{hi}\|_0}{\|\nabla\boldsymbol{u}\|_0}$		$\frac{\|p-p_{hi}\|_0}{\|p\|_0}$		$\|\nabla_h \cdot \boldsymbol{u}_{hi}\|_{0,\infty}$
		Error	Rate	Error	Rate	Error	Rate	Error
1	4×4	1.5314e-02	-	4.3800e-02	-	1.6032e-01	-	2.4925e-15
	8×8	1.9010e-03	3.01	1.1121e-02	1.98	3.4978e-02	2.15	1.0206e-14
	12×12	5.6253e-04	3.00	4.9640e-03	1.99	1.4918e-02	2.09	2.6064e-14
	16×16	2.3740e-04	3.00	2.7977e-03	1.99	8.2361e-03	2.06	1.4461e-13
2	4×4	1.4859e-02	-	4.3805e-02	-	1.4305e-01	-	5.3069e-14
	8×8	1.8806e-03	2.99	1.1123e-02	1.98	2.9207e-02	2.23	3.6426e-14
	12×12	5.5839e-04	3.00	4.9648e-03	1.99	1.2228e-02	2.13	1.6748e-15
	16×16	2.3593e-04	2.99	2.7981e-03	1.99	6.6989e-03	2.08	5.7779e-14

From the numerical results listed in Tables 1-4, we have the following observations:

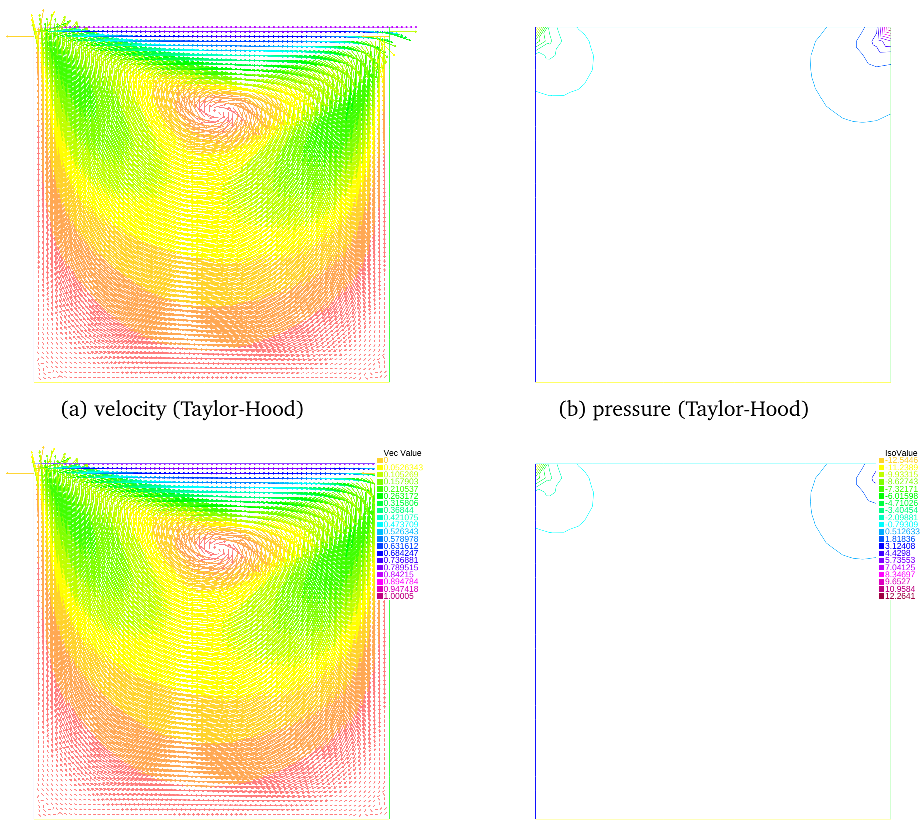
- The convergence rates of $\|\nabla \boldsymbol{u} - \nabla_h \boldsymbol{u}_{hi}\|_0$ and $\|p - p_{hi}\|_0$ for the WG scheme are m-th orders in the cases of $m = 1, 2$ and $k = m, m-1$. These are conformable to the theoretical results in Theorem 4.1.

- The convergence rate of $\|\boldsymbol{u} - \boldsymbol{u}_{hi}\|_0$ is $(m+1)$-th order, which is conformable to the theoretical result in Theorem 5.1.

- The results of $\|\nabla_h \cdot \boldsymbol{u}_{hi}\|_{0,\infty}$ are almost zero. This means that the discrete velocity is globally divergence-free, which is consistent with Theorem 2.1.

Example 7.3 (The lid-driven cavity flow problem)**.** This problem is used to test the influence of damping parameters α and r on the solution of the WG scheme. Take $\Omega = [0, 1] \times [0, 1]$, $\nu = 0.1$ and $\boldsymbol{f} = \boldsymbol{0}$. The boundary conditions are as follows:

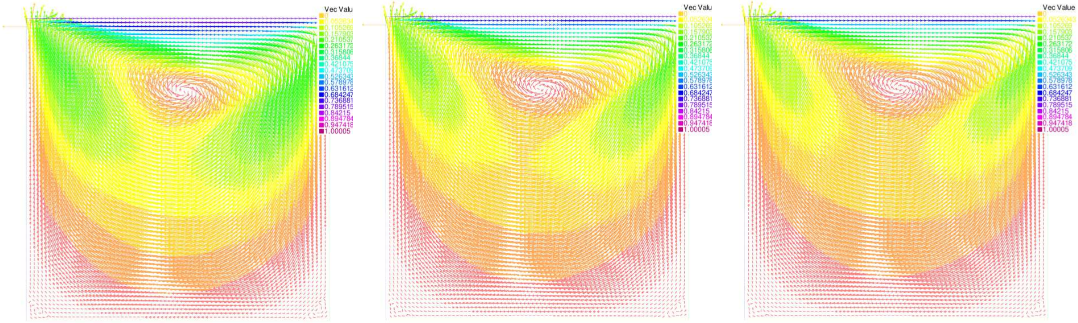
$$\boldsymbol{u}|_{x=0} = \boldsymbol{u}|_{x=1} = \boldsymbol{u}|_{y=0} = \boldsymbol{0}, \quad \boldsymbol{u}|_{y=1} = (1, 0)^\top.$$



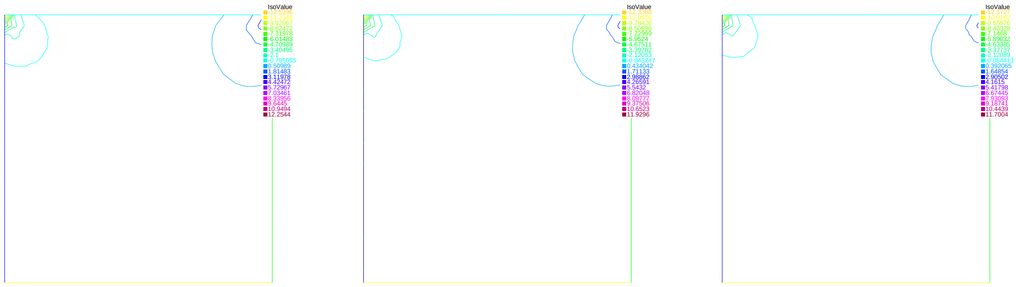
(a) velocity (Taylor-Hood) (b) pressure (Taylor-Hood)

(c) velocity (WG) (d) pressure (WG)

Figure 2: The velocity streamlines and pressure contours for Example 7.3: $\alpha = 0$.



(a) velocity: $\alpha = 1$ (b) velocity: $\alpha = 50$ (c) velocity: $\alpha = 100$



(d) pressure: $\alpha = 1$ (e) pressure: $\alpha = 50$ (f) pressure: $\alpha = 100$

Figure 3: The velocity streamlines and pressure contours for Example 7.3: $r = 5$ and $\alpha = 1, 50, 100$.

We compute the WG scheme (2.6) with $m = k = 2$ on the 25×25 uniform triangular mesh (cf. Fig. 1) in the following cases:

(I) $\alpha = 0$, i.e. the case of the Navier-Stokes equations;

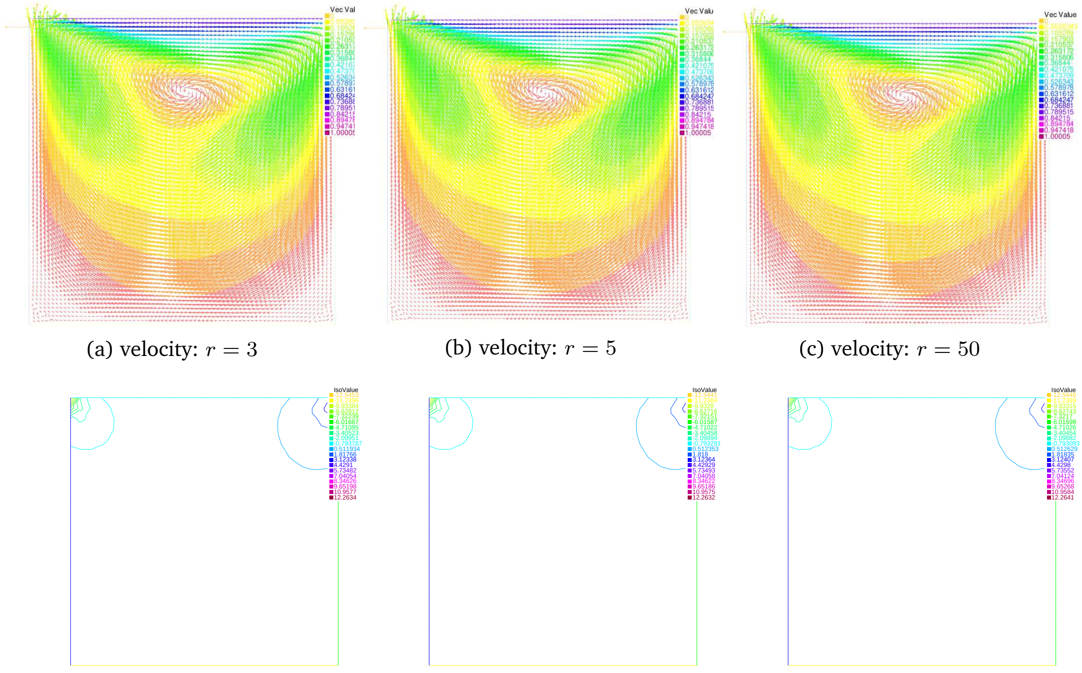
(II) $r = 5$ and $\alpha = 1, 50, 100$;

(III) $\alpha = 5$ and $r = 3, 5, 50$.

The velocity streamlines and the pressure contours are displayed in Figs. 2-4. As a comparison, the referenced numerical solutions obtained with the Taylor-Hood element are also shown for $\alpha = 0$; see Figs. 2(a) and 2(b).

From Fig. 3 we can see that the shape and size of the vortex change evidently, which means that the damping effect becomes greater for the velocity as the damping parameter α increases. We can also see that the pressure approximation is not significantly affected by α. On the other hand, as shown in Fig. 4, the velocity and pressure approximations are not significantly effected by the number r.

Example 7.4 (The problem of flow around a circular cylinder)**.** The flow around a circular cylinder is examined with the Brinkman-Forchheimer model (1.1) and the WG method. We take $\Omega = [0,6] \times [0,1] \setminus O_d(1, 0.5)$, $\nu = 0.002$ and $\boldsymbol{f} = \boldsymbol{0}$, where $O_d(1, 0.5)$ is a disk with center $(1, 0.5)$ and diameter $d = 0.3$; see Fig. 5 for the domain and its finite



(a) velocity: $r=3$ (b) velocity: $r=5$ (c) velocity: $r=50$

(d) pressure: $r=3$ (e) pressure: $r=5$ (f) pressure: $r=50$

Figure 4: The velocity streamlines and pressure contours for Example 7.3: $\alpha=5$ and $r=3,5,50$.

element mesh. The boundary conditions are as follows:

$$\begin{aligned}
&\boldsymbol{u}|_{y=0}=\boldsymbol{u}|_{y=1}=\boldsymbol{u}|_{\partial O_d}=\boldsymbol{0},\\
&\boldsymbol{u}|_{x=0}=\big(6y(1-y),0\big)^\top,\\
&(-p\boldsymbol{I}+\nu\nabla\boldsymbol{u})\,\boldsymbol{n}\big|_{x=6}=0,
\end{aligned}$$

where $\boldsymbol{I}$ and $\boldsymbol{n}$ are the unit matrix and the outward unit normal vector, respectively. We compute the WG scheme (2.6) with $m=k=2$ in the following cases:

(I) $\alpha=0$, i.e. the case of the Navier-Stokes equations;

(II) $r=3.5$ and $\alpha=0.1,1,10$;

(III) $\alpha=1$ and $r=3,4,5$.

The obtained velocity, vorticity and pressure approximations are shown in Figs. 6-8, respectively. As a comparison, the referenced numerical solutions obtained with the Taylor-Hood element are also shown for $\alpha=0$; see Figs. 6(a)-6(c). We can see that our method is effective and the damping effect is gradually enhanced as the parameters α and r increase.

Example 7.5 (The backward-facing step flow problem). We consider a backward-facing step flow problem in $\Omega=\Omega_1\setminus\Omega_2$, with $\Omega_1=[-4,16]\times[-1,2]$ and $\Omega_2=$

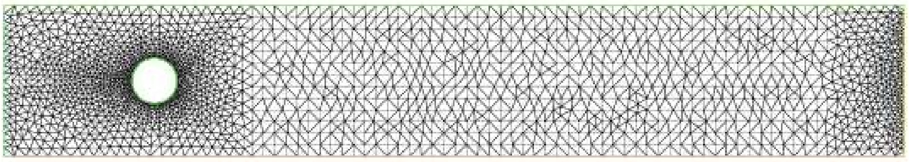
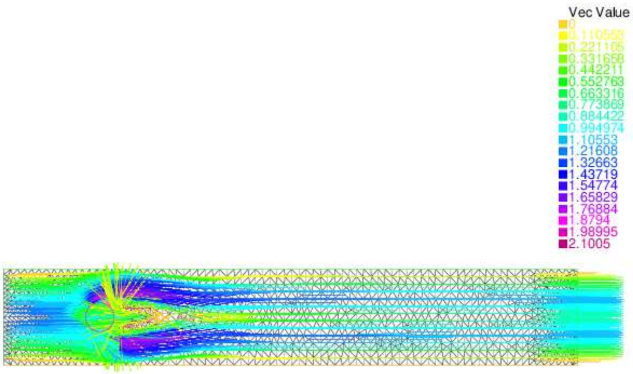


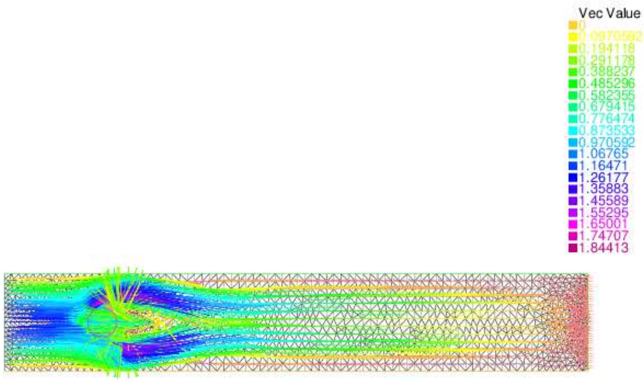
Figure 5: The domain and finite element mesh for Example 7.4.



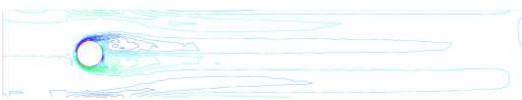
(a) velocity (Taylor-Hood)

(b) vorticity (Taylor-Hood)

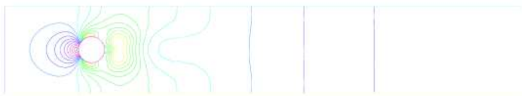
(c) pressure (Taylor-Hood)



(d) velocity (WG)

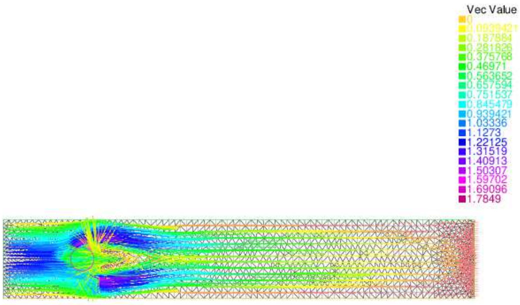


(e) vorticity (WG)

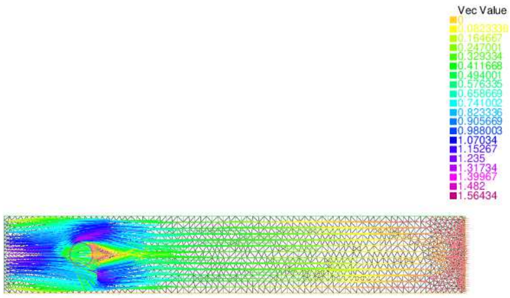


(f) pressure (WG)

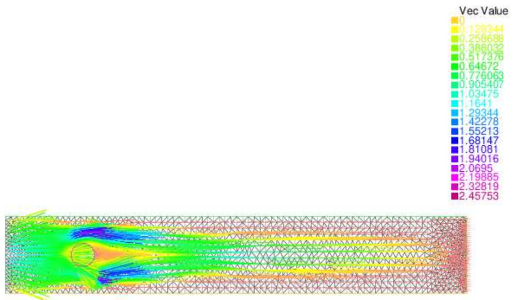
Figure 6: The velocity streamlines, vortex lines and pressure contours for Example 7.4: $\alpha = 0$.



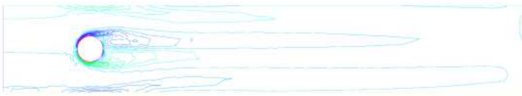
(a) velocity: $\alpha = 0.1$



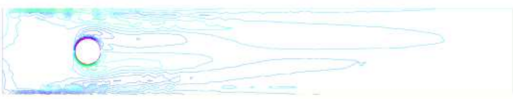
(b) velocity: $\alpha = 1$



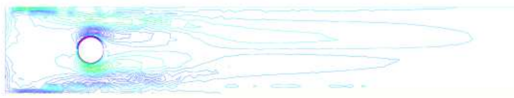
(c) velocity: $\alpha = 10$



(d) vorticity: $\alpha = 0.1$



(e) vorticity: $\alpha = 1$



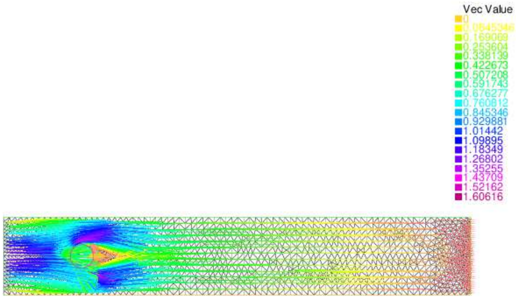
(f) vorticity: $\alpha = 10$

(g) pressure: $\alpha = 0.1$

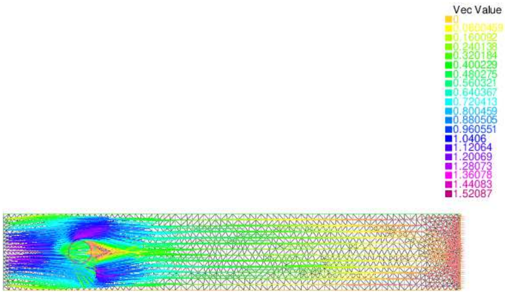
(h) pressure: $\alpha = 1$

(i) pressure: $\alpha = 10$

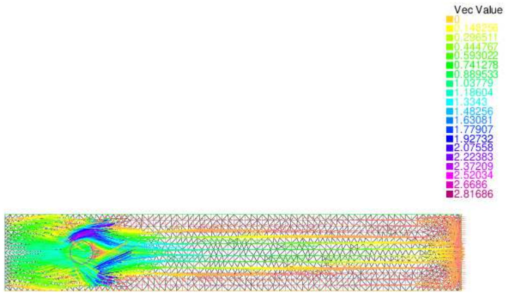
Figure 7: The velocity streamlines, vortex lines and pressure contours for Example 7.4: $r = 3.5$ and $\alpha = 0.1, 1, 10$.



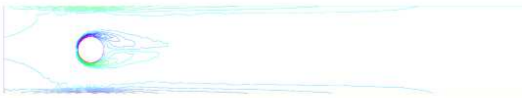
(a) velocity: $r = 3$



(b) velocity: $r = 4$



(c) velocity: $r = 5$



(d) vorticity: $r = 3$

(e) vorticity: $r = 4$

(f) vorticity: $r = 5$

(g) pressure: $r = 3$

(h) pressure: $r = 4$

(i) pressure: $r = 5$

Figure 8: The velocity streamlines, vortex lines and pressure contours for Example 7.4: $\alpha = 1$ and $r = 3, 4, 5$.

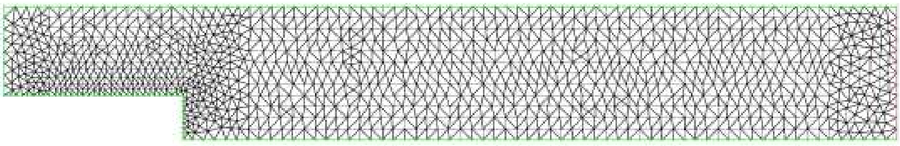


Figure 9: The domain and finite element mesh for Example 7.5.

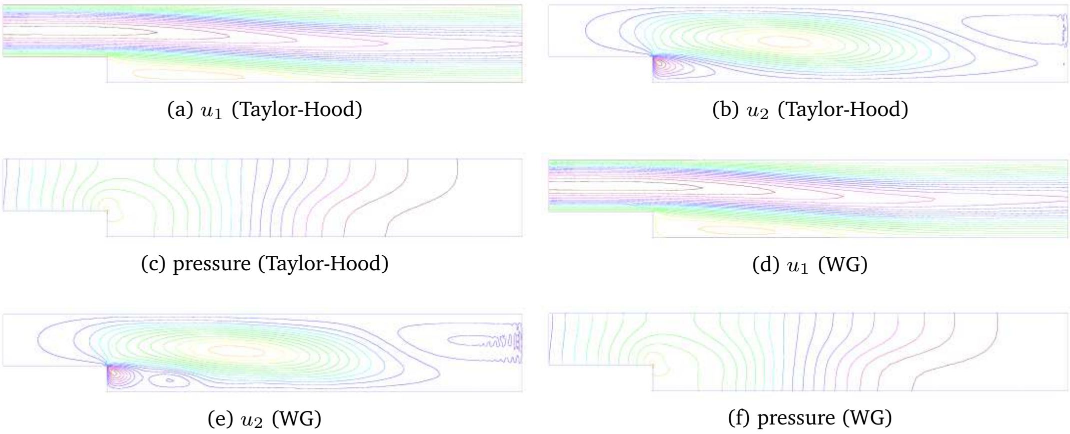


Figure 10: The velocity $\boldsymbol{u}_h = (u_1, u_2)^\top$ and pressure contours for Example 7.5: $\alpha = 0$.

$[-4,0]\times[-1,0]$; see Fig. 9 for the domain and its finite element mesh. We take $\nu = 0.005$ and $\boldsymbol{f} = \boldsymbol{0}$. The boundary conditions are as follows:

$$\boldsymbol{u}|_{y=-1} = \boldsymbol{u}|_{y=2} = \boldsymbol{u}|_{-4\le x\le 0, y=0} = \boldsymbol{u}|_{x=0,-1\le y\le 0} = \boldsymbol{0},$$
$$\boldsymbol{u}|_{x=-4} = \big(y(2-y),0\big)^\top, \quad \left(-p+\nu\frac{\partial u_1}{\partial x}\right)|_{x=16} = 0, \quad u_2|_{x=16} = 0.$$

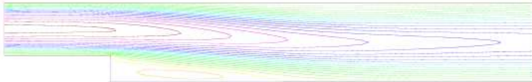
We compute the WG scheme (2.6) with $m = k = 2$ in the following cases:

(I) $\alpha = 0$, i.e. the case of the Navier-Stokes equations;

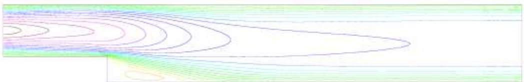
(II) $r = 3.5$ and $\alpha = 0.01, 0.1, 1$;

(III) $\alpha = 1$ and $r = 5, 10, 50$.

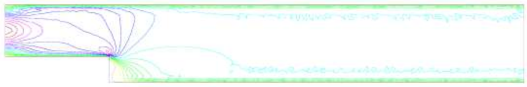
The obtained velocity and pressure approximations are shown in Figs. 10-12. As a comparison, the numerical solutions obtained with the Taylor-Hood element are also shown for $\alpha = 0$; see Figs. 10(a)-10(c). Similar to Example 7.4, we can see that our method is effective and the damping effect is gradually enhanced as the parameters α and r increase.



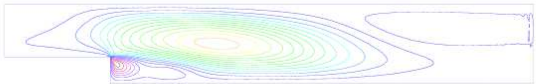
(a) u_1: $\alpha = 0.01$



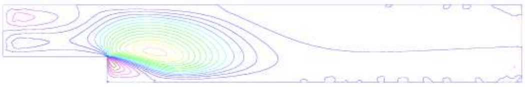
(b) u_1: $\alpha = 0.1$



(c) u_1: $\alpha = 1$

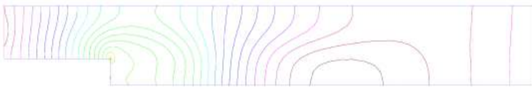


(d) u_2: $\alpha = 0.01$



(e) u_2: $\alpha = 0.1$

(f) u_2: $\alpha = 1$



(g) pressure: $\alpha = 0.01$

(h) pressure: $\alpha = 0.1$

(i) pressure: $\alpha = 1$

Figure 11: The velocity $\boldsymbol{u}_h = (u_1, u_2)^\top$ and pressure contours for Example 7.5: $r = 3.5$ and $\alpha = 0.01, 0.1, 1$.

8. Conclusion

We have developed a class of WG methods of arbitrary order for the steady Brinkman-Forchheimer equations. The methods yield globally divergence-free velocity and are pressure robust. Optimal error estimates have been derived for the velocity and pressure approximations. The proposed Oseen type iteration algorithm is unconditionally convergent. Numerical experiments have verified the theoretical analysis and demonstrated the robustness of the methods.

Acknowledgments

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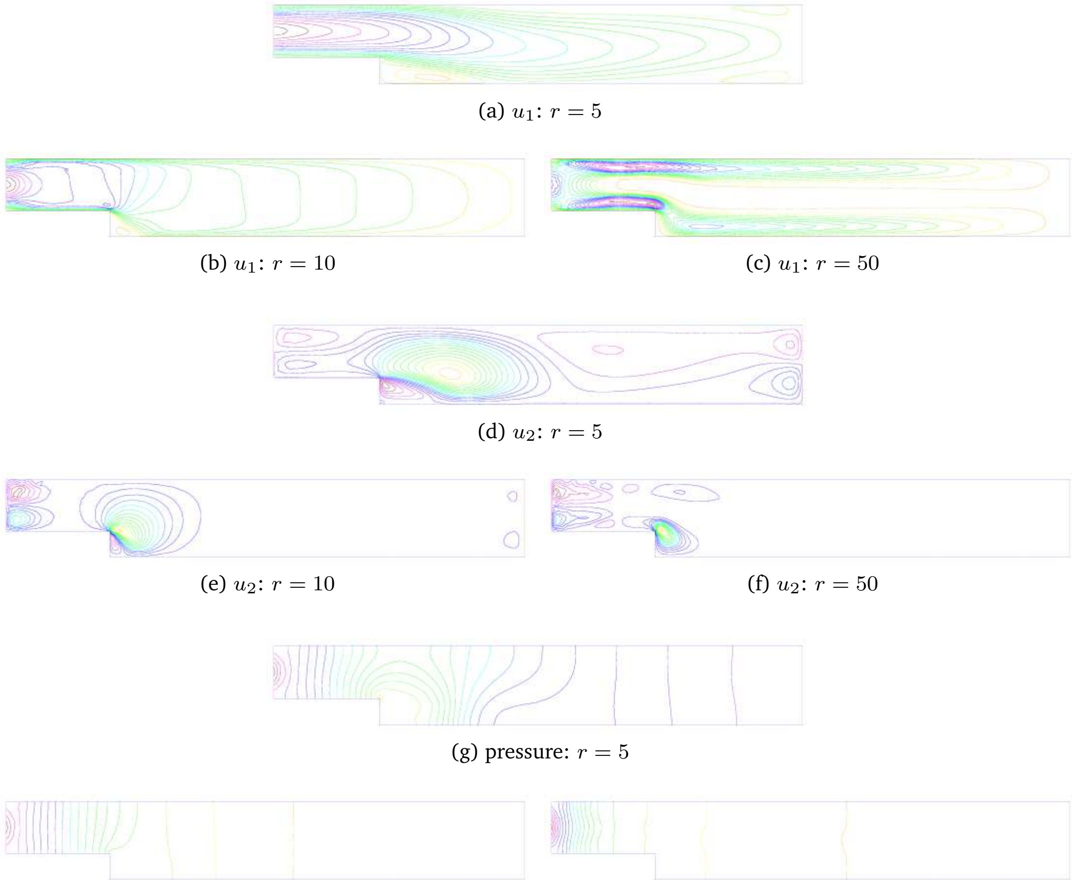

(a) u_1: $r = 5$

(b) u_1: $r = 10$ (c) u_1: $r = 50$

(d) u_2: $r = 5$

(e) u_2: $r = 10$ (f) u_2: $r = 50$

(g) pressure: $r = 5$

(h) pressure: $r = 10$ (i) pressure: $r = 50$

Figure 12: The velocity $\boldsymbol{u}_h = (u_1, u_2)^\top$ and pressure contours for Example 7.5: $\alpha = 1$ and diverse $r = 5, 10, 50$.

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