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# A Divergence-Free Pk CDG Finite Element for the Stokes Equations on Triangular and Tetrahedral Meshes

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**Abstract.** In the conforming discontinuous Galerkin method, the standard bilinear form for the conforming finite elements is applied to discontinuous finite elements without adding any inter-element nor penalty form. The $P_k$ $(k \geq 1)$ discontinuous finite elements and the $P_{k-1}$ weak Galerkin finite elements are adopted to approximate the velocity and the pressure respectively, when solving the Stokes equations on triangular or tetrahedral meshes. The discontinuous finite element solutions are divergence-free and surprisingly H-div functions on the whole domain. The optimal order convergence is achieved for both variables and for all $k \geq 1$. The theory is verified by numerical examples.

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**Key words**: Finite element, conforming discontinuous Galerkin method, Stokes equations, stabilizer free, divergence free.

## 1. Introduction

In this paper, we introduce a divergence-free conforming discontinuous Galerkin method to solve the Stokes equations, finding unknown velocity $\mathbf{u}$ and pressure $p$ such that

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where viscosity $\mu > 0$ is a constant, and $\Omega$ is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$. The weak form of the Stokes equations is: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$

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such that

$$(\mu\nabla\mathbf{u},\nabla\mathbf{v})-(p,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad \forall\,\mathbf{v}\in H_0^1(\Omega), \tag{1.4}$$

$$(q,\nabla\cdot\mathbf{u})=0,\qquad \forall\,q\in L_0^2(\Omega), \tag{1.5}$$

where $L_0^2(\Omega)$ is the $L^2$ space with mean value $0$ on the domain.

To get divergence-free finite element solutions, the conforming finite element method employs the continuous $P_k$ finite elements and the discontinuous $P_{k-1}$ finite elements approximating the velocity and the pressure respectively on triangular or tetrahedral meshes, using the bilinear forms in (1.4)-(1.5). However, most such combinations are not stable. The first such a working method was discovered by Scott and Vogelius [34, 35], on 2D triangular meshes for all $k\ge 4$, provided that the underlying meshes have no nearly-singular vertex. Such divergence-free finite elements are studied on special 2D and 3D meshes, or with high-order polynomial and/or macro bubbles, in [3, 7, 8, 10–12, 14–16, 18, 22, 27, 30–33, 44, 60–68, 68].

Another method, to get divergence-free finite element solutions, is to employ $H(\text{div})$ finite element functions on triangular and tetrahedral meshes. That is, the $H(\text{div})$-$P_k$/discontinuous-$P_{k-1}$ mixed element also produces divergence-free solutions, cf. [26, 37, 39, 42]. However, as the $H(\mathrm{div})$ finite element is not $H^1$-conforming, in most cases, additional inter-element and penalty forms are added to the variational formulation (1.4)-(1.5). One can get rid of all such stabilizers by using proper weak gradients and weak divergences [28, 29, 49, 50, 69].

A completely new method, to get divergence-free finite element solutions, is to be proposed in this work. We start with totally discontinuous $P_k$ ($k\ge 1$) polynomials to approximate the velocity. The resulting discontinuous $P_k$ solutions are no longer discontinuous, but are continuous in the normal direction on all edges/triangles. That is, the finite element solutions are in the $H(\text{div},\Omega)$ space and are point-wise divergence-free. Let $\mathcal{T}_h=\{T\}$ be a quasi-uniform triangular or tetrahedral mesh with mesh-size $h$. The conforming discontinuous Galerkin (CDG) finite element spaces are defined by

$$\mathbf{V}_h=\left\{\mathbf{u}_h\in L^2(\Omega):\mathbf{u}_h|_T\in\mathbf{P}_k(T),\ T\in\mathcal{T}_h\right\}, \tag{1.6}$$

where $\mathbf{P}_k(T)$ is the space of 2D or 3D vector polynomials of degree $k\ge 1$ or less on $T$. Some references on the CDG methods can be found in [9, 47, 48, 51–56]. The weak Galerkin (WG) finite element spaces are defined by

$$\begin{aligned}P_h=\big\{q_h=\{q_0,q_b\}:q_0|_T\in P_{k-1}(T),\ T\in\mathcal{T};\ q_b|_e\in P_k(e),\\ e\in\mathcal{E}_h;(q_0,1)_{\mathcal{T}_h}+\langle q_b,1\rangle_{\partial\mathcal{T}_h}=0\big\},\end{aligned} \tag{1.7}$$

where $\mathcal{E}_h$ is the set of edges or face-triangles in mesh $\mathcal{T}_h$, $(\cdot,\cdot)_{\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}(\cdot,\cdot)_T$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial T}$. Some references for the WG methods are [1, 2, 4, 6, 13, 17, 19–21, 23–25, 36, 38, 40, 41, 43, 46, 57–59, 70]. The proposed CDG finite element method for the Stokes equations reads: Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times P_h$ such that

$$(\mu\nabla_w\mathbf{u}_h,\nabla_w\mathbf{v}_h)+(\nabla_w p_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h),\quad\forall\,\mathbf{v}_h\in\mathbf{V}_h, \tag{1.8}$$

$$(\nabla_w q_h,\mathbf{u}_h)=0,\qquad\forall\,q_h\in P_h, \tag{1.9}$$

where $\mathbf{V}_h$ and $P_h$ are defined in (1.6) and (1.7) respectively, and the weak gradients $\nabla_w \mathbf{u}_h$ and $\nabla_w p_h$ are defined in (3.2) and (2.3) below, respectively. It is shown that both the solutions $\mathbf{u}_h$ and $p_h$ of (1.8)-(1.9) converge quasi-optimally. Additionally, the solution $\mathbf{u}_h$ is an $H(\text{div})$ function and is divergence-free. Consequently, the method is pressure robust, i.e., both the velocity error and the pressure error are independent of $p$ and $\mu$. We verify the theory with some numerical examples in 2D and 3D.

An $H(\text{div})$-conforming HDG method is proposed in [5], where, comparing to our CDG-WG method, an inter-element trace velocity $\mathbf{u}_b$ space $[P_k(\mathcal{E}_h)]^d$ and its two Lagrange multiplier spaces $[P_k(\mathcal{E}_h)]$ for $(\mathbf{u}-\mathbf{u}_b)\cdot\mathbf{n}=0$, are added. Here $\mathcal{E}_h$ is the set of edges/triangles in a triangular/tetrahedral mesh $\mathcal{T}_h$. Thus the normal jump of the discontinuous velocity $\mathbf{u}_h$ is forced to zero by these five (in 3D) $P_k$ trace spaces so that the resulting solution is also in $H(\text{div})$ and consequently also divergence-free. The method here is to choose a proper pressure finite element space $P_h$ so that the gradient space is precisely the BDM$_k$ space, i.e., $\nabla P_h = \{\mathbf{v}_h \in H(\text{div},\Omega) : \mathbf{v}_h|_T \in [P_k(T)]^d\}$. If the pressure space is a little larger, there is no solution as the inf-sup condition would fail. If the pressure space is a little smaller, it would not force the velocity solution $\mathbf{u}_h$ to be in $H(\text{div})$, neither divergence-free.

The rest of the paper is organized as follows. In Section 2, the auxiliary WG finite element method is defined and the known results on the WG method are quoted. In Section 3, the CDG method is defined and the uniqueness of the solution, its $H(\text{div})$ conformity and its inf-sup condition are also proved. In Section 4, pressure-robust error estimates for the velocity in $H^1$-norm and for the pressure in $L^2$-norm are established. In Section 5, optimal order and pressure-robust convergence for the velocity in $L^2$-norm is proved. In Section 6, we provide several numerical examples in 2D and 3D.

## 2. Preliminary on WG

For the purpose of error analysis, we define a weak Galerkin finite element space as follows for approximating the velocity,

$$\tilde{\mathbf{V}}_h = \big\{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0|_T \in \mathbf{P}_k(T),\ T \in \mathcal{T}_h, \mathbf{v}_b|_e \in \mathbf{P}_{k+1}(e),\\ e \in \mathcal{E}_h,\ \mathbf{v}_b|_{\partial\Omega} = \mathbf{0}\big\}. \tag{2.1}$$

For a weak function $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{\mathbf{V}}_h$, its weak gradient $\nabla_w \mathbf{v}$ is a piecewise polynomial such that $\nabla_w \mathbf{v}|_T \in [P_{k+1}(T)]^{d\times d}$ and satisfies the following equation:

$$(\nabla_w \mathbf{v}_h, \tau)_T = -(\mathbf{v}_0, \nabla\cdot\tau)_T + \langle \mathbf{v}_b,\ \tau\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\, \tau \in [P_{k+1}(T)]^{d\times d}, \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial T$.

For a weak pressure function $q = \{q_0, q_b\} \in P_h$ in (1.7), its weak gradient $\nabla_w q$ is defined as a piecewise vector-valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w q \in \mathbf{P}_k(T)$ satisfies

$$(\nabla_w q_h, \boldsymbol{\varphi})_T = -(q_0, \nabla\cdot\boldsymbol{\varphi})_T + \langle q_b, \boldsymbol{\varphi}\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\, \boldsymbol{\varphi} \in [P_k(T)]^d. \tag{2.3}$$

We introduce two semi-norms $|||\mathbf{v}|||$ and $\|\mathbf{v}\|_{1,h}$ for any $\mathbf{v} \in \tilde{\mathbf{V}}_h$ as follows:

$$|||\mathbf{v}|||^2 = (\nabla_w \mathbf{v}, \nabla_w \mathbf{v}), \tag{2.4}$$

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla \mathbf{v}_0\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2, \tag{2.5}$$

and two semi-norms $|||q|||$ and $\|q\|_{1,h}$ for $q \in P_h$ as follows:

$$|||q|||^2 = (\nabla_w q, \nabla_w q), \tag{2.6}$$

$$\|q\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla q_0\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1} \|q_0 - q_b\|_{\partial T}^2. \tag{2.7}$$

The following norm equivalence inequalities have been proved in [2]:

$$C_1 \|\mathbf{v}\|_{1,h} \le |||\mathbf{v}||| \le C_2 \|\mathbf{v}\|_{1,h}, \quad \forall\, \mathbf{v} \in \tilde{\mathbf{V}}_h, \tag{2.8}$$

$$C_1 \|q\|_{1,h} \le |||q||| \le C_2 \|q\|_{1,h}, \quad \forall\, q \in P_h. \tag{2.9}$$

## 3. The CDG method

We extend the CDG finite element space $\mathbf{V}_h$ to a subspace of the WG finite element space $\tilde{\mathbf{V}}_h$. For $T \in \mathcal{T}_h$, we define the jump of $\tau$ as $[\tau]_e = (\tau|_T - \tau|_{T_n})/2$ and the average of $\tau$ as $\{\tau\}_e = (\tau|_T + \tau|_{T_n})/2$ and $T_n$ denotes the elements neighboring to $T$ at an edge/triangle $e$. For $e \in \partial T \cap \partial\Omega$, we define $[\mathbf{v}_h]_e = \mathbf{v}_h$ and $\{\mathbf{v}_h\}_e = \mathbf{0}$ as a free-variable on the edge for all $\mathbf{v}_h \in \mathbf{V}_h$.

Define $E_h : \mathbf{V}_h \to \tilde{\mathbf{V}}_h$ such that for $\mathbf{v} \in \mathbf{V}_h$

$$E_h \mathbf{v} = \{\mathbf{v}, \{\mathbf{v}\}\} \in \tilde{\mathbf{V}}_h. \tag{3.1}$$

For $\mathbf{v} \in \mathbf{V}_h$, we define the $\nabla_w \mathbf{v}$ naturally by

$$\nabla_w \mathbf{v} = \nabla_w E_h \mathbf{v}, \tag{3.2}$$

where $\nabla_w E_h \mathbf{v}$ is defined by (2.2).

**Lemma 3.1.** *The CDG finite element problem* (1.8)-(1.9) *has a unique solution.*

*Proof.* It suffices to show that zero is the only solution of (1.8)-(1.9) if $\mathbf{f} = \mathbf{0}$. To this end, let $\mathbf{f} = \mathbf{0}$ and take $\mathbf{v}_h = \mathbf{u}_h$ in (1.8) and $q_h = p_h$ in (1.9). The difference of the two resulting equations gives

$$(\nabla_w \mathbf{u}_h,\ \nabla_w \mathbf{u}_h) = 0,$$

which implies that $\nabla_w \mathbf{u}_h = \nabla_w E_h \mathbf{u}_h = \mathbf{0}$ on each element $T$. By (2.8), we have $\|E_h \mathbf{u}_h\|_{1,h} = 0$ which implies that $\nabla \mathbf{u}_h = \mathbf{0}$ on $T$ and $\mathbf{u}_h - \{\mathbf{u}_h\} = [\mathbf{u}_h] = 0$ on $\partial T$. Thus, we obtain $\mathbf{u}_h = \mathbf{0}$.

Since $\mathbf{u}_h = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$, the Eq. (1.8) becomes $(\mathbf{v}_h, \nabla_w p_h) = 0$ for any $\mathbf{v}_h \in \mathbf{V}_h$. Letting $\mathbf{v}_h = \nabla_w p_h \in \mathbf{P}_k(T)$, we have $\nabla_w p_h = \mathbf{0}$. It follows from (2.9) and $p_h \in L_0^2(\Omega)$ that $p_h = 0$. We have proved the lemma. □

**Theorem 3.1.** *The CDG finite element solution* $\mathbf{u}_h$ *in* (1.8)-(1.9) *is an* $H_0(\mathrm{div},\Omega)$ *function and is divergence-free, i.e.,*

$$\mathbf{u}_h \in H_0(\mathrm{div},\Omega) \quad \textit{and} \quad \nabla\cdot\mathbf{u}_h = 0. \tag{3.3}$$

*Proof.* In (1.9), letting $q_0 = 0$ and $q_b = 0$ everywhere except on one edge/triangle $e$ where $q_b|_b = [\mathbf{u}_h]_e$, we get, by (2.3),

$$0 = (\nabla q_h, \mathbf{u}_h) = 2\big\langle[\mathbf{u}_h\cdot\mathbf{n}], [\mathbf{u}_h\cdot\mathbf{n}]\big\rangle_e.$$

Thus,

$$[\mathbf{u}_h\cdot\mathbf{n}]_e = 0 \quad \text{on all} \quad e\in\mathcal{E}_h \quad \text{and} \quad \{\mathbf{u}_h\cdot\mathbf{n}\}|_e = 0, \quad \forall\, e\in\mathcal{E}_h\cap\partial\Omega,$$

i.e., $\mathbf{u}_h \in H_0(\mathrm{div},\Omega)$.

In (1.9), letting $q_0 = -\nabla\cdot\mathbf{u}_h$ and $q_b = 0$, we get, by (2.3),

$$0 = (\nabla q_h, \mathbf{u}_h) = -(q_0, \nabla\cdot\mathbf{u}_h) + \langle q_b, [\mathbf{u}_h]\rangle_{\partial\mathcal{T}_h} = (\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{u}_h),$$

which proves (3.3). □

**Lemma 3.2.** *There exists a positive constant* $\beta$ *independent of* $h$ *such that for all* $q = \{q_0, q_b\} \in P_h$*,*

$$\sup_{\mathbf{v}\in\mathbf{V}_h} \frac{(\mathbf{v}, \nabla_w q)}{|\!|\!|\mathbf{v}|\!|\!|} \ge \beta\|q_0\|, \tag{3.4}$$

*where* $|\!|\!|\mathbf{v}|\!|\!| = \|\nabla_w E_h\mathbf{v}\|_0$ *defined by* (3.1)*,* (3.2) *and* (2.4).

*Proof.* For a $q_0 \in L_0^2(\Omega)$ (because $(\mathbf{v}, \nabla_w q_b) = 0$ when we choose $\mathbf{v}$ in the $H(\mathrm{div})$ subspace), by (2.3), there is a BDM $H_0(\mathrm{div})$ interpolation function $\mathbf{v}$ such that

$$\nabla\cdot\mathbf{v} = -q_0, \quad |\mathbf{v}|_{1,h} \le C\|q_0\|_0, \quad \mathbf{v}|_T \in \mathbf{P}_k(T).$$

Thus, $\mathbf{v}\in\mathbf{V}_h$ and, by (2.5) and (2.8),

$$(\mathbf{v}, \nabla_w q) = \|q_0\|_0^2 \ge C^{-1}|\mathbf{v}|_{1,h}\|q_0\|_0 \ge \beta|\!|\!|\mathbf{v}|\!|\!|\,\|q_0\|_0.$$

The lemma is proved. □

## 4. Error estimates

Corresponding to the CDG method (1.8)-(1.9), we introduce a WG method. Let $\tilde{\mathbf{u}}_h = \{\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_b\} \in \tilde{\mathbf{V}}_h$ and $\tilde{p}_h \in P_h$ be the weak Galerkin finite element solution for (1.1)-(1.3) such that for all $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{\mathbf{V}}_h$ and $q \in P_h$,

$$(\mu\nabla_w\tilde{\mathbf{u}}_h,\ \nabla_w\mathbf{v}) + (\nabla_w\tilde{p}_h, \mathbf{v}_0) = (\mathbf{f},\ \mathbf{v}_0), \tag{4.1}$$

$$(\nabla_w q, \tilde{\mathbf{u}}_0) = 0. \tag{4.2}$$

Let $\Pi_k$ be a generic element-wise defined $L^2$ projection onto $[P_k(T)]^j$ where $j = 1, d, d\times d$ and $T\in\mathcal{T}_h$. Let $\Pi_k^b$ be a generic edge/face-wise defined $L^2$ projection onto $[P_k(e)]^j$ for $e\in\partial T$. Define

$$\mathbf{Q}_h\mathbf{u} = \left\{\Pi_k\mathbf{u}, \Pi_{k+1}^b\mathbf{u}\right\} \in \tilde{\mathbf{V}}_h, \quad Q_h p = \left\{\Pi_{k-1}p, \Pi_k^b p\right\} \in P_h. \tag{4.3}$$

**Theorem 4.1** ([45, Theorems 5.1 and 6.1]). *Let* $\tilde{\mathbf{u}}_h = \{\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_b\} \in \tilde{\mathbf{V}}_h$ *and* $\tilde{p}_h \in P_h$ *be the WG finite element solution of* (4.1)-(4.2). *Then*

$$h\|\nabla_w(\mathbf{Q}_h\mathbf{u} - \tilde{\mathbf{u}}_h)\|_0 + \|\mathbf{u} - \tilde{\mathbf{u}}_0\|_0 \le Ch^{k+1}|u|_{k+1}, \tag{4.4}$$

$$\|\Pi_{k-1}p - \tilde{p}_h\|_0 \le Ch^k|u|_{k+1}, \tag{4.5}$$

*where* $\mathbf{Q}_h$ *and* $\Pi_{k-1}$ *are defined in* (4.3).

For any function $\varphi\in H^1(T)$, the following trace inequality holds true:

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{4.6}$$

**Lemma 4.1.** *Let* $\mathbf{u}\in[H^{k+1}(\Omega)]^d$. *Then we have*

$$\|\nabla_w(\mathbf{Q}_h\mathbf{u} - \Pi_k\mathbf{u})\|_0 \le Ch^k|\mathbf{u}|_{k+1}, \tag{4.7}$$

*where* $\mathbf{Q}_h$ *and* $\Pi_k$ *are defined in* (4.3).

*Proof.* Recall $\mathbf{Q}_h\mathbf{u} = \{\Pi_k\mathbf{u}, \Pi_{k+1}^b\mathbf{u}\}$ and $E_h\Pi_k\mathbf{u} = \{\Pi_k\mathbf{u}, \{\Pi_k\mathbf{u}\}\}$. Letting $\mathbf{q} = \nabla_w(\mathbf{Q}_h\mathbf{u} - \Pi_k\mathbf{u})$ and using (2.2), the trace inequality (4.6) and inverse inequality yield

$$\begin{aligned}
&\|\nabla_w(\mathbf{Q}_h\mathbf{u} - \Pi_k\mathbf{u})\|_0^2\\
=\,&\|\nabla_w(\mathbf{Q}_h\mathbf{u} - E_h\Pi_h\mathbf{u})\|_0^2\\
=\,&\big(\nabla_w(\mathbf{Q}_h\mathbf{u} - E_h\Pi_k\mathbf{u}), \mathbf{q}\big)\\
=\,&\sum_{T\in T_h}\left\langle \Pi_{k+1}^b\mathbf{u} - \{\Pi_k\mathbf{u}\}, \mathbf{q}\cdot\mathbf{n}\right\rangle_{\partial T}\\
=\,&\sum_{T\in T_h}\left\langle \Pi_{k+1}^b\mathbf{u} - \mathbf{u} - \{\Pi_k\mathbf{u} - \mathbf{u}\}, \mathbf{q}\cdot\mathbf{n}\right\rangle_{\partial T}\\
\le\,& C\Big(\sum_{T\in T_h} h_T^{-1}\|\Pi_k\mathbf{u} - \mathbf{u}\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in T_h} h_T\|\mathbf{q}\|_{\partial T}^2\Big)^{1/2}\\
\le\,& Ch^k|\mathbf{u}|_{k+1}\|\nabla_w(\mathbf{Q}_h\mathbf{u} - \Pi_k\mathbf{u})\|_0.
\end{aligned}$$

We complete the proof of the lemma. □

The differences of (4.1)-(4.2) and (1.8)-(1.9) give

$$\big(\mu\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h), \nabla_w\mathbf{v}\big) + \big(\nabla_w(\tilde{p}_h - p_h), \mathbf{v}\big) = 0, \quad \forall\,\mathbf{v}\in\mathbf{V}_h, \tag{4.8}$$

$$(\tilde{\mathbf{u}}_0 - \mathbf{u}_h,\ \nabla_w q) = 0, \quad \forall\, q\in P_h. \tag{4.9}$$

Let $\mathbf{v} = \tilde{\mathbf{u}}_0 - \mathbf{u}_h$ in (4.8) and $q = \tilde{p}_h - p_h$ in (4.9). Subtracting (4.8) from (4.9) implies

$$\big(\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h), \nabla_w(\tilde{\mathbf{u}}_0 - \mathbf{u}_h)\big) = 0. \tag{4.10}$$

**Lemma 4.2.** *Let* $\mathbf{u} \in H^{k+1}(\Omega)$ *and* $\tilde{\mathbf{u}}_h = \{\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_b\} \in \tilde{\mathbf{V}}_h$ *be the WG solution of* (4.1)-(4.2). *Then we have*

$$\|\nabla_w(\tilde{\mathbf{u}}_h - \tilde{\mathbf{u}}_0)\|_0 \le Ch^k|\mathbf{u}|_{k+1}. \tag{4.11}$$

*Proof.* For $\mathbf{q} \in [P_{k+1}(T)]^{d\times d}$, using $\{\tilde{\mathbf{u}}_0\} = \tilde{\mathbf{u}}_0 - [\tilde{\mathbf{u}}_0]$, $[\mathbf{u}] = 0$, $\|\Pi_{k+1}^b\mathbf{u} - \mathbf{u}\|_{\partial T} \le \|\Pi_k\mathbf{u} - \mathbf{u}\|_{\partial T}$, (2.2), (3.2), (4.6), (2.8) and (4.4), we have

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h} \langle \tilde{\mathbf{u}}_b - \{\tilde{\mathbf{u}}_0\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\Big| \\
&= \Big|\sum_T \Big\langle \big((\tilde{\mathbf{u}}_b - \Pi_{k+1}^b\mathbf{u}) - (\tilde{\mathbf{u}}_0 - \Pi_k\mathbf{u})\big) + (\Pi_{k+1}^b\mathbf{u} - \mathbf{u}) \\
&\qquad\qquad + (\mathbf{u} - \Pi_k\mathbf{u}) + [\tilde{\mathbf{u}}_0 - \mathbf{u}], \mathbf{q}\cdot\mathbf{n}\Big\rangle_{\partial T}\Big| \\
&\le \Big(\sum_T h^{-1}\|(\tilde{\mathbf{u}}_b - \Pi_{k+1}^b\mathbf{u}) - (\tilde{\mathbf{u}}_0 - \Pi_k\mathbf{u})\|_{\partial T}^2 + 2\|\Pi_k\mathbf{u} - \mathbf{u}\|_{\partial T}^2 + 2\|\tilde{\mathbf{u}}_0 - \mathbf{u}\|_{\partial T}^2\Big)^{1/2} \\
&\quad \times \Big(\sum_T h\|\mathbf{q}\|_{\partial T}^2\Big)^{1/2} \\
&\le C\Big(\|\nabla_w(\mathbf{Q}_h\mathbf{u} - \tilde{\mathbf{u}}_h)\|_0 + \sqrt{2}(h^{-1}\|\Pi_k\mathbf{u} - \mathbf{u}\|_0 + \|\nabla(\Pi_k\mathbf{u} - \mathbf{u})\|_0) \\
&\qquad + \sqrt{2}(h^{-1}\|\Pi_k\mathbf{u} - \tilde{\mathbf{u}}_0\|_0 + 2h^{-1}\|\Pi_k\mathbf{u} - \mathbf{u}\|_0 + 2\|\nabla(\Pi_k\mathbf{u} - \mathbf{u})\|_0)\Big)\|\mathbf{q}\|_0 \\
&\le Ch^k|\mathbf{u}|_{k+1}\|\mathbf{q}\|_0,
\end{aligned} \tag{4.12}$$

where in the last but one step we inserted another $\Pi_k\mathbf{u}$ and used the inverse inequality

$$\|\Pi_k\mathbf{u} - \tilde{\mathbf{u}}_0\|_{\partial T} \le h^{-1/2}\|\Pi_k\mathbf{u} - \tilde{\mathbf{u}}_0\|_T$$

instead of the trace inequality, to avoid the error estimate on $\|\nabla(\mathbf{u} - \tilde{\mathbf{u}}_0)\|_0$ (it can be easily done too.)

Let $\mathbf{q} = \nabla_w(\tilde{\mathbf{u}}_h - \tilde{\mathbf{u}}_0)$. Using (2.2) and (4.12), we have

$$\begin{aligned}
\|\nabla_w(\tilde{\mathbf{u}}_h - \tilde{\mathbf{u}}_0)\|_0^2 &= \big(\nabla_w(\tilde{\mathbf{u}}_h - \tilde{\mathbf{u}}_0), \mathbf{q}\big) \\
&= \sum_{T\in T_h} \langle \tilde{\mathbf{u}}_b - \{\tilde{\mathbf{u}}_0\}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T} \le Ch^k|\mathbf{u}|_{k+1}\|\mathbf{q}\|_0,
\end{aligned}$$

which proves the lemma. □

**Lemma 4.3.** *Let* $\mathbf{u} \in H^{k+1}(\Omega)$. *Then we have*

$$\|\nabla_w(\tilde{\mathbf{u}}_h - \mathbf{u}_h)\|_0 \le Ch^k|\mathbf{u}|_{k+1}, \tag{4.13}$$

$$\|\tilde{p}_h - p_h\| \le Ch^k|\mathbf{u}|_{k+1}. \tag{4.14}$$

*Proof.* By (4.10) and (4.11), we have

$$\begin{aligned}&\big(\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h),\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\big)\\=&\big(\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h),\nabla_w(\tilde{\mathbf{u}}_h-\tilde{\mathbf{u}}_0)\big)\\\le&\|\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\|_0\|\nabla_w(\tilde{\mathbf{u}}_h-\tilde{\mathbf{u}}_0)\|_0\\\le&Ch^k|\mathbf{u}|_{k+1}\|\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\|_0,\end{aligned}$$

which implies (4.13). The estimate (4.14) follows from (4.8), (4.13) and (3.4) and we proved the lemma.

**Theorem 4.2.** *Let* $\mathbf{u}\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$ *be the exact solution of* (1.1) *and* (1.3). *Let* $\mathbf{u}_h\in\mathbf{V}_h$ *and* $p_h\in P_h$ *be the finite element solution of* (1.8)-(1.9). *Then*

$$\|\nabla_w(\Pi_k\mathbf{u}-\mathbf{u}_h)\|_0\le Ch^k|\mathbf{u}|_{k+1},\tag{4.15}$$

$$\|\Pi_{k-1}p-p_h\|_0\le Ch^k|\mathbf{u}|_{k+1}.\tag{4.16}$$

*Proof.* By (4.4), (4.13) and (4.7), we have

$$\begin{aligned}\|\nabla_w(\Pi_k\mathbf{u}-\mathbf{u}_h)\|_0&\le\|\nabla_w(\Pi_k\mathbf{u}-\mathbf{Q}_h\mathbf{u})\|_0+\|\nabla_w(\mathbf{Q}_h\mathbf{u}-\tilde{\mathbf{u}}_h)\|_0+\|\nabla_w(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\|_0\\&\le Ch^k|\mathbf{u}|_{k+1},\end{aligned}$$

which yields (4.15). The estimates (4.14) and (4.5) imply (4.16). We have finished the proof of the theorem. ☐

## 5. Error estimate in the L2 norm

Consider

$$-\mu\Delta\boldsymbol{\psi}+\nabla\xi=\tilde{\mathbf{u}}_0-\mathbf{u}_h\quad\text{in }\Omega,\tag{5.1}$$

$$\nabla\cdot\boldsymbol{\psi}=0\quad\text{in }\Omega,\tag{5.2}$$

$$\boldsymbol{\psi}=0\quad\text{on }\partial\Omega,\tag{5.3}$$

where $\tilde{\mathbf{u}}_0$ and $\mathbf{u}_h$ are the solutions in (4.1) and (1.8) respectively. Assume that the dual problem (5.1)-(5.3) has the $[H^2(\Omega)]^d\times H^1(\Omega)$-regularity property in the sense that the solution $(\boldsymbol{\psi},\xi)\in[H^2(\Omega)]^d\times H^1(\Omega)$ and the following a priori estimate holds true:

$$\mu\|\boldsymbol{\psi}\|_2+\|\xi\|_1\le C\|\tilde{\mathbf{u}}_0-\mathbf{u}_h\|.\tag{5.4}$$

**Theorem 5.1.** *Let* $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times P_h$ *be the solution of* (1.8)-(1.9). *Assume that* (5.4) *holds true. Then, we have*

$$\|\Pi_k\mathbf{u}-\mathbf{u}_0\|\le Ch^{k+1}|\mathbf{u}|_{k+1}.\tag{5.5}$$

*Proof.* Let $\boldsymbol{\psi}_h = \{\boldsymbol{\psi}_0, \boldsymbol{\psi}_b\} \in \tilde{\mathbf{V}}_h$ and $\xi_h \in P_h$ be the solution of the WG finite element method (4.1)-(4.2) such that for all $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{\mathbf{V}}_h$ and $q \in P_h$

$$(\mu \nabla_w \boldsymbol{\psi}_h,\ \nabla_w \mathbf{v}) + (\nabla_w \xi_h, \mathbf{v}_0) = (\tilde{\mathbf{u}}_0 - \mathbf{u}_h, \mathbf{v}_0), \tag{5.6}$$

$$(\boldsymbol{\psi}_0, \nabla_w q) = 0. \tag{5.7}$$

Letting $\mathbf{v} = \tilde{\mathbf{u}}_h - E_h \mathbf{u}_h \in \tilde{\mathbf{V}}_h$ in (5.6) and using (4.9) gives

$$\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0^2 = \big(\mu \nabla_w \boldsymbol{\psi}_h,\ \nabla_w (\tilde{\mathbf{u}}_h - \mathbf{u}_h)\big). \tag{5.8}$$

Letting $\mathbf{v} = \boldsymbol{\psi}_0 \in \mathbf{V}_h$ in (4.8) and using (5.7) give

$$\big(\nabla_w (\tilde{\mathbf{u}}_h - \mathbf{u}_h), \nabla \boldsymbol{\psi}_0\big) = 0. \tag{5.9}$$

Using (5.8), (5.9) and (4.11), we have

$$\begin{aligned}\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0^2 &= \big(\mu \nabla_w (\boldsymbol{\psi}_h - \boldsymbol{\psi}_0),\ \nabla_w (\tilde{\mathbf{u}}_h - \mathbf{u}_h)\big) \\ &\le C h^{k+1} \mu \|\boldsymbol{\psi}\|_2 \|\mathbf{u}\|_{k+1}.\end{aligned} \tag{5.10}$$

It follows from (5.10) and (5.4),

$$\|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0 \le h^{k+1} \|\mathbf{u}\|_{k+1}. \tag{5.11}$$

The triangle inequality, (5.11) and (4.4) imply

$$\|\Pi_k \mathbf{u} - \mathbf{u}_h\|_0 \le \|\Pi_k \mathbf{u} - \tilde{\mathbf{u}}_0\|_0 + \|\tilde{\mathbf{u}}_0 - \mathbf{u}_h\|_0 \le C h^{k+1} \|\mathbf{u}\|_{k+1}.$$

We have proved the theorem. □

## 6. Numerical experiments

In the numerical computation in 2D, the domain is $\Omega = (0,1) \times (0,1)$. We choose an $\mathbf{f}$ (depending on $\mu$) in (1.1) so that the exact solution of (1.1)-(1.3) is

$$\begin{aligned}\mathbf{u} &= \begin{pmatrix} (2y - 6y^2 + 4y^3)(x^2 - 2x^3 + x^4) \\ -(2x - 6x^2 + 4x^3)(y^2 - 2y^3 + y^4) \end{pmatrix}, \\ p &= -2x^3 + 3x^2 - x.\end{aligned} \tag{6.1}$$

We compute the solution (6.1) on triangular grids shown in Fig. 1 by the $P_k$-CDG/$P_{k-1}$-WG mixed finite elements for $k = 1,2,3,4,5$. The results are listed in Tables 1-5. The optimal order of convergence is achieved for all solutions in all norms. From the data, we can see the method is pressure robust that the error is independent of viscosity $\mu$.

We note that on some high level grids the computer round-off error exceeds the truncation error, when $\mu = 10^{-6}$, in Tables 3-5.

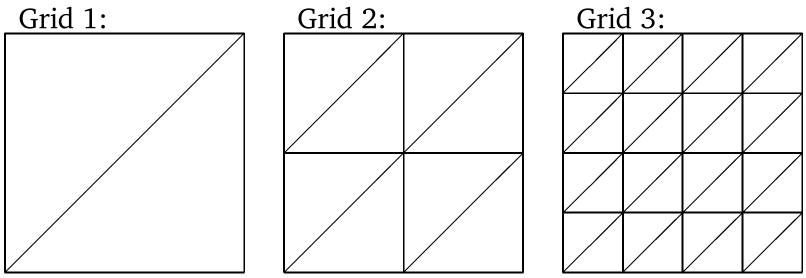


Figure 1: The first three grids for the computation in Tables 1-5.

Table 1: The error and the computed order of convergence by the $P_1$ element for the solution (6.1) on Fig. 1 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 5 | 0.9385E-03 | 1.77 | 0.2850E-01 | 0.99 | 0.2115E-01 | 1.01 |
| 6 | 0.2498E-03 | 1.91 | 0.1418E-01 | 1.01 | 0.1067E-01 | 0.99 |
| 7 | 0.6409E-04 | 1.96 | 0.7064E-02 | 1.00 | 0.5369E-02 | 0.99 |
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 5 | 0.9385E-03 | 1.77 | 0.2850E-01 | 0.99 | 0.2113E-07 | 1.01 |
| 6 | 0.2498E-03 | 1.91 | 0.1418E-01 | 1.01 | 0.1067E-07 | 0.99 |
| 7 | 0.6409E-04 | 1.96 | 0.7064E-02 | 1.00 | 0.5371E-08 | 0.99 |

Table 2: The error and the computed order of convergence by the $P_2$ element for the solution (6.1) on Fig. 1 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 4 | 0.1080E-03 | 0.00 | 0.7306E-02 | 0.00 | 0.1453E-01 | 0.00 |
| 5 | 0.1258E-04 | 3.10 | 0.1842E-02 | 1.99 | 0.4717E-02 | 1.62 |
| 6 | 0.1524E-05 | 3.04 | 0.4595E-03 | 2.00 | 0.1329E-02 | 1.83 |
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 4 | 0.1080E-03 | 3.12 | 0.7306E-02 | 1.92 | 0.1456E-07 | 1.13 |
| 5 | 0.1258E-04 | 3.10 | 0.1842E-02 | 1.99 | 0.4725E-08 | 1.62 |
| 6 | 0.1524E-05 | 3.04 | 0.4595E-03 | 2.00 | 0.1373E-08 | 1.78 |

We compute the 2D solution (6.1) again on slightly perturbed triangular grids shown in Fig. 2 by the $P_k$-CDG/$P_{k-1}$-WG mixed finite elements for $k=1,2,3,4,5$. The results are listed in Tables 6-10. The optimal order of convergence is achieved for all solutions in all norms.

Table 3: The error and the computed order of convergence by the $P_3$ element for the solution (6.1) on Fig. 1 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 4 | 0.7562E-05 | 3.89 | 0.6979E-03 | 2.85 | 0.1852E-02 | 2.57 |
| 5 | 0.4798E-06 | 3.98 | 0.8938E-04 | 2.97 | 0.2634E-03 | 2.81 |
| 6 | 0.3004E-07 | 4.00 | 0.1125E-04 | 2.99 | 0.3497E-04 | 2.91 |
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 4 | 0.7562E-05 | 3.89 | 0.6979E-03 | 2.85 | 0.1929E-08 | 2.51 |
| 5 | 0.4798E-06 | 3.98 | 0.8938E-04 | 2.97 | 0.3937E-09 | 2.29 |
| 6 | 0.3004E-07 | 4.00 | 0.1125E-04 | 2.99 | 0.3334E-09 | 0.24 |

Table 4: The error and the computed order of convergence by the $P_4$ element for the solution (6.1) on Fig. 1 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_4$-CDG/$P_3$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 3 | 0.1583E-04 | 4.49 | 0.8399E-03 | 3.59 | 0.7350E-03 | 3.40 |
| 4 | 0.5330E-06 | 4.89 | 0.5711E-04 | 3.88 | 0.5016E-04 | 3.87 |
| 5 | 0.1701E-07 | 4.97 | 0.3700E-05 | 3.95 | 0.3191E-05 | 3.97 |
| | By the $P_4$-CDG/$P_3$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 3 | 0.1583E-04 | 4.49 | 0.8399E-03 | 3.59 | 0.8573E-09 | 3.18 |
| 4 | 0.5330E-06 | 4.89 | 0.5711E-04 | 3.88 | 0.4274E-09 | 1.00 |
| 5 | 0.1701E-07 | 4.97 | 0.3701E-05 | 3.95 | 0.4401E-09 | 0.00 |

Table 5: The error and the computed order of convergence by the $P_5$ element for the solution (6.1) on Fig. 1 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_5$-CDG/$P_4$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 2 | 0.6597E-04 | 4.15 | 0.2216E-02 | 3.40 | 0.1255E-02 | 3.16 |
| 3 | 0.1285E-05 | 5.68 | 0.8154E-04 | 4.76 | 0.4852E-04 | 4.69 |
| 4 | 0.2218E-07 | 5.86 | 0.2737E-05 | 4.90 | 0.1652E-05 | 4.88 |
| | By the $P_5$-CDG/$P_4$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 2 | 0.6597E-04 | 4.15 | 0.2216E-02 | 3.40 | 0.1430E-08 | 3.00 |
| 3 | 0.1285E-05 | 5.68 | 0.8154E-04 | 4.76 | 0.3918E-09 | 1.87 |
| 4 | 0.2218E-07 | 5.86 | 0.2738E-05 | 4.90 | 0.5148E-09 | 0.00 |

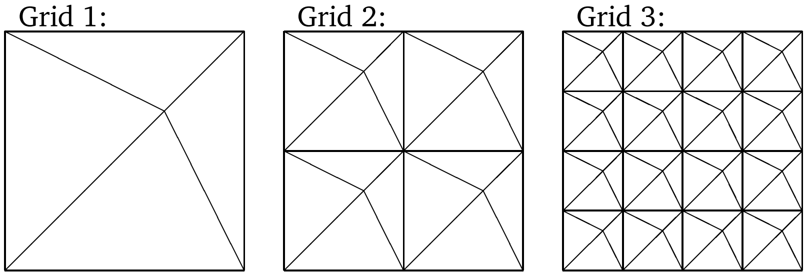


Figure 2: The first three grids for the computation in Tables 6-10.

Table 6: The error and the computed order of convergence by the $P_1$ element for the solution (6.1) on Fig. 2 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 4 | 0.1331E-02 | 1.74 | 0.3608E-01 | 0.95 | 0.1617E-01 | 1.13 |
| 5 | 0.3531E-03 | 1.91 | 0.1818E-01 | 0.99 | 0.7686E-02 | 1.07 |
| 6 | 0.9023E-04 | 1.97 | 0.9104E-02 | 1.00 | 0.3821E-02 | 1.01 |
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 4 | 0.1331E-02 | 1.74 | 0.3608E-01 | 0.95 | 0.1617E-07 | 1.14 |
| 5 | 0.3531E-03 | 1.91 | 0.1818E-01 | 0.99 | 0.7692E-08 | 1.07 |
| 6 | 0.9023E-04 | 1.97 | 0.9104E-02 | 1.00 | 0.3836E-08 | 1.00 |

Table 7: The error and the computed order of convergence by the $P_2$ element for the solution (6.1) on Fig. 2 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 4 | 0.2849E-04 | 3.00 | 0.2592E-02 | 1.92 | 0.1695E-02 | 1.32 |
| 5 | 0.3518E-05 | 3.02 | 0.6589E-03 | 1.98 | 0.5933E-03 | 1.51 |
| 6 | 0.4381E-06 | 3.01 | 0.1657E-03 | 1.99 | 0.1728E-03 | 1.78 |
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 4 | 0.2849E-04 | 3.00 | 0.2592E-02 | 1.92 | 0.1711E-08 | 1.32 |
| 5 | 0.3518E-05 | 3.02 | 0.6589E-03 | 1.98 | 0.6852E-09 | 1.32 |
| 6 | 0.4381E-06 | 3.01 | 0.1657E-03 | 1.99 | 0.3558E-09 | 0.95 |

In the 3D numerical computation, the domain is $\Omega=(0,1)\times(0,1)\times(0,1)$. We choose an $\mathbf{f}$ in (1.1) so that the exact solution is

$$\mathbf{u}=\begin{pmatrix} -2^{10}(x-1)^2x^2(y-1)^2y^2(z-3z^2+2z^3) \\ 2^{10}(x-1)^2x^2(y-1)^2y^2(z-3z^2+2z^3) \\ 2^{10}[(x-3x^2+2x^3)(y^2-y)^2-(x^2-x)^2(y-3y^2+2y^3)](z^2-z)^2 \end{pmatrix}, \tag{6.2}$$
$$p=-10(3y^2-2y^3-y).$$

Table 8: The error and the computed order of convergence by the $P_3$ element for the solution (6.1) on Fig. 2 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 3 | 0.1821E-04 | 3.31 | 0.1234E-02 | 2.56 | 0.1475E-02 | 2.76 |
| 4 | 0.1234E-05 | 3.88 | 0.1634E-03 | 2.92 | 0.1542E-03 | 3.26 |
| 5 | 0.7930E-07 | 3.96 | 0.2084E-04 | 2.97 | 0.1622E-04 | 3.25 |
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 3 | 0.1821E-04 | 3.31 | 0.1234E-02 | 2.56 | 0.1570E-08 | 2.67 |
| 4 | 0.1234E-05 | 3.88 | 0.1634E-03 | 2.92 | 0.4175E-09 | 1.91 |
| 5 | 0.7930E-07 | 3.96 | 0.2084E-04 | 2.97 | 0.3651E-09 | 0.19 |

Table 9: The error and the computed order of convergence by the $P_4$ element for the solution (6.1) on Fig. 2 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_4$-CDG/$P_3$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 2 | 0.3561E-04 | 3.66 | 0.1743E-02 | 2.87 | 0.1885E-02 | 3.19 |
| 3 | 0.1193E-05 | 4.90 | 0.1090E-03 | 4.00 | 0.1854E-03 | 3.35 |
| 4 | 0.3911E-07 | 4.93 | 0.7003E-05 | 3.96 | 0.1422E-04 | 3.70 |
| | By the $P_4$-CDG/$P_3$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 2 | 0.3561E-04 | 3.66 | 0.1743E-02 | 2.87 | 0.1985E-08 | 3.11 |
| 3 | 0.1193E-05 | 4.90 | 0.1090E-03 | 4.00 | 0.5254E-09 | 1.92 |
| 4 | 0.3911E-07 | 4.93 | 0.7003E-05 | 3.96 | 0.4155E-09 | 0.34 |

Table 10: The error and the computed order of convergence by the $P_5$ element for the solution (6.1) on Fig. 2 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_5$-CDG/$P_4$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 2 | 0.3155E-05 | 5.79 | 0.1881E-03 | 4.80 | 0.1420E-03 | 4.88 |
| 3 | 0.5300E-07 | 5.90 | 0.6291E-05 | 4.90 | 0.4695E-05 | 4.92 |
| 4 | 0.1202E-08 | 5.46 | 0.3843E-06 | 4.03 | 0.1514E-06 | 4.95 |
| | By the $P_5$-CDG/$P_4$-WG elements, $\mu=10^{-6}$ in (1.1). | | | | | |
| 2 | 0.3155E-05 | 5.79 | 0.1881E-03 | 4.80 | 0.5111E-09 | 3.05 |
| 3 | 0.5299E-07 | 5.90 | 0.6289E-05 | 4.90 | 0.4639E-09 | 0.14 |
| 4 | 0.1197E-08 | 5.47 | 0.3832E-06 | 4.04 | 0.5125E-09 | 0.00 |

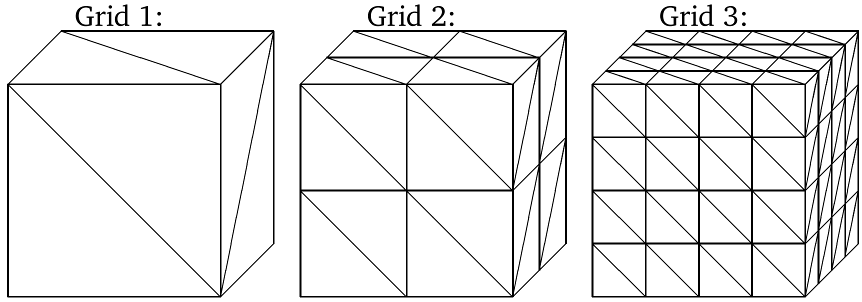


Figure 3: The first three tetrahedral grids for the computation in Tables 11-13.

Table 11: The error and the computed order of convergence by the $P_1$ element for the solution (6.2) on Fig. 3 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 3 | 0.452E-01 | 0.91 | 0.564E+00 | 0.88 | 0.434E+00 | 2.25 |
| 4 | 0.137E-01 | 1.72 | 0.226E+00 | 1.32 | 0.108E+00 | 2.01 |
| 5 | 0.365E-02 | 1.91 | 0.971E-01 | 1.22 | 0.227E-01 | 2.25 |
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=10^{-3}$ in (1.1). | | | | | |
| 3 | 0.568E-01 | 0.72 | 0.751E+00 | 0.64 | 0.635E-03 | 2.20 |
| 4 | 0.145E-01 | 1.97 | 0.235E+00 | 1.67 | 0.144E-03 | 2.14 |
| 5 | 0.371E-02 | 1.97 | 0.974E-01 | 1.27 | 0.260E-04 | 2.47 |

Table 12: The error and the computed order of convergence by the $P_2$ element for the solution (6.2) on Fig. 3 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 3 | 0.593E-02 | 3.08 | 0.172E+00 | 2.18 | 0.131E+00 | 2.03 |
| 4 | 0.570E-03 | 3.38 | 0.392E-01 | 2.14 | 0.231E-01 | 2.50 |
| 5 | 0.558E-04 | 3.35 | 0.923E-02 | 2.08 | 0.319E-03 | 6.18 |
| | By the $P_2$-CDG/$P_1$-WG elements, $\mu=10^{-3}$ in (1.1). | | | | | |
| 3 | 0.592E-02 | 3.09 | 0.172E+00 | 2.19 | 0.132E-03 | 2.03 |
| 4 | 0.646E-03 | 3.19 | 0.374E-01 | 2.20 | 0.177E-04 | 2.89 |
| 5 | 0.689E-04 | 3.23 | 0.908E-02 | 2.04 | 0.238E-05 | 2.90 |

We numerically compare the CDG-WG divergence-free finite element method with an HDG divergence-free finite element method, in [5, (2.6)], in Table 14. The HDG finite element method [5] adds a $\mathbf{u}_b$ $[P_k(\mathcal{E}_h)]^d$ velocity space to the CDG velocity space, where $\mathcal{E}_h$ is the set of all triangles in the tetrahedral mesh $\mathcal{T}_h$. And it also adds two

Table 13: The error and the computed order of convergence by the $P_3$ element for the solution (6.2) on Fig. 3 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\!\|\!\|\mathbf{u}-\mathbf{u}_h\|\!\|\!\|$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ |
|---|---|---|---|---|---|---|
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=1$ in (1.1). | | | | | |
| 3 | 0.688E-03 | 3.90 | 0.353E-01 | 2.87 | 0.215E-01 | 3.66 |
| 4 | 0.427E-04 | 4.01 | 0.478E-02 | 2.89 | 0.117E-02 | 4.19 |
| 5 | 0.268E-05 | 3.99 | 0.615E-03 | 2.96 | 0.220E-04 | 5.74 |
| | By the $P_3$-CDG/$P_2$-WG elements, $\mu=10^{-3}$ in (1.1). | | | | | |
| 3 | 0.701E-03 | 3.92 | 0.357E-01 | 2.86 | 0.247E-04 | 3.30 |
| 4 | 0.425E-04 | 4.04 | 0.477E-02 | 2.90 | 0.933E-06 | 4.72 |
| 5 | 0.267E-05 | 3.99 | 0.614E-03 | 2.96 | 0.311E-07 | 4.91 |

Table 14: The error and the computed order of convergence for (6.2) on Fig. 3 meshes.

| Grid | $\|\mathbf{u}-\mathbf{u}_h\|_0$ | $\mathcal{O}(h^r)$ | $\|\Pi_{k-1}p-p_0\|_0$ | $\mathcal{O}(h^r)$ | d.o.f. |
|---|---|---|---|---|---|
| | By the $P_1$-CDG/$P_0$-WG elements, $\mu=1$. | | | | |
| 1 | 0.897E-01 | 0.0 | 0.653E+00 | 0.0 | 132 |
| 2 | 0.705E-01 | 0.3 | 0.142E-01 | 5.5 | 984 |
| 3 | 0.452E-01 | 0.9 | 0.434E+00 | 2.3 | 7584 |
| 4 | 0.137E-01 | 1.7 | 0.108E+00 | 2.0 | 59520 |
| 5 | 0.365E-02 | 1.9 | 0.227E-01 | 2.2 | 471552 |
| | By the $P_1$-HDG method (2.6) in [5], $\mu=1$. | | | | |
| 1 | 0.872E-01 | 0.0 | 0.793E+00 | 0.0 | 276 |
| 2 | 0.877E-01 | 0.0 | 0.295E+01 | 0.0 | 1992 |
| 3 | 0.516E-01 | 0.8 | 0.635E+00 | 2.2 | 15072 |
| 4 | 0.122E-01 | 2.1 | 0.186E+00 | 1.8 | 117120 |
| 5 | 0.278E-02 | 2.1 | 0.413E-01 | 2.2 | 923136 |

$P_k(\mathcal{E}_h)$ Lagrange multiplier spaces, on the two sides of $\mathbf{u}_b\cdot\mathbf{n}$, to enforce the $H(\text{div})$ continuity of the discrete velocity solution. Thus the number of degrees of freedom, i.e., the number of unknowns, of the HDG method [5] nearly doubles that of the CDG-WG method proposed in this manuscript, cf. Table 14. The two methods are about equally accurate, as shown in Table 14.

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