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Finite Volume Element Method for a Nonlinear Parabolic Equation

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Abstract. In this article, we study a parabolic equation with both nonlinear time-derivative term and nonlinear diffusion term by the finite volume element method. The optimal error estimate in H^1-norm is proved for fully discrete scheme. The sub-optimal error estimate in L^2-norm is proved both for semi-discrete scheme and fully discrete scheme. We prove the existence of solution for the fully discrete scheme. Numerical results show the effectiveness of our method.

AMS subject classifications: 65N08, 65N15

Key words: Nonlinear parabolic equation, error estimate, finite volume element method.

1 Introduction

The theory of numerical methods for linear equations is relatively mature. In contrast, the research findings of nonlinear equations are inadequate. Based on the different positions of nonlinear terms, nonlinear equations are divided into different types. For the nonlinear parabolic equation, a common case is that the diffusion term or the convective term is nonlinear, and many authors have done a lot of work in this area. However, only a few researchers pay attention to the situation that the time derivative is nonlinear. In fact, this type of equations are also of great importance in describing phenomenons in natural and social science. For the radiation diffusion equation [28], the energy can be a function of the temperature in some cases. In this condition, the radiation diffusion system can be transformed into a equation with nonlinear time derivative. In [1], the authors used an equation with nonlinear time derivative to simulate the flow in root-soil system. In

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this paper. we consider the following nonlinear parabolic equation with both nonlinear diffusion term and nonlinear time derivative term:

$$\begin{cases} \dfrac{\partial a(u)}{\partial t} - \nabla\cdot(b(u))\nabla u) = f(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times I, \\ u(\mathbf{x},t) = g(\mathbf{x},t), & (\mathbf{x},t)\in\partial\Omega\times I, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^2$ is a bounded domain and $I=(0,T]$ is the time interval. We assume that there exists constants α_i, $i=1,2,3,4$ and β_i, $i=1,2,3$, satisfying

$$0<\alpha_1\le a(u)\le\alpha_2, \quad 0<\alpha_3\le a'(u)\le\alpha_4, \quad |a''(u)|\le\alpha_5, \tag{1.2a}$$

$$0<\beta_1\le b(u)\le\beta_2, \quad |b'(u)|\le\beta_3. \tag{1.2b}$$

For the elliptic equation with nonlinear diffusion term, Douglas and Dupont [14] proposed a Galerkin method for the nonlinear Dirichlet problem in 1974. Later, Liu [19] et al. also used finite element method (FEM) to solve the nonlinear elliptic equation, and extended the coefficient matrix to a more general case. Due to the property of local conservation, finite volume element method [11, 16, 20, 27] (FVEM) is attracting more and more attention. In terms of FVEM, Li [18] gave a linear element finite volume method for this equation and obtained the error estimate in H^1-norm. Chatzipantelidis [5] et al. provided a new proof and got the optimal error estimate both in H^1-norm and L^2-norm. Bi [3] et al. discussed a two-grid finite volume element method for nonlinear elliptic equation. Recently, Du [15] et al. discussed a quadratic FVEM for the nonlinear elliptic equation and established the optimal error estimates.

For the parabolic equation with nonlinear diffusion term and linear time derivative, there are lots of articles both in FEM and FVEM. For the FEM, Douglas and Dupont [13] discussed some linear and nonlinear parabolic equations with Galerkin method and obtained optimal H^1-norm error estimate in 1970. On this basis, Wheeler [21] used Galerkin method to solve a nonlinear parabolic problem and derived L^2-norm error estimate. Chen [9] et al. analyzed a two-grid expanded mixed finite element method for the nonlinear parabolic equation. Yang [23] used the least-squares mixed finite element to solve the nonlinear convection-diffusion equation. Cannon and Lin [4] studied finite element method for the nonlinear diffusion equation with memory and gave error estimate in L^2-norm. For the FVEM, Wu [22] solved a nonlinear parabolic equation by the generalized difference method and obtained the optimal H^1-norm error estimate in 1987. Chatzipantelidis and Ginting [6] studied a nonlinear parabolic equation by the finite volume element method and got the error estimate under a mild mesh condition. Chen and Liu [8] used the finite volume element method solving a nonlinear parabolic problem, where the diffusion term is nonlinear. Zhang [25] presented a semi-discrete finite volume element scheme to solve a nonlinear parabolic equation, where the convection term, diffusion term and reaction term are all nonlinear. Zhang [26] et al. gave a full-discrete two-grid finite volume element scheme to solve the nonlinear parabolic equation. Yang and

Yuan [24] used a quadratic finite volume element method to solve nonlinear parabolic systems.

For the parabolic equation with nonlinear diffusion term and nonlinear time derivative, Arbogast [1] et al. used it to simulate the flow in root-soil system in 1993. They devised the finite element scheme for this equation and proved the error estimate. Then, Arbogast and Wheeler [2] applied the mixed finite element method to solve this equation and obtained the error estimate. Clement [12] et al. studied a two-dimensional finite difference scheme for this equation. Eymard [17] et al. applied the finite volume method to solve the Richards equation. However, we haven't found articles about finite volume element for this equation.

In this article, we use the finite volume element method to solve a parabolic equation with both nonlinear time-derivative term and nonlinear diffusion term. For the nonlinear time derivative $\partial a(u)/\partial t$, a common method is to convert it into $a'(u)\partial u/\partial t$. The theoretical analysis of the converted equation is mature. However, that scheme can't keep the conservation of physical variable. We devise a scheme to solve the original equation which is conservative in physics. We firstly construct the semi-discrete and fully discrete schemes for this equation. Then, the optimal error estimate in H^1-norm and suboptimal error estimate in L^2-norm are proved. Finally, some numerical examples are presented to confirm the theoretical results. In error estimation, we use a similar procedure in [22] to estimate the nonlinear diffusion term. The main difficulty for the error estimation is to handle the nonlinear time derivative. We divide the inner product of nonlinear time derivative and test function into two parts. For the first part, we refer to the results for the same equation by FEM [1, 2]. The results of [1] is used as a bridge to estimate the error of FVEM. For the second part, we use the principle of interpolation to estimate the difference between test functions of FEM and FVEM. Combining these two parts, we can obtain the error estimate.

The rest of the paper is organized as follows. In Section 2, we present the numerical schemes and some necessary lemmas. In Section 3, the error estimates for semi-discrete scheme is obtained. Similarly, the error estimates for fully discrete scheme is presented in Section 4. We prove the existence of the fully discrete scheme in Section 5. In Section 6, we use iteration method to solve this equation and get the numerical results. Finally, we give a brief conclusion in Section 7.

2 Finite volume discretization

The finite volume method need two partitions, the primary partition and the dual partition. We divide the domain into some triangles and the primary $\mathcal{T}_h = \{K\}$ is obtained where $h = \max_{K\in\mathcal{T}_h} h_K$ and h_K is the diameter of element K. We denote $\mathcal{N}_h$ the set of Lagrange interpolation nodes on all K. For each node in $\mathcal{N}_h$, we create a corresponding control volume K^*. Then, we get the dual partition $\mathcal{T}_h^* = \{K^*\}$. For the time interval, we set $\Delta t = \frac{T}{N}$ and $0 = t_0 < t_1 \cdots < t_N = T$ be the partition of I.

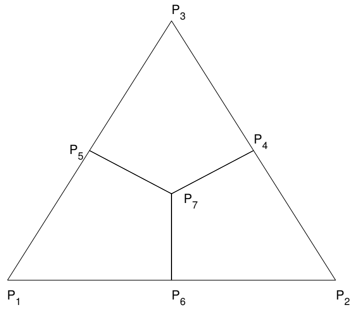

Figure 1: Dual partition in an element.

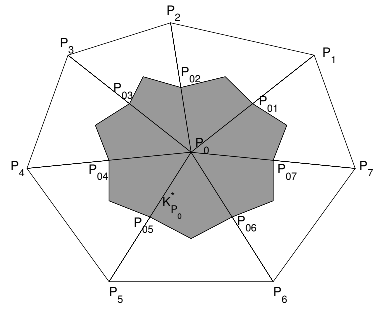

Figure 2: Control volume around a vertex.

Fig. 1 shows the dual partition in an element, P_i $(1\leq i\leq 3)$ is the vertex of the triangle. P_i $(4\leq i\leq 6)$ is the midpoint of an edge, and P_7 is the barycenter of a triangle. In Fig. 2, we present a control volume around a vertex. P_i $(0\leq i\leq 7)$ is the vertex of the triangle, P_{0i} $(1\leq i\leq 7)$ is the midpoint between P_0 and P_i. The grey shaded area is control volume $K^*_{P_0}$ around P_0.

We assume that the primary partition is quasi uniform, then there exists a positive constant C such that

$$C^{-1}h^2\leq meas(K)\leq Ch^2, \quad \forall K\in\mathcal{T}_h. \tag{2.1}$$

Based on the two partitions, we can define the trial function space and the test function space. Take standard Lagrange linear finite element space as trial function space U_h

$$U_h=\{u_h|u_h\in C(\Omega),\, u_h|_K\in P^1(K),\, \forall K\in\mathcal{T}_h\}, \tag{2.2}$$

and define piecewise constant space as test function space V_h

$$V_h=\{v_h|v_h\in L^2(\Omega),\, v_h|_{K^*_P}=\text{constant},\, \forall K^*_P\in\mathcal{T}^*_h,\, v_h|_{K^*_P}=0,\, P\in\partial\Omega\cap\mathcal{N}_h\}. \tag{2.3}$$

In order to estimate the errors between exact solution and numerical solution, two projections are defined. The piecewise Lagrange interpolation projection from $H^1(\Omega)$ to standard Lagrange finite element space is called Π_h. Similarly, we get the piecewise constant interpolation Π^*_h from H^1 to V_h. According to the principle of interpolation in Sobolev space, we get

$$\|v-\Pi_h v\|_m\leq Ch^{2-m}\|v\|_2, \quad m=0,1, \quad \forall v\in H^2, \tag{2.4}$$

and

$$\|v-\Pi^*_h v\|_0\leq Ch\|v\|_1, \quad \forall v\in H^1. \tag{2.5}$$

For $z,u\in H^1$ and $v_h\in V_h$, we define the bilinear form as follow

$$B(z;u,v_h)=-\sum_{K^*}\int_{\partial K^*}(b(z)\nabla u)\cdot\mathbf{n}v_h ds. \tag{2.6}$$

The weak formulation of (1.1) can be written as

$$(a(u)_t,v_h)+B(u;u,v_h)=(f,v_h),\quad \forall v_h\in V_h, \tag{2.7}$$

where

$$(a(u)_t,v_h)=\Big(\frac{\partial a(u)}{\partial t},v_h\Big)=\sum_{K^*}\int_{K^*}\frac{\partial a(u)}{\partial t}v_h dxdy, \tag{2.8a}$$

$$(f,v_h)=\sum_{K^*}\int_{K^*}fv_h dxdy. \tag{2.8b}$$

By means of the projection Π_h^*, (2.7) can also be written into

$$(a(u)_t,\Pi_h^*v_h)+B(u;u,\Pi_h^*v_h)=(f,\Pi_h^*v_h),\quad \forall v_h\in U_h. \tag{2.9}$$

Similarly, the semi-discrete finite volume element scheme of (1.1) is to find $u_h\in U_h$ such that

$$(a(u_h)_t,\Pi_h^*v_h)+B(u_h;u_h,\Pi_h^*v_h)=(f,\Pi_h^*v_h),\quad \forall v_h\in U_h. \tag{2.10}$$

Using the backward Euler scheme to approximate the time derivative term, the fully discrete finite volume scheme of (1.1) is to find $u_h^n\in U_h$ such that

$$(\partial_t a(u_h^n),\Pi_h^*v_h)+B(u_h^n;u_h^n,\Pi_h^*v_h)=(f^n,\Pi_h^*v_h),\quad \forall v_h\in U_h, \tag{2.11}$$

where

$$\partial_t a(u_h^n)=\frac{a(u_h^n)-a(u_h^{n-1})}{\Delta t}. \tag{2.12}$$

The semi-discrete finite element scheme of (1.1) is to find $\tilde{u}_h\in U_h$, such that

$$(a(\tilde{u}_h)_t,v_h)+B_h(\tilde{u}_h;\tilde{u}_h,v_h)=(f,v_h),\quad \forall v_h\in U_h, \tag{2.13}$$

where

$$B_h(\tilde{u}_h;\tilde{u}_h,v_h)=\int_\Omega b(\tilde{u}_h)\nabla\tilde{u}_h\nabla v_h dxdy. \tag{2.14}$$

Obviously, there is a positive constant κ such that

$$B_h(z_h;u_h,u_h)\geq\kappa\|u_h\|_1^2,\quad \forall z_h,u_h\in U_h. \tag{2.15}$$

Following [10], we introduce the standard Ritz projection.

Lemma 2.1. *Supposing that R_h is a projection from H^1 to U_h and satisfies*

$$B_h(u;R_hu,v_h)=B_h(u;u,v_h), \quad \forall v_h\in U_h. \tag{2.16}$$

Then, there exists a positive constant C such that

$$\|u-R_hu\|_0+h\|u-R_hu\|_1\le Ch^2\|u\|_2, \tag{2.17a}$$

$$\|(u-R_hu)_t\|_0+h\|(u-R_hu)_t\|_1\le Ch^2\|u_t\|_2. \tag{2.17b}$$

In addition, there exists a positive constant M_0 independent of h, such that

$$\|\nabla R_hu\|_\infty+\|\nabla R_hu_t\|_\infty\le M_0 \quad \textit{for} \ \ t\le T. \tag{2.18}$$

For error analysis we define two error functionals

$$\varepsilon_h(f,\chi)=(f,\chi)-(f,\Pi_h^*\chi), \qquad \forall \chi\in U_h, \tag{2.19a}$$

$$\varepsilon_a(w;\chi,\psi)=B_h(w;\chi,\psi)-B(w;\chi,\Pi_h^*\psi), \qquad \forall w,\chi,\psi\in U_h. \tag{2.19b}$$

The bounds of (2.19) are proved in [5,7] and shown as the following lemma.

Lemma 2.2. *Let $\chi\in U_h$, then*

$$|\varepsilon_h(f,\chi)|\le Ch^{i+j}\|f\|_i\|\chi\|_j, \qquad f\in H^i(\Omega), \qquad i,j=0,1, \tag{2.20a}$$

$$|\varepsilon_a(w;R_hv,\chi)|\le Ch^{i+j}\|v\|_{i+1}\|\chi\|_j, \qquad v\in H^{i+1}\cap H^1(\Omega), \qquad i,j=0,1. \tag{2.20b}$$

Further, [5] proved the following estimation for ε_a.

Lemma 2.3. *Let $\chi,\psi\in U_h$, there exists a positive constant C such that*

$$|\varepsilon_a(v;\psi,\chi)-\varepsilon_a(w;\psi,\chi)|\le Ch\|\nabla\psi\|_\infty(1+\|\nabla w\|_\infty)\|\nabla(v-w)\|_0\|\nabla\chi\|_0. \tag{2.21}$$

For $M=\max(2M_0,1)$, we define

$$\mathcal{B}_M=\{\chi\in U_h: \|\nabla\chi\|_\infty\le M\}. \tag{2.22}$$

For the bilinear form, the following coercive ellipticity with barycenter dual partition is proved in [5]. We only list the conclusion and will not repeat the proof process.

Lemma 2.4. *Supposing that $z_h\in U_h\cap\mathcal{B}_M$ and $u_h\in U_h$, there exists a positive constant γ such that*

$$B(z_h;u_h,\Pi_h^*u_h)\ge\gamma\|u_h\|_1^2. \tag{2.23}$$

The following Lemma 2.5 illustrates that the bilinear form is bounded, which has been proved in [18].

Lemma 2.5. *Supposing that $z_h,u_h,v_h\in U_h$, there exists a positive constant C such that*

$$B(z_h;u_h,\Pi_h^*v_h)\le C\|u_h\|_1\|v_h\|_1. \tag{2.24}$$

3 Error estimates for semi-discrete scheme

Theorem 3.1. *Let $u\in H^2\cap H^1$ and $u_h\in U_h\cap\mathcal{B}_M$ be the solutions of Eq.* (2.9) *and Eq.* (2.10)*, respectively. Supposing that $u_{ht}\in\mathcal{B}_M$, there exists a constant C such that*

$$\|u-u_h\|_0^2+C\int_0^t\|u(\xi)-u_h(\xi)\|_1^2d\xi\le Ch^2. \tag{3.1}$$

Proof. Subtracting (2.10) from (2.9), we have

$$(a(u)_t-a(u_h)_t,\Pi_h^*v_h)+B(u;u,\Pi_h^*v_h)-B(u_h;u_h,\Pi_h^*v_h)=0,\quad\forall v_h\in U_h. \tag{3.2}$$

Define two variables

$$e=R_hu-u_h,\quad \rho=u-R_hu. \tag{3.3}$$

Then, Eq. (3.2) can be written as

$$\begin{aligned}&(a(u)_t-a(u_h)_t,\Pi_h^*v_h)+B(u_h;e,\Pi_h^*v_h)\\=&-B(u;u,\Pi_h^*v_h)+B(u_h;R_hu,\Pi_h^*v_h),\quad\forall v_h\in U_h.\end{aligned} \tag{3.4}$$

According to Lemma 2.1 and Eq. (2.19), we have

$$\begin{aligned}&-B(u;u,\Pi_h^*v_h)+B(u_h;R_hu,\Pi_h^*v_h)\\=&(a(u)_t-f,\Pi_h^*v_h)+B(u_h;R_hu,\Pi_h^*v_h)\\=&(a(u)_t-f,\Pi_h^*v_h)+B(u_h;R_hu,\Pi_h^*v_h)-B_h(u_h;R_hu,v_h)+B_h(u_h;R_hu,v_h)\\&-B_h(u;R_hu,v_h)-(a(u)_t-f,v_h)\\=&[(a(u)_t-f,\Pi_h^*v_h)-(a(u)_t-f,v_h)]+[B(u_h;R_hu,\Pi_h^*v_h)-B_h(u_h;R_hu,v_h)]\\&+[B_h(u_h;R_hu,v_h)-B_h(u;R_hu,v_h)]\\=&\varepsilon_h(f-a(u)_t,v_h)-\varepsilon_a(u_h;R_hu,v_h)+[B_h(u_h;R_hu,v_h)-B_h(u;R_hu,v_h)].\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) and setting $v_h=e$ in (3.4) yields

$$\begin{aligned}&(a(u)_t-a(u_h)_t,\Pi_h^*e)+B(u_h;e,\Pi_h^*e)\\=&\varepsilon_h(f-a(u)_t,e)-\varepsilon_a(u_h;R_hu,e)+[B_h(u_h;R_hu,e)-B_h(u;R_hu,e)].\end{aligned} \tag{3.6}$$

For the first term on the left hand side of Eq. (3.6), we have

$$\begin{aligned}&(a(u)_t-a(u_h)_t,\Pi_h^*e)\\=&(a(u)_t-a(u_h)_t,\Pi_h^*e-e)+(a(u)_t-a(u_h)_t,e).\end{aligned} \tag{3.7}$$

Using the Leibniz formula, we get

$$\begin{aligned}(a(u)_t-a(u_h)_t,e)=&(a(u)_t-a(u_h)_t,u-u_h)-(a(u)_t-a(u_h)_t,\rho)\\=&(a(u)_t-a(u_h)_t,u-u_h)-(a(u)-a(u_h),\rho)_t+(a(u)-a(u_h),\rho_t),\end{aligned} \tag{3.8}$$

and

$$(a(u)_t - a(u_h)_t, \Pi_h^* e - e) = (a(u) - a(u_h), \Pi_h^* e - e)_t - (a(u) - a(u_h), (\Pi_h^* e - e)_t). \tag{3.9}$$

Substituting (3.7)-(3.9) into (3.6), we obtain

$$\begin{aligned}&(a(u)_t - a(u_h)_t, u - u_h) + B(u_h; e, \Pi_h^* e)\\ =&\varepsilon_h(f - a(u)_t, e) - \varepsilon_a(u_h; R_h u, e) + [B_h(u_h; R_h u, e) - B_h(u; R_h u, e)] + (a(u) - a(u_h), \rho)_t\\ &- (a(u) - a(u_h), \rho_t) - (a(u) - a(u_h), \Pi_h^* e - e)_t + (a(u) - a(u_h), (\Pi_h^* e - e)_t). \end{aligned}\tag{3.10}$$

For the first term on the left hand side of Eq. (3.10), we know

$$\begin{aligned}&(a(u) - a(u_h))_t (u - u_h)\\ =&\Big(\int_{u_h}^{u} (a(\zeta) - a(u_h)) d\zeta\Big)_t - (a(u) - a(u_h)) u_t + a(u)_t (u - u_h).\end{aligned}\tag{3.11}$$

According to the mean value theorem, we get

$$\begin{aligned}&\int_\Omega (a(u) - a(u_h)) u_t - a(u)_t (u - u_h) dxdy\\ =&\int_\Omega a'(\eta_1)(u - u_h) u_t - a'(u) u_t (u - u_h) dxdy\\ =&\int_\Omega a''(\eta_2)(\eta_1 - u)(u - u_h) u_t dxdy\\ \le& C\|u - u_h\|_0^2,\end{aligned}\tag{3.12}$$

where η_1 is a value between (u, u_h) and η_2 is a value between (u, η_1). Substitute (3.11) and (3.12) into (3.10) to have

$$\begin{aligned}&\int_\Omega \Big(\int_{u_h}^{u} (a(\zeta) - a(u_h)) d\zeta\Big)_t dxdy + B(u_h; e, \Pi_h^* e)\\ \le&\varepsilon_h(f - a(u)_t, e) - \varepsilon_a(u_h; R_h u, e) + [B_h(u_h; R_h u, e) - B_h(u; R_h u, e)]\\ &+ C\|u - u_h\|_0^2 + ((a(u) - a(u_h)), \rho)_t - ((a(u) - a(u_h)), \rho_t)\\ &- ((a(u) - a(u_h)), \Pi_h^* e - e)_t + ((a(u) - a(u_h)), (\Pi_h^* e - e)_t).\end{aligned}\tag{3.13}$$

Since $a'(u)$ is bounded, there exists a positive constant λ such that

$$\lambda (u - u_h)^2 \le \int_{u_h}^{u} (a(\zeta) - a(u_h)) d\zeta. \tag{3.14}$$

Substituting (3.14) into (3.13) and integrating (3.13) from 0 to t gives

$$\begin{aligned}&\lambda \|u - u_h\|_0^2 + \int_0^t B(u_h; e, \Pi_h^* e) d\xi\\ \le& \int_0^t \varepsilon_h(f - a(u)_t, e) - \varepsilon_a(u_h; R_h u, e) + [B_h(u_h; R_h u, e) - B_h(u; R_h u, e)] d\xi\end{aligned}$$

$$+C\int_0^t \|u-u_h\|_0^2 d\xi+((a(u)-a(u_h)),\rho)-\int_0^t ((a(u)-a(u_h)),\rho_t)d\xi$$
$$-((a(u)-a(u_h)),\Pi_h^* e-e)+\int_0^t ((a(u)-a(u_h)),(\Pi_h^* e-e)_t)d\xi. \tag{3.15}$$

According to Lemma 2.2 and Cauchy-Schwarz inequality, one can obtain

$$\int_0^t \varepsilon_h(f-a(u)_t,e)d\xi \le Ch^2\int_0^t \|f-a(u)_t\|_1\|e\|_1 d\xi, \tag{3.16a}$$
$$\int_0^t \varepsilon_a(u_h;R_h u,e)d\xi \le Ch^2\int_0^t \|u\|_2\|e\|_1 d\xi, \tag{3.16b}$$
$$\begin{aligned}&\int_0^t [B_h(u_h;R_h u,e)-B_h(u;R_h u,e)]d\xi\\ \le& C\int_0^t \|b(u_h)-b(u)\|\|\nabla(R_h u)\|_\infty\|\nabla e\|d\xi\\ \le& C\int_0^t \|u-u_h\|_0\|e\|_1 d\xi.\end{aligned} \tag{3.16c}$$

Using the Cauchy-Schwarz inequality, we have

$$((a(u)-a(u_h)),\rho)\le C\|u-u_h\|_0\|\rho\|_0, \tag{3.17a}$$
$$\int_0^t ((a(u)-a(u_h)),\partial_t\rho)d\xi \le \int_0^t C(\|u-u_h\|_0^2+\|\rho_t\|_0^2)d\xi, \tag{3.17b}$$
$$((a(u)-a(u_h)),\Pi_h^* e-e)\le Ch\|u-u_h\|_0\|e\|_1, \tag{3.17c}$$
$$\int_0^t ((a(u)-a(u_h)),(\Pi_h^* e-e)_t)d\xi \le Ch\int_0^t \|u-u_h\|_0\|e_t\|_1 d\xi. \tag{3.17d}$$

Substituting (3.16) and (3.17) into (3.15) and using Lemma 2.4, one can obtain

$$\begin{aligned}&\lambda\|u-u_h\|_0^2+\gamma\int_0^t \|e\|_1^2 d\xi\\ \le& C\int_0^t (h^2\|f-a(u)_t\|_1+h^2\|u\|_2+\|u-u_h\|_0)\|e\|_1 d\xi+C\int_0^t \|u-u_h\|_0^2 d\xi\\ &+C\|u-u_h\|_0(\|\rho\|_0+h\|e\|_1)+C\int_0^t \|\rho_t\|_0^2 d\xi+Ch\int_0^t \|u-u_h\|_0\|e_t\|_1 d\xi.\end{aligned} \tag{3.18}$$

Using the Young inequality, we have

$$\begin{aligned}&\lambda\|u-u_h\|_0^2+\gamma\int_0^t \|e\|_1^2 d\xi\\ \le& C\int_0^t (h^4+\|u-u_h\|_0^2)d\xi+C\|\rho\|_0^2+C\int_0^t \|\rho_t\|_0^2 d\xi\\ &+Ch^2\|e\|_1^2+Ch^2\int_0^t \|e_t\|_1^2 d\xi+\varepsilon_1\|u-u_h\|_0^2+\varepsilon_2\int_0^t \|e\|_1^2 d\xi.\end{aligned} \tag{3.19}$$

The equation above can be written as

$$\begin{aligned}&(\lambda-\varepsilon_1)\|u-u_h\|_0^2+(\gamma-\varepsilon_2)\int_0^t\|e(\xi)\|_1^2d\xi\\ \leq& C\int_0^t(h^4+\|u-u_h\|_0^2+\|\rho_t\|_0^2)d\xi+C\|\rho\|_0^2+Ch^2\|e\|_1^2+Ch^2\int_0^t\|e_t\|_1^2d\xi.\end{aligned}\tag{3.20}$$

Since $\|e\|_1$ and $\|e_t\|_1$ are bounded, we have

$$\begin{aligned}&(\lambda-\varepsilon_1)\|u-u_h\|_0^2+(\gamma-\varepsilon_2)\int_0^t\|e(\xi)\|_1^2d\xi\\ \leq& C\int_0^t(h^4+\|u-u_h\|_0^2+\|\rho_t\|_0^2)d\xi+C\|\rho\|_0^2+Ch^2.\end{aligned}\tag{3.21}$$

Then, the inequality (3.21) can be written as

$$\|u-u_h\|_0^2+C\int_0^t\|e(\xi)\|_1^2d\xi\leq C\int_0^t(h^4+\|u-u_h\|_0^2)d\xi+Ch^2.\tag{3.22}$$

Using the Gronwall inequality and Lemma 2.1, we get the conclusion

$$\begin{aligned}&\|u-u_h\|_0^2+C\int_0^t\|u(\xi)-u_h(\xi)\|_1^2d\xi\\ \leq&\|u-u_h\|_0^2+C\int_0^t(\|e(\xi)\|_1^2+\|\rho(\xi)\|_1^2)d\xi\leq Ch^2.\end{aligned}\tag{3.23}$$

This completes the proof. □

4 Error estimates for fully discrete scheme

Theorem 4.1. *Let $u^n\in H^2\cap H^1$ and $u_h^n\in U_h\cap\mathcal{B}_M$ be the solutions of Eq.* (2.9) *and Eq.* (2.11)*, respectively. There exists a constant C such that*

$$\|u^l-u_h^l\|_1^2\leq C(h^2+\Delta t^2)\tag{4.1}$$

and

$$\|R_hu^l-u_h^l\|_1\leq C(\Delta t+\Delta t^{-1/2}h^2).\tag{4.2}$$

Proof. Taking $u=u^n$ in (2.9) and subtracting (2.11) from (2.9), we have

$$(a(u^n)_t-\partial_ta(u_h^n),\Pi_h^*v_h)+B(u^n;u^n,\Pi_h^*v_h)-B(u_h^n;u_h^n,\Pi_h^*v_h)=0,\quad\forall v_h\in U_h.\tag{4.3}$$

The equation above can be rewritten as

$$\begin{aligned}&(\partial_ta(u^n)-\partial_ta(u_h^n),\Pi_h^*v_h)+B(u^n;u^n,\Pi_h^*v_h)-B(u_h^n;u_h^n,\Pi_h^*v_h)\\ =&(E_1,\Pi_h^*v_h),\quad\forall v_h\in U_h,\end{aligned}\tag{4.4}$$

where

$$E_1 = \partial_t a(u^n) - a(u^n)_t, \tag{4.5}$$

and we can get $E_1 = \mathcal{O}(\Delta t)$ according to Taylor's expansion. Define two variables

$$e^n = R_h u^n - u_h^n, \quad \rho^n = u^n - R_h u^n. \tag{4.6}$$

Then, Eq. (4.3) can be rewritten as

$$\begin{aligned}&(\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* v_h) + B(u_h^n; e^n, \Pi_h^* v_h)\\ =& -B(u^n; u^n, \Pi_h^* v_h) + B(u_h^n; R_h u^n, \Pi_h^* v_h) + (E_1, \Pi_h^* v_h), \quad \forall v_h \in U_h.\end{aligned} \tag{4.7}$$

Similar to Eq. (3.5), we obtain

$$\begin{aligned}&-B(u^n; u^n, \Pi_h^* v_h) + B(u_h^n; R_h u^n, \Pi_h^* v_h)\\ =&\varepsilon_h(f^n - a(u^n)_t, v_h) - \varepsilon_a(u_h^n; R_h u^n, v_h) + [B_h(u_h^n; R_h u^n, v_h) - B_h(u^n; R_h u^n, v_h)].\end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.7) and setting $v_h = \partial_t e^n$ in (4.7) yields

$$\begin{aligned}&(\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* \partial_t e^n) + B(u_h^n; e^n, \Pi_h^* \partial_t e^n)\\ =&\varepsilon_h(f^n - a(u^n)_t, \partial_t e^n) - \varepsilon_a(u_h^n; R_h u^n, \partial_t e^n)\\ &+ [B_h(u_h^n; R_h u^n, \partial_t e^n) - B_h(u^n; R_h u^n, \partial_t e^n)] + (E_1, \Pi_h^* \partial_t e^n).\end{aligned} \tag{4.9}$$

The equation above can be written into

$$\begin{aligned}&(\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* \partial_t e^n) + B_h(u_h^n; e^n, \partial_t e^n)\\ =&\varepsilon_h(f^n - a(u^n)_t, \partial_t e^n) - \varepsilon_a(u_h^n; R_h u^n, \partial_t e^n) + \varepsilon_a(u_h^n; e^n, \partial_t e^n)\\ &+ [B_h(u_h^n; R_h u^n, \partial_t e^n) - B_h(u^n; R_h u^n, \partial_t e^n)] + (E_1, \Pi_h^* \partial_t e^n).\end{aligned} \tag{4.10}$$

For the first term on the left hand side of Eq. (4.10), we have

$$\begin{aligned}&(\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* \partial_t e^n)\\ =&(a'(\nu_1^n)\partial_t u^n - a'(\nu_2^n)\partial_t u_h^n, \Pi_h^* \partial_t e^n)\\ =&(a'(\nu_1^n)\partial_t u^n - a'(\nu_2^n)\partial_t u^n + a'(\nu_2^n)\partial_t u^n - a'(\nu_2^n)\partial_t u_h^n, \Pi_h^* \partial_t e^n)\\ =&((a'(\nu_1^n) - a'(\nu_2^n))\partial_t u^n + a'(\nu_2^n)(\partial_t \rho^n + \partial_t e^n), \Pi_h^* \partial_t e^n).\end{aligned} \tag{4.11}$$

For the second term on the left hand side of Eq. (4.10), we have

$$2\Delta t B_h(u_h^n; e^n, \partial_t e^n) = B_h(u_h^n; e^n, e^n) - B_h(u_h^n; e^{n-1}, e^{n-1}) + \Delta t^2 B_h(u_h^n; \partial_t e^n, \partial_t e^n). \tag{4.12}$$

Multiplying (4.10) by $2\Delta t$ and using (4.11) and (4.12), we get

$$\begin{aligned}&2\Delta t(a'(\nu_2^n)\partial_t e^n, \Pi_h^* \partial_t e^n) + B_h(u_h^n; e^n, e^n) + \Delta t^2 B_h(u_h^n; \partial_t e^n, \partial_t e^n)\\ =&-2\Delta t((a'(\nu_1^n) - a'(\nu_2^n))\partial_t u^n + a'(\nu_2^n)\partial_t \rho^n, \Pi_h^* \partial_t e^n) + B_h(u_h^n; e^{n-1}, e^{n-1})\\ &+ 2\Delta t \varepsilon_h(f^n - a(u^n)_t, \partial_t e^n) - 2\Delta t \varepsilon_a(u_h^n; R_h u^n, \partial_t e^n) + 2\Delta t \varepsilon_a(u_h^n; e^n, \partial_t e^n)\\ &+ 2\Delta t[B_h(u_h^n; R_h u^n, \partial_t e^n) - B_h(u^n; R_h u^n, \partial_t e^n)] + 2\Delta t(E_1, \Pi_h^* \partial_t e^n).\end{aligned} \tag{4.13}$$

Subtracting $B_h(u_h^{n-1};e^{n-1},e^{n-1})$ from both sides of (4.13) and summing on n from 1 to l, we have

$$\begin{aligned}
&\sum_{n=1}^{l}2\Delta t(a'(\nu_2^n)\partial_t e^n,\Pi_h^*\partial_t e^n)+B_h(u_h^l;e^l,e^l)-B_h(u_h^0;e^0,e^0)+\sum_{n=1}^{l}\Delta t^2 B_h(u_h^n;\partial_t e^n,\partial_t e^n)\\
=&-\sum_{n=1}^{l}2\Delta t((a'(\nu_1^n)-a'(\nu_2^n))\partial_t u^n,\Pi_h^*\partial_t e^n)-\sum_{n=1}^{l}2\Delta t(a'(\nu_2^n)\partial_t\rho^n,\Pi_h^*\partial_t e^n)\\
&+\sum_{n=1}^{l}[B_h(u_h^n;e^{n-1},e^{n-1})-B_h(u_h^{n-1};e^{n-1},e^{n-1})]+\sum_{n=1}^{l}2\Delta t(E_1,\Pi_h^*\partial_t e^n)\\
&+\sum_{n=1}^{l}2\Delta t\varepsilon_h(f^n-a(u^n)_t,\partial_t e^n)-\sum_{n=1}^{l}2\Delta t\varepsilon_a(u_h^n;R_h u^n,\partial_t e^n)\\
&+\sum_{n=1}^{l}2\Delta t\varepsilon_a(u_h^n;e^n,\partial_t e^n)+\sum_{n=1}^{l}2\Delta t[B_h(u_h^n;R_h u^n,\partial_t e^n)-B_h(u^n;R_h u^n,\partial_t e^n)]. \qquad (4.14)
\end{aligned}$$

For the first two terms on the right-hand side of Eq. (4.14), we have

$$\begin{aligned}
&((a'(\nu_1^n)-a'(\nu_2^n))\partial_t u^n,\Pi_h^*\partial_t e^n)+(a'(\nu_2^n)\partial_t\rho^n,\Pi_h^*\partial_t e^n)\\
\le& C(\|\nu_1^n-\nu_2^n\|_0+\|\partial_t\rho^n\|_0)\|\partial_t e^n\|_0\\
\le& C(\|u_h^n-u^n\|_0+\|u_h^{n-1}-u^{n-1}\|_0+\|u^n-u^{n-1}\|_0+\|\partial_t\rho^n\|_0)\|\partial_t e^n\|_0\\
\le& C(\|e^n\|_0+\|\rho^n\|_0+\|e^{n-1}\|_0+\|\rho^{n-1}\|_0+\Delta t^2+\|\partial_t\rho^n\|_0)\|\partial_t e^n\|_0. \qquad (4.15)
\end{aligned}$$

For the third and fourth terms on the right-hand side of Eq. (4.14), we get

$$\begin{aligned}
&|B_h(u_h^n;e^{n-1},e^{n-1})-B_h(u_h^{n-1};e^{n-1},e^{n-1})|\\
\le&\Delta t\||\partial_t u_h^n|\cdot|\nabla e^{n-1}|\|_0\|e^{n-1}\|_1\\
\le&\Delta t(\||\partial_t e^n|\cdot|\nabla e^{n-1}|\|_0+\||R_h\partial_t u^n|\cdot|\nabla e^{n-1}|\|_0)\|e^{n-1}\|_1\\
\le& C\Delta t(\|\partial_t e^n\|_0+\|e^{n-1}\|_1)\|e^{n-1}\|_1. \qquad (4.16)
\end{aligned}$$

For the fifth to eighth terms, using Cauchy-Schwarz inequality and Lemma 2.2, we obtain

$$(E_1,\Pi_h^*\partial_t e^n)\le\|E_1\|_0\|\partial_t e^n\|_0, \qquad (4.17a)$$
$$\varepsilon_h(f^n-a(u^n)_t,\partial_t e^n)\le Ch\|f^n-a(u^n)_t\|_1\|\partial_t e^n\|_0, \qquad (4.17b)$$
$$\varepsilon_a(u_h^n;R_h u^n,\partial_t e^n)\le Ch\|u^n\|_2\|\partial_t e^n\|_0, \qquad (4.17c)$$
$$\varepsilon_a(u_h^n;e^n,\partial_t e^n)\le C\|e^n\|_1\|\partial_t e^n\|_0. \qquad (4.17d)$$

For the last two terms on the right-hand side of Eq. (4.14), we know that

$$\begin{aligned}
&[B_h(u_h^n;R_h u^n,\partial_t e^n)-B_h(u^n;R_h u^n,\partial_t e^n)]\\
=&((b(u_h^n)-b(u^n))\nabla R_h u^n,\nabla\partial_t e^n)\\
=&(-\nabla\cdot[(b(u_h^n)-b(u^n))\nabla R_h u^n],\partial_t e^n) \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}&\le C\|b'(u_h^n)\nabla u_h^n - b'(u^n)\nabla u^n\|_0\|\partial_t e^n\|_0\\&\le C(\|b'(u_h^n)(\nabla u_h^n-\nabla u^n)\|_0+\|(b'(u_h^n)-b'(u^n))\nabla u^n\|_0)\|\partial_t e^n\|_0\\&\le C(\|\rho^n\|_1+\|e^n\|_1)\|\partial_t e^n\|_0.\end{aligned}\tag{4.19}$$

Substituting (4.15)-(4.18) into Eq. (4.14) and using the coercive ellipticity of B_h, we have

$$\begin{aligned}&\sum_{n=1}^{l}2\Delta t\sigma\|\partial_t e^n\|_0^2+\kappa\|e^l\|_1^2+\kappa\sum_{n=1}^{l}\Delta t^2\|\partial_t e^n\|_1^2\\\le& C\sum_{n=1}^{l}\Delta t(\|e^n\|_0+\|\rho^n\|_0+\|e^{n-1}\|_0+\|\rho^{n-1}\|_0+\Delta t^2+\|\partial_t\rho^n\|_0+\|e^{n-1}\|_1+\|e^n\|_1\\&\quad+\|\rho^n\|_1+h+\|E_1\|_0)\|\partial_t e^n\|_0+C\sum_{n=1}^{l}\Delta t\|e^{n-1}\|_1^2,\end{aligned}\tag{4.20}$$

where σ is a positive constant related to α_3.

According to the Young inequality, we have

$$\begin{aligned}\kappa\|e^l\|_1^2+\kappa\sum_{n=1}^{l}\Delta t^2\|\partial_t e^n\|_1^2\le& C\sum_{n=1}^{l}\Delta t(\|\rho^{n-1}\|_0^2+\Delta t^4+\|\partial_t\rho^n\|_0^2+\|\rho^n\|_1^2+h^2+\|E_1\|_0^2)\\&+C\sum_{n=1}^{l}\Delta t(\|e^{n-1}\|_1^2+\|e^n\|_1^2).\end{aligned}\tag{4.21}$$

Using the Gronwall inequality and dropping the nonnegative terms, we get

$$\kappa\|e^l\|_1^2\le C\sum_{n=1}^{l}\Delta t(\|\rho^{n-1}\|_0^2+\Delta t^4+\|\partial_t\rho^n\|_0^2+\|\rho^n\|_1^2+h^2+\|E_1\|_0^2).\tag{4.22}$$

According to Lemma 2.1, we have the conclusion (4.1)

$$\|u^l-u_h^l\|_1^2\le\|e^l\|_1^2+\|\rho^l\|_1^2\le C(h^2+\Delta t^2).\tag{4.23}$$

In order to get (4.2), we have a proof different from (4.17)-(4.23). using Cauchy-Schwarz inequality and Lemma 2.2, we obtain

$$\varepsilon_h(f^n-a(u^n)_t,\partial_t e^n)\le Ch^2\|f^n-a(u^n)_t\|_1\|\partial_t e^n\|_1,\tag{4.24a}$$

$$\varepsilon_a(u_h^n;R_hu^n,\partial_t e^n)\le Ch^2\|u^n\|_2\|\partial_t e^n\|_1,\tag{4.24b}$$

and

$$\begin{aligned}&[B_h(u_h^n;R_hu^n,\partial_t e^n)-B_h(u^n;R_hu^n,\partial_t e^n)]\\=&((b(u_h^n)-b(R_hu^n)+b(R_hu^n)-b(u^n))\nabla R_hu^n,\nabla\partial_t e^n)\\=&(-\nabla\cdot[(b(u_h^n)-b(R_hu^n))\nabla R_hu^n],\partial_t e^n)+((b(R_hu^n)-b(u^n))\nabla R_hu^n,\nabla\partial_t e^n)\\\le& C\|b'(u_h^n)\nabla u_h^n-b'(R_hu^n)\nabla R_hu^n\|_0\|\partial_t e^n\|_0+Ch^2\|\partial_t e^n\|_1\\\le& C(\|b'(u_h^n)(\nabla u_h^n-\nabla R_hu^n)\|_0+\|(b'(u_h^n)-b'(R_hu^n))\nabla R_hu^n\|_0)\|\partial_t e^n\|_0\\\le& C\|e^n\|_1\|\partial_t e^n\|_0+Ch^2\|\partial_t e^n\|_1.\end{aligned}\tag{4.25}$$

Substituting (4.15)-(4.16) and (4.24)-(4.25) into Eq. (4.14) and using the coercive ellipticity of B_h, we have

$$\begin{aligned}&\sum_{n=1}^{l}2\Delta t\sigma\|\partial_t e^n\|_0^2+\kappa\|e^l\|_1^2+\kappa\sum_{n=1}^{l}\Delta t^2\|\partial_t e^n\|_1^2\\ \le& C\sum_{n=1}^{l}\Delta t(\|e^n\|_0+\|\rho^n\|_0+\|e^{n-1}\|_0+\|\rho^{n-1}\|_0+\Delta t^2+\|\partial_t\rho^n\|_0+\|e^{n-1}\|_1\\ &+\|e^n\|_1+\|E_1\|_0)\|\partial_t e^n\|_0+C\sum_{n=1}^{l}\Delta t h^2\|\partial_t e^n\|_1+C\sum_{n=1}^{l}\Delta t\|e^{n-1}\|_1^2. \end{aligned}\tag{4.26}$$

According to the Young inequality, we have

$$\begin{aligned}\kappa\|e^l\|_1^2\le& C\sum_{n=1}^{l}\Delta t(\|\rho^{n-1}\|_0^2+\Delta t^4+\|\partial_t\rho^n\|_0^2+\Delta t^{-1}h^4+\|E_1\|_0^2)\\ &+C\sum_{n=1}^{l}\Delta t(\|e^{n-1}\|_1^2+\|e^n\|_1^2).\end{aligned}\tag{4.27}$$

Using the Gronwall inequality, we get

$$\|e^l\|_1^2\le C(\Delta t^{-1}h^4+\Delta t^2),\tag{4.28}$$

which implies the conclusion (4.2). □

Theorem 4.2. *Let $u^n\in H^2\cap H^1$ and $u_h^n\in U_h\cap\mathcal{B}_M$ be the solutions of Eq.* (2.9) *and Eq.* (2.11)*, respectively. Supposing that $u_{ht}\in\mathcal{B}_M$, there exists a constant C such that*

$$\|u^l-u_h^l\|_0^2+\sum_{n=1}^{l}\Delta t\|u^n-u_h^n\|_1^2\le C(h^2+\Delta t^2).\tag{4.29}$$

Proof. Setting $v_h=e^n$ in Eq. (4.7) yields

$$\begin{aligned}&(\partial_t a(u^n)-\partial_t a(u_h^n),\Pi_h^*e^n)+B(u_h^n;e^n,\Pi_h^*e^n)\\ =&\varepsilon_h(f^n-a(u^n)_t,e^n)-\varepsilon_a(u_h^n;R_hu^n,e^n)\\ &+[B_h(u_h^n;R_hu^n,e^n)-B_h(u^n;R_hu^n,e^n)]+(E_1,\Pi_h^*e^n).\end{aligned}\tag{4.30}$$

For the first term on the left hand side of Eq. (4.30), we have

$$(\partial_t a(u^n)-\partial_t a(u_h^n),\Pi_h^*e^n)=(\partial_t a(u^n)-\partial_t a(u_h^n),\Pi_h^*e^n-e^n+e^n),\tag{4.31}$$

and

$$(\partial_t a(u^n)-\partial_t a(u_h^n),e^n)=(\partial_t a(u^n)-\partial_t a(u_h^n),u^n-u_h^n)-(\partial_t a(u^n)-\partial_t a(u_h^n),\rho^n).\tag{4.32}$$

Inserting (4.31) and (4.32) into (4.30), we get

$$\begin{aligned}&(\partial_t a(u^n)-\partial_t a(u_h^n),u^n-u_h^n)+B(u_h^n;e^n,\Pi_h^* e^n)\\=&\varepsilon_h(f^n-a(u^n)_t,e^n)-\varepsilon_a(u_h^n;R_h u^n,e^n)\\&+[B_h(u_h^n;R_h u^n,e^n)-B_h(u^n;R_h u^n,e^n)]+(E_1,\Pi_h^* e^n)\\&+(\partial_t a(u^n)-\partial_t a(u_h^n),\rho^n)-(\partial_t a(u^n)-\partial_t a(u_h^n),\Pi_h^* e^n-e^n).\end{aligned}\tag{4.33}$$

For the first term on the left hand side of Eq. (4.33), we have

$$(\partial_t a(u^n)-\partial_t a(u_h^n),u^n-u_h^n)=\int_\Omega\Big(\partial_t\Big(\int_{u_h}^u a(\mu)-a(u_h)d\mu\Big)^n-E_2\Big)dxdy,\tag{4.34}$$

where

$$\begin{aligned}E_2=&(a(u^n)-a(u_h^n))\partial_t u^n-\partial_t a(u)^n(u^n-u_h^n)+(a(u_h^n)-a(u_h^{n-1}))\partial_t u^n\\&-\frac{1}{\Delta t}\int_{u^{n-1}}^{u^n}(a(u^n)-a(\mu))d\mu-\frac{1}{\Delta t}\int_{u_h^{n-1}}^{u_h^n}(a(\mu)-a(u_h^{n-1}))d\mu.\end{aligned}\tag{4.35}$$

The estimate of E_2 appears in [1]. We reproduce it here for completeness. For the first two terms of Eq. (4.35), there are ν_1 and ν_2 such that

$$\begin{aligned}&(a(u^n)-a(u_h^n))\partial_t u^n-\partial_t a(u)^n(u^n-u_h^n)\\=&(a'(\nu_1)-a'(\nu_2))(u^n-u_h^n)\partial_t u^n\\\le& C|(\nu_1-\nu_2)(u^n-u_h^n)|\\\le& C[(u^n-u_h^n)^2+(\Delta t)^2].\end{aligned}\tag{4.36}$$

For the first integral term in (4.35), there are ν_3 and ν_4 such that

$$\int_{u^{n-1}}^{u^n}(a(u^n)-a(\mu))d\mu=\int_{u^{n-1}}^{u^n}a'(\nu_3(\mu))(u^n-\mu)d\mu=\frac{1}{2}a'(\nu_4)(u^n-u^{n-1})^2.\tag{4.37}$$

Similarly, there is ν_5 such that

$$\int_{u_h^{n-1}}^{u_h^n}(a(\mu)-a(u_h^{n-1}))d\mu=\frac{1}{2}a'(\nu_5)(u_h^n-u_h^{n-1})^2.\tag{4.38}$$

Then, the last three terms of (4.35) are

$$\begin{aligned}&\frac{1}{2\Delta t}[2a'(\nu_6)(u_h^n-u_h^{n-1})(u^n-u^{n-1})-a'(\nu_4)(u^n-u^{n-1})^2-a'(\nu_5)(u_h^n-u_h^{n-1})^2]\\=&\frac{1}{2\Delta t}[2(a'(\nu_6)-a'(\nu_5))(u_h^n-u_h^{n-1})(u^n-u^{n-1})+2a'(\nu_5)(u_h^n-u_h^{n-1})(u^n-u^{n-1})\\&-a'(\nu_4)(u^n-u^{n-1})^2-a'(\nu_5)(u_h^n-u_h^{n-1})^2]\\\le&\frac{1}{2\Delta t}[2(a'(\nu_6)-a'(\nu_5))(u_h^n-u_h^{n-1})(u^n-u^{n-1})+(a'(\nu_5)-a'(\nu_4))(u^n-u^{n-1})^2]\\\le& C[|\nu_6-\nu_5||u_h^n-u_h^{n-1}|+|\nu_5-\nu_4|\Delta t]\\\le& C[(u^n-u_h^n)^2+(u^{n-1}-u_h^{n-1})^2+\Delta t^2].\end{aligned}\tag{4.39}$$

Combining (4.36) and (4.39), we have

$$E_2 \le C[(u^n - u_h^n)^2 + (u^{n-1} - u_h^{n-1})^2 + \Delta t^2]. \tag{4.40}$$

Then, Eq. (4.33) can be rewritten into

$$\begin{aligned}
&\int_\Omega \partial_t \Big(\int_{u_h}^{u} a(\mu) - a(u_h) d\mu \Big)^n dxdy + B(u_h^n; e^n, \Pi_h^* e^n) \\
=&\varepsilon_h (f^n - a(u^n)_t, e^n) - \varepsilon_a (u_h^n; R_h u^n, e^n) + [B_h(u_h^n; R_h u^n, e^n) \\
&- B_h(u^n; R_h u^n, e^n)] + (E_1, \Pi_h^* e^n) + \int_\Omega E_2 dxdy \\
&+ (\partial_t a(u^n) - \partial_t a(u_h^n), \rho^n) - (\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* e^n - e^n).
\end{aligned} \tag{4.41}$$

Multiplying (4.41) by Δt and summing on n from 1 to l, we have

$$\begin{aligned}
&\int_\Omega \Big(\int_{u_h}^{u} a(\mu) - a(u_h) d\mu \Big)^l dxdy + \sum_{n=1}^{l} \Delta t B(u_h^n; e^n, \Pi_h^* e^n) \\
=&\sum_{n=1}^{l} \Delta t [\varepsilon_h (f^n - a(u^n)_t, e^n) - \varepsilon_a (u_h^n; R_h u^n, e^n)] + \sum_{n=1}^{l} \Delta t (E_1, \Pi_h^* e^n) \\
&+ \sum_{n=1}^{l} \Delta t [B_h(u_h^n; R_h u^n, e^n) - B_h(u^n; R_h u^n, e^n)] + \sum_{n=1}^{l} \Delta t \int_\Omega E_2 dxdy \\
&+ \sum_{n=1}^{l} \Delta t (\partial_t a(u^n) - \partial_t a(u_h^n), \rho^n) - \sum_{n=1}^{l} \Delta t (\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* e^n - e^n).
\end{aligned} \tag{4.42}$$

For the fifth term on the right hand-side of Eq. (4.42), we get

$$\begin{aligned}
&\sum_{n=1}^{l} \Delta t (\partial_t a(u^n) - \partial_t a(u_h^n), \rho^n) \\
=&(a(u^l) - a(u_h^l), \rho^l) - (a(u^0) - a(u_h^0), \rho^1) - \sum_{n=1}^{l-1} \Delta t (a(u^n) - a(u_h^n), \partial_t \rho^{n+1}).
\end{aligned} \tag{4.43}$$

Similarly, the last term on the right hand-side of Eq. (4.42) can be written as

$$\begin{aligned}
&\sum_{n=1}^{l} \Delta t (\partial_t a(u^n) - \partial_t a(u_h^n), \Pi_h^* e^n - e^n) \\
=&(a(u^l) - a(u_h^l), \Pi_h^* e^l - e^l) - (a(u^0) - a(u_h^0), \Pi_h^* e^1 - e^1) \\
&- \sum_{n=1}^{l-1} \Delta t (a(u^n) - a(u_h^n), \partial_t (\Pi_h^* e^{n+1} - e^{n+1})).
\end{aligned} \tag{4.44}$$

Using the results of [1], we know that

$$\lambda(u^l-u_h^l)^2\le\int_{u_h^l}^{u^l}(a(\zeta)-a(u_h^l))d\zeta. \tag{4.45}$$

According to (4.43)-(4.45), the initial condition and the ellipticity of $B(u_h^n;e^n,\Pi_h^*e^n)$, one can have

$$\begin{aligned}&\lambda\|u^l-u_h^l\|_0^2+\gamma\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\\ \le&\sum_{n=1}^{l}\Delta t[\varepsilon_h(f^n-a(u^n)_t,e^n)-\varepsilon_a(u_h^n;R_hu^n,e^n)]+\sum_{n=1}^{l}\Delta t(E_1,\Pi_h^*e^n)\\&+\sum_{n=1}^{l}\Delta t[B_h(u_h^n;R_hu^n,e^n)-B_h(u^n;R_hu^n,e^n)]+\sum_{n=1}^{l}\Delta t\int_\Omega E_2dxdy\\&+(a(u^l)-a(u_h^l),\rho^l)-\sum_{n=1}^{l-1}\Delta t(a(u^n)-a(u_h^n),\partial_t\rho^{n+1})\\&-(a(u^l)-a(u_h^l),\Pi_h^*e^l-e^l)+\sum_{n=1}^{l-1}\Delta t(a(u^n)-a(u_h^n),\partial_t(\Pi_h^*e^{n+1}-e^{n+1})).\end{aligned} \tag{4.46}$$

According to Lemma 2.2 and Cauchy-Schwarz inequality, one can obtain

$$\varepsilon_h(f^n-a(u^n)_t,e^n)\le Ch^2\|f^n-a(u^n)_t\|_1\|e^n\|_1, \tag{4.47a}$$

$$\varepsilon_a(u_h^n;R_hu^n,e^n)\le Ch^2\|u^n\|_2\|e^n\|_1, \tag{4.47b}$$

$$\begin{aligned}&[B_h(u_h^n;R_hu^n,e^n)-B_h(u^n;R_hu^n,e^n)]\\&\le C\|b(u_h^n)-b(u^n)\|\|\nabla(R_hu^n)\|_\infty\|\nabla e^n\|\\&\le C\|u^n-u_h^n\|_0\|e^n\|_1.\end{aligned} \tag{4.47c}$$

Using the Cauchy-Schwarz inequality, we know that

$$(a(u^l)-a(u_h^l),\rho^l)\le C\|u^l-u_h^l\|_0\|\rho^l\|_0, \tag{4.48a}$$

$$(a(u^n)-a(u_h^n),\partial_t\rho^{n+1})\le C\|u^n-u_h^n\|_0\|\partial_t\rho^{n+1}\|_0, \tag{4.48b}$$

$$(a(u^l)-a(u_h^l),\Pi_h^*e^l-e^l)\le Ch\|u^l-u_h^l\|_0\|e^l\|_1, \tag{4.48c}$$

$$(a(u^n)-a(u_h^n),\partial_t(\Pi_h^*e^{n+1}-e^{n+1}))\le Ch\|u^n-u_h^n\|_0\|\partial_te^{n+1}\|_1. \tag{4.48d}$$

Substituting Eq. (4.47) and Eq. (4.48) into Eq. (4.46), we have

$$\begin{aligned}&\lambda\|u^l-u_h^l\|_0^2+\gamma\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\\ \le& C\sum_{n=1}^{l}\Delta t[(h^2\|f^n-a(u^n)_t\|_1+h^2\|u^n\|_2+\|u^n-u_h^n\|_0)\|e^n\|_1+\|u^n-u_h^n\|_0^2\\&+\|u^{n-1}-u_h^{n-1}\|_0^2+\Delta t^2]+C\|u^l-u_h^l\|_0(\|\rho^l\|_0+h\|e^l\|_1)\end{aligned}$$

$$+C\sum_{n=1}^{l-1}\Delta t\|u^n-u_h^n\|_0\|\partial_t\rho^{n+1}\|_0+Ch\sum_{n=1}^{l-1}\Delta t\|u^n-u_h^n\|_0\|\partial_t e^{n+1}\|_1. \tag{4.49}$$

According to Young inequality, we get

$$\begin{aligned}
&\lambda\|u^l-u_h^l\|_0^2+\gamma\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\\
\le& C\sum_{n=1}^{l}\Delta t[\|u^n-u_h^n\|_0^2+\|u^{n-1}-u_h^{n-1}\|_0^2+h^4+\Delta t^2]+C\|\rho^l\|_0^2\\
&+C\sum_{n=1}^{l-1}\Delta t(\|u^n-u_h^n\|_0^2+\|\partial_t\rho^{n+1}\|_0^2)+Ch^2\|e^l\|_1^2\\
&+Ch^2\Delta t\sum_{n=1}^{l-1}\|\partial_t e^{n+1}\|_1^2+\varepsilon_1\|u^l-u_h^l\|_0^2+\varepsilon_2\sum_{n=1}^{l}\Delta t\|e^n\|_1^2.
\end{aligned} \tag{4.50}$$

The equation above can be written as

$$\begin{aligned}
&(\lambda-\varepsilon_1)\|u^l-u_h^l\|_0^2+(\gamma-\varepsilon_2)\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\\
\le& C\sum_{n=1}^{l}\Delta t[\|u^n-u_h^n\|_0^2+\|\partial_t\rho^{n+1}\|_0^2+h^4+\Delta t^2]+C\|\rho^l\|_0^2\\
&+Ch^2\|e^l\|_1^2+Ch^2\Delta t\sum_{n=1}^{l-1}\|\partial_t e^{n+1}\|_1^2.
\end{aligned} \tag{4.51}$$

Since $\|e^l\|_1$ and $\|e_t\|_1$ are bounded, we have

$$\begin{aligned}
&(\lambda-\varepsilon_1)\|u^l-u_h^l\|_0^2+(\gamma-\varepsilon_2)\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\\
\le& C\sum_{n=1}^{l}\Delta t[\|u^n-u_h^n\|_0^2+\|\partial_t\rho^{n+1}\|_0^2+h^4+\Delta t^2]+C\|\rho^l\|_0^2+Ch^2.
\end{aligned} \tag{4.52}$$

Then, the inequality (4.52) can be rewritten as

$$\|u^l-u_h^l\|_0^2+\sum_{n=1}^{l}\Delta t\|e^n\|_1^2\le C\sum_{n=1}^{l}\Delta t[\|u^n-u_h^n\|_0^2+h^4+\Delta t^2]+Ch^2. \tag{4.53}$$

Using the Gronwall inequality and Lemma 2.1, we get the conclusion

$$\begin{aligned}
&\|u^l-u_h^l\|_0^2+\sum_{n=1}^{l}\Delta t\|u^n-u_h^n\|_1^2\\
\le&\|u^l-u_h^l\|_0^2+\sum_{n=1}^{l}\Delta t(\|e^n\|_1^2+\|\rho^n\|_1^2)\le C(h^2+\Delta t^2).
\end{aligned} \tag{4.54}$$

Thus, we complete the proof. □

5 Existence of the fully discrete scheme

In this section, we prove the existence of the solution of the fully discrete scheme. We refer to the results of [6]. We assume $u_h^0=R_hu^0$ and $\Delta t=\mathcal{O}(h^{1+\varepsilon})$ with $0<\varepsilon<1$.

Let $G_n:U_h\to U_h$ be defined by

$$(a(G_nv)-a(u_h^{n-1}),\Pi_h^*v_h)+\Delta tB(v;G_nv,\Pi_h^*v_h)=\Delta t(f^n,\Pi_h^*v_h),\quad \forall v_h\in U_h. \tag{5.1}$$

Obviously, if G_n has a fixed point v, then $u_h^n=v$ is the solution.

If $u_h^{n-1}\in\mathcal{B}_M$, according to Theorem 4.1, we have

$$\|\nabla(u_h^{n-1}-R_hu^{n-1})\|_0\le C(\Delta t+\Delta t^{-1/2}h^2). \tag{5.2}$$

Using the stability property of R_h, we get

$$\begin{aligned}\|\nabla(u_h^{n-1}-R_hu^n)\|_0\le&\|\nabla(u_h^{n-1}-R_hu^{n-1})\|_0+\Delta t\|\nabla R_h\partial_tu^n\|_0\\ \le&C(\Delta t+\Delta t^{-1/2}h^2)+\Delta t\|\nabla R_h\partial_tu^n\|_0\le Ch^{1+\tilde\varepsilon},\end{aligned} \tag{5.3}$$

where $\tilde\varepsilon=\min(\varepsilon,\frac{1-\varepsilon}{2})$. Then, the following lemma holds.

Lemma 5.1. *Let $u_h^{n-1},v\in\mathcal{B}_M$ such that* (5.3) *holds. If*

$$\|\nabla(v-R_hu^n)\|_0\le Ch^{1+\tilde\varepsilon},$$

then there exists a positive constant C such that

$$\|\nabla(G_nv-R_hu^n)\|_0\le Ch^{1+\tilde\varepsilon}.$$

Proof. Taking $u=u^n$ in (2.9) and multiplying it by Δt, we have

$$\Delta t(a(u^n)_t,\Pi_h^*v_h)+\Delta tB(u^n;u^n,\Pi_h^*v_h)=\Delta t(f^n,\Pi_h^*v_h),\quad\forall v_h\in U_h. \tag{5.4}$$

Subtracting (5.1) from (5.4) and denoting

$$\xi^n=G_nv-R_hu^n\quad\text{ and }\quad \xi^{n-1}=u_h^{n-1}-R_hu^{n-1},$$

we get

$$\begin{aligned}&(a(u^n)-a(u^{n-1})-(a(G_nv)-a(u_h^{n-1})),\Pi_h^*v_h)+\Delta tB(v;\xi^n,\Pi_h^*v_h)\\ =&\Delta t\varepsilon_h(f^n-a(u^n)_t,v_h)-\Delta t\varepsilon_a(v;R_hu^n,v_h)+\Delta t(E_1,\Pi_h^*v_h)\\ &+\Delta t[B_h(v;R_hu^n,v_h)-B_h(u^n;R_hu^n,v_h)],\quad\forall v_h\in U_h.\end{aligned} \tag{5.5}$$

Setting $v_h=\partial_t\xi^n$ in (5.5), the equation can be written into

$$\begin{aligned}&(a(u^n)-a(u^{n-1})-(a(G_nv)-a(u_h^{n-1})),\Pi_h^*\partial_t\xi^n)+\Delta tB_h(v;\xi^n,\Pi_h^*\partial_t\xi^n)\\ =&\Delta t\varepsilon_h(f^n-a(u^n)_t,\partial_t\xi^n)-\Delta t\varepsilon_a(v;R_hu^n,\partial_t\xi^n)+\Delta t\varepsilon_a(v;\xi^n,\Pi_h^*\partial_t\xi^n)\\ &+\Delta t[B_h(v;R_hu^n,\partial_t\xi^n)-B_h(u^n;R_hu^n,\partial_t\xi^n)]+\Delta t(E_1,\Pi_h^*\partial_t\xi^n).\end{aligned} \tag{5.6}$$

We adopt steps similar to (4.11)-(4.13) to get

$$\begin{aligned}&2\Delta t(a'(\nu_2^n)\partial_t\xi^n,\Pi_h^*\partial_t\xi^n)+B_h(v;\xi^n,\xi^n)+\Delta t^2B_h(v;\partial_t\xi^n,\partial_t\xi^n)\\=&B_h(v;\xi^{n-1},\xi^{n-1})-2\Delta t((a'(\nu_1^n)-a'(\nu_2^n))\partial_tu^n+a'(\nu_2^n)\partial_t\rho^n,\Pi_h^*\partial_t\xi^n)\\&+2\Delta t\varepsilon_h(f^n-a(u^n)_t,\partial_t\xi^n)-2\Delta t\varepsilon_a(v;R_hu^n,\partial_t\xi^n)+2\Delta t\varepsilon_a(v;\xi^n,\Pi_h^*\partial_t\xi^n)\\&+2\Delta t[B_h(v;R_hu^n,\partial_t\xi^n)-B_h(u^n;R_hu^n,\partial_t\xi^n)]+2\Delta t(E_1,\Pi_h^*\partial_t\xi^n).\end{aligned}\tag{5.7}$$

Using methods similar to (4.15) and (4.24)-(4.26), we obtain

$$\begin{aligned}&2\Delta t\sigma\|\partial_t\xi^n\|_0^2+\kappa\|\xi^n\|_1^2+\kappa\Delta t^2\|\partial_t\xi^n\|_1^2\\\le& C\|\xi^{n-1}\|_1^2+C\Delta t(\|\xi^n\|_0+\|\rho^n\|_0+\|\xi^{n-1}\|_0+\|\rho^{n-1}\|_0+\Delta t^2+\|\partial_t\rho^n\|_0\\&+\|\xi^n\|_1+\|E_1\|_0+\|v-R_hu^n\|_1)\|\partial_t\xi^n\|_0+C\Delta th^2\|\partial_t\xi^n\|_1.\end{aligned}\tag{5.8}$$

According to the Young inequality, we have

$$\begin{aligned}\kappa\|\xi^n\|_1^2\le& C\|\xi^{n-1}\|_1^2+C\Delta t(\|\xi^n\|_0^2+\|\rho^n\|_0^2+\|\xi^{n-1}\|_0^2+\|\rho^{n-1}\|_0^2+\Delta t^4\\&+\|\partial_t\rho^n\|_0^2+\|\xi^n\|_1^2+\|E_1\|_0^2+\Delta t^{-1}h^4+\|v-R_hu^n\|_1^2).\end{aligned}\tag{5.9}$$

Using Lemma 2.1, one can get

$$\|\xi^n\|_1^2\le C\|\xi^{n-1}\|_1^2+C\Delta t(\Delta t^{-1}h^4+\Delta t^2+\|v-R_hu^n\|_1^2).\tag{5.10}$$

Combining (5.10) and the condition of this lemma, we get the conclusion

$$\|\nabla(G_nv-R_hu^n)\|_0\le\|\xi^n\|_1\le Ch^{1+\tilde\varepsilon}.\tag{5.11}$$

This completes the proof. □

On the basis of the above lemma, we prove the existence for Δt and h sufficiently small.

Theorem 5.1. *If $\mathcal{T}_h$ is quasiuniform and inverse inequality holds. Let $u_h^{n-1},v\in\mathcal{B}_M$ such that* (5.3) *holds. Then for h sufficiently small and $\Delta t=\mathcal{O}(h^{1+\varepsilon})$ with $0<\varepsilon<1$, there exists $u_h^n\in\mathcal{B}_M$ satisfying* (2.11).

Proof. We take $v_0=u_h^{n-1}$ and $v_{j+1}=G_nv_j$. According to Lemma 5.1,

$$\|\nabla(G_nv_j-R_hu^n)\|_0\le Ch^{1+\tilde\varepsilon},\quad j\ge0.$$

Considering $\tilde\varepsilon>0$ and $M>M_0$, for h sufficiently small, we get

$$\|\nabla G_nv_j\|_\infty\le\|\nabla R_hu^n\|_\infty+Ch^{-1}\|\nabla(G_nv_j-R_hu^n)\|_0\le M,\quad j\ge0,\tag{5.12}$$

which implies that $G_nv_j\in\mathcal{B}_M$.

Now, we prove the existence of $u_h^n \in \mathcal{B}_M$. The key is to prove that G_n is a compressed mapping. By means of (5.1), for $v,w \in \mathcal{B}_M$ and $v_h \in U_h$, we have

$$(a(G_n v)-a(G_n w),\Pi_h^* v_h)+\Delta t B(v;G_n v,\Pi_h^* v_h)-\Delta t B(w;G_n w,\Pi_h^* v_h)=0. \tag{5.13}$$

Setting $v_h=\chi=G_n v-G_n w$, the equation above can be written into

$$\begin{aligned}&(a(G_n v)-a(G_n w),\Pi_h^*\chi)+\Delta t B(w;\chi,\Pi_h^*\chi)\\ =&\Delta t(B_h(w;G_n v,\chi)-B_h(v;G_n v,\chi))+\Delta t(\varepsilon_a(v;G_n v,\chi)-\varepsilon_a(w;G_n v,\chi))\\ =&I_1+I_2.\end{aligned} \tag{5.14}$$

We use the Cauchy-Schwarz inequality and the fact that $G_n v \in \mathcal{B}_M$ to get

$$|I_1| \le C\Delta t\|\nabla G_n v\|_\infty \|v-w\|_0\|\chi\|_1 \le C\Delta t\|v-w\|_0\|\chi\|_1. \tag{5.15}$$

Using Lemma 2.3, the inverse inequality and the fact $v,G_n v \in \mathcal{B}_M$, we have

$$|I_2| \le C\Delta t h\|\nabla(v-w)\|_0\|\chi\|_1 \le C\Delta t\|v-w\|_0\|\chi\|_1. \tag{5.16}$$

According to Lemma 2.4, (1.2) and (5.15)-(5.16), we know that

$$\sigma\|\chi\|_0^2+\gamma\Delta t\|\chi\|_1^2 \le C\Delta t\|v-w\|_0\|\chi\|_1 \le C\Delta t\|v-w\|_0^2+\gamma\Delta t\|\chi\|_1^2. \tag{5.17}$$

When Δt is sufficiently small, the equation above means that G_n is a compressed mapping and has a fixed point. This fixed point is the solution of (2.11). □

6 Numerical results

In order to testify the correctness of our theoretical results, we complete two numerical experiments. Since the equation is a nonlinear, we use the Newton method to accomplish the numerical experiment. The results of the following two examples show that our method is correct.

6.1 Example I

Consider the following two-dimensional equation

$$\begin{cases} \dfrac{\partial(u^4+u)}{\partial t}-\nabla\cdot\left(\left(\dfrac{1}{2}u^2+1\right)\nabla u\right)=f, & (x,y)\in\Omega, \quad t\in(0,T],\\ u(x,y,t)=g, & (x,y)\in\partial\Omega, \quad t\in[0,T],\\ u(x,y,0)=0, & (x,y)\in\Omega,\end{cases} \tag{6.1}$$

where $\Omega=(0,1)\times(0,1)$ and $T=1$. We take the analytical solution

$$u(x,y)=t^2\sin(\pi x)\sin(\pi y).$$

Table 1: The numerical results of FVE method when $\tau=1/1000$.

$(h,\Delta t)$	$\|u_h-u\|_{L^2}$	Convergence order	$\|u_h-u\|_{H^1}$	rate
(1/4,1/1000)	5.7103e-002	-	6.5858e-001	-
(1/8,1/1000)	1.4133e-002	2.0145	3.2669e-001	1.0114
(1/16,1/1000)	3.4459e-003	2.0361	1.6280e-001	1.0048
(1/32,1/1000)	7.7508e-004	2.1525	8.1323e-002	1.0014

Table 2: The numerical results of FVE method when $h=1/150$.

(h,τ)	$\|u_h-u\|_{L^2}$	Convergence order	$\|u_h-u\|_{H^1}$	rate
(1/150,1/5)	2.6383e-002	-	1.2170e-001	-
(1/150,1/10)	1.4126e-002	0.9013	6.6909e-002	0.8631
(1/150,1/15)	9.6077e-003	0.9506	4.7325e-002	0.8541
(1/150,1/20)	7.2713e-003	0.9685	3.7620e-002	0.7978

Table 3: The numerical results of FVE method when $\Delta t=h^2$.

$(h,\Delta t)$	$\|u_h-u\|_{L^2}$	Convergence order	$\|u_h-u\|_{H^1}$	Convergence order
(1/4,1/16)	5.1465e-002	-	6.6670e-001	-
(1/8,1/64)	1.2511e-002	2.0404	3.2778e-001	1.0243
(1/12,1/144)	5.5404e-003	2.0089	2.1757e-001	1.0107
(1/16,1/256)	3.1133e-003	2.0035	1.6291e-001	1.0057

In Table 1, we present the errors in L^2-norm and H^1-norm with linear element on uniform triangular meshes. We take $\Delta t=1/1000$ and $h=1/4,1/8,1/16,1/32$, respectively. We find that the convergence order of errors in L^2-norm is second-order and the convergence order of error in H^1-norm is first-order. The numerical results are in accordance with theoretical results. In Table 2, we take $h=1/150$ and $\Delta t=1/5,1/10,1/15,1/20$. We can see that the convergence order of errors in L^2-norm and H^1-norm are both nearly first-order. The numerical results are identical with the theoretical results. In Table 3, we take $\Delta t=h^2$ and $h=1/4,1/8,1/12,1/16$. The convergence orders of errors in L^2-norm and H^1-norm are second-order and first-order, which are accord with our theoretical proof.

6.2 Example II

Consider the following parabolic equation

$$\begin{cases} \dfrac{\partial \exp(u)}{\partial t}-\nabla\cdot((\sin(u)+1)\nabla u)=f, & (x,y)\in\Omega, \quad t\in(0,T], \\ u(x,y,t)=g, & (x,y)\in\partial\Omega, \quad t\in[0,T], \\ u(x,y,0)=0, & (x,y)\in\Omega, \end{cases} \tag{6.2}$$

Table 4: The numerical results of FVE method when $\Delta t = 1/5000$.

$(h,\Delta t)$	$\|u_h - u\|_{L^2}$	Convergence order	$\|u_h - u\|_{H^1}$	Convergence order
(1/4,1/5000)	1.8469e-002	-	1.3539e-001	-
(1/8,1/5000)	4.3839e-003	2.0748	6.1491e-002	1.1387
(1/12,1/5000)	1.9534e-003	1.9937	4.0030e-002	1.0587
(1/16,1/5000)	1.1195e-003	1.9351	2.9754e-002	1.0312

Table 5: The numerical results of FVE method when $h = 1/150$.

$(h,\Delta t)$	$\|u_h - u\|_{L^2}$	Convergence order	$\|u_h - u\|_{H^1}$	Convergence order
(1/150,1/5)	8.6461e-002	-	4.9508e-001	-
(1/150,1/10)	4.5080e-002	0.9396	2.6114e-001	0.9228
(1/150,1/15)	3.0399e-002	0.9718	1.7669e-001	0.9635
(1/150,1/20)	2.2921e-002	0.9815	1.3343e-001	0.9762

Table 6: The numerical results of FVE method when $\Delta t = h^2$.

$(h,\Delta t)$	$\|u_h - u\|_{L^2}$	Convergence order	$\|u_h - u\|_{H^1}$	Convergence order
(1/4,1/16)	3.8116e-002	-	1.4207e-001	-
(1/8,1/64)	1.0149e-002	1.9091	6.3797e-002	1.1550
(1/12,1/144)	4.5690e-003	1.9683	4.0907e-002	1.0960
(1/16,1/256)	2.5821e-003	1.9838	3.0180e-002	1.0571

where $\Omega = (0,1) \times (0,1)$ and $T = 1$. The analytical solution is

$$u(x,y) = t^2(x^2 + y^2 + 1).$$

In Table 4, we present the error in L^2-norm and H^1-norm. We take $\Delta t = 1/5000$ and $h = 1/4, 1/8, 1/16, 1/32$. Numerical results show that the convergence order in L^2-norm is second order and the convergence order in H^1-norm is first order. The numerical results are in accordance with the theoretical results. In Table 5, we take $h = 1/150$ and $\Delta t = 1/5, 1/10, 1/15, 1/20$. The convergence orders of error in L^2-norm and H^1-norm are both first order, which is accordance with the theoretical resutls. In Table 6, we take $\Delta t = h^2$ and $h = 1/4, 1/8, 1/12, 1/16$. We find the convergence orders of errors in L^2-norm and H^1-norm are second-order and first-order, respectively.

7 Conclusions

In this paper, we propose a finite volume element method for a parabolic equation with nonlinear time derivative term and nonlinear diffusion term. Firstly, we construct semi-discrete and fully discrete schemes for this equation. Then, we prove the suboptimal error estimate in L^2-norm scheme and optimal error estimate in H^1-norm. Finally, some numerical examples are presented to testify the effectiveness of our method.

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