CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0056

Vol. **x**, No. x, pp. 1-20
xx 2025

A Mathematical Analysis for the Dynamics of Multiple Languages

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Received 1 December 2024; Accepted 5 April 2025

Abstract. In many regions of the world, languages coexist in daily life, but often one tongue increases its use at the expense of another. In the present paper, we build a large compartmental system of differential equations that meets the situation of two "prestigious" tongues and many local languages, whose use is reduced by social interaction. The focus is on the preferred language in social relationships for communicating, rather than mere knowledge. We aim at stating and proving theorems on the qualitative behavior of the system. Numerical simulations illustrate the results, giving rise to distinct dynamics.

AMS subject classifications: 34A34, 37M05, 91D10

Key words: Language competition, mathematical model, dynamical system, simulation.

1 Introduction

There is no unique theory on the origin of languages, neither spoken nor signed. Two examples are the "continuity theories" and "discontinuity theories". In the first case, the main idea is that language is so complex that it cannot appear from nothing, and therefore it is supposed that language evolves from some prelinguistic systems used by primate ancestors. This position is defended by Ulbaek [31], who argues that language evolves from primate cognition. In the second case, language is supposed to appear as a single chance mutation, as suggested by Chomsky [6], dated about 100,000 years ago. On the other hand, the "usage-based theory of language acquisition" was introduced by Tomasello [30], who states that language structure appears from language use, because tongue acquisition is done with general cognitive processes.

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It was only quite recently that UNESCO started to consider preserving Intangible Cultural Heritage. The text of the convention [32] was approved during the General Conference held in Paris in 2003. This convention divides the Intangible Cultural Heritage into five domains, being the first one "Oral traditions and expressions, including language as a vehicle of the Intangible Cultural Heritage". The key point of this domain refers to language being a vehicle of transmission of culture. Following [33], language as culture also has three important aspects, namely: a) culture is a product of history which it in turn reflects; b) the second aspect of language as culture is as an image-forming agent in the mind of a child; and c) culture transmits or imparts those images of the world and reality through the spoken and written language, that is, through a specific language.

The implementation of foreign languages has occurred all over the world and can be studied from many points of view. The main aim of our work is to mathematically analyze the influence of two major languages (understood as "prestigious" by high society) into a community using a number J of their own local languages, by employing abstract compartmental models of ordinary differential equations and dynamical-systems theory.

A compartmental system of ordinary differential equations models a continuous-time phenomenon, such as that of language shift within a community, by analyzing rates of change and fluxes of individuals between groups. It is built on a macroscopic framework, where processes are studied from an aggregate and averaged point of view. The prototypical compartmental model is the SIR formulation, first developed in 1927 [14]. In the simplest case, it divides a population suffering a contagious disease into susceptible (S), infected (I), and recovered (R) persons, and fluxes represent contagions and recoveries [4, 34]. The output is not at individual level, but a mean value for the region (for which a probabilistic interval could be incorporated if stochastic ordinary differential equations were used). Extensions of the SIR equations have been utilized for different diseases, such as HIV/AIDS, COVID-19, etc. [23,27,28]. However, some of these types of models are not restricted to epidemics. Social interactions do not only transmit viruses and bacteria, but serve for spreading opinions, habits, ideologies, etc. [3, 11, 15, 29]. Since tongues and attitudes on them can be "transmitted" by contacts, some compartmental models should be useful tools to capture the mechanisms of language acquisition. These ideas have been used for various social phenomena, such as drug consumption [35], criminality [10, 18], and telecommunications [5].

The seminal one-page paper [1] initiated the work on the mathematical modeling of language interaction through ordinary differential equations and dynamical systems. The authors proposed a planar system to study the coexistence of two monolingual groups in a region, with contacts represented by probabilities of shift and with a parameter of language's status. Some data were fitted for Scottish Gaelic, Quechua, and Welsh. Later, reference [19] extended the theory to three equations, giving the possibility of stable bilingualism. Data on Castilian and Galician in Spain were fitted. Other alternative developments were given in [25], where spatial diffusion was added to the model of [1] to situate the tongues in different geographical areas, and in [26], where a logistic natural growth of languages was suggested. In Spain, the Galician language was further discussed in [20, 22], and the situation of the Catalan language was also analyzed [21], with these mathematical tools. Other countries were also addressed, such as India with "Hinglish" speakers [24] and Mexico with Yucatec Maya and Spanish [2]. A recent contribution conducted a theoretical study of a dynamical compartmental model of languages, with three groups defined by the dominant language, bilingualism, and the underrepresented tongue; equilibrium points and stability were investigated [7]. There are parts of the world with more complex realities due to a huge variety of languages that have not been studied mathematically, such as Africa. In regions of this continent, there may be many local languages, that are being removed by major tongues. For example, in Cameroon, both French and English are the official languages, but there are at least 250 local languages in the country [8,13,16]. Thus, there should be more than three compartments in the model's formulation, with many parameters capturing language abandonment or adoption, on an aggregate fashion. The use of multiple competing languages in models has been proposed in some articles of statistical physics, such as [9, 17, 36]. In [17], languages had attractiveness and the number of speakers was modeled in a compartmental system. In [36], an extended Abrams-Strogatz model was utilized to fit the data in Singapore and Hong Kong, by means of a Bayesian computation method. In [9], a theoretical study of an extended Abrams-Strogatz model was conducted among a general number of languages, based on adequate simulations. Our contribution is focused on the mathematical analysis for a model of multiple languages.

The plan of the paper is the following. In Section 2, we propose a compartmental model of ordinary differential equations where J local tongues compete with two dominant languages, which is related to previous formulations in the literature. We consider the possibility that an individual knows and uses more than one language, being one of the tongues the preferred one, and the resulting system is of size $3J+2$. In Section 3, which represents the main part of the work, we state and prove theoretical results on well-posedness and dynamics, giving rise to global existence and positivity of solutions and, under certain parameters' inequalities, extinction of the local languages. Many of our arguments for the study of dynamics are new in the mathematics literature, as far as we know. In Section 4, some numerical simulations are performed to illustrate the theoretical results on language dynamics. Finally, in Section 5, conclusions of this work are presented.

2 Proposed model

Let us assume that in a community we have two "prestigious" languages, denoted by P_1 and P_2, as well as J local tongues, L_j, with $j=1,\dots,J$. We shall denote by $p_i(t)$ the number of persons who speak P_i more in social relations, with $i\in\{1,2\}$. Similarly, $l_j(t)$ stands for the number of individuals that speak L_j more in social relations, with $j\in\{1,\dots,J\}$. The time is $t\geq 0$ (years, months, etc.). For the proposed model, we shall need some extra notation: $b_j(t)$ represents the number of individuals that, in social interactions and at time t, communicate equally via L_j and P_1, and finally, $\tilde{b}_j(t)$ represents the number of individuals who communicate both with L_j and P_2, equally in social interactions. The scheme of language adoption is

$$l_j \quad\to\; b_j \;\to\; p_1 \quad \text{or} \quad l_j \;\to\; \tilde{b}_j \;\to\; p_2.$$

Notice that we are not dealing with "knowledge" of a language, but with "preferred language in social relations for communicating", which best reflects the strength of a tongue. Hence, the used compartments do not intersect: although there could be bilingual persons, one of the languages is more used in social relations, and this is the tongue kept by the model.

We used the term "prestigious" instead of dominant because, at the beginning of the implementation in the region, it might not still be dominant. The quotation marks indicate that the adjective prestigious is actually subjective, the way the population views the tongues.

With the aforementioned notations and interpretations, the proposed model is expressed in terms of a system of $3J+2$ differential equations

$$l_j' = -\beta_j l_j(b_j+p_1) - \tilde{\beta}_j l_j(\tilde{b}_j+p_2) + \delta_j l_j, \quad j=1,\ldots,J, \tag{2.1a}$$
$$b_j' = \beta_j l_j(b_j+p_1) - \alpha_j b_j p_1 + \epsilon_j b_j, \qquad j=1,\ldots,J, \tag{2.1b}$$
$$\tilde{b}_j' = \tilde{\beta}_j l_j(\tilde{b}_j+p_2) - \tilde{\alpha}_j \tilde{b}_j p_2 + \tilde{\epsilon}_j \tilde{b}_j, \qquad j=1,\ldots,J, \tag{2.1c}$$
$$p_1' = \gamma + \delta p_1 + \sum_{j=1}^{J} \alpha_j b_j p_1, \tag{2.1d}$$
$$p_2' = \tilde{\gamma} + \tilde{\delta} p_2 + \sum_{j=1}^{J} \tilde{\alpha}_j \tilde{b}_j p_2. \tag{2.1e}$$

We assume that the parameters $\beta_j, \tilde{\beta}_j, \alpha_j, \tilde{\alpha}_j, \gamma$ and $\tilde{\gamma}$ are positive. The remaining coefficients $\delta_j, \epsilon_j, \tilde{\epsilon}_j, \delta$ and $\tilde{\delta}$ are real numbers. The interpretation of the parameters is the following: β_j is the force of transition from l_j to b_j; $\tilde{\beta}_j$ is the force of transition from l_j to $\tilde{b}_j$; α_j is the force of transition from b_j to p_1; $\tilde{\alpha}_j$ is the force of transition from b_j to p_2; γ is the immigration rate, that adopts the "prestigious" language P_1; $\tilde{\gamma}$ is the immigration rate, that adopts the "prestigious" language P_2; and, respectively, $\delta_j, \epsilon_j, \tilde{\epsilon}_j, \delta$ and $\tilde{\delta}$ represent the birth rate minus the death rate and the emigration rate in the corresponding group. There are initial conditions

$$\begin{aligned} &l_j(0) = l_{j,0} > 0, \qquad b_j(0) = b_{j,0} > 0, \qquad \tilde{b}_j(0) = \tilde{b}_{j,0} > 0, \\ &p_1(0) = p_{1,0} > 0, \quad p_2(0) = p_{2,0} > 0. \end{aligned} \tag{2.2}$$

Observe that, if the total population is

$$N = \sum_{j=1}^{J} \left(l_j + b_j + \tilde{b}_j\right) + p_1 + p_2,$$

then

$$N'(t)=\gamma+\tilde{\gamma}+\sum_{j=1}^{J}\delta_j l_j(t)+\sum_{j=1}^{J}\epsilon_j b_j(t)+\sum_{j=1}^{J}\tilde{\epsilon}_j\tilde{b}_j(t)+\delta p_1(t)+\tilde{\delta}p_2(t), \tag{2.3}$$

after adding the equations in (2.1). Thus, considering the interpretation of the parameters, N changes depending on immigration, emigration, births and deaths. When $\delta_j=\epsilon_j=\tilde{\epsilon}_j=\delta=\tilde{\delta}=\Lambda$ for all j, we can derive an ordinary differential equation for N of Malthusian type, given by $N'=\gamma+\tilde{\gamma}+\Lambda N$; otherwise N is not described by a linear differential equation.

Finally, we notice that (2.1) would be of application when comparing different dialects, instead of different languages. In a tongue, there may be variants that could be considered more "prestigious" than others, for example, the standard language by convention. Speakers' interactions shape language evolution and the acquirement of certain linguistic characteristics, hence changing language dynamics and dialectal stability [12].

3 Results on well-posedness and dynamics

We state and prove results on (2.1)-(2.2). Compactly, we can write the system (2.1) as $x'=f(x)$, where

$$x=(l_1,\ldots,l_J,b_1,\ldots,b_J,\tilde{b}_1,\ldots,\tilde{b}_J,p_1,p_2)$$

is the state variable,

$$x(0)=(x_{1,0},\ldots,x_{3J+2,0})$$

is the initial condition, and

$$f=(f_1,\ldots,f_{3J+2}):\mathbb{R}^{3J+2} \to \mathbb{R}^{3J+2}$$

is the vector field. Theorem 3.1 demonstrates that (2.1)-(2.2) is well-posed and consistent. Theorem 3.2 shows that, if δ and $\tilde{\delta}$ are non-negative, then the local languages disappear and the "prestigious" languages grow towards infinity (of course, this is an idealization that simply represents the increase of the new tongues). A simple consequence of the proof, Corollary 3.1, is given, for the cases $\delta\geq 0$ and $\tilde{\delta}\geq 0$ separately (i.e., for the "prestigious" tongues separately). Theorem 3.3 is concerned with the situation $\delta\geq 0,\tilde{\delta}<0$, with an additional inequality for the parameters, which renders extinction of the local languages, convergence to infinity for P_1, and convergence to a finite equilibrium point for P_2. The assumptions of Corollary 3.1 and Theorem 3.3 are tight. Theorem 3.4 illustrates that, when δ and $\tilde{\delta}$ are both negative, then the local languages die out too under certain parameters' inequalities, with the two "prestigious" languages approaching positive and finite equilibrium points. This last Theorem 3.4 only demonstrates local asymptotic stability, based on eigenvalues of the Jacobian matrix, rather than global. The symbol $0_J=(0,\ldots,0)$ denotes a vector of zeros of length J.

Theorem 3.1. *The solution of* (2.1)-(2.2) *is given by unique functions*

$$l_j,b_j,\tilde{b}_j,p_1,p_2:[0,\infty) \to (0,\infty).$$

Proof. Note that f is smooth. By the Picard-Lindelöf theorem, there exists a unique solution $x:[0,T)\to\mathbb{R}^{3J+2}$ of (2.1), where $T\le\infty$ is the maximum possible value. The goal is to prove positivity on $[0,T)$ and $T=\infty$.

1) Positivity on $[0,T)$. In (2.1), the equation for l_j has the structure

$$l_j' = a_j(t)l_j(t), \tag{3.1}$$

where $a_j:[0,T)\to\mathbb{R}$ is continuous. Then,

$$l_j(t) = l_{j,0}\exp\left(\int_0^t a_j(\tau)\mathrm{d}\tau\right) > 0, \quad \forall t\in[0,T). \tag{3.2}$$

Suppose that $p_1(t^*)=0$ for some $t^*\in(0,T)$. Then, $p_1'(t^*)=\gamma>0$, which is a contradiction. Hence, t^* does not exist and $p_1(t)>0$ for all $t\in[0,T)$. Another proof of this fact consists in noticing that

$$p_1'(t)=\gamma+h_1(t)p_1(t),$$

where $h_1:[0,T)\to\mathbb{R}$ is continuous, so the solution possesses the form

$$p_1(t)=p_{1,0}\exp\left(\int_0^t h_1(\tau)\mathrm{d}\tau\right)+\gamma\int_0^t\exp\left(\int_r^t h_1(\tau)\mathrm{d}\tau\right)\mathrm{d}r>0.$$

An analogous results follows for p_2: We have $p_2(t)>0$ for every $t\in[0,T)$.

The same technique works for b_j and $\tilde{b}_j$. Suppose that $b_j(t^*)=0$ for some $t^*\in(0,T)$. Then, $b_j'(t^*)=\beta_j l_j(t^*)p_1(t^*)>0$, which is a contradiction. Hence, $b_j(t)>0$ for every $t\in[0,T)$. The other proof of this result consists in rewriting

$$b_j'(t)=y_j(t)+z_j(t)b_j(t),$$

where $y_j:[0,T)\to(0,\infty)$ and $z_j:[0,T)\to\mathbb{R}$ are continuous; the solution has the form

$$b_j(t)=b_{j,0}\exp\left(\int_0^t z_j(\tau)\mathrm{d}\tau\right)+\int_0^t\exp\left(\int_r^t z_j(\tau)\mathrm{d}\tau\right)y_j(r)\mathrm{d}r>0.$$

An analogous result holds for $\tilde{b}_j$.

2) The horizon is $T=\infty$. Suppose that $T<\infty$. Notice that, with (2.3),

$$\begin{aligned}\sum_{j=1}^{3J+2}x_j'(t)&=\sum_{j=1}^{J}l_j'(t)+\sum_{j=1}^{J}b_j'(t)+\sum_{j=1}^{J}\tilde{b}_j'(t)+p_1'(t)+p_2'(t)\\&=\gamma+\tilde{\gamma}+\sum_{j=1}^{J}\delta_j l_j(t)+\sum_{j=1}^{J}\epsilon_j b_j(t)+\sum_{j=1}^{J}\tilde{\epsilon}_j\tilde{b}_j(t)+\delta p_1(t)+\tilde{\delta}p_2(t)\\&\le\gamma^*+\delta^*\sum_{j=1}^{3J+2}x_j(t),\end{aligned}$$

where

$$\begin{aligned}&\gamma^*=\max\{\gamma,\tilde{\gamma}\}>0,\\&\delta^*=\max\{|\delta_1|,\dots,|\delta_J|,|\epsilon_1|,\dots,|\epsilon_J|,|\tilde{\epsilon}_1|,\dots,|\tilde{\epsilon}_J|,|\delta|,|\tilde{\delta}|\}\geq 0.\end{aligned}$$

By Grönwall's inequality,

$$\sum_{j=1}^{3J+2}x_j(t)\leq\sum_{j=1}^{3J+2}x_{j,0}+\gamma^*\frac{t^2}{2}+\int_0^t\left(\sum_{j=1}^{3J+2}x_{j,0}+\gamma^*\frac{\tau^2}{2}\right)\delta^*\exp\big(\delta^*(t-\tau)\big)\mathrm{d}\tau=h(t),$$

where $h:[0,\infty)\to\mathbb{R}$ is a continuous function. Consider any $T<\xi<\infty$. Let

$$\mathcal{H}=\left\{(t,y):t\in[0,\xi],y\in\mathbb{R}^{3J+2},y_j\geq 0,\forall j,\sum_{j=1}^{3J+2}y_j\leq h(t)\right\}.$$

We know that the solution satisfies

$$(t,x(t))=\Big(t,\big(l_j(t)\big)_{j=1}^J,\big(b_j(t)\big)_{j=1}^J,\big(\tilde{b}_j(t)\big)_{j=1}^J,p_1(t),p_2(t)\Big)\in\mathcal{H},\quad\forall t\in[0,T).$$

The set $\mathcal{H}$ is compact, so $(t,x)\mapsto f(x)$ is bounded on $\mathcal{H}$ with respect to any norm $|\cdot|$. Let $f_{\max}$ be the bound for $|f|$ on $\mathcal{H}$. Since

$$x(t)=x(0)+\int_0^t f\big(x(s)\big)\mathrm{d}s,$$

we derive

$$|x(t)-x(t')|\leq f_{\max}|t-t'|$$

for $t,t'\in\mathcal{H}$. Hence, with Cauchy sequences, the limit $\lim_{t\to T^-}x(t)$ exists. This implies that x has a prolongation to T. □

Theorem 3.2. *Given* (2.1), *if* $\delta\geq 0$ *and* $\tilde{\delta}\geq 0$, *then*

$$\begin{aligned}&\lim_{t\to\infty}p_1(t)=\lim_{t\to\infty}p_2(t)=\infty,\\&\lim_{t\to\infty}l_j(t)=\lim_{t\to\infty}b_j(t)=\lim_{t\to\infty}\tilde{b}_j(t)=0\end{aligned}$$

for all $j=1,\dots,J$, *irrespective of the positive initial conditions* (2.2).

Proof. Since $p_1'\geq\gamma$ and $p_2'\geq\tilde{\gamma}$ (because $\delta\geq 0$ and $\tilde{\delta}\geq 0$), we deduce

$$p_1(t)\geq\gamma t+p_{1,0}\overset{t\to\infty}{\longrightarrow}\infty,\tag{3.3}$$

$$p_2(t)\geq\gamma t+p_{2,0}\overset{t\to\infty}{\longrightarrow}\infty.\tag{3.4}$$

Such growths are of increasing form.

On the other hand, we have the differential equation (3.1), where

$$\lim_{t\to\infty} a_j(t) = -\infty \tag{3.5}$$

by (3.3) or (3.4). Then, fixed any $M<0$, we can ensure that $a_j(t)\le -M$ for all $t\ge t_M>0$. As a result, from (3.2),

$$\begin{aligned} l_j(t) &= l_{j,0}\exp\left(\int_0^{t_M} a_j(\tau)\mathrm{d}\tau + \int_{t_M}^{t} a_j(\tau)\mathrm{d}\tau\right) \\ &\le l_{j,0}\exp\left(\int_0^{t_M} a_j(\tau)\mathrm{d}\tau - M(t-t_M)\right) \ \stackrel{t\to\infty}{\longrightarrow}\ 0, \end{aligned} \tag{3.6}$$

exponentially fast.

In fact, let us see the stronger limits

$$\lim_{t\to\infty} p_1(t) l_j(t) = 0, \tag{3.7}$$

$$\lim_{t\to\infty} p_2(t) l_j(t) = 0. \tag{3.8}$$

By (3.1) and (3.5), we derive that $l_j'(t)<0$ for all sufficiently large t. With (3.6), we then obtain

$$\lim_{t\to\infty} l_j'(t) = 0. \tag{3.9}$$

From the Eq. (2.1a), properties (3.6) and (3.9) imply in consequence that

$$\lim_{t\to\infty} l_j(t)\big(b_j(t)+p_1(t)\big) = \lim_{t\to\infty} l_j(t)\big(\tilde{b}_j(t)+p_2(t)\big) = 0.$$

In particular, both (3.7) and (3.8) are verified.

Conditions (3.7) and (3.8) are very relevant to prove that b_j and $\tilde{b}_j$, respectively, tend to 0 as $t\to\infty$. Indeed, rewrite

$$b_j'(t) = q_j(t) b_j(t) + w_j(t), \tag{3.10}$$

where

$$q_j(t) = \beta_j l_j(t) - \alpha_j p_1(t) + \epsilon_j \ \stackrel{t\to\infty}{\longrightarrow}\ -\infty, \tag{3.11}$$

because of (3.3), and

$$w_j(t) = \beta_j l_j(t) p_1(t) \ \stackrel{t\to\infty}{\longrightarrow}\ 0 \tag{3.12}$$

by (3.7). The three expressions (3.10)-(3.12) imply that

$$\lim_{t\to\infty} b_j(t) = 0.$$

Analogously, with (3.4) and (3.8),

$$\lim_{t\to\infty} \tilde{b}_j(t) = 0$$

is true as well. $\square$

Remark 3.1. Notice that we work with the independent variable $t \geq 0$. On $(-\infty,0]$, the result presented in Theorem 3.1 could be different. For example, the solution of $x'=1$ is negative for small t. In this sense, there could a bounded domain $(\tilde{T},0]$ where the solution blows-up or tends to zero at $\tilde{T}<0$.

Corollary 3.1. *Given* (2.1)*, if* $\delta \geq 0$*, then*

$$\begin{aligned} &\lim_{t\to\infty} p_1(t) = \infty, \\ &\lim_{t\to\infty} l_j(t) = \lim_{t\to\infty} b_j(t) = 0 \end{aligned}$$

for all $j=1,\ldots,J$*, irrespective of the positive initial conditions* (2.2)*. When* $\delta<0$*, the result is never true, irrespective of the positive initial conditions* (2.2)*.*

If $\tilde{\delta} \geq 0$*, then*

$$\begin{aligned} &\lim_{t\to\infty} p_2(t) = \infty, \\ &\lim_{t\to\infty} l_j(t) = \lim_{t\to\infty} \tilde{b}_j(t) = 0 \end{aligned}$$

for all $j=1,\ldots,J$*, irrespective of the positive initial conditions* (2.2)*. When* $\tilde{\delta}<0$*, the result is never true, irrespective of the positive initial conditions* (2.2)*.*

Proof. We consider the case $\delta \geq 0$. The assumption implies $p_1' \geq \gamma$ and (3.3). The properties (3.5), (3.6), and (3.9) hold as well. The development to prove (3.7) and (3.8) is satisfied. We also have (3.10)-(3.12).

Now suppose that $\delta<0$, with $p_1(t)\to\infty$ as well. The reasoning of the proof of Theorem 3.2 is valid here, so that $l_j(t)$ and $b_j(t)$ tend to zero with t. However, from (2.1),

$$p_1'(t) = \gamma - \left(-\delta + \sum_{j=1}^{J} \alpha_j b_j(t)\right) p_1(t) \overset{t\to\infty}{\longrightarrow} -\infty,$$

since $b_j(t)\to 0$. This is impossible considering that $p_1(t)\to\infty$. $\square$

Remark 3.2. The infinite value of the limit in Theorem 3.2 and Corollary 3.1 is a consequence of immigration and the larger or equal value of births compared to deaths and emigrations. Since this demography is constant along $[0,\infty)$, population increases forever and infinite values emerge. This is an abstract result. In practice, intervals are actually bounded, and one should then consider these mathematical theorems under the appropriate sense, simply meaning that the “prestigious” languages grow in reality, at the expense of the original languages. The model should be taken as a representation for short-time data, which informs about the possible future from the demography observed in the past.

Theorem 3.3. *Given* (2.1), *if* $\delta\geq 0, \tilde{\delta}<0$, *and* $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j<0$, *then*

$$\lim_{t\to\infty} p_1(t)=\infty, \quad \lim_{t\to\infty} p_2(t)=\frac{\tilde{\gamma}}{-\tilde{\delta}},$$
$$\lim_{t\to\infty} l_j(t)=\lim_{t\to\infty} b_j(t)=\lim_{t\to\infty} \tilde{b}_j(t)=0$$

for all $j=1,\ldots,J$, *irrespective of the positive initial conditions* (2.2). *When* $\delta\geq 0, \tilde{\delta}<0$, *and* $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j>0$, *the result is never true, irrespective of the positive initial conditions* (2.2).

If $\tilde{\delta}\geq 0, \delta<0$, *and* $\alpha_j\gamma/\delta+\epsilon_j<0$, *then*

$$\lim_{t\to\infty} p_1(t)=\frac{\gamma}{-\delta}, \quad \lim_{t\to\infty} p_2(t)=\infty,$$
$$\lim_{t\to\infty} l_j(t)=\lim_{t\to\infty} b_j(t)=\lim_{t\to\infty} \tilde{b}_j(t)=0$$

for all $j=1,\ldots,J$, *irrespective of the positive initial conditions* (2.2). *When* $\tilde{\delta}\geq 0, \delta<0$, *and* $\alpha_j\gamma/\delta+\epsilon_j>0$, *the result is never true, irrespective of the positive initial conditions* (2.2).

Proof. We suppose the first case, i.e. $\delta\geq 0, \tilde{\delta}<0$, and $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j<0$. By Corollary 3.1, we only need to prove that $\tilde{b}_j(t)\to 0$ and $p_2(t)\to\tilde{\gamma}/(-\tilde{\delta})$ when $t\to\infty$. Observe that, from the Eq. (2.1e),

$$p_2'(t)\geq\tilde{\gamma}+\tilde{\delta}p_2(t).$$

With the integrating factor $\exp(-\tilde{\delta}t)$, we have

$$\left(\exp(-\tilde{\delta}t)p_2\right)'\geq\exp(-\tilde{\delta}t)\tilde{\gamma},$$

which implies the Grönwall-type inequality

$$p_2(t)\geq\exp(\tilde{\delta}t)p_{2,0}+\frac{1-\exp(\tilde{\delta}t)}{-\tilde{\delta}}\tilde{\gamma}$$

and

$$\liminf_{t\to\infty} p_2(t)\geq\frac{\tilde{\gamma}}{-\tilde{\delta}}. \tag{3.13}$$

Recall also that (3.8) holds, by (3.6) and (3.9).

Write

$$\tilde{b}_j'(t)=\tilde{q}_j(t)\tilde{b}_j(t)+\tilde{w}_j(t), \tag{3.14}$$

where

$$\tilde{q}_j(t)=\tilde{\beta}_j l_j(t)-\tilde{\alpha}_j p_2(t)+\tilde{\epsilon}_j<0 \tag{3.15}$$

for all sufficiently large t, by (3.13) and the assumption, and

$$\tilde{w}_j(t)=\tilde{\beta}_j l_j(t)p_2(t) \overset{t\to\infty}{\longrightarrow} 0 \tag{3.16}$$

with (3.8). The three expressions (3.14)-(3.16) imply that

$$\lim_{t\to\infty}\tilde{b}_j(t)=0,$$

as wanted.

Now, suppose that the function p_2 is unbounded on $[0,\infty)$. For each natural number k, choose the first t_k such that $p_2(t_k)\geq k$. Note that $t_k\to\infty$ and $p_2(t_k)\to\infty$, as k grows. As a result,

$$p_2'(t_k)=\tilde{\gamma}-\left(-\tilde{\delta}+\sum_{j=1}^{J}\tilde{\alpha}_j\tilde{b}_j(t_k)\right)p_2(t_k)\ \stackrel{t_k\to\infty}{\longrightarrow}\ -\infty,$$

because $\tilde{b}_j(t_k)\to 0$. In particular, $p_2'(t_k)<0$ from a certain k, but this fact contradicts the minimality of t_k satisfying $p(t_k)\geq k$. In conclusion: p_2 is bounded on $[0,\infty)$.

Suppose that p_2 is not a monotonic function on any interval $[t^*,\infty)$, for $t^*>0$. Then there exists a subsequence $s_m\to\infty$ such that $p_2'(s_m)=0$ for all m, because the function exhibits oscillations forever. By (2.1),

$$p_2(s_m)=\frac{\tilde{\gamma}}{(-\tilde{\delta})-\sum_{j=1}^{J}\tilde{\alpha}_j\tilde{b}_j(s_m)}\ \stackrel{m\to\infty}{\longrightarrow}\ \frac{\tilde{\gamma}}{-\tilde{\delta}}>0.$$

This is a contradiction, hence p_2 is monotonic from a certain point. As it is also bounded, there must exist $\ell=\lim_{t\to\infty}p_2(t)\in[0,\infty)$. By taking the limit in the Eq. (2.1e), we obtain $0=\tilde{\gamma}+\tilde{\delta}\ell$, and we are done.

Finally, suppose that $\delta\geq 0,\tilde{\delta}<0$, and $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{e}_j>0$. Assume also that $\tilde{\gamma}/(-\tilde{\delta})$ is the limit of p_2. Expression (3.14) is still true. By Corollary 3.1, $l_j(t)\to 0$. In (3.15), we now have $\tilde{q}_j(t)>\eta>0$, for a number $\eta>0$ and all sufficiently large t, $t\geq t_\eta$. In consequence, from (3.14),

$$\begin{aligned}\tilde{b}_j(t)&\geq\tilde{b}_{j,0}\exp\left(\int_0^t\tilde{q}_j(\tau)\mathrm{d}\tau\right)\\&\geq\tilde{b}_{j,0}\exp\left(\int_0^{t_\eta}\tilde{q}_j(\tau)\mathrm{d}\tau\right)\exp\left(\eta(t-t_\eta)\right)\ \stackrel{t\to\infty}{\longrightarrow}\ \infty.\end{aligned}$$

Hence, $\tilde{b}_j(t)$ does not tend to 0. □

Remark 3.3. When $\delta\geq 0,\tilde{\delta}<0$, and $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{e}_j=0$, the proof of Theorem 3.3 is not conclusive. If we aimed at proving that the limits hold, then in the first part of the proof, we would have $\limsup_{t\to\infty}\tilde{q}_j(t)\leq 0$. As a result, we could not ensure that such a limsup is negative and $\tilde{b}_j(t)\to\infty$. On the other hand, if we aimed at proving that the limits are not true, assuming that $\tilde{\gamma}/(-\tilde{\delta})$ is the limit of p_2, we would have $\lim_{t\to\infty}\tilde{q}_j(t)=0$, which would not be of use for $\tilde{b}_j$ again.

Theorem 3.4. *Given* (2.1), *if $\delta<0$ and $\tilde{\delta}<0$, then $D=(0_J,0_J,0_J,\gamma/(-\delta),\tilde{\gamma}/(-\tilde{\delta}))$ is a locally asymptotically stable equilibrium point if, for all $j\in\{1,\ldots,J\}$, the numbers*

$$\beta_j\frac{\gamma}{\delta}+\tilde{\beta}_j\frac{\tilde{\gamma}}{\tilde{\delta}}+\delta_j, \tag{3.17}$$

$$\alpha_j\frac{\gamma}{\delta}+\epsilon_j, \tag{3.18}$$

$$\tilde{\alpha}_j\frac{\tilde{\gamma}}{\tilde{\delta}}+\tilde{\epsilon}_j \tag{3.19}$$

are negative. For example, when $\delta_j\le 0,\epsilon_j\le 0$ and $\tilde{\epsilon}_j\le 0$ for all $j\in\{1,\ldots,J\}$.

Those values (3.17)-(3.19), *together with δ and $\tilde{\delta}$, form the eigenvalues of the Jacobian matrix $\mathcal{J}f(D)$. If* (3.17) *or* (3.18) *or* (3.19) *takes a positive value for some j, then D is unstable.*

Proof. Simple calculations show that $\mathcal{J}f(D)$ is lower triangular, where the diagonal is formed by (3.17)-(3.19), δ, and $\tilde{\delta}$. Then, use the known theory about dynamical systems and stability. □

Remark 3.4. When some of the quantities (3.17) or (3.18) or (3.19) are zero, we cannot ensure anything about the behavior of D in Theorem 3.4, according to the standard theory of dynamical systems.

4 Numerical simulations

We simulate (2.1) and (2.2) for several values of the parameters, to illustrate the theory of the preceding section. We set $J=3$ local tongues, that compete with the two dominant languages. The models are complex and contain many parameters. Interesting dynamics emerge, with divergence, stability, and instability cases. We recall Remark 3.2, observing that these results and graphics are of mathematical nature to understand the dynamical system, and applications in reality would correspond to short-term data where demography (immigration, emigration, births, deaths) remains approximately constant; the model informs about the possible future values for the short-term data. We use the software Mathematica®, version 12.0, with the standard `NDSolveValue` routine for solving differential equations.

Example 4.1. Let

$$\begin{array}{lll}
\beta_1=10^{-9}, & \beta_2=5\times 10^{-9}, & \beta_3=10^{-9},\\
\tilde{\beta}_1=10^{-10}, & \tilde{\beta}_2=0.7\times 10^{-10}, & \tilde{\beta}_3=10^{-9},\\
\delta_1=10^{-2}, & \delta_2=0.008, & \delta_3=0.009,\\
\alpha_1=10^{-9}, & \alpha_2=2\times 10^{-9}, & \alpha_3=0.5\times 10^{-9},\\
\tilde{\alpha}_1=3\times 10^{-9}, & \tilde{\alpha}_2=4\times 10^{-9}, & \tilde{\alpha}_3=7\times 10^{-9},
\end{array}$$

$$\begin{array}{lll}
\epsilon_1 = 0.02, & \epsilon_2 = 0.008, & \epsilon_3 = 0.009, \\
\tilde{\epsilon}_1 = 0.03, & \tilde{\epsilon}_2 = 0.01, & \tilde{\epsilon}_3 = 0.01, \\
\delta = 0, & \tilde{\delta} = 0.004, & \\
\gamma = 1000, & \tilde{\gamma} = 500, & \\
l_{1,0} = 10^6, & l_{2,0} = 3.5 \times 10^6, & l_{3,0} = 2 \times 10^6, \\
b_{1,0} = 4 \times 10^6, & b_{2,0} = 2.5 \times 10^6, & b_{3,0} = 10^5, \\
\tilde{b}_{1,0} = 10^6, & \tilde{b}_{2,0} = 10^6, & \tilde{b}_{3,0} = 10^5, \\
p_{1,0} = 5 \times 10^6, & p_{2,0} = 4 \times 10^6. &
\end{array}$$

Since $\delta \geq 0$ and $\tilde{\delta} \geq 0$, Theorem 3.2 applies. Fig. 1 confirms the stated behavior, where p_1 and p_2 grow and the other compartments become extinct. For the local tongues, there may be a phase of increase, but they decay from a certain time and die out.

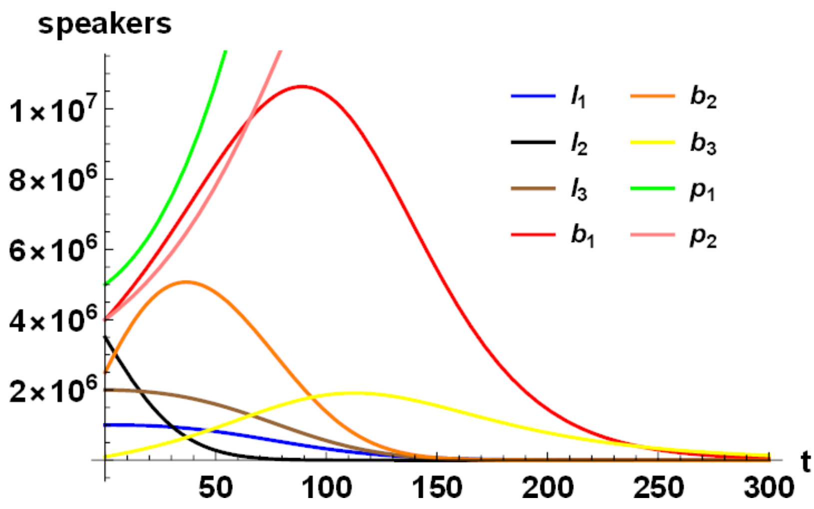

Figure 1: Language evolution in Example 4.1.

Example 4.2. Let

$$\begin{array}{lll}
\beta_1 = 10^{-9}, & \beta_2 = 5 \times 10^{-9}, & \beta_3 = 10^{-9}, \\
\tilde{\beta}_1 = 10^{-10}, & \tilde{\beta}_2 = 0.7 \times 10^{-10}, & \tilde{\beta}_3 = 10^{-9}, \\
\delta_1 = 10^{-2}, & \delta_2 = 0.008, & \delta_3 = 0.009, \\
\alpha_1 = 10^{-9}, & \alpha_2 = 2 \times 10^{-9}, & \alpha_3 = 0.5 \times 10^{-9}, \\
\tilde{\alpha}_1 = 3 \times 10^{-9}, & \tilde{\alpha}_2 = 4 \times 10^{-9}, & \tilde{\alpha}_3 = 7 \times 10^{-9}, \\
\epsilon_1 = 0.02, & \epsilon_2 = 0.008, & \epsilon_3 = 0.009, \\
\tilde{\epsilon}_1 = 0.03, & \tilde{\epsilon}_2 = 0.01, & \tilde{\epsilon}_3 = 0.01,
\end{array}$$

$$\begin{array}{lll} \delta=0, & \tilde{\delta}=-0.5, & \\ \gamma=1000, & \tilde{\gamma}=5000, & \\ l_{1,0}=10^6, & l_{2,0}=3.5\times 10^6, & l_{3,0}=2\times 10^6, \\ b_{1,0}=4\times 10^6, & b_{2,0}=2.5\times 10^6, & b_{3,0}=10^5, \\ \tilde{b}_{1,0}=10^6, & \tilde{b}_{2,0}=10^6, & \tilde{b}_{3,0}=10^5, \\ p_{1,0}=5\times 10^6, & p_{2,0}=4\times 10^6. & \end{array}$$

Since $\delta \geq 0$ and $\tilde{\delta}<0$, we use Theorem 3.3. The values of $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j$ are positive, hence the asymptotic behavior is not obvious. Indeed, as Fig. 2 shows, $p_2(t)$ oscillates without a limit value. It is unclear whether this oscillatory behavior appears in empirical studies or it corresponds to a theoretical artifact of the Lotka-Volterra-type model; it would represent a situation where language use increases and decreases with time by people's elections.

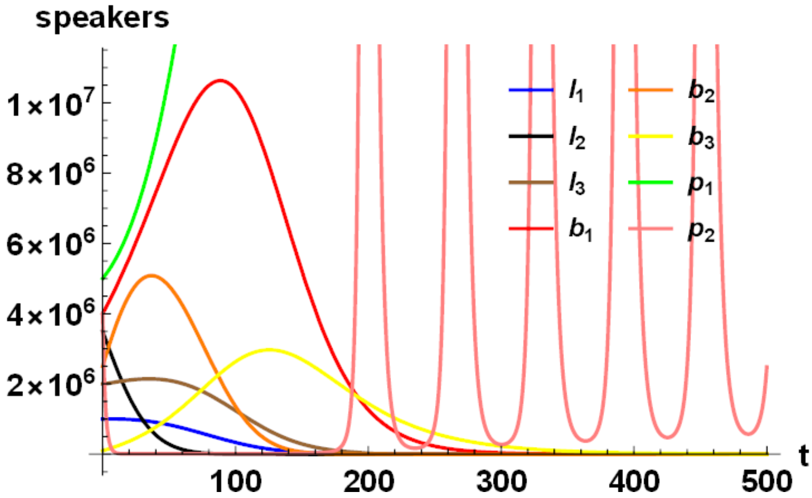

Figure 2: Language evolution in Example 4.2.

Example 4.3. Let

$$\begin{array}{lll} \beta_1=10^{-9}, & \beta_2=5\times 10^{-9}, & \beta_3=10^{-9}, \\ \tilde{\beta}_1=10^{-10}, & \tilde{\beta}_2=0.7\times 10^{-10}, & \tilde{\beta}_3=10^{-9}, \\ \delta_1=10^{-2}, & \delta_2=0.008, & \delta_3=0.009, \\ \alpha_1=10^{-9}, & \alpha_2=2\times 10^{-9}, & \alpha_3=0.5\times 10^{-9}, \\ \tilde{\alpha}_1=3\times 10^{-9}, & \tilde{\alpha}_2=4\times 10^{-9}, & \tilde{\alpha}_3=7\times 10^{-9}, \\ \epsilon_1=0.02, & \epsilon_2=0.008, & \epsilon_3=0.009, \\ \tilde{\epsilon}_1=0.015, & \tilde{\epsilon}_2=0.024, & \tilde{\epsilon}_3=0.07, \end{array}$$

$$\begin{array}{lll}
\delta=0, & \tilde{\delta}=-0.0005, & \\
\gamma=1000, & \tilde{\gamma}=5000, & \\
l_{1,0}=10^6, & l_{2,0}=3.5\times 10^6, & l_{3,0}=2\times 10^6, \\
b_{1,0}=4\times 10^6, & b_{2,0}=2.5\times 10^6, & b_{3,0}=10^5, \\
\tilde{b}_{1,0}=10^6, & \tilde{b}_{2,0}=10^6, & \tilde{b}_{3,0}=10^5, \\
p_{1,0}=5\times 10^6, & p_{2,0}=4\times 10^6. &
\end{array}$$

As $\delta\geq 0$ and $\tilde{\delta}<0$, we employ Theorem 3.3. The values of $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j$ are negative for $j\in\{1,2\}$ and zero for $j=3$, hence the asymptotic behavior is unclear again. As Fig. 3 illustrates, now $p_2(t)$ increases and does not approach an equilibrium.

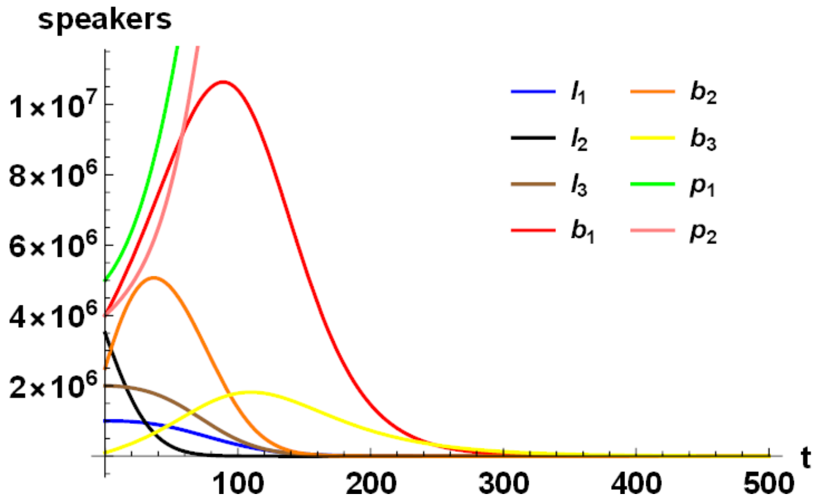

Figure 3: Language evolution in Example 4.3.

Example 4.4. Let

$$\begin{array}{lll}
\beta_1=10^{-9}, & \beta_2=5\times 10^{-9}, & \beta_3=10^{-9}, \\
\tilde{\beta}_1=10^{-10}, & \tilde{\beta}_2=0.7\times 10^{-10}, & \tilde{\beta}_3=10^{-9}, \\
\delta_1=10^{-2}, & \delta_2=0.008, & \delta_3=0.009, \\
\alpha_1=10^{-9}, & \alpha_2=2\times 10^{-9}, & \alpha_3=0.5\times 10^{-9}, \\
\tilde{\alpha}_1=3\times 10^{-9}, & \tilde{\alpha}_2=4\times 10^{-9}, & \tilde{\alpha}_3=7\times 10^{-9}, \\
\epsilon_1=0.02, & \epsilon_2=0.008, & \epsilon_3=0.009, \\
\tilde{\epsilon}_1=0.015, & \tilde{\epsilon}_2=0.024, & \tilde{\epsilon}_3=7\times 10^{-7}, \\
\delta=0, & \tilde{\delta}=-0.005, & \\
\gamma=1000, & \tilde{\gamma}=5000, &
\end{array}$$

$$
\begin{array}{lll}
l_{1,0}=10^6, & l_{2,0}=3.5\times 10^6, & l_{3,0}=2\times 10^6,\\
b_{1,0}=4\times 10^6, & b_{2,0}=2.5\times 10^6, & b_{3,0}=10^5,\\
\tilde{b}_{1,0}=10^6, & \tilde{b}_{2,0}=10^6, & \tilde{b}_{3,0}=10^5,\\
p_{1,0}=5\times 10^6, & p_{2,0}=4\times 10^6. &
\end{array}
$$

As $\delta\geq 0$ and $\tilde{\delta}<0$, we use Theorem 3.3. The values of $\tilde{\alpha}_j\tilde{\gamma}/\tilde{\delta}+\tilde{\epsilon}_j$ are all negative, therefore the limits are clear: p_1 grows and p_2 goes to a steady-state quantity, equal to $\tilde{\gamma}/(-\tilde{\delta})=10^6$. The result is confirmed by Fig. 4.

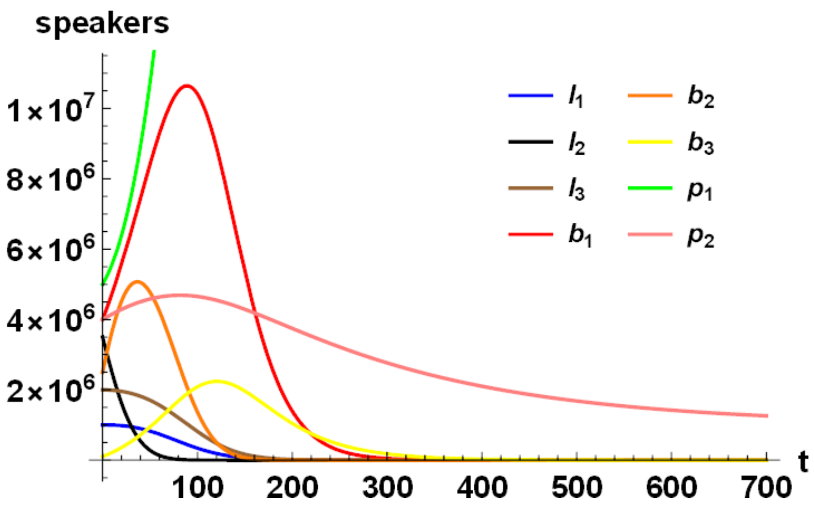

Figure 4: Language evolution in Example 4.4.

Example 4.5. We consider the same inputs as in Example 4.4, but with $\delta=-0.005$. We are in the context of Theorem 3.4. Since (3.18) is positive, the equilibrium point D is unstable. The result agrees with Fig. 5, where b_1,b_3 and p_1 oscillate, while p_2 converges to a non-zero value.

Example 4.6. Let

$$
\begin{array}{lll}
\beta_1=10^{-9}, & \beta_2=5\times 10^{-9}, & \beta_3=10^{-9},\\
\tilde{\beta}_1=10^{-10}, & \tilde{\beta}_2=0.7\times 10^{-10}, & \tilde{\beta}_3=10^{-9},\\
\delta_1=-0.00015, & \delta_2=-0.000535, & \delta_3=-0.0006,\\
\alpha_1=10^{-9}, & \alpha_2=2\times 10^{-9}, & \alpha_3=0.5\times 10^{-9},\\
\tilde{\alpha}_1=3\times 10^{-9}, & \tilde{\alpha}_2=4\times 10^{-9}, & \tilde{\alpha}_3=7\times 10^{-9},\\
\epsilon_1=0.0006, & \epsilon_2=0.00096, & \epsilon_3=0.00036,\\
\tilde{\epsilon}_1=0.015, & \tilde{\epsilon}_2=0.024, & \tilde{\epsilon}_3=7\times 10^{-7},\\
\delta=-0.005, & \tilde{\delta}=-0.005, &
\end{array}
$$

$$
\begin{array}{lll}
\gamma = 6000, & \tilde{\gamma} = 5000, & \\
l_{1,0} = 10^6, & l_{2,0} = 3.5\times 10^6, & l_{3,0} = 2\times 10^6, \\
b_{1,0} = 4\times 10^6, & b_{2,0} = 2.5\times 10^6, & b_{3,0} = 10^5, \\
\tilde{b}_{1,0} = 10^6, & \tilde{b}_{2,0} = 10^6, & \tilde{b}_{3,0} = 10^5, \\
p_{1,0} = 5\times 10^6, & p_{2,0} = 4\times 10^6. &
\end{array}
$$

In Theorem 3.4, the eigenvalues (3.17)-(3.19) are negative, so the equilibrium point D is stable. See Fig. 6, where p_1 and p_2 converge to a non-zero value.

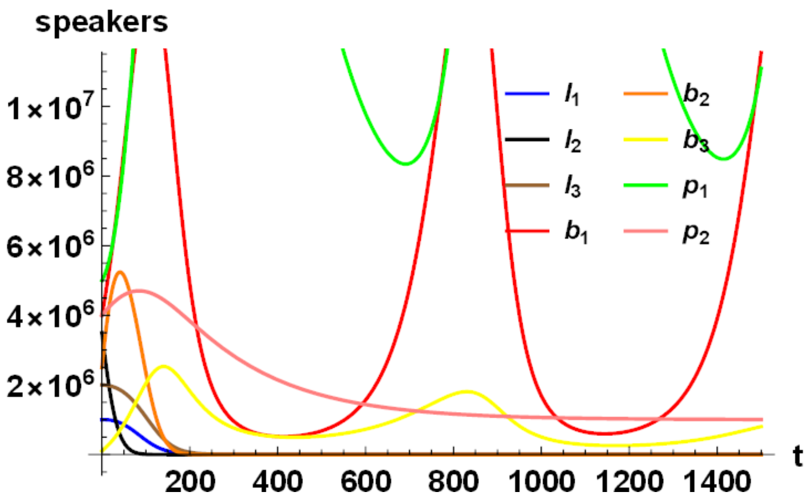

Figure 5: Language evolution in Example 4.5.

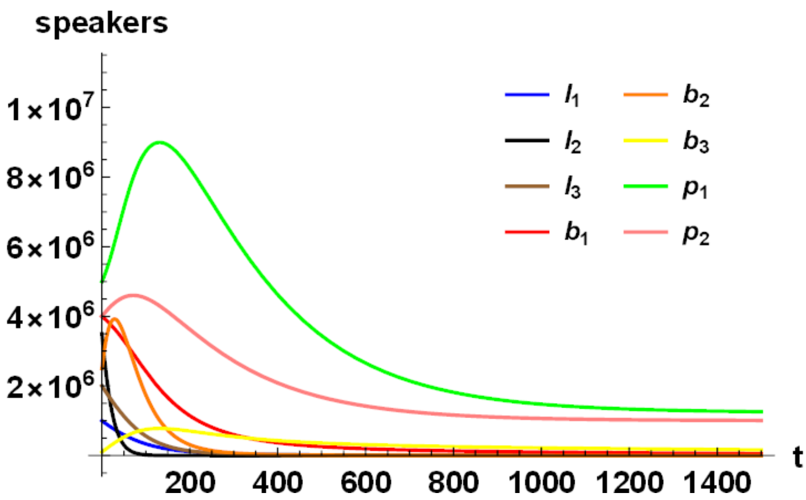

Figure 6: Language evolution in Example 4.6.

5 Conclusions

The compartmental system (2.1)-(2.2) models the competition between two principal languages and J local languages, in daily social interactions. The local languages tend to be reduced in communication. Theorems 3.1-3.4 give well-posedness and asymptotic dynamics of the model, with a full range of possibilities depending on the inequalities of the parameters. Examples 4.1-4.6 illustrate the theory for $J=3$, with rich dynamics that show diverging, convergent, and oscillating evolution curves depending on the values set for the parameters.

We notice that compartmental models like the one presented here neglect spatial or network-based constraints on agent-agent interactions, which could influence predictions. Stochastic variations, which may play a significant role in language dynamics, are also being omitted. On the other hand, logistic growths could be included in the formulation to achieve bounded asymptotic values, although the mathematical results and proofs would be more complex and further analysis would be required. Finally, we remark that, in [1] and follow-up contributions, there was a parameter "a" that appeared in the exponent for some terms in the differential equations; this parameter adds more flexibility to the nonlinearity of the interactions. In our paper, "a" is taken to be 1, and it would be of relevance to extend the theoretical results of the paper to the case $a\neq 1$, considering the higher nonlinearity.

Acknowledgments

The work I. Area has been partially supported by the Agencia Estatal de Investigación, Spain, Grant PID2020-113275GB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe" of the European Union.

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