On a Rayleigh-Faber-Krahn Inequality for the Regional Fractional Laplacian

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Abstract. We study a Rayleigh-Faber-Krahn inequality for regional fractional Laplacian operators. In particular, we show that there exists a compactly supported nonnegative Sobolev function u_0 that attains the infimum (which will be a positive real number) of the set

$$\left\{ \iint_{\{u>0\}\times\{u>0\}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y : u\in \mathring{H}^\sigma(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} u^2=1, \quad |\{u>0\}|\le 1 \right\}.$$

Unlike the corresponding problem for the usual fractional Laplacian, where the domain of the integration is $\mathbb{R}^n\times\mathbb{R}^n$, symmetrization techniques may not apply here. Our approach is instead based on the direct method and new *a priori* diameter estimates. We also present several remaining open questions concerning the regularity and shape of the minimizers, and the form of the Euler-Lagrange equations.

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1   Introduction

Let $n \geq 1$, $\sigma \in (0,1)$ (with the additional assumption that $\sigma < 1/2$ if $n = 1$), and $\Omega \subset \mathbb{R}^n$ be an open set. There are two natural fractional Sobolev norms which may be defined for $u \in C_c^\infty(\Omega)$:

$$I_{n,\sigma,\mathbb{R}^n}[u] := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y$$

and

$$I_{n,\sigma,\Omega}[u] := \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y.$$

Depending on the choices of n, σ and Ω, these two norms may or may not be equivalent. Even when they are equivalent (see Lemma 2.1), there are still subtle differences in how they depend on the domain Ω.

One significant difference is the behavior of their corresponding best Sobolev constants:

$$S_{n,\sigma}(\Omega) := \inf\left\{ I_{n,\sigma,\Omega}[u] : u \in C_c^\infty(\Omega), \quad \int_\Omega |u|^{\frac{2n}{n-2\sigma}} \mathrm{d}x = 1 \right\}$$

and

$$\widetilde{S}_{n,\sigma}(\Omega) := \inf\left\{ I_{n,\sigma,\mathbb{R}^n}[u] : u \in C_c^\infty(\Omega), \quad \int_\Omega |u|^{\frac{2n}{n-2\sigma}} \mathrm{d}x = 1 \right\}.$$

Clearly, $\widetilde{S}_{n,\sigma}(\Omega) \geq \widetilde{S}_{n,\sigma}(\mathbb{R}^n)$ and, in fact, using the dilation or translation invariance of $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$, it is not difficult to see that

$$\widetilde{S}_{n,\sigma}(\Omega) = \widetilde{S}_{n,\sigma}(\mathbb{R}^n) = S_{n,\sigma}(\mathbb{R}^n).$$

Moreover, a result of Lieb [15], classifies all minimizers for $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$ and shows that they do not vanish anywhere on $\mathbb{R}^n$. Therefore, the infimum $\widetilde{S}_{n,\sigma}(\Omega)$ is not attained unless $\Omega = \mathbb{R}^n$.

However, in [10], two of the authors with R. Frank discovered that the minimization problem for $S_{n,\sigma}(\Omega)$ behaves differently from $\widetilde{S}_{n,\sigma}(\Omega)$. Let us first recall some qualitative results about whether the constant $S_{n,\sigma}(\Omega)$ is positive or zero:

- For $n \geq 2$ and $\sigma > 1/2$, one has $S_{n,\sigma}(\Omega) > 0$ for any open set Ω. This follows from Dyda-Frank [8], which even shows that $\underline{S}_{n,\sigma} := \inf_\Omega S_{n,\sigma}(\Omega) > 0$.

- When $n \geq 1$ and $\sigma < 1/2$, one has $S_{n,\sigma}(\Omega) = 0$ for any open set Ω of finite measure with sufficiently regular boundary; see Lemma 16 in [10].

- However, one has $S_{n,\sigma}(\Omega)>0$ for $n\geq 1$ and $\sigma<1/2$ if Ω is the complement of the closure of a bounded Lipschitz domain or a domain above the graph of a Lipschitz function. This follows from the Sobolev inequality on $\mathbb{R}^n$ and the Hardy inequality from Dyda [7].

More quantitatively speaking, it was shown in [10] that the best constant $S_{n,\sigma}(\Omega)$ depends on the domain Ω, and it can be achieved in many cases assuming that $n\geq 4\sigma$:

- If the complement Ω^c has an interior point, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n)$.

- If $\sigma\neq 1/2$, then $S_{n,\sigma}(\mathbb{R}^n_+)$ is achieved (see also Musina-Nazarov [19]).

- If $\sigma>1/2$, Ω is a bounded domain such that $B_1^+\subset\Omega\subset\mathbb{R}^n_+$, then $S_{n,\sigma}(\Omega)<S_{n,\sigma}(\mathbb{R}^n_+)$. Moreover, if $\partial\Omega$ is smooth then $S_{n,\sigma}(\Omega)$ is achieved.

The discrepancy between the $S_{n,\sigma}(\Omega)$ problem and the $\widetilde{S}_{n,\sigma}(\Omega)$ problem can be explained as a Brézis-Nirenberg [4] effect:

$$I_{n,\sigma,\Omega}[u]=I_{n,\sigma,\mathbb{R}^n}[u]-2\int_\Omega u^2(x)\mathrm{d}x\int_{\mathbb{R}^n\setminus\Omega}\frac{1}{|x-y|^{n+2\sigma}}\mathrm{d}y$$

$$\approx I_{n,\sigma,\mathbb{R}^n}[u]-c_{n,\sigma}\int_\Omega\frac{u^2(x)}{\mathrm{dist}(x,\partial\Omega)^{2\sigma}}\mathrm{d}x,\quad\forall u\in C_c^\infty(\Omega).$$

Therefore, the $S_{n,\sigma}(\Omega)$ problem is the $\widetilde{S}_{n,\sigma}(\Omega)$ problem with an additional negative term, and it is this term that for $n\geq 4\sigma$ lowers the value of the infimum and produces a minimizer. This fact was first observed by Brézis-Nirenberg [4] in the Laplacian setting.

In probability, $I_{n,\sigma,\Omega}$ is called the Dirichlet form of the censored 2σ-stable process [3] in Ω. Its generator

$$(-\Delta)_\Omega^\sigma u:=2\lim_{\varepsilon\to 0}\int_{\{y\in\Omega:\ |y-x|\geq\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\mathrm{d}y \tag{1.1}$$

is usually called the regional fractional Laplacian operator [11,12]. Therefore, the difference between $S_{n,\sigma}(\Omega)$ and $\widetilde{S}_{n,\sigma}(\Omega)$ is also related to the difference between this regional fractional Laplacian and the "full" fractional Laplacian on $\mathbb{R}^n$, and in turn by the nonlocal Hardy-type term's dependence on Ω.

The nontrivial dependence of $S_{n,\sigma}(\Omega)$ on Ω leads to interesting and natural questions of a shape-optimization nature. For example: it was asked in [10] that

Open Question 1.1. Assume $n\geq 2$ and $\sigma>\frac{1}{2}$. What are all the open sets $\Omega\subset\mathbb{R}^n$ with $S_{n,\sigma}(\Omega)=\underline{S}_{n,\sigma}$, where, as mentioned before,

$$\underline{S}_{n,\sigma}=\inf\{S_{n,\sigma}(V):V\subset\mathbb{R}^n\text{ is an open set}\}>0?$$

As far as we are aware, it is unknown whether there are any such sets. A tempting conjecture would be that all such sets are balls, but the nature of the underlying equations does not support the arguments usually used to prove such symmetry results (symmetrization, moving planes; see the discussion below).

We do not attack this question here: there are several difficulties in applying calculus of variations, shape optimization, and free boundary techniques to it, and we wish to first focus on the challenges related to the nonlocality of the equation and the subtle dependence of $I_{n,\sigma,\Omega}$ on Ω in isolation. To do so, we formulate an optimization problem with similar characteristics but which avoids the criticality and extra scale invariance associated with the optimal Sobolev exponent $\frac{2n}{n-2\sigma}$.

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\sigma \in (\frac{1}{2}, 1)$ and $\mathring{H}^\sigma(\Omega)$ be the completion of $C_c^1(\Omega)$ with respect to the quadratic form $I_{n,\sigma,\Omega}$. Then we have the compact fractional Sobolev embedding $\mathring{H}^\sigma(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, the eigenvalues defined as

$$\begin{cases} (-\Delta)_\Omega^\sigma u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

consist a sequence that can be ordered (counting the multiplicities) as

$$0 < \lambda_{1,\sigma}(\Omega) \leq \lambda_{2,\sigma}(\Omega) \leq \cdots \leq \lambda_{k,\sigma}(\Omega) \leq \cdots \to \infty.$$

Moreover, we have for the first eigenvalue that

$$\lambda_{1,\sigma}(\Omega) = \min\{I_{n,\sigma,\Omega}[u] : u \in \mathring{H}^\sigma(\Omega), \ \|u\|_{L^2(\Omega)} = 1\}.$$

Let $u \in \mathring{H}^\sigma(\Omega)$ be an eigenfunction for $\lambda_{1,\sigma}(\Omega)$. Then by the fractional Sobolev inequality and Hölder inequality, we have

$$\lambda_{1,\sigma}(\Omega)\|u\|_{L^2(\Omega)}^2 = I_{n,\sigma,\Omega}[u] \geq S_{n,\sigma}(\Omega)\|u\|_{L^{\frac{2n}{n-2\sigma}}(\Omega)}^2 \geq S_{n,\sigma}(\Omega)|\Omega|^{-\frac{2\sigma}{n}}\|u\|_{L^2(\Omega)}^2.$$

Therefore,

$$\lambda_{1,\sigma}(\Omega) \geq S_{n,\sigma}(\Omega)|\Omega|^{-\frac{2\sigma}{n}} \geq \underline{S}_{n,\sigma} \cdot |\Omega|^{-\frac{2\sigma}{n}}.$$

Inspired by the classical Rayleigh-Faber-Krahn inequality [9, 13, 21], we would like to study the variational problem

$$\inf\{\lambda_{1,\sigma}(\Omega) : \Omega \subset \mathbb{R}^n \text{ a bounded open set such that } |\Omega| = |B_1|\}, \tag{1.3}$$

where $|\Omega|$ is the Lebesgue measure of Ω.

One notable difficulty of this variational problem is that radial symmetrization rearrangement may *not* work. Recall that for $u \in C_c^1(\Omega)$, the Pólya-Szegö inequality [20] states that

$$\int_\Omega |\nabla u|^p \geq \int_{\Omega^*} |\nabla u^*|^p \tag{1.4}$$

for $1 \leq p < \infty$, where Ω^* is the symmetric rearrangement of the set Ω and u^* is the symmetric decreasing rearrangement of the function u (for the definitions of rearrangements we refer to the book of Lieb-Loss [16]). Meanwhile, Pólya-Szegö type inequality also holds for $I_{n,\sigma,\mathbb{R}^n}[u]$. Namely, for $0 < \sigma < 1$ and $1 \leq p < \infty$, it was shown in Theorem 9.2 in Almgren-Lieb [2] that for every $u \in C_c^1(\mathbb{R}^n)$

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}} \mathrm{d}x \mathrm{d}y \geq \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{n+\sigma p}} \mathrm{d}x \mathrm{d}y. \tag{1.5}$$

However, in the interesting paper [14], it was showed that there exists a nonnegative $u \in C_c^\infty(B_1)$ such that

$$\iint_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\sigma p}} \mathrm{d}x \mathrm{d}y < \iint_{B_1 \times B_1} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{n+\sigma p}} \mathrm{d}x \mathrm{d}y. \tag{1.6}$$

This failure of Pólya-Szegö type inequality for $I_{n,\sigma,\Omega}[u]$ is another notable difference between the two norms $I_{n,\sigma,\mathbb{R}^n}[u]$ and $I_{n,\sigma,\Omega}[u]$ on the set $C_c^\infty(\Omega)$.

If one considers the first eigenvalue of the classical fractional Laplacian

$$\begin{cases} (-\Delta)^\sigma u = \widetilde{\lambda}_{1,\sigma}(\Omega) u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.7}$$

then, using the Pólya-Szegö type inequality (1.5) for $I_{n,\sigma,\mathbb{R}^n}[u]$, we know that $\widetilde{\lambda}_{1,\sigma}(\Omega) \geq \widetilde{\lambda}_{1,\sigma}(B_1)$ for all open set Ω of the same measure as B_1, see Sire-Vázquez-Volzone [22].

To remove the measure constraint in (1.3), one can consider the following equivalent problem (up to scaling):

$$\inf\{\lambda_{1,\sigma}(\Omega) + |\Omega| : \Omega \subset \mathbb{R}^n \text{ is a bounded open set}\}. \tag{1.8}$$

As we will show later in Proposition 4.1,

$$\inf\{\lambda_{1,\sigma}(\Omega) + |\Omega| : \Omega \subset \mathbb{R}^n \text{ is a bounded open set}\}$$

$$= \inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|_{L^2(\mathbb{R}^n)}^2} + |\{u > 0\}| : u \in \mathring{H}^\sigma(\mathbb{R}^n), \ u \not\equiv 0, \ u \geq 0 \text{ in } \mathbb{R}^n\right\}.$$

In this paper, we prove the following existence result:

Theorem 1.1. *Let $n \geq 2$ and $\sigma \in (\frac{1}{2}, 1)$. There exists $u_0 \in \mathring{H}^\sigma(\mathbb{R}^n)$, $u_0 \not\equiv 0$, $u_0 \geq 0$ in $\mathbb{R}^n$ such that $\{u_0 > 0\}$ is bounded, and*

$$\inf \left\{ \frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|_{L^2(\mathbb{R}^n)}^2} + |\{u > 0\}| : u \in \mathring{H}^\sigma(\mathbb{R}^n), \quad u \not\equiv 0, \ u \geq 0 \ in \ \mathbb{R}^n \right\} \tag{1.9}$$

is achieved by u_0.

The key difficulty in proving this is to obtain an *a priori* bound on the diameter of $\{u_0 > 0\}$. In the shape optimization and free boundary literature, such bounds are common and are closely tied with lower bounds on the growth of u_0 from the boundary of $\partial\{u_0 > 0\}$, or perhaps of auxiliary functions related to u_0: the idea is that if u_0 grows at a prescribed rate from $\partial\{u_0 > 0\}$, its support cannot have long, thin necks or many small connected components. Such growth estimates are usually obtained by using sets like $\{u_0 > 0\} \setminus B_R(x)$ as competitors (with appropriately chosen functions). We proceed along these lines here as well, but carrying out the argument requires various non-standard modifications, some decidedly nonlocal multi-scale iteration procedures, and in the end does not result in "uniform" growth estimates on u_0.

For related reasons, we are unable to show much more than stated in Theorem 1.1 concerning the nature of the minimizing u_0 (though see Lemma 5.3, which shows u_0 is bounded). Of particular interest is this question:

Open Question 1.2. Is (1.9) achieved by a continuous function $u_0 \in \mathring{H}^\sigma(\mathbb{R}^n)$?

An affirmative answer would imply that problems (1.9) and (1.8) have identical minimizers (in the sense that the support of a minimizer of (1.9) is a minimizer of (1.8), and vice versa). The difficulty of proving this continuity is in estimating the energy difference between u_0 and its competitor whose support is slightly enlarged. See Section 7 for further discussion and additional open problems.

If one restricts the problem (1.3) to the class of convex sets, then we have

Theorem 1.2. *Let $n \geq 2$ and $\sigma \in (\frac{1}{2}, 1)$. There exists a convex open set that achieves*

$$\inf\{\lambda_{1,\sigma}(\Omega) : \Omega \subset \mathbb{R}^n \ is \ a \ convex \ open \ set \ and \ |\Omega| = 1\}.$$

This paper is organized as follows. In Section 2, we recall some inequalities that are needed here. In Section 3, we prove the existence of minimizers for the convex case: this is much simpler, but illustrates the concepts in play. In Section 4, we reformulate the variational problem (1.8) as (1.9). In Section 5, we show a local pointwise upper bound on the minimizer. In Section 6, we show a lower bound of the minimizer, use it to estimate the diameter of the support of u_0, and conclude the proof of Theorem 1.1. In the last section, we discuss Open Question 1.2 and related topics.

2 Some inequalities

Let us first recall a sharp Hardy inequality for fractional integrals on general domains. Let $\Omega \subsetneq \mathbb{R}^n$ be an open set. For a direction $\omega \in \mathbb{S}^{n-1}$, we define

$$d_{\omega,\Omega}(x) = \inf\{|t| : x + t\omega \notin \Omega\}$$

and for $\alpha > 1$ that

$$m_\alpha(x) = \left(\frac{2\pi^{\frac{n-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{S}^{n-1}} \frac{1}{d_{\omega,\Omega}(x)^\alpha} d\omega \right)^{-\frac{1}{\alpha}}.$$

Theorem 2.1 (Theorem 1.2 in Loss-Sloane [18]). *Let $\frac{1}{2} < \sigma < \frac{p}{2} < \infty$ and $n \geq 1$. Then for any $u \in C_c^\infty(\Omega)$,*

$$\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))^p}{|x-y|^{n+2\sigma}} dx dy \geq C_{n,p,\sigma} \int_\Omega \frac{|u(x)|^p}{m_{2\sigma}(x)^{2\sigma}} dx,$$

where

$$C_{n,p,\sigma} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+2\sigma}{2})}{\Gamma(\frac{N+2\sigma}{2})} \int_0^1 \frac{|1-r^{\frac{2\sigma-1}{p}}|^p}{(1-r)^{1+2\sigma}} dr$$

is sharp.

Consequently, we can show that $I_{n,\sigma,\mathbb{R}^n}[u]$ can be controlled by $I_{n,\sigma,\Omega}[u]$.

Lemma 2.1. *Let $n \geq 2$ and $\sigma \in (1/2,1)$. There exists a constant $C = C(n,\sigma) > 0$ such that for all open sets $\Omega \subset \mathbb{R}^n$ and all $u \in \mathring{H}^\sigma(\Omega)$,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} dx dy \leq C \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} dx dy. \tag{2.1}$$

Proof. By a density argument, we only need to show this for $u \in C_c^1(\Omega)$. Suppose $\Omega \subsetneq \mathbb{R}^n$. We have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}} dx dy$$

$$= \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}} dx dy + 2 \int_\Omega u^2(x) \left(\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+2\sigma}} dy \right) dx. \tag{2.2}$$

Also, for $x \in \Omega$, we have

$$\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+2\sigma}} dy \leq \int_{\mathbb{S}^{n-1}} d\omega \int_{d_{\omega,\Omega}(x)}^\infty \frac{1}{r^{1+2\sigma}} dr = \int_{\mathbb{S}^{n-1}} \frac{1}{2\sigma d_{\omega,\Omega}(x)^{2\sigma}} d\omega = \frac{C(n,\sigma)}{(m_{2\sigma}(x))^{2\sigma}}$$

for some constant $C(n,\sigma)$ depending only on n, σ, but *not* on Ω. Thus, we have

$$\int_\Omega u^2(x)\left(\int_{\mathbb{R}^n\setminus\Omega}\frac{1}{|x-y|^{n+2\sigma}}\mathrm{d}y\right)\mathrm{d}x$$

$$\leq C(n,\sigma)\int_\Omega\frac{u^2(x)}{(m_{2\sigma}(x))^{2\sigma}}\mathrm{d}x$$

$$\leq C(n,\sigma)\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y, \tag{2.3}$$

where we used Theorem 2.1 in the last inequality. $\square$

We rewrite Lemma 2.1 into another form for convenience.

Lemma 2.2. *There exists $C=C(n,\sigma)$ such that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y\leq C\iint_{\{u>0\}\times\{u>0\}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$

for every nonnegative function $u\in\mathring{H}^\sigma(\mathbb{R}^n)$.

Proof. Without loss of generality, we assume that $|\mathbb{R}^n\setminus\{u>0\}|>0$. We also assume that

$$\iint_{\{u>0\}\times\{u>0\}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y=1.$$

Using only the outer regularity of Lebesgue measure, we can find a sequence of open sets $\Omega_u^k\subset\mathbb{R}^n$ such that

$$\{u>0\}\subset\Omega_u^k\subset\Omega_u^{k-1} \quad \text{and} \quad |\Omega_u^k\setminus\{u>0\}|\to 0.$$

Since $u\in\mathring{H}^\sigma(\mathbb{R}^n)$, the monotone convergence theorem gives that

$$\lim_{k\to\infty}\iint_{\Omega_u^k\times\Omega_u^k}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y=\iint_{\{u>0\}\times\{u>0\}}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y=1.$$

By (2.1) applied to each Ω_u^k,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y\leq C(n,\sigma)\inf_k\iint_{\Omega_u^k\times\Omega_u^k}\frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y=C(n,\sigma),$$

giving the conclusion. $\square$

3   Convex domains

We first consider the special case of the minimizing problem among all convex sets.

Proof of Theorem 1.2. We will adapt the proof in Lin [17] for the Laplacian case. First, because of (2.1), we have

$$\widetilde{\lambda}_{1,\sigma}(\Omega) \leq C\lambda_{1,\sigma}(\Omega), \tag{3.1}$$

where $\widetilde{\lambda}_{1,\sigma}$ is defined in (1.7).

Let $\{\Omega_k\}$ be a minimizing sequence. By John's Lemma, we have that either (A) Ω_k converges in the Hausdorff distance to a bounded convex set Ω_∞ with $|\Omega_\infty|=1$ (we may assume that Ω_∞ is open since the boundary of Ω_∞ is of measure zero), or (B) there is a subsequence, still denoted Ω_k, such that Ω_k are contained in strips (after suitable rotations and translations, noting that $\lambda_{1,\sigma}(\Omega)$ is invariant under them) of form $[-\delta_k,\delta_k]\times[-L_k,L_k]^{n-1}$ such that $\delta_k\to 0^+$ and $L_k\to\infty$.

We will show that in case (B), $\widetilde{\lambda}_{1,\sigma}(\Omega_k)\to\infty$, which contradicts (3.1) and the fact that Ω_k is a minimizing sequence. Denote $Q_k=[-\delta_k,\delta_k]\times[-L_k,L_k]^{n-1}$. Since $C_c^\infty(\Omega_k)\subset C_c^\infty(Q_k)$, it is clear that

$$\widetilde{\lambda}_{1,\sigma}(Q_k)\leq\widetilde{\lambda}_{1,\sigma}(\Omega_k).$$

Therefore, we only need to show that $\widetilde{\lambda}_{1,\sigma}(Q_k)\to\infty$.

Let $Q_0=[-1,1]\times\mathbb{R}^{n-1}$. We claim that there exists $C>0$ such that

$$\|\varphi\|_{L^2(\mathbb{R}^n)}\leq C\|\varphi\|_{\mathring{H}^\sigma(\mathbb{R}^n)} \quad \text{for all} \quad \varphi\in C_c^\infty(Q_0). \tag{3.2}$$

Indeed, for every $x'\in\mathbb{R}^{n-1}$, we have

$$\int_{-1}^1 \varphi(x_1,x')^2\mathrm{d}x_1 \leq C\|\varphi(\cdot,x')\|_{\mathring{H}^\sigma(\mathbb{R}^1)}^2 = C\int_\mathbb{R}|\xi_1|^{2\sigma}\phi(\xi_1,x')^2\mathrm{d}\xi_1,$$

where $\phi(\xi_1,x')$ is the (full) Fourier transform of $\varphi(x_1,x')$ in the x_1 variable. Integrating the above inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\varphi(x)^2\mathrm{d}x &\leq C\int_\mathbb{R}|\xi_1|^{2\sigma}\mathrm{d}\xi_1\int_{\mathbb{R}^{n-1}}\phi(\xi_1,x')^2\mathrm{d}x' \\
&= C\int_\mathbb{R}|\xi_1|^{2\sigma}\mathrm{d}\xi_1\int_{\mathbb{R}^{n-1}}\hat{\varphi}(\xi_1,\xi')^2\mathrm{d}\xi' \\
&= C\int_{\mathbb{R}^n}|\xi_1|^{2\sigma}\hat{\varphi}(\xi)^2\mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}^n}|\xi|^{2\sigma}\hat{\varphi}(\xi)^2\mathrm{d}\xi = C\|\varphi\|_{\mathring{H}^\sigma(\mathbb{R}^n)},
\end{aligned}$$

where $\hat{\varphi}$ is the Fourier transform of φ, and we used the Plancherel theorem in the first equality. This proves (3.2).

Then we have

$$
\begin{aligned}
\widetilde{\lambda}_{1,\sigma}(Q_k) &\geq \inf_{\varphi \in C_c^\infty([-\delta_k,\delta_k]\times\mathbb{R}^{n-1})} \frac{\|\varphi\|_{\mathring{H}^\sigma(\mathbb{R}^n)}^2}{\|\varphi\|_{L^2(\mathbb{R}^n)}^2} \\
&= \frac{1}{\delta_k^{2\sigma}} \inf_{\varphi \in C_c^\infty([-1,1]\times\mathbb{R}^{n-1})} \frac{\|\varphi\|_{\mathring{H}^\sigma(\mathbb{R}^n)}^2}{\|\varphi\|_{L^2(\mathbb{R}^n)}^2} \\
&\geq \frac{1}{C\delta_k^{2\sigma}} \to \infty \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

We reached a contradiction, and thus case (A) holds: Ω_k converges in the Hausdorff distance to a bounded convex open set Ω_∞ with $|\Omega_\infty| = 1$.

Suppose that

$$
\lambda_{1,\sigma}(\Omega_k) = I_{n,\sigma,\Omega_k}[u_k]
$$

for some $u_k \in \mathring{H}^\sigma(\Omega_k)$ such that $\|u_k\|_{L^2(\Omega_k)} = 1$. We extend u_k to be zero in $\mathbb{R}^n \setminus \Omega_k$. By (2.1), we know that u_k is bounded in $\mathring{H}^\sigma(\mathbb{R}^n)$. Then subject to a subsequence, there exists $u \in \mathring{H}^\sigma(\mathbb{R}^n)$ such that $u_k \rightharpoonup u$ weakly in $\mathring{H}^\sigma(\mathbb{R}^n)$ and $u_k \to u$ in $L^2(\mathbb{R}^n)$. Hence, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega_\infty$ and $\|u\|_{L^2(\Omega_\infty)} = 1$. Morevoer, by Fatou's Lemma, we have

$$
\liminf_{k\to\infty} \iint_{\Omega_k\times\Omega_k} \frac{(u_k(x)-u_k(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y \geq \iint_{\Omega_\infty\times\Omega_\infty} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y.
$$

Hence,

$$
\lambda_{1,\sigma}(\Omega_\infty) = \inf\{\lambda_{1,\sigma}(\Omega) : \Omega \subset \mathbb{R}^n \text{ is a convex open set and } |\Omega| = 1\}.
$$

This finishes the proof of this theorem.   □

4   A formulation for functions

In this section, we would like reformulate the variational problem (1.8) for domains to a variational problem for functions. Since $||u(x)|-|u(y)|| \leq |u(x)-u(y)|$, we know that the first eigenfunction, that are the solutions of (1.2), do not change signs in Ω. The next lemma states that they do not vanish in Ω.

Lemma 4.1. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a open set, $\sigma \in (\frac{1}{2},1)$. Let $u \in \mathring{H}^\sigma(\Omega)$, $u \not\equiv 0$, be a nonnegative weak solution of (1.2), that is,*

$$
\iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y = \lambda_{1,\Omega} \int_\Omega u(x)\varphi(x)\mathrm{d}x
$$

for every $\varphi \in \mathring{H}^\sigma(\Omega)$. Then u is smooth and positive in Ω.

Proof. For $u, \varphi \in \mathring{H}^\sigma(\Omega)$, we extend u and φ to be identically zero in $\mathbb{R}^n \setminus \Omega$, and by Lemma 2.1, they are functions in $\mathring{H}^\sigma(\mathbb{R}^n)$. We can rewrite the integral as

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y - \int_\Omega c(x)u(x)\varphi(x)\mathrm{d}x = \lambda_{1,\Omega} \int_\Omega u(x)\varphi(x)\mathrm{d}x,$$

where

$$c(x) = 2 \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x-y|^{n+2\sigma}} \mathrm{d}y.$$

That is, u is a weak solution of

$$(-\Delta)^\sigma u(x) - c(x)u(x) = \lambda_{1,\Omega} u(x) \qquad \text{on} \ \ \Omega,$$
$$u = 0 \qquad \qquad \qquad \qquad \text{in} \ \ \mathbb{R}^n \setminus \Omega.$$

Since $c(x)$ is smooth in Ω, by the standard regularity theory for fractional Laplacian equations, u is smooth in Ω. Hence, Eq. (1.2) holds pointwise, that is,

$$2 \lim_{\varepsilon \to 0} \int_{\{y \in \Omega: \ |y-x| \geq \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}} \mathrm{d}y = \lambda_{1,\Omega} u(x) \quad \text{for every } x \in \Omega.$$

Therefore, if there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, then since u is nonnegative in Ω, u must be identically zero. This is a contradiction. $\square$

Let $N > 0$ be a positive real number. Let us consider an alternative variational problem

$$m(N) := \inf \left\{ \frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|_{L^2(B_N)}^2} + |\{u > 0\}| : u \in \mathring{H}^\sigma(B_N), \ u \not\equiv 0, \ u \geq 0 \ \ \text{in} \ \ B_N \right\}. \quad (4.1)$$

Lemma 4.2. $m(N) = \inf \{ \lambda_{1,\sigma}(\Omega) + |\Omega| : \Omega \subset B_N \ \text{is an open set} \}.$

Proof. Let $\Omega \subset B_N$ be an open set, and let $u \in \mathring{H}^\sigma(\Omega)$ be such that $\lambda_{1,\sigma}(\Omega) = I_{n,\sigma,\Omega}[u]$, $\|u\|_{L^2} = 1$. Then, by Lemma 4.1, we can choose that $u > 0$ in Ω. Hence,

$$m(N) \leq \lambda_{1,\sigma}(\Omega) + |\Omega|.$$

Taking the infimum over all open sets, we obtain

$$m(N) \leq \inf \{ \lambda_{1,\sigma}(\Omega) + |\Omega| : \Omega \subset B_N \ \text{is an open set} \}.$$

On the other hand, let $u \in \mathring{H}^{\sigma}(B_N)$ with $u \geq 0$ in B_N. For every $\varepsilon > 0$, there exists an open set $\Omega_{\varepsilon} \subset B_N$ (otherwise just considering $\Omega_{\varepsilon} \cap B_N$) such that $\{u > 0\} \subset \Omega_{\varepsilon}$, $|\Omega_{\varepsilon} \setminus \{u > 0\}|$ is small, and

$$\iint_{\Omega_{\varepsilon} \times \Omega_{\varepsilon}} \frac{(u(x) - u(y))^2}{|x - y|^{n + 2\sigma}} \mathrm{d}x \mathrm{d}y \leq \iint_{\{u > 0\} \times \{u > 0\}} \frac{(u(x) - u(y))^2}{|x - y|^{n + 2\sigma}} \mathrm{d}x \mathrm{d}y + \varepsilon.$$

Moreover, there exists a sequence of functions $u_j \in C_c^{\infty}(\Omega_{\varepsilon})$ with $u_j \geq 0$ such that $u_j \to u$ in $H^{\sigma}(\Omega_{\varepsilon})$. Since $\{u_j > 0\}$ is open, we have

$$\frac{I_{n, \sigma, \{u_j > 0\}}[u_j]}{\|u_j\|_{L^2(B_N)}^2} + |\{u_j > 0\}| \geq \inf \{\lambda_{1, \sigma}(\Omega) + |\Omega| : \Omega \subset B_N \text{ is an open set}\}.$$

Since

$$\frac{I_{n, \sigma, \{u_j > 0\}}[u_j]}{\|u_j\|_{L^2(B_N)}^2} \leq \frac{I_{n, \sigma, \Omega_{\varepsilon}}[u_j]}{\|u_j\|_{L^2(B_N)}^2} \leq \frac{I_{n, \sigma, \Omega_{\varepsilon}}[u] + \varepsilon}{\|u_j\|_{L^2(B_N)}^2} \leq \frac{I_{n, \sigma, \{u > 0\}}[u] + 2\varepsilon}{\|u_j\|_{L^2(B_N)}^2}$$

for all large j, we can send $j \to \infty$ and then $\varepsilon \to 0$ to obtain

$$m(N) \geq \inf \{\lambda_{1, \sigma}(\Omega) + |\Omega| : \Omega \subset B_N \text{ is an open set}\}.$$

Thus, we complete the proof. $\qquad \square$

Lemma 4.3. *The $m(N)$ defined in (4.1) is attained by a function $u \in \mathring{H}^{\sigma}(B_N)$.*

Proof. Let u_k be a minimizing sequence of m in (4.1). Suppose that $\|u_k\|_{L^2(B_N)} = 1$. Since $u_k \in \mathring{H}^{\sigma}(B_N)$, we know from (2.1) that $u_k \in \mathring{H}^{\sigma}(\mathbb{R}^n)$, and

$$\iint_{\{u_k > 0\} \times \{u_k > 0\}} \frac{(u_k(x) - u_k(y))^2}{|x - y|^{n + 2\sigma}} \mathrm{d}x \mathrm{d}y \leq C.$$

By Lemma 2.2, we know that $\{u_k\}$ is a bounded sequence in $\mathring{H}^{\sigma}(\mathbb{R}^n)$. Then passing to a subsequence, there exists $u \in \mathring{H}^{\sigma}(\mathbb{R}^n)$ such that $u_k \rightharpoonup u$ weakly in $\mathring{H}^{\sigma}(\mathbb{R}^n)$ and $u_k \to u$ strongly in $L^2(B_N)$. Hence, $u \geq 0$ in B_N, $u \equiv 0$ in $\mathbb{R}^n \setminus B_N$ and $\|u\|_{L^2(B_N)} = 1$. Morevoer, by Fatou's Lemma, we have

$$\liminf_{k \to \infty} \iint_{\{u_k > 0\} \times \{u_k > 0\}} \frac{(u_k(x) - u_k(y))^2}{|x - y|^{n + 2\sigma}} \mathrm{d}x \mathrm{d}y \geq \iint_{\{u > 0\} \times \{u > 0\}} \frac{(u(x) - u(y))^2}{|x - y|^{n + 2\sigma}} \mathrm{d}x \mathrm{d}y$$

and

$$\liminf_{k \to \infty} |\{u_k > 0\}| \geq |\{u > 0\}|.$$

Thus, $m(N)$ is attained by u. $\qquad \square$

Now we can show that the variational problem (1.8) is equivalent to (1.9).

Proposition 4.1. *We have*

$$\inf\{\lambda_{1,\sigma}(\Omega)+|\Omega|:\ \Omega\subset\mathbb{R}^n\ \text{a bounded open set}\}$$

$$=\inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}}+|\{u>0\}|:u\in\mathring{H}^\sigma(\mathbb{R}^n),\ \ u\not\equiv0,\ u\geq0\ \ in\ \ \mathbb{R}^n\right\}.$$

Proof. First, it is clear from Lemma 4.2 that

$$\inf\{\lambda_{1,\sigma}(\Omega)+|\Omega|:\ \Omega\subset\mathbb{R}^n\ \text{a bounded open set}\}$$
$$=\inf_{N\in\mathbb{N}}\big(\inf\{\lambda_{1,\sigma}(\Omega)+|\Omega|:\Omega\subset B_N\ \text{is an open set}\}\big)$$
$$=\inf_{N\in\mathbb{N}}m(N),$$

while from the definition of $m(N)$,

$$\inf_{N\in\mathbb{N}}m(N)\geq\inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}}+|\{u>0\}|:u\in\mathring{H}^\sigma(\mathbb{R}^n),\ \ u\not\equiv0,\ u\geq0\ \ in\ \ \mathbb{R}^n\right\}.$$

Second, given $u\in\mathring{H}^\sigma(\mathbb{R}^n)$, it is elementary to check that $\eta_N u\in\mathring{H}^\sigma(\mathbb{R}^n)$ and $\eta_N u\to u$ in $\mathring{H}^\sigma(\mathbb{R}^n)$ as $N\to\infty$, where $\eta_N(x)=\eta(x/N)$ and η is a standard radial cut-off function supported in B_2 and equal to 1 in B_1. Hence,

$$\inf_{N\in\mathbb{N}}m(N)\leq\inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}}+|\{u>0\}|:u\in\mathring{H}^\sigma(\mathbb{R}^n),\ \ u\not\equiv0,\ u\geq0\ \ in\ \ \mathbb{R}^n\right\}.$$

This gives

$$\inf_{N\in\mathbb{N}}m(N)=\inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}}+|\{u>0\}|:u\in\mathring{H}^\sigma(\mathbb{R}^n),\ \ u\not\equiv0,\ u\geq0\ \ in\ \ \mathbb{R}^n\right\},\quad(4.2)$$

which implies the conclusion. □

5   An upper bound

Now we turn our attention to the regularity of minimizing functions u for $m(N)$. At present, we only know that they exist and are nonnegative functions in the space $H^\sigma(\mathbb{R}^n)$, supported on B_N. In this section, we will show they are bounded and admit a kind of local maximum principle.

Lemma 5.1. *Let* $u \in \mathring{H}^\sigma(\mathbb{R}^n)$ *be a nonnegative minimizer of* $m(N)$. *Then it satisfies*

$$(-\Delta)^\sigma_{\{u>0\}} u \leq \lambda u \quad in \ \ \mathbb{R}^n \tag{5.1}$$

in the weak sense, where

$$\lambda = \|u\|^{-2}_{L^2(\{u>0\})} \iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y. \tag{5.2}$$

More precisely,

$$\iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y \leq \lambda \int_{\mathbb{R}^n} u(x)\varphi(x)\mathrm{d}x \tag{5.3}$$

for every nonnegative function $\varphi \in \mathring{H}^\sigma(\mathbb{R}^n)$.

Proof. Let $\varphi \in \mathring{H}^\sigma(\mathbb{R}^n)$ be a nonnegative function and $u_t = (u-t\varphi)^+$ for any small $t > 0$. Then we have

$$\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(B_N)}} + |\{u>0\}| \leq \frac{I_{n,\sigma,\{u_t>0\}}[u_t]}{\|u_t\|^2_{L^2(B_N)}} + |\{u_t>0\}|.$$

Since $\{u_t>0\} \subset \{u>0\}$, we have

$$\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(B_N)}} \leq \frac{I_{n,\sigma,\{u_t>0\}}[u_t]}{\|u_t\|^2_{L^2(B_N)}} \leq \frac{I_{n,\sigma,\{u>0\}}[u_t]}{\|u_t\|^2_{L^2(B_N)}}. \tag{5.4}$$

Since $u = (u-t\varphi)^+ - (u-t\varphi)^- + t\varphi$, we obtain

$$\begin{aligned}
&(u(x)-u(y))^2\\
&=(u_t(x)-u_t(y))^2 + \Big((u(x)-t\varphi(x))^- - (u(y)-t\varphi(y))^-\Big)^2\\
&\quad + 2(u(y)-t\varphi(y))^+(u(x)-t\varphi(x))^- + 2(u(x)-t\varphi(x))^+(u(y)-t\varphi(y))^-\\
&\quad + 2t(\varphi(x)-\varphi(y))(u(x)-u(y)) - t^2(\varphi(x)-\varphi(y))^2\\
&\geq (u_t(x)-u_t(y))^2 + 2t[\varphi(x)-\varphi(y)][u(x)-u(y)] - t^2(\varphi(x)-\varphi(y))^2
\end{aligned}$$

and

$$\begin{aligned}
u^2 &= u_t^2 + [(u-t\varphi)^-]^2 + t^2\varphi^2 + 2t\varphi(u-t\varphi)\\
&\leq u_t^2 + t^2\varphi^2 + t^2\varphi^2 + 2t\varphi(u-t\varphi)\\
&= u_t^2 + 2t\varphi u.
\end{aligned}$$

Applying these two inequalities in (5.4), we obtain

$$\|u\|_{L^2(B_N)}^2 \iint_{\{u>0\}\times\{u>0\}} \frac{2t[\varphi(x)-\varphi(y)][u(x)-u(y)]-t^2(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y$$
$$\leq 2t \int_{B_N} \varphi(x)u(x)\mathrm{d}x \iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y.$$

By canceling t and sending $t\to 0$, we obtain

$$\iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y \leq \lambda \int_{B_N} u(x)\varphi(x)\mathrm{d}x$$

with λ as in (5.2). $\qquad\square$

Next is a Cacciopoli inequality, based on u being a subsolution of this nonlocal equation. Set

$$\mathscr{B}_u[f,g] = \iint_{\{u>0\}\times\{u>0\}} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y. \qquad (5.5)$$

Lemma 5.2. *Let $0<r<\ell<R$, $\eta:\mathbb{R}^n\to[0,1]$ be a smooth radial cutoff function which is 1 on B_r, vanishes outside of B_ℓ, and satisfies $|\nabla\eta|\leq 2/(\ell-r)$. Then*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$
$$\leq C\int_{B_\ell} u(x)\mathrm{d}x \int_{\mathbb{R}^n\setminus B_R} \frac{u(y)}{|y-x|^{n+2\sigma}}\mathrm{d}y + C\int_{B_R} u^2 + \frac{CR^{2-2\sigma}}{(R-r)^2}\int_{B_R} u^2.$$

Proof. We use $\eta^2 u$ as a test function, to get that

$$\mathscr{B}_u[u,\eta^2 u] \leq C\int_{\mathbb{R}^n} u^2\eta^2 \leq C\int_{B_\ell} u^2.$$

The quantity on the left may be subdivided as

$$\mathscr{B}_u[u,\eta^2 u]$$
$$= \left(\int_{(B_R\cap\{u>0\})^2} + 2\int_{B_R\cap\{u>0\}\times\{u>0\}\setminus B_R}\right) \frac{[u(x)-u(y)][\eta^2 u(x)-\eta^2 u(y)]}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y.$$

The integral over the second region can be estimated using the fact that the support of η is a distance $R-\ell$ from the complement of B_R:

$$-\iint_{B_R\cap\{u>0\}\times\{u>0\}\backslash B_R}\frac{[u(x)-u(y)][\eta^2u(x)-\eta^2u(y)]}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$
$$\leq C\int_{B_R}\eta^2u(x)\mathrm{d}x\int_{\{u>0\}\backslash B_R}\frac{u(y)}{|x-y|^{n+2\sigma}}\mathrm{d}y$$
$$\leq C\int_{B_\ell}u(x)\mathrm{d}x\int_{\mathbb{R}^n\backslash B_R}\frac{u(y)}{|y-x|^{n+2\sigma}}\mathrm{d}y.$$

Let us focus on the much more difficult estimate on the other region. We claim that,

$$|u(x)-u(y)|^2(\eta^2(x)+\eta^2(y))$$
$$\leq 4(u(x)-u(y))(\eta^2u(x)-\eta^2u(y))+4(u^2(x)+u^2(y))(\eta(x)-\eta(y))^2.$$

To see this, start with the more obvious identity

$$|u(x)-u(y)|^2\eta^2(x)$$
$$=(u(x)-u(y))(\eta^2(x)u(x)-\eta^2(y)u(y))-u(y)(u(x)-u(y))(\eta^2(x)-\eta^2(y)).$$

Now reverse the role of x and y, add, factor, and apply Cauchy's inequality to the last term:

$$|u(x)-u(y)|^2(\eta^2(x)+\eta^2(y))$$
$$=2(u(x)-u(y))(\eta^2(x)u(x)-\eta^2(y)u(y))-(u(y)+u(x))(u(x)-u(y))(\eta^2(x)-\eta^2(y))$$
$$\leq 2(u(x)-u(y))(\eta^2(x)u(x)-\eta^2(y)u(y))$$
$$\quad+|u(x)+u(y)|^2|\eta(x)-\eta(y)|^2+\frac{1}{4}|u(x)-u(y)|^2|\eta(x)+\eta(y)|^2$$
$$\leq 2(u(x)-u(y))(\eta^2(x)u(x)-\eta^2(y)u(y))$$
$$\quad+2|u^2(x)+u^2(y)||\eta(x)-\eta(y)|^2+\frac{1}{2}|u(x)-u(y)|^2|\eta^2(x)+\eta^2(y)|.$$

Reabsorb the rightmost term and multiply by 2 to recover the claimed inequality.

Also, from Cauchy's inequality, we have

$$|u(x)\eta(x)-u(y)\eta(y)|^2\leq 2|u(x)-u(y)|^2\eta^2(x)+2|\eta(x)-\eta(y)|^2u^2(y).$$

Switching the role of x and y, we obtain

$$|u(x)\eta(x)-u(y)\eta(y)|^2$$
$$\leq|u(x)-u(y)|^2(\eta^2(x)+\eta^2(y))+|\eta(x)-\eta(y)|^2(u^2(x)+u^2(y)).$$

Therefore, we obtain

$$|u(x)\eta(x)-u(y)\eta(y)|^2$$
$$\leq 4(u(x)-u(y))(\eta^2 u(x)-\eta^2 u(y))+5(u^2(x)+u^2(y))(\eta(x)-\eta(y))^2.$$

Using this, we immediately get that

$$\iint_{(\{u>0\}\cap B_R)^2}\frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$
$$\leq 4\iint_{(\{u>0\}\cap B_R)^2}\frac{(u(x)-u(y))(\eta^2 u(x)-\eta^2 u(y))}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$
$$+C(R-r)^{-2}\int_{\{u>0\}\cap B_R}u^2(x)\int_{\{u>0\}\cap B_R}|x-y|^{2-n-2\sigma}\mathrm{d}y\mathrm{d}x.$$

For the second term on the right, we have

$$C(R-r)^{-2}\int_{\{u>0\}\cap B_R}u^2(x)\int_{\{u>0\}\cap B_R}|x-y|^{2-n-2\sigma}\mathrm{d}y\mathrm{d}x\leq\frac{CR^{2-2\sigma}}{(R-r)^2}\int_{B_R}u^2.$$

Combining with our earlier estimates, we obtain

$$\iint_{(\{u>0\}\cap B_R)^2}\frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$
$$\leq C\int_{B_\ell}u(x)\mathrm{d}x\int_{\mathbb{R}^n\setminus B_R}\frac{u(y)}{|y-x|^{n+2\sigma}}\mathrm{d}y+C\int_{B_R}u^2+\frac{CR^{2-2\sigma}}{(R-r)^2}\int_{B_R}u^2.$$

Since ηu is supported $\{u>0\}\cap B_R$, then the conclusion follows from the Hardy inequality in Lemma 2.1 (and also Lemma 2.2). □

We can now obtain a local bound for minimizers, similar to Theorem 1.1 of Di Castro–Kuusi–Palatucci [5]. Their work concerns (sub)solutions to the *full* fractional Laplacian $(-\Delta)_{\mathbb{R}^n}^\sigma$. In our setting,

$$(-\Delta)_{\mathbb{R}^n}^\sigma u>(-\Delta)_{\{u>0\}}^\sigma u,$$

so these results can not applied directly.

Lemma 5.3. *Let $0<R\leq 1$. Then there exists $C=C(n,\sigma)$ such that*

$$\sup_{B_{R/2}}u\leq\varepsilon R^{2\sigma}\int_{\mathbb{R}^n}\frac{u(y)}{(R+|y|)^{n+2\sigma}}\mathrm{d}y+C\varepsilon^{-\frac{n}{4\sigma}}R^{-\frac{n}{2}}\|u\|_{L^2(B_R)}$$

for all $\varepsilon\in(0,1]$.

Proof. Let $L>0$, $0<r<\ell<R$, $\eta:\mathbb{R}^n\to[0,1]$ be a radial cutoff function which is 1 on B_r, vanishes outside of B_ℓ, and satisfies $|\nabla\eta|\leq 2/(\ell-r)$. As in the proof of Lemma 5.2, we use $\eta^2(u-L)^+$ as the test function. Then we have

$$\mathscr{B}_u[u,\eta^2(u-L)^+]\leq\int_{\mathbb{R}^n}\eta^2 u(u-L)^+\leq\int_{\mathbb{R}^n}\eta^2[(u-L)^+]^2+L\int_{\mathbb{R}^n}\eta^2(u-L)^+.$$

Using

$$u(x)-u(y)=(u(x)-L)^+-(u(y)-L)^--(u(x)-L)^-+(u(y)-L)^-,$$

one has

$$\mathscr{B}_u[(u-L)^+,\eta^2(u-L)^+]\leq\mathscr{B}_u[u,\eta^2(u-L)^+].$$

Following from the proof of Lemma 5.2, one has

$$
\begin{aligned}
&\|\eta(u-L)^+\|^2_{\mathring{H}^\sigma(\mathbb{R}^n)}\\
&\leq C\int_{B_\ell}(u(x)-L)^+\mathrm{d}x\int_{\mathbb{R}^n\setminus B_R}\frac{(u(y)-L)^+}{|y-x|^{n+2\sigma}}\mathrm{d}y+C\int_{B_R}[(u-L)^+]^2\\
&\quad+\frac{CR^{2-2\sigma}}{(R-r)^2}\int_{B_R}[(u-L)^+]^2+CL\int_{B_R}[(u-L)^+].
\end{aligned}
$$

By the Sobolev inequality, we have

$$
\begin{aligned}
&\|(u-L)^+\|^2_{L^{\frac{2n}{n-2\sigma}}(B_r)}\\
&\leq C\int_{B_\ell}(u(x)-L)^+\mathrm{d}x\int_{\mathbb{R}^n\setminus B_R}\frac{(u(y)-L)^+}{|y-x|^{n+2\sigma}}\mathrm{d}y+C\int_{B_R}[(u-L)^+]^2\\
&\quad+\frac{CR^{2-2\sigma}}{(R-r)^2}\int_{B_R}[(u-L)^+]^2+CL\int_{B_R}[(u-L)^+].
\end{aligned}
$$

For $k=0,1,\cdots$, let $L_k=(1-2^{-k})M$, $R_k=\frac{1}{2}(1+2^{-k})R$, where $M>0$ to be fixed in the end, $\ell_k=\frac{1}{2}(R_k+R_{k+1})$, and

$$U_k=\|(u-L_k)^+\|_{L^2(B_{R_k})}.$$

We have

$$
\begin{aligned}
U_{k+1}&\leq\|(u-L_{k+1})^+\|_{L^{\frac{2n}{n-2\sigma}}(B_{R_{k+1}})}|\{u>L_{k+1}\}\cap B_{R_{k+1}}|^{\frac{\sigma}{n}}\\
&\leq\|(u-L_{k+1})^+\|_{L^{\frac{2n}{n-2\sigma}}(B_{R_{k+1}})}(U_k 2^{k+1}/M)^{\frac{2\sigma}{n}}.
\end{aligned}
$$

Also,

$$\int_{B_{R_k}} [(u - L_{k+1})^+]^2 \leq U_k^2,$$

$$L_{k+1} \int_{B_{R_k}} [(u - L_{k+1})^+] \leq \frac{M}{L_{k+1} - L_k} \int_{B_{R_k}} [(u - L_k)^+]^2 \leq 2^{k+1} U_k^2,$$

and

$$\int_{B_{\ell_k}} (u(x) - L_{k+1})^+ \mathrm{d}x \int_{\mathbb{R}^n \setminus B_{R_k}} \frac{(u(y) - L_{k+1})^+}{|y - x|^{n+2\sigma}} \mathrm{d}y$$

$$\leq \frac{2^{k+1}}{M} U_k^2 \, 2^{(k+3)(n+2\sigma)} \int_{\mathbb{R}^n} \frac{u(y)}{(R + |y|)^{n+2\sigma}} \mathrm{d}y.$$

Denote

$$A = R^{2\sigma} \int_{\mathbb{R}^n} \frac{u(y)}{(R + |y|)^{n+2\sigma}} \mathrm{d}y \quad \text{and} \quad \widetilde{U}_k = R^{-\frac{n}{2}} U_k.$$

Then we obtain

$$\widetilde{U}_{k+1} \leq (C^k \widetilde{U}_k)^{1 + \frac{2\sigma}{n}} \left(\sqrt{\frac{A}{M}} + 1 \right) M^{-2\sigma/n}.$$

Taking

$$M \geq \varepsilon A,$$

then

$$\widetilde{U}_{k+1} \leq (C^k \widetilde{U}_k)^{1 + \frac{2\sigma}{n}} \varepsilon^{-1/2} M^{-2\sigma/n}.$$

Hence $\widetilde{U}_k \to 0$ as $k \to \infty$ provided that

$$M \geq C \varepsilon^{-\frac{n}{4\sigma}} \widetilde{U}_0.$$

Hence, we can take

$$M = \varepsilon A + C \varepsilon^{-\frac{n}{4\sigma}} \widetilde{U}_0,$$

and thus,

$$\sup_{B_{R/2}} u \leq \varepsilon R^{2\sigma} \int_{\mathbb{R}^n} \frac{u(y)}{(R + |y|)^{n+2\sigma}} \mathrm{d}y + C \varepsilon^{-\frac{n}{4\sigma}} R^{-\frac{n}{2}} \|u\|_{L^2(B_R)}.$$

Thus, we complete the proof. □

Remark 5.1. The ε in Lemma 5.3 interpolates between the local and nonlocal terms. It plays an essential role when we prove a lower bound for u in Section 6.

Consequently, we have

Theorem 5.1. *There exists a positive constant $C(n,\sigma)$ such that*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n,\sigma)\|u\|_{L^2(\mathbb{R}^n)}.$$

Proof. This follows from Lemma 5.3 and the Hölder inequality. □

Lemma 5.4. *Let $u\in\mathring{H}^\sigma(\mathbb{R}^n)$ be a nonnegative minimizer of $m(N)$. Then it satisfies*

$$(-\Delta)^\sigma_{\{u>0\}}u \geq \lambda u \quad in \ \{u>0\} \tag{5.6}$$

in the weak sense, with λ as in (5.2). In other words,

$$\iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y \geq \lambda \int_{\mathbb{R}^n} u(x)\varphi(x)\mathrm{d}x \tag{5.7}$$

for every nonnegative function $\varphi\in\mathring{H}^\sigma(\mathbb{R}^n)$ such that

$$|\mathrm{supp}\varphi\setminus\{u>0\}|=0.$$

Proof. Let $\varphi\in\mathring{H}^\sigma(\mathbb{R}^n)$ such that $|\mathrm{supp}\varphi\setminus\{u>0\}|=0$, and $u_t=u+t\varphi$ for any small $t>0$. Then we have

$$\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(B_N)}}+|\{u>0\}| \leq \frac{I_{n,\sigma,\{u_t>0\}}[u_t]}{\|u_t\|^2_{L^2(B_N)}}+|\{u_t>0\}|.$$

Since $\{u_t>0\}=\{u>0\}$, we have

$$\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(B_N)}} \leq \frac{I_{n,\sigma,\{u>0\}}[u_t]}{\|u_t\|^2_{L^2(B_N)}}. \tag{5.8}$$

Then

$$\int_{\{u>0\}}(2tu\varphi+t^2\varphi^2)\cdot\iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y$$

$$\leq\|u\|^2_{L^2(B_N)}\iint_{\{u>0\}\times\{u>0\}} \frac{2t[\varphi(x)-\varphi(y)][u(x)-u(y)]+t^2(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y.$$

By canceling t and sending $t\to0$, we obtain

$$\iint_{\{u>0\}\times\{u>0\}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}}\mathrm{d}x\mathrm{d}y \geq \lambda \int_{B_N} u(x)\varphi(x)\mathrm{d}x$$

with λ as in (5.2). □

The following corollary follows from Lemma 5.1 and Lemma 5.4, making precise the fact that a minimizer u solves the eigenvalue equation on its domain of positivity.

Corollary 5.1. *Let $u \in \mathring{H}^\sigma(\mathbb{R}^n)$ be a nonnegative minimizer of $m(N)$. Let $\varphi \in \mathring{H}^\sigma(\mathbb{R}^n)$ such that*

$$|\{\varphi \neq 0\} \setminus \{u > 0\}| = 0.$$

Then

$$\iint_{\{u>0\} \times \{u>0\}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y = \lambda \int_{\mathbb{R}^n} u(x)\varphi(x)\mathrm{d}x, \qquad (5.9)$$

where λ as in (5.2).

6 A bound from below and existence

We also can derive a lower bound for minimizers. The precise statement of it below takes on an unusual nonlocal form: it guarantees the existence of some (large) scale R, possibly depending on the point considered, where $\sup_{B_R(x)} u$ is at least R^σ. We note that this is not a "uniform" estimate (i.e., true for all R), nor do we expect a uniform estimate of this form to be valid: we believe that the exponent σ here is actually not the correct rate of growth for u near $\partial\{u > 0\}$. Nonetheless, this estimate will be enough for concentration compactness arguments in proving existence below, while an optimal, uniform estimate appears to require new ideas.

Lemma 6.1. *Let u be a nonnegative minimizer (normalized so that $\int_{\mathbb{R}^n} u^2 = 1$) of $m(N)$ defined in (4.1). Then there are small numbers c_*, R_* (depending only on n and σ) such that if x is a Lebesgue point of $\{u > 0\}$, then for some $R \in [R_*, 1]$,*

$$\sup_{B_R(x)} u \geq c_* R^\sigma.$$

Proof. After a translation we may assume $x = 0$. Take $\eta : \mathbb{R}^n \to [0, 1]$ to be a radial cutoff function with $\eta = 1$ outside $B_{R/2}$, $\eta = 0$ precisely on $B_{R/10}$, and $|\nabla \eta| \leq 4/R$. We will use $w = \eta u$ as a competitor for u. Note that $\{w > 0\} \subset \{u > 0\}$.

First, reducing the domain of positivity decreased the associated Gagliardo norm:

$$\mathscr{B}_w[w, w] \leq \mathscr{B}_u[w, w],$$

where $\mathscr{B}.[\cdot, \cdot]$ is defined in (5.5). We may rewrite the right as

$$\mathscr{B}_u[w, w] = \mathscr{B}_u[u, u] - 2\mathscr{B}_u[v, u] + \mathscr{B}_u[v, v],$$

where $v = u - w = (1 - \eta)u$. Now apply Lemma 5.4 with $\phi = v$ to give

$$\mathscr{B}_u[v, u] \geq \lambda \int_{\mathbb{R}^n} vu \geq 0,$$

and thus,

$$\mathscr{B}_w[w, w] \leq \mathscr{B}_u[u, u] + \mathscr{B}_u[v, v].$$

It follows from Lemma 5.2 that

$$\mathscr{B}_u[v, v] \leq C \int_{B_R} u(x)\mathrm{d}x \int_{\mathbb{R}^n} \frac{u(y)}{(R + |y|)^{n+2\sigma}} \mathrm{d}y + CR^{-2\sigma} \int_{B_R} u^2.$$

Now from the minimality of u, we get

$$\mathscr{B}_u[u, u] + |\{u > 0\}| \leq \frac{\mathscr{B}_w[w, w]}{\int w^2} + |\{w > 0\}|$$

$$\leq \frac{\mathscr{B}_u[u, u] + C \int_{B_R} u(x)\mathrm{d}x \int_{\mathbb{R}^n} \frac{u(y)}{(R+|y|)^{n+2\sigma}}\mathrm{d}y + CR^{-2\sigma}\int_{B_R}u^2}{1 - 2\int_{\mathbb{R}^n} uv}$$

$$+ |\{u > 0\} \setminus B_{R/10}|.$$

Since u is bounded (see Theorem 5.1), and

$$\int_{\mathbb{R}^n} uv \leq \int_{B_{R/2}} u^2 \leq CR^2,$$

which can be small if $R \leq R_0$ for some small R_0, we obtain

$$|\{u > 0\} \cap B_{R/10}| \leq C \int_{B_R} u(x)\mathrm{d}x \int_{\mathbb{R}^n} \frac{u(y)}{(R + |y|)^{n+2\sigma}} \mathrm{d}y + CR^{-2\sigma} \int_{B_R} u^2.$$

We now argue as follows. Assume that for all $S \in [R, R']$, where $R \leq (R')^2 \leq 1$, we have that

$$\sup_{B_S} |u| \leq c_0 S^\sigma,$$

where $c_0 \leq c_*$ is small. We will show that this guarantees that

$$\sup_{B_{R/20}} u \leq \frac{1}{2} c_0 \left(\frac{R}{20}\right)^\sigma + \left(\frac{R}{20}\right)^{2\sigma} (R')^{-2\sigma}.$$

Indeed, we have that in this case,

$$\int_{B_R} u^2 \le c_0^2 R^{n+2\sigma}, \quad \int_{B_R} u(x)\mathrm{d}x \int_{\mathbb{R}^n} \frac{u(y)}{(R+|y|)^{n+2\sigma}}\mathrm{d}y \le Cc_0 R^n[c_0+R^\sigma(R')^{-2\sigma}], \quad (6.1)$$

so

$$|\{u>0\}\cap B_{R/10}| \le Cc_0 R^n. \quad (6.2)$$

Applying Lemma 5.3,

$$\sup_{B_{R/20}} u \le \varepsilon R^{2\sigma}\int_{\mathbb{R}^n} \frac{u(y)}{(R+|y|)^{n+2\sigma}}\mathrm{d}y + C\varepsilon^{-\frac{n}{4\sigma}}R^{-\frac{n}{2}}\|u\|_{L^2(B_{R/10})}$$

$$\le C\varepsilon R^{2\sigma}(c_0 R^{-\sigma}+(R')^{-2\sigma})+C\varepsilon^{-\frac{n}{4\sigma}}c_0^{3/2}R^\sigma$$

$$= C\varepsilon c_0 R^\sigma + C\varepsilon R^{2\sigma}(R')^{-2\sigma}+C\varepsilon^{-\frac{n}{4\sigma}}c_0^{3/2}R^\sigma. \quad (6.3)$$

By choosing $\varepsilon = 1/(2C(20)^\sigma)$, and c_* sufficiently small, this gives the promised estimate.

Let $4R_0^\sigma \le c_* < 1/20$. Now assume that for $S\in[R_0,1]$, $\sup_{B_S} u \le c_* S^\sigma$. We argue that this is a contradiction. We may apply our claim with $c_0 = c_*$ for every R in $[R_0, 20R_0]$ (with $R'=1$) to give that

$$\sup_{B_S} u \le \frac{1}{2}c_* S^\sigma + S^{2\sigma} \le \frac{1}{2}c_* S^\sigma + R_0^\sigma S^\sigma \le \frac{3}{4}c_* S^\sigma \quad \text{for all} \quad S\in[R_0/20, R_0]. \quad (6.4)$$

We may continue applying the claim with $c_0 = c_*$ and $R\in[R_0/20^k, R_0/20^{k-1}]$ with $R'=1$ to get, inductively, that this holds on $R\in[R_0/20^{k+1}, R_0/20^k]$. Hence, we have

$$\sup_{B_S} u \le \frac{1}{2}c_* S^\sigma + S^{2\sigma} \le \frac{3}{4}c_* S^\sigma \quad \text{for all} \quad S\le R_0. \quad (6.5)$$

Next, we may slightly improve this estimate. To do so, we set $R\le 20R_1 \ll R_0$ and $R'=R_0$, and apply the claim with $c_0 = \frac{3}{4}c_*$ instead. This gives

$$\sup_{B_S} u \le \frac{3}{8}c_* S^\sigma + S^{2\sigma}R_0^{-2\sigma} \le \left[\frac{3}{8}c_* + (R_1/R_0^2)^\sigma\right]S^\sigma \quad \text{for any} \quad S\le R_1.$$

If

$$(R_1/R_0^2)^\sigma = \frac{1}{4}\cdot\frac{3}{4}c_*,$$

this gives that

$$\sup_{B_S} u \le \left(\frac{3}{4}\right)^2 c_* S^\sigma \quad \text{for all} \quad S\le R_1.$$

We may continue on in this fashion, choosing

$$(R_k/R_{k-1}^2)^\sigma = \frac{1}{4}\left(\frac{3}{4}\right)^k c_*,$$

to get that if for

$$\sup_{B_S} u \le (3/4)^k c_* S^\sigma \quad \text{for all} \ \ S \le R_{k-1},$$

then

$$\sup_{B_S} u \le \left[\frac{1}{2}(3/4)^k c_* + (R_k/R_{k-1}^2)^\sigma\right] S^\sigma \le (3/4)^{k+1} S^\sigma c_* \quad \text{for all} \ \ S \le R_k.$$

Combining with the intermediate estimate (6.2), it gives

$$|\{u>0\}\cap B_{R_k/10}| \le C\left(\frac{3}{4}\right)^k c_* R_k^n.$$

This gives that the Lebesgue density of $\{u>0\}$ at 0 is 0, which contradicts the assumptions. $\qquad\square$

Corollary 6.1. *Let u be as in Lemma* 6.1. *There exists $c = c(n,\sigma)$ such that*

$$\int_{B_2(x)} u^2 \ge c$$

for every Lebesgue point x of $\{u>0\}$.

Proof. Let R and R_* be the ones in Lemma 6.1. Assume $x=0$. Combining the results in Lemma 5.3 and Lemma 6.1, we have

$$R^\sigma \le \varepsilon R^{2\sigma} \int_{\mathbb{R}^n} \frac{u(y)}{(R+|y|)^{n+2\sigma}} dy + C\varepsilon^{-\frac{n}{4\sigma}} R^{-\frac{n}{2}} \|u\|_{L^2(B_{2R})}$$
$$\le C\varepsilon + C\varepsilon^{-\frac{n}{4\sigma}} R^{-\frac{n}{2}} \|u\|_{L^2(B_{2R})}.$$

By choosing $\varepsilon = R^\sigma/(2C)$, we have

$$\|u\|_{L^2(B_{2R})} \ge CR^{\frac{3n}{4}+\sigma} \ge CR_*^{\frac{3n}{4}+\sigma}.$$

This finishes the proof. $\qquad\square$

This leads to:

Lemma 6.2. *Let u be as in Lemma* 6.1. *Then there is a large number $K = K(n,\sigma)$ and another nonnegative $v \in \mathring{H}^\sigma(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} v^2 = 1$ such that v is supported on B_K and*

$$\mathscr{B}_v[v,v] + |\{v > 0\}| \leq \mathscr{B}_u[u,u] + |\{u > 0\}|.$$

Proof. Select a representative of u which vanishes except at Lebesgue points of $\{u > 0\}$. By the Besicovitch covering theorem, from the collection of balls $\{B_2(x): x \in \{u > 0\}\}$, we can select a finite-overlapping subcover of $\{u > 0\}$, $U = \{B_2(x_i)\}_{i \in I}$. Together with Corollary 6.1, we have from above that

$$\int_{B_2(x_i)} u^2 \geq c,$$

but

$$\sum_i \int_{B_2(x_i)} u^2 \leq C \int_{\{u > 0\}} u^2 = C.$$

Let $M = \#(I)$ be the number of balls in U: this gives that $M < \infty$ and bounded in terms of n and σ only.

Assume that u has the following property: $u = \sum_i u_i$, where u_i are nonzero and $|\{u_i > 0\} \cap \{u_j > 0\}| = 0$ for all $i \neq j$. Then we see that

$$\sum_i \mathscr{B}_{u_i}[u_i, u_i] = \sum_i \mathscr{B}_{u_i}[u, u] < \mathscr{B}_u[u, u],$$

as all the cross terms are positive. Also

$$\sum_i \int_{\mathbb{R}^n} u_i^2 = 1, \quad \sum_i |\{u_i > 0\}| = |\{u > 0\}|.$$

Set

$$v_i(x) = R_i^{-n/2} u_i(R_i x) \left(\int_{\mathbb{R}^n} u_i^2 \right)^{-1/2}$$

(this has $\int_{\mathbb{R}^n} v_i^2 = 1$) and compute the energy:

$$\mathscr{B}_{v_i}[v_i, v_i] + |\{v_i > 0\}| = R_i^{2s} \frac{\mathscr{B}_{u_i}[u_i, u_i]}{\int_{\mathbb{R}^n} u_i^2} + R_i^{-n} |\{u_i > 0\}|.$$

Choose

$$R_i = (|\{u_i > 0\}| / |\{u > 0\}|)^{1/n},$$

noting that $R_i^{2s} < 1$. Then we have that

$$\sum_i \int_{\mathbb{R}^n} u_i^2 [\mathscr{B}_{v_i}[v_i,v_i] + |\{v_i > 0\}|]$$
$$= |\{u > 0\}| + \sum_i R_i^{2s} \mathscr{B}_{u_i}[u_i,u_i] < \mathscr{B}_u[u,u] + |\{u > 0\}|.$$

Since

$$\sum_i \int_{\mathbb{R}^n} u_i^2 = 1,$$

then at least one of the v_i has to have

$$\mathscr{B}_{v_i}[v_i,v_i] + |\{v_i > 0\}| < \mathscr{B}_u[u,u] + |\{u > 0\}|.$$

Let $\{U_j\}_{j \in J}$ be the connected components of the set $\cup_{i \in I} B_2(x_i)$. If there is only one connected component, then the function $v(x) = u(x - x_1)$ is supported on B_{M+1} and we may conclude. If not, let $u_j = u|_{U_j}$ and apply the above construction to find one v_i with energy less than u. Note that as

$$|\{u_i > 0\}| \geq c \int_{U_1} u^2 \geq c,$$

we have that $R_i \geq c$, so a translate of v_i will be supported on $B_{(M+1)/c}$. We use this v_i to conclude. $\qquad\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $K(n,\sigma)$ be the one in Lemma 6.2, and $N_1, N_2 > K(n,\sigma)$. Let u_1 and u_2 be minimizers of $m(N_1)$ and $m(N_2)$. Due to the translation invariance of $m(N)$, u_1 is a valid competitor of u_2 for $m(N_2)$, and u_2 is a valid competitor of u_1 for $m(N_1)$. Therefore, $m(N_1) = m(N_2)$. Because of (4.2), u_1 is a desired minimzier. $\qquad\square$

7   Discussions and open questions

We know from Theorem 1.1 that the minimizing set is bounded with bounded support. A natural question is whether the minimizing eigenfunction u_0 is continuous (and in particular, whether $\{u_0 > 0\}$ admits an open representative); this was stated as Open Question 1.2. We do not know how to prove this, and in this section, we would like to discuss the difficulties and present some related open questions.

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a fixed set such that $B_{2r}(x_0) \subset \Omega$. Let $v \in \mathring{H}^\sigma(\mathbb{R}^n)$ be weak solution of $(-\Delta)^\sigma_\Omega v = 0$ in $B_r(x_0)$. Assume that $|v| \leq 1$ on $\mathbb{R}^n$. Then for every $\alpha \in (0,1)$, there exists $C > 0$ depending only on n, α and σ, such that*

$$[v]_{C^\alpha(B_{r/2}(x_0))} \leq Cr^{-\alpha}.$$

Proof. We first assume that $r = 1$ and $x_0 = 0$. Since $B_2 \subset \Omega$, we know that

$$\int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x-y|^{n+2\sigma}} \mathrm{d}y$$

is a smooth function in $B_{3/2}$ and all its derivative are bounded in $B_{3/2}$ independent of Ω. Then it follows from standard estimates for the fractional Laplacian in the whole space that for every $\alpha \in (0,1)$,

$$[v]_{C^\alpha(B_{1/2})} \leq C,$$

where C depends only on n, α and σ.

In general, we let $w(x) = v(x_0 + rx)$. Then

$$(-\Delta)^\sigma_{\widetilde{\Omega}} w = 0 \quad \text{in} \quad B_1,$$

where $\widetilde{\Omega} = (\Omega - x)/r$ containing B_2. Then we have

$$[w]_{C^\alpha(B_{1/2})} \leq C.$$

Rescaling back, we have

$$[v]_{C^\alpha(B_{r/2}(x_0))} \leq Cr^{-\alpha}.$$

Thus, we complete the proof. $\square$

Let u_0 be a minimizer in Theorem 1.1. Let us normalize it so that $\|u_0\|_{L^2(\mathbb{R}^n)} = 1$, and hence, $u_0 \leq C$ in $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$, $R > 0$, and

$$\Omega = \{u_0 > 0\} \cup B_{2R}(x_0).$$

Let $h \in \mathring{H}(\mathbb{R}^n)$ be the solution of $(-\Delta)^\sigma_\Omega h = 0$ in $B_R(x_0)$ and $h \equiv u_0$ in $\mathbb{R}^n \setminus B_R(x_0)$. Since $u_0 \geq 0$, we have that $h \geq 0$ in Ω. Since $u_0 \leq C$, we have $h \leq C$.

Moreover,

$$\begin{aligned}
\mathscr{B}_\Omega[u_0 - h, u_0 - h] &= \mathscr{B}_\Omega[u_0 - h, u_0 + h] - 2\mathscr{B}_\Omega[u_0 - h, h] \\
&= \mathscr{B}_\Omega[u_0, u_0] - \mathscr{B}_\Omega[h, h] \\
&\leq \mathscr{B}_\Omega[u_0, u_0] - \mathscr{B}_{\{h>0\}}[h, h],
\end{aligned} \tag{7.1}$$

where

$$\mathscr{B}_\Omega[v,\varphi] = \iint_{\Omega\times\Omega} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\mathrm{d}y,$$

and we used the equation of h in the second inequality so that $\mathscr{B}_\Omega[u_0-h,h]=0$. On the other hand, using h as a competitor for u_0, we know that

$$\mathscr{B}_{\{u_0>0\}}[u_0,u_0] + |\{u_0>0\}| \le \frac{\mathscr{B}_{\{h>0\}}[h,h]}{\|h\|_{L^2(\Omega)}} + |\Omega|. \tag{7.2}$$

Since

$$\int_{\mathbb{R}^n} |h^2-u_0^2| \le CR^n,$$

we have

$$\int_{\mathbb{R}^n} h^2 \ge 1-CR^n.$$

From $\mathscr{B}_\Omega[u_0-h,h]=0$, we have

$$\mathscr{B}_\Omega[h,h] \le \mathscr{B}_\Omega[u_0,u_0] \le \mathscr{B}_{\mathbb{R}^n}[u_0,u_0] \le C.$$

Then, from (7.2), we obtain

$$\mathscr{B}_{\{u_0>0\}}[u_0,u_0] \le \mathscr{B}_{\{h>0\}}[h,h] + CR^n.$$

Therefore, from (7.1), we obtain

$$\mathscr{B}_\Omega[u_0-h,u_0-h] \le \mathscr{B}_\Omega[u_0,u_0] - \mathscr{B}_{\{u_0>0\}}[u_0,u_0] + CR^n.$$

Using the Poincaré inequality, we have

$$\int_{\mathbb{R}^n} |u_0-h|^2 = \int_{B_R(x_0)} |u_0-h|^2 \le CR^{2\sigma}\mathscr{B}_{B_R(x_0)}[u_0-h,u_0-h].$$

Hence,

$$\int_{\mathbb{R}^n} |u_0-h|^2 \le CR^{2\sigma}(\mathscr{B}_\Omega[u_0,u_0] - \mathscr{B}_{\{u_0>0\}}[u_0,u_0]) + CR^{n+2\sigma}$$

$$= CR^{2\sigma}\int_{\{u_0>0\}} u_0(x)^2 \int_{B_{2R}(x_0)\backslash\{u_0>0\}} \frac{\mathrm{d}y}{|x-y|^{n+2\sigma}}\mathrm{d}x + CR^{n+2\sigma}.$$

Open Question 7.1. Is it true that

$$\int_{\{u_0>0\}} u_0(x)^2 \int_{B_{2R}(x_0)\setminus\{u_0>0\}} \frac{\mathrm{d}y}{|x-y|^{n+2\sigma}} \mathrm{d}x \le CR^N$$

for some $N>n-2\sigma$?

This is the difficulty of the above attempt of proving the minimizer's global continuity. We do not know how to obtain such a sufficient decay estimate. If the solution to this open question is true, then it will lead to an affirmative answer to Open Question 1.2.

The boundary behavior of solutions to homogeneous Dirichlet problems for the regional fractional Laplace equation (on sufficiently regular domains) was studied by Chen-Kim-Song [6], culminating in two-sided heat kernel estimates (and consequently, the two-sided Green's function estimates for the regional fractional Laplacian in Proposition 4.2 of [10]). These results show that solutions on Ω grow like $d^{2s-1}(\cdot,\partial\Omega)$ from the boundary. It is natural, then, to ask whether such growth estimates can be established for our minimizers u_0 near $\partial\{u_0>0\}$:

Open Question 7.2. Does there exist $C>0$ such that

$$u_0(x) \le C[\mathrm{dist}(x,\partial\{u_0>0\})]^{2\sigma-1}, \quad \forall x \in \{u_0>0\}?$$

The point here is that we do not know that $\partial\{u_0>0\}$ is regular, so this is does not follow directly from [6]. A related question is whether there is a complementary uniform *lower* bound on the growth of solutions:

Open Question 7.3. Does there exist $c>0$ such that for all $R\in(0,1)$ and all $x\in\partial\{u_0>0\}$,

$$\sup_{B_R(x)} u_0 \ge cR^{2\sigma-1}?$$

This form of a lower bound is common in the free boundary literature. Note that the scaling here, R^{2s-1}, is different from what was used in Lemma 6.1 to obtain a non-uniform lower bound: indeed, this difference in scaling explains why the estimate there could not be uniform in R. It also suggests new ideas, and possibly new competitor sets and functions, are needed to answer Open Questions 7.2 and 7.3, as most of the arguments in this paper do not detect this R^{2s-1} scaling.

The Euler-Lagrange equation for the minimizer is another interesting open question. We have already seen that u_0 satisfies

$$(-\Delta)^{\sigma}_{\{u_0>0\}} u_0 = \lambda u_0 \quad \text{in} \ \ \{u_0>0\},$$

but by analogy to the second order case [1] we expect another pointwise equation to be satisfied by u_0 along the boundary $\partial\{u_0>0\}$, perhaps of the following form:

Open Question 7.4. Does $\partial\{u_0 > 0\}$ have some approximate normal vectors ν_x in an appropriate sense, and does

$$\lim_{t \to 0} \frac{u_0(x - t\nu_x)}{t^\alpha} = const?$$

Finally, let us suggest the following question about the shape of minimizers. Unlike the previous sequence of questions, a positive answer cannot be attained by local regularity or competitor arguments, and requires some new global approach.

Open Question 7.5. Are minimizers u_0 radial and unique up to translations?

Acknowledgements

References

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