Piecewise Spectral Collocation Method for Second Order Volterra Integro-Differential Equations with Nonvanishing Delay

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Abstract. In this paper, the piecewise spectral-collocation method is used to solve the second-order Volterra integral differential equation with nonvanishing delay. In this collocation method, the main discontinuity point of the solution of the equation is used to divide the partitions to overcome the disturbance of the numerical error convergence caused by the main discontinuity of the solution of the equation. Derivative approximation in the sense of integral is constructed in numerical format, and the convergence of the spectral collocation method in the sense of the $L^\infty$ and $L^2$ norm is proved by the Dirichlet formula. At the same time, the error convergence also meets the effect of spectral accuracy convergence. The numerical experimental results are given at the end also verify the correctness of the theoretically proven results.

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Key words: Second-order Volterra type integro-differential equation, delay function, piecewise spectral-collocation method.

1 Introduction

Second-order Volterra integro-differential equations (VIDEs) have long appeared in mathematical models of physical phenomena and biological phenomena, which has led many scholars to develop a theoretical and numerical analysis of these equations. For some early research results, such as the general linear method [7], the linear multistep

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method [8, 9], the Runge-Kutta method [10, 12]. Over the years, polynomial spline collocation method [13–16, 27, 28], spectral Galerkin method [20] and Bologna [29] have developed an asymptotic solution for first-order and second-order VIDEs containing arbitrary kernels. In [24, 37], the collocation method is used to approximate second-order VIDEs. In the recent papers [42–44], an $hp$-version of the spectral method has been proposed and analyzed for VIDEs. The spectral method has been used in applied mathematics and scientific calculations to numerically solve certain partial differential equations (PDEs) [31–34]. In practice, the spectral method has high-precision convergence, the so-called “exponential convergence”.

Delay VIDEs have many practical applications, such as competitive ecosystem [1], biology and Species model [2–4], virus transmission [5], and so on. For more information on the application of VIDEs with delay in species models, please refer to reference [6].

Remove the integral term in the delayed VIDEs to get the delayed differential equation (DDEs). The DDEs model appears in many practical problems, such as tumor growth models [38], species dynamics systems [39], hepatitis virus infection models [40], and toxic species presence diffusion models [41]. Reference [39] contains a lot of literature on the application of DDEs.

The literatures using spectral methods for solving delay VIDEs are [17–19, 21, 22, 24–26, 35]. The numerical methods used in these articles are used to solve the vanishing delay VIDEs. So far, there are few numerical methods for solving the nonvanishing delay type VIDEs that can make the numerical error converge to the spectral accuracy. When the VIDEs with nonvanishing delay are numerically solved, it is necessary to overcome that the solution of the equation has a main discontinuity point, which is inconsistent with the spectral method’s requirement for the global smoothness of the equation solution. In the literature [36], the author gives a method of slicing the spectrum for solving VIDEs with nonvanishing delay first-order VIDEs, and gives proof of convergence. So far, few scholars have studied the spectral approximation of second-order VIDEs for the piecewise spectral-collocation method.

The second-order Volterra integro-differential equation with nonvanishing delay considered here is as follows:

$$\begin{align*}
y''(t) &= a_1(t)y(t) + a_2(t)y'(t) + b_1(t)y(\theta(t)) + b_2(t)y'(\theta(t)) + g(t) \\
&+ \int_0^t K_1(t, s)y(s)ds + \int_0^t K_2(t, s)y'(s)ds \\
&+ \int_0^{\theta(t)} R_1(t, s)y(s)ds + \int_0^{\theta(t)} R_2(t, s)y'(s)ds,
\end{align*}$$

$$(1.1a)$$

$$y(t) = \phi(t), \quad y'(t) = \varphi(t), \quad t \in [\theta(0), 0].$$

$$(1.1b)$$

The functions $y(t), t \in (0, T]$ are unknown functions, and $y \in C^{m+1}([0, T])$. Assume func-
tions (1.1) are \( m \geq 1 \) order continuous differentiable functions in their domain, i.e.,

\[
a_1(t), a_2(t), b_1(t), b_2(t), g(t) \in C^m([0,T]), \quad \phi(t) \in C^m([\theta(0),0]),
\]

\[
K_1(t,s), K_2(t,s) \in C^m(\Omega_1), \quad \Omega_1 := \{(t,s) : 0 \leq s \leq t \leq T\},
\]

\[
R_1(t,s), R_2(t,s) \in C^m(\Omega_2), \quad \Omega_2 := \{(t,s) : \theta(0) \leq s \leq \theta(t), 0 \leq t \leq T\}.
\]

And the delay function \( \theta \) satisfies the following conditions:

\[
\theta(t) := t - \tau(t), \quad \tau \in C^m([0,T]), \quad \theta \text{ is strictly increasing on } [0,T],
\]

\[
\tau(t) \geq \tau(0) > 0 \quad \text{for all } t \in [0,T].
\]

In Eq. (1.1), the non-decaying hysteresis function \( \theta \) causes the solution of the Volterra integro-differential equation to have major discontinuities \( \{\xi_\mu\} \). The recurrence relation of these major discontinuities for

\[
\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 0(\xi_{-1} := \theta(0), \xi_0 = 0).
\]

And these main discontinuities have the characteristics of consistent interval, that is

\[
\xi_\mu - \xi_{\mu-1} = \tau > 0, \quad \mu \geq 0.
\]

For the convenience of discussion and elaboration in the following papers, we assume

\[
T = \xi_{N+1}, \quad M \geq 1.
\]

The rest of the paper is organized as follows: in Section 2, a numerical method of piecewise spectral-collocation is proposed to solve the second-order Volterra integro-differential equation with nonvanishing delay. Some concepts and several useful lemmas are described in detail in Section 3. The convergence proof of the method is presented in Section 4 and the experimental results in Section 5 also verify the effectiveness of the proposed method. Some concluding remarks are given in Section 6.

**2 Piecewise spectral-collocation method**

In this section, our main goal is to obtain the numerical format of the piecewise spectral-collocation method for Eq. (1.1). We use the main discontinuity point of the solution of the equation as the interval node, and use Legendre on each subinterval approximation of Equations by spectral collocation method. By transformation:

\[
t(x) = \frac{T}{2}(x+1), \quad s(z) = \frac{T}{2}(z+1).
\]
Convert Eq. (1.1) into a new equation defined on \([-1,1]\).

\[
\begin{align*}
u''(x) &= A_1(x) u(x) + A_2(x) u'(x) + B_1(x) u(\theta(x)) + B_2(t) u'(\theta(x)) + f(x) \\
&\quad + \int_{-1}^{x} S_1(x,z) u(z) dz + \int_{-1}^{x} S_2(x,z) u'(z) dz \\
&\quad + \int_{-1}^{\theta(x)} Q_1(x,z) u(z) dz + \int_{-1}^{\theta(x)} Q_2(x,z) u'(z) dz,
\end{align*}
\]

where

\[
\begin{align*}
u(x) &= \psi(x), \quad u'(x) = \chi(x), \quad x \in [\theta(-1),-1],
\end{align*}
\]

\(2.2a\)

\(2.2b\)

Next, we divide the interval \([-1,1]\) into \(M+1\) subintervals,

\[
[-1,1] = \bigcup_{\mu=0}^{M} \delta_{\mu}, \quad \mu = 1, \ldots, M,
\]

\(2.4\)

where

\[
\delta_0 := [-1,\eta_1], \quad \eta_{M+1} = 1, \quad \delta_{\mu} := (\eta_{\mu},\eta_{\mu+1}).
\]

We choose the following collocation points on the interval \([-1,1]\):

\[
X_N := \bigcup_{\mu=0}^{M} X^\mu, \quad X^\mu := \{x_i^\mu : \eta_{\mu} = x_0^\mu < x_1^\mu < \cdots < x_N^\mu = \eta_{\mu+1}\},
\]

\(2.5\)

which

\[
\begin{align*}
h_{\mu} &:= \frac{\eta_{\mu+1} - \eta_{\mu}}{2} \\
x_i^\mu &:= h_{\mu} x_i + \frac{\eta_{\mu+1} + \eta_{\mu}}{2}.
\end{align*}
\]
Then Eq. (2.2) is true at $x_i^{(1)}$, that is,

$$u''(x_i^{(1)}) = A_1(x_i^{(1)})u(x_i^{(1)}) + A_2(x_i^{(1)})u'(x_i^{(1)}) + B_1(x_i^{(1)})u(\theta(x_i^{(1)})) + B_2(x_i^{(1)})u'(\theta(x_i^{(1)}))$$

$$+ \int_{-1}^{x_i^{(1)}} S_1(x_i^{(1)}, z)u(z)dz + \int_{-1}^{x_i^{(1)}} S_2(x_i^{(1)}, z)u'(z)dz$$

$$+ \int_{-1}^{\theta(x_i^{(1)})} Q_1(x_i^{(1)}, z)u(z)dz + \int_{-1}^{\theta(x_i^{(1)})} Q_2(x_i^{(1)}, z)u'(z)dz + f(x_i^{(1)}),$$

$$i = 0, 1, \ldots, N, \quad \mu = 0, 1, \ldots, M. \quad (2.6)$$

Next, we use $\nu_i^{(1)}$, $\sigma_i^{(1)}$ and $\xi_i^{(1)}$ to approximate $u(\theta(x_i^{(1)}))$, $u'(\theta(x_i^{(1)}))$ and $u''(x_i^{(1)})$ respectively. We have

$$\varrho_\mu(x) := \sum_{j=0}^{N} \varrho_j^{(1)} F_j^{(1)}(x), \quad x \in [\eta_\mu, \eta_{\mu+1}].$$

Therefore, $u''|_{\delta_\mu}(x)$ can be approximated by $\varrho_\mu(x)$. Then $u''(x)$ can be approximated by $\varrho(x)$, that is,

$$\varrho(x) := \varrho_\mu(x), \quad \text{if} \ x \in [\eta_\mu, \eta_{\mu+1}].$$

In order to facilitate the proof of the theoretical part, we make the following definitions:

$$\rho(x) = \zeta(-1) + \int_{-1}^{x} \varrho(z)dz,$$

the derivative relationship is

$$\frac{d}{dx} \rho(x) = \varrho(x).$$

Then function $u'(x)$ can be approximated by $\rho(x)$.

Similarly, we define:

$$\zeta(x) = \chi(-1) + \int_{-1}^{x} \rho(z)dz,$$

the derivative relationship is:

$$\frac{d}{dx} \zeta(x) = \rho(x).$$

Then function $u(x)$ can be approximated by $\zeta(x)$. Thus (2.6) has the following approximation:

$$\varrho_i^{(1)} \approx A_1(x_i^{(1)})\zeta(x_i^{(1)}) + A_2(x_i^{(1)})\rho(x_i^{(1)}) + B_1(x_i^{(1)})\nu_i^{(1)} + B_2(x_i^{(1)})\sigma_i^{(1)} + f(x_i^{(1)})$$

$$+ \int_{-1}^{x_i^{(1)}} S_1(x_i^{(1)}, z)\zeta(z)dz + \int_{-1}^{x_i^{(1)}} S_2(x_i^{(1)}, z)\rho(z)dz$$

$$+ \int_{-1}^{\theta(x_i^{(1)})} Q_1(x_i^{(1)}, z)\zeta(z)dz + \int_{-1}^{\theta(x_i^{(1)})} Q_2(x_i^{(1)}, z)\rho(z)dz. \quad (2.7)$$
This can be rewritten as

$$
\phi_{i}^{\mu} \approx A_{1}(x_{i}^{\mu}) \zeta(x_{i}^{\mu}) + A_{2}(x_{i}^{\mu}) \rho(x_{i}^{\mu}) + B_{1}(x_{i}^{\mu}) v_{i}^{\mu} + B_{2}(x_{i}^{\mu}) \sigma_{i}^{\mu} + f(x_{i}^{\mu})
$$

$$
+ \sum_{r=0}^{\mu-1} \int_{\eta_{r}}^{\eta_{r+1}} S_{1}(x_{i}^{\mu}, z(z)) \zeta(z) dz + \int_{\eta_{r}}^{\eta_{r}} S_{1}(x_{i}^{\mu}, z(z)) \zeta(z) dz 
$$

$$
+ \sum_{r=0}^{\mu-1} \int_{\eta_{r}}^{\eta_{r+1}} S_{2}(x_{i}^{\mu}, z(z)) \rho(z) dz + \int_{\eta_{r}}^{\eta_{r}} S_{2}(x_{i}^{\mu}, z(z)) \rho(z) dz
$$

$$
+ \sum_{r=0}^{\mu-2} \int_{\eta_{r}}^{\eta_{r+1}} Q_{1}(x_{i}^{\mu}, z(z)) \zeta(z) dz + \int_{\eta_{r}}^{\eta_{r+1}} Q_{1}(x_{i}^{\mu}, z(z)) \zeta(z) dz
$$

$$
+ \sum_{r=0}^{\mu-2} \int_{\eta_{r}}^{\eta_{r+1}} Q_{2}(x_{i}^{\mu}, z(z)) \rho(z) dz + \int_{\eta_{r}}^{\eta_{r+1}} Q_{2}(x_{i}^{\mu}, z(z)) \rho(z) dz.
$$

(2.8)

To approximate the integral term using the Gauss quadrature formula, we replace the integration interval $[a, b]$ to the standard interval $[-1, 1]$ with the following variables instead

$$
z(a, b, v) := \frac{b-a}{2} v + \frac{b+a}{2}, \quad v \in [-1, 1].
$$

(2.9)

For the convenience of the following derivation, remember

$$
z_{r}(v) := z(\eta_{r}, \eta_{r+1}, v), \quad v \in [-1, 1), \quad r > 0.
$$

(2.10)

Using the Gauss quadrature formula to approximate the integral term in (2.8), we can get

$$
\phi_{i}^{\mu} = A_{1}(x_{i}^{\mu}) \zeta(x_{i}^{\mu}) + A_{2}(x_{i}^{\mu}) \rho(x_{i}^{\mu}) + B_{1}(x_{i}^{\mu}) v_{i}^{\mu} + B_{2}(x_{i}^{\mu}) \sigma_{i}^{\mu} + f(x_{i}^{\mu})
$$

$$
+ \sum_{r=0}^{\mu-1} h_{r} \sum_{k=0}^{N} S_{1}(x_{i}^{\mu}, z_{r}(v_{k})) \zeta(z_{r}(v_{k})) \omega_{k}
$$

$$
+ h_{\mu} \frac{x_{i}+1}{2} \sum_{k=0}^{N} S_{1}(x_{i}^{\mu}, z_{\mu}(z(-1, x_{i}, v_{k}))) \zeta(z_{\mu}(z(-1, x_{i}, v_{k}))) \omega_{k}
$$

$$
+ \sum_{r=0}^{\mu-1} h_{r} \sum_{k=0}^{N} S_{2}(x_{i}^{\mu}, z_{r}(v_{k})) \rho(z_{r}(v_{k})) \omega_{k}
$$

$$
+ h_{\mu} \frac{x_{i}+1}{2} \sum_{k=0}^{N} S_{2}(x_{i}^{\mu}, z_{\mu}(z(-1, x_{i}, v_{k}))) \rho(z_{\mu}(z(-1, x_{i}, v_{k}))) \omega_{k}
$$

$$
+ \sum_{r=0}^{\mu-2} h_{r} \sum_{k=0}^{N} Q_{1}(x_{i}^{\mu}, z_{r}(v_{k})) \zeta(z_{r}(v_{k})) \omega_{k} + h_{\mu-1} \frac{s_{(x_{i}^{\mu})} + 1}{2}
$$

$$
\times \sum_{k=0}^{N} Q_{1}(x_{i}^{\mu}, z_{\mu-1}(z(-1, s_{(x_{i}^{\mu})}, v_{k}))) \zeta(z_{\mu-1}(z(-1, s_{(x_{i}^{\mu})}, v_{k}))) \omega_{k}
$$

(2.9)
\[ + \sum_{r=0}^{\mu-2} h_r \sum_{k=0}^{N} Q_2(x_i^\mu, z_r(v_k)) \rho(z_r(v_k)) \omega_k + h_{\mu-1} \frac{\tilde{\varphi}(x_i^\mu)+1}{2} \times \sum_{k=0}^{N} Q_2(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\varphi}(x_i^\mu), v_k))) \rho(z_{\mu-1}(z(-1, \tilde{\varphi}(x_i^\mu), v_k))) \omega_k, \tag{2.11} \]

where \( v_k, k = 0, 1, \cdots, N \) are \( N + 1 \) Legendre Gauss-Lobatto integral nodes on the interval \([-1,1]\), and the corresponding weight function is \( \omega_k, k = 0, 1, \cdots, N \). Additionally,

\[ \tilde{\varphi}(x_i^\mu) := \frac{1}{h_{\mu-1}} \varphi(x_i^\mu) - \frac{\eta_\mu+\eta_{\mu-1}}{\eta_\mu-\eta_{\mu-1}}, \quad \mu > 0. \tag{2.12} \]

Simplifying (2.11) gives

\[ \theta_i^\mu = A_1(x_i^\mu)\zeta(x_i^\mu) + A_2(x_i^\mu)\rho(x_i^\mu) + B_1(x_i^\mu)\psi(x_i^\mu) + B_2(x_i^\mu)\sigma(x_i^\mu), \]

\[ + \beta_2(x_i^\mu) + \lambda_1(x_i^\mu) + \lambda_2(x_i^\mu), \quad i = 0, 1, \cdots, N, \quad \mu = 0, 1, \cdots, M, \tag{2.13} \]

where \( \beta_1(x_i^\mu) \) and \( \beta_2(x_i^\mu) \) can be expressed as:

\[ \beta_1(x_i^\mu) := \sum_{r=0}^{\mu-1} h_r \sum_{k=0}^{N} S_1(x_i^\mu, z_r(v_k)) \zeta(z_r(v_k)) \omega_k \]

\[ + h_{\mu} \frac{x_i+1}{2} \sum_{k=0}^{N} S_1(x_i^\mu, z_{\mu}(z(-1, x_i, v_k))) \zeta(z_{\mu}(z(-1, x_i, v_k))) \omega_k, \]

\[ \beta_2(x_i^\mu) := \sum_{r=0}^{\mu-1} h_r \sum_{k=0}^{N} S_2(x_i^\mu, z_r(v_k)) \rho(z_r(v_k)) \omega_k \]

\[ + h_{\mu} \frac{x_i+1}{2} \sum_{k=0}^{N} S_2(x_i^\mu, z_{\mu}(z(-1, x_i, v_k))) \rho(z_{\mu}(z(-1, x_i, v_k))) \omega_k, \]

\( \lambda_1(x_i^\mu) \) and \( \lambda_2(x_i^\mu) \) can be expressed as:

\[ \lambda_1(x_i^\mu) := \begin{cases} \frac{\varphi(x_i^\mu)+1}{2} \sum_{k=0}^{N} Q_1(x_i^\mu, z(-1, \varphi(x_i^\mu), v_k)) \psi(z(-1, \varphi(x_i^\mu), v_k)) \omega_k, \quad \mu = 0, \\
\sum_{r=0}^{\mu-2} h_r \sum_{k=0}^{N} Q_1(x_i^\mu, z_r(v_k)) \zeta(z_r(v_k)) \omega_k + h_{\mu-1} \frac{\tilde{\varphi}(x_i^\mu)+1}{2} \\
\times \sum_{k=0}^{N} Q_1(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\varphi}(x_i^\mu), v_k))) \zeta(z_{\mu-1}(z(-1, \tilde{\varphi}(x_i^\mu), v_k))) \omega_k, \quad \mu > 0, \end{cases} \]
Then (2.15) has the following approximation

\[
\lambda_2(x_i^\mu) := \begin{cases} 
\frac{\theta(x_i^0) + 1}{2} \sum_{k=0}^N Q_2(x_i^0, z(-1, \theta(x_i^0), v_k)) \chi(z(-1, \theta(x_i^0), v_k)) \omega_k, & \mu = 0, \\
- \sum_{r=0}^{\mu-2} h_r \sum_{k=0}^N Q_2(x_i^\mu, z_r(v_k)) \rho(z_r(v_k)) \omega_k + \frac{\tilde{\theta}(x_i^\mu) + 1}{2} \\
\times \sum_{k=0}^N Q_2(x_i^\mu, z_{\mu-1}(z(-1, \tilde{\theta}(x_i^\mu), v_k))) \rho(z_{\mu-1}(z(-1, \tilde{\theta}(x_i^\mu), v_k))) \omega_k, & \mu > 0. 
\end{cases}
\]

Similarly, we can get

\[
\zeta(x_i^\mu) = \psi(-1) + \int_{-1}^{x_i^\mu} \rho(z) dz, \quad i = 0, 1, \cdots, N, \quad \mu = 0, 1, \cdots, M. \tag{2.16}
\]

Relying on the discrete system (2.13) alone is not enough to solve the unknown. We need another four about \(\zeta(x_i^\mu), \rho(x_i^\mu), v_i^\mu, \epsilon_i^\mu, \sigma_i^\mu, i = 0, 1, \cdots, N, \mu = 0, 1, \cdots, M\) for discrete systems

\[
\rho(x) = \chi(-1) + \int_{-1}^{x} \epsilon(z) dz = \chi(-1) + \sum_{r=0}^{\mu-1} \int_{\eta_r}^{\eta_{r+1}} \epsilon_r(z) dz + \int_{\eta_{\mu}}^{x} \epsilon_{\mu}(z) dz = \chi(-1) + \sum_{r=0}^{\mu-1} h_r \sum_{k=0}^N \epsilon_r(z_r(v_k)) \omega_k + \frac{x+1}{2} \sum_{k=0}^N \epsilon_{\mu}(z(-1, x, v_k)) \omega_k
\]

\[
= \chi(-1) + \sum_{r=0}^{\mu-1} h_r \sum_{k=0}^N \epsilon_r(z_r(v_k)) \omega_k + \frac{\tilde{x} + 1}{2} \sum_{k=0}^N \epsilon_{\mu}(z(-1, \tilde{x}, v_k)) \omega_k, \quad x \in [\eta_{\mu}, \eta_{\mu+1}], \tag{2.14}
\]

where

\[
\tilde{x} := \frac{2}{\eta_{\mu+1} - \eta_{\mu}} \left( \tilde{x} - \frac{\eta_{\mu+1} + \eta_{\mu}}{\eta_{\mu+1} - \eta_{\mu}} \right).
\]

According to (2.14), we get

\[
u(x_i^\mu) = u(-1) + \int_{-1}^{x_i^\mu} u'(z) dz. \tag{2.15}
\]

Then (2.15) has the following approximation

\[
\zeta(x_i^\mu) = \psi(-1) + \int_{-1}^{x_i^\mu} \rho(z) dz, \quad i = 0, 1, \cdots, N, \quad \mu = 0, 1, \cdots, M. \tag{2.16}
\]

Similarly, we can get

\[
u(\tilde{\theta}(x_i^\mu)) = u(-1) + \int_{-1}^{\tilde{\theta}(x_i^\mu)} u'(z) dz
\]

\[
= \psi(-1) + \sum_{r=0}^{\mu-2} h_r \int_{-1}^{1} u'(z_r(v)) dv + h_{\mu-1} \frac{\tilde{\theta}(x_i^\mu) + 1}{2} \int_{-1}^{1} u'(z_{\mu-1}(z(-1, \tilde{\theta}(x_i^\mu), v))) dv. \tag{2.17}
\]
Then (2.17) has the following approximation:

\[\psi_i^\mu = \psi(-1) + a_2(x_i^\mu), \quad i = 0, 1, \ldots, N, \quad \mu = 0, 1, \ldots, M, \tag{2.18}\]

where

\[a_2(x_i^\mu) = \begin{cases} 
\psi(\bar{\phi}(x_i^0)) - \psi(-1), & \mu = 0, \\
\sum_{r = 0}^{\mu - 2} h_r \sum_{k = 0}^{N} \rho_k^r \omega_k + h_{\mu - 1} \frac{\bar{\phi}(x_i^\mu) + 1}{2} \sum_{j = 0}^{N} \rho_j^{\mu - 1} \sum_{k = 0}^{N} F_j(z(-1, \bar{\phi}(x_i^\mu), v_k)) \omega_k, & \mu > 0.
\end{cases}\]

Similarly, we have

\[u'(x_i^\mu) = u'(-1) + \int_{-1}^{x_i^\mu} u''(z)dz = \chi(-1) + \sum_{r = 0}^{\mu - 1} h_r \int_{-1}^{1} u''(z_r(v))dv + h_{\mu} \frac{x_i + 1}{2} \int_{-1}^{1} u''(z_{\mu}(z(-1, x_i, v)))dv. \tag{2.19}\]

Then (2.19) has the following approximation:

\[\rho(x_i^\mu) = \chi(-1) + a_3(x_i^\mu), \quad i = 0, 1, \ldots, N, \quad \mu = 0, 1, \ldots, M, \tag{2.20}\]

where

\[a_3(x_i^\mu) = \sum_{r = 0}^{\mu - 1} h_r \sum_{k = 0}^{N} \rho_k^r \omega_k + h_{\mu} \frac{x_i + 1}{2} \sum_{j = 0}^{N} \rho_j^\mu \sum_{k = 0}^{N} F_j(z(-1, x_i, v_k)) \omega_k. \tag{2.21}\]

Finally, we have

\[u'(\bar{\phi}(x_i^\mu)) = u'(-1) + \int_{-1}^{\bar{\phi}(x_i^\mu)} u''(z)dz = \chi(-1) + \sum_{r = 0}^{\mu - 2} h_r \int_{-1}^{1} u''(z_r(v))dv + h_{\mu - 1} \frac{\bar{\phi}(x_i^\mu) + 1}{2} \int_{-1}^{1} u''(z_{\mu - 1}(z(-1, \bar{\phi}(x_i^\mu), v)))dv. \tag{2.22}\]

Then (2.22) has the following approximation:

\[\sigma_i^\mu = \chi(-1) + a_4(x_i^\mu), \quad i = 0, 1, \ldots, N, \quad \mu = 0, 1, \ldots, M, \tag{2.23}\]

where

\[a_4(x_i^\mu) = \begin{cases} 
\chi(\bar{\phi}(x_i^0)) - \chi(-1), & \mu = 0, \\
\sum_{r = 0}^{\mu - 2} h_r \sum_{k = 0}^{N} \rho_k^r \omega_k + h_{\mu - 1} \frac{\bar{\phi}(x_i^\mu) + 1}{2} \sum_{j = 0}^{N} \rho_j^{\mu - 1} \sum_{k = 0}^{N} F_j(z(-1, \bar{\phi}(x_i^\mu), v_k)) \omega_k, & \mu > 0.
\end{cases}\]

The other four discrete systems we are looking for are (2.16), (2.20), (2.18) and (2.23). Our method is to solve for \(\bar{\phi}(x_i^\mu), \rho(x_i^\mu)\) and \(\zeta(x_i^\mu)\) \((i = 0, 1, \ldots, N, \mu = 0, 1, \ldots, M)\) through (2.13), (2.16), (2.20), (2.18) and (2.23).

The approximation of \(y(t)\) is \(\zeta(\frac{2}{T}t - 1)\) and the approximation of \(y'(t)\) is \(\frac{2}{T}\rho(\frac{2}{T}t - 1)\), the approximation of \(y''(t)\) is \((\frac{2}{T})^2\sigma(\frac{2}{T}t - 1)\).
3 Some preliminaries and useful lemmas

In this section, we give the lemmas needed in the theoretical proof.

Let \((a, b)\) be a bounded interval on the real number field, and \(\omega(x)\) be the Legendre weight function on \((a, b)\). \(L^2_\omega(a, b)\) is the measurable function space, and its corresponding norm is

\[
\|u\|_{L^2_\omega(a, b)} := \left( \int_a^b |u(x)|^2 \omega(x) \, dx \right)^{1/2}.
\]

\(L^\infty(a, b)\) is the measurable function space, and its corresponding norm is

\[
\|u\|_{L^\infty(a, b)} := \text{ess sup}_{x \in (a, b)} |u(x)|.
\]

Let \(\Lambda_h\) denote the collection of subintervals \(\delta_\mu, \mu = 0, 1, \ldots, M\). We define the broken Sobolev space \(H^m(\Lambda_h)\) as

\[
H^m(\Lambda_h) := \left\{ u : u|_{\delta_\mu} \in H^m(\delta_\mu), \mu = 0, 1, \ldots, M \right\}.
\]

The associated norm is

\[
\|u\|_{H^m(\Lambda_h)} := \left( \sum_{k=0}^m \|u^{(k)}\|_{L^2(\Lambda_h)}^2 \right)^{1/2},
\]

where

\[
\|u^{(k)}\|_{L^2(\Lambda_h)}^2 := \sum_{\mu=0}^M \|\partial_x^k (u|_{\delta_\mu})\|_{L^2(\delta_\mu)}^2, \quad k = 0, 1, \ldots, m.
\]

**Lemma 3.1** ([23]). Let \(0 \leq M < +\infty\). If the non-negative integrable function \(e(x)\) satisfies

\[
e(x) \leq v(x) + M \int_1^x e(z) \, dz,
\]

where \(v(x)\) is also a non-negative integrable function, then

\[
\|e\|_{L^p(-1, 1)} \leq C \|v\|_{L^p(-1, 1)}, \quad p = 2, +\infty.
\]

**Lemma 3.2** ([36]). Let \(u \in C([-1, 1]) \cap H^m(\Lambda_h)\). Define \(I_N\) as the interpolation operator. Note \(h\) as \(\max\{h_\mu : \mu = 0, 1, \ldots, M\}\). Then when \(N \geq m - 1\), we have

\[
\begin{align*}
\|u - I_N u\|_{L^2(-1, 1)} &\leq C h^m N^{-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \\
\|u - I_N u\|_{L^\infty(-1, 1)} &\leq C h^{m-1/2} N^{1/2-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \\
\|I_N\|_{L^\infty(-1, 1)} &\leq C \log(N + 1), \\
\|I_N u\|_{L^2(-1, 1)} &\leq C \|u\|_{L^\infty(-1, 1)}.
\end{align*}
\]
Lemma 3.3 ([11]). Let \( u \in H^m(-1,1), m \geq 1 \) and \( \varphi \in \mathcal{P}_N \), \( \mathcal{P}_N \) be a set of polynomials of all degrees up to \( N \). The existence of a constant \( C \) that has nothing to do with \( N \) makes it possible to have \( N \geq m - 1 \)

\[
\left| \int_{-1}^{1} u(x) \varphi(x) - \sum_{j=0}^{N} u(x_j) \varphi(x_j) \omega_j \right| \leq CN^{-m} \| \partial_x^m u \|_{L^2(-1,1)} \| \varphi \|_{L^2(-1,1)},
\]

where \( x_j \) is the Legendre Gauss-Lobatto quadrature node point, and the corresponding weight function is \( \omega_j, j = 0, 1, \ldots, N \).

Lemma 3.4 ([36]). Let \( u \in H^m(-1,1), m \geq 1 \) and \( \varphi|_{\delta} \in \mathcal{P}_N(\delta \mu) \), that is, \( \varphi \) is a polynomial whose degree does not exceed \( N \) in the interval \( \delta \mu \). When \( N > m - 1 \), there is a constant \( C \) that is unrelated to \( N \) is established as follows

\[
\left| \int_{-1}^{x} u(z) \varphi(z) - S(x) \right| \leq Ch^mN^{-m} \| \partial_x^m u \|_{L^2(-1,x)} \| \varphi \|_{L^2(-1,x)}, \quad x \in \delta \mu,
\]

where

\[
S(x) := \sum_{r=0}^{\mu-1} h_r \sum_{j=0}^{N} u|_{\delta_r}(z_r(v_j)) \varphi|_{\delta_r}(z_r(v_j)) \omega_j + h_\mu \frac{\bar{x} + 1}{2} \sum_{j=0}^{N} u|_{\delta_\mu}(z_\mu(z(-1,v_j))) \varphi|_{\delta_\mu}(z_\mu(z(-1,\bar{x},v_j))) \omega_j,
\]

\[
\bar{x} := \frac{2}{\eta_{\mu+1} - \eta_\mu} x - \frac{\eta_{\mu+1} + \eta_\mu}{\eta_{\mu+1} - \eta_\mu},
\]

\( v_j \) is the Legendre Gauss-Lobatto point, the corresponding weight is \( \omega_j, j = 0, 1, \ldots, N \).

4 Convergence analysis

Theorem 4.1. Let \( u(x) \) be the solution of Eq. (2.2). Use the numerical format (2.13), (2.16), (2.20), (2.18) and (2.23) The approximation of \( u(x) \) is \( \zeta(x) \), the approximation of its derivative is \( \rho(x) \), and the approximation of its second derivative is \( \varphi(x) \). Then for a sufficiently large \( N \geq m - 1 \) there is

\[
\| \xi_i \|_{L^\infty(-1,1)} \leq CN^{1/2-m}R \left( \| u \|_{L^\infty(-1,1)} + \| u' \|_{L^\infty(-1,1)} + \| u^{(m+2)} \|_{L^2(\Lambda_h)} + 1 \right), \quad i = 0, 1, 2,
\]

(4.1)
where

\[ e_0(x) := \begin{cases} 0, & x \in [\hat{\theta}(-1), -1], \\ u(x) - \zeta(x), & x \in (-1, 1], \end{cases} \]

\[ e_1(x) := \begin{cases} 0, & x \in [\hat{\theta}(-1), -1], \\ u'(x) - \rho(x), & x \in (-1, 1], \end{cases} \]

\[ e_2(x) := \begin{cases} 0, & x \in [\hat{\theta}(-1), -1], \\ u''(x) - \varphi(x), & x \in (-1, 1], \end{cases} \]

\( R \) is a constant related only to the \( m \) order derivative of \( S_j(x,z), Q_j(x,z), \psi(z), \chi(z) \) (\( j = 1, 2 \)).

**Proof.** During the derivation of the numerical format part, we can get

\[ u(x_i^\mu) - \zeta(x_i^\mu) = \int_{-1}^{x_i^\mu} e_1(z)dz, \quad u'(x_i^\mu) - \rho(x_i^\mu) = \int_{-1}^{x_i^\mu} e_2(z)dz, \quad (4.2a) \]

\[ u(\theta(x_i^\mu)) - \nu_i^\mu = \int_{-1}^{\theta(x_i^\mu)} e_1(z)dz, \quad u'(\theta(x_i^\mu)) - \sigma_i^\mu = \int_{-1}^{\theta(x_i^\mu)} e_2(z)dz. \quad (4.2b) \]

Subtracting (2.6) from (2.13) gives

\[ u''(x_i^\mu) - \varphi_i^\mu = A_1(x_i^\mu) \int_{-1}^{x_i^\mu} e_1(z)dz + A_2(x_i^\mu) \int_{-1}^{x_i^\mu} e_2(z)dz + B_1(x_i^\mu) \int_{-1}^{\theta(x_i^\mu)} e_1(z)dz + B_2(x_i^\mu) \int_{-1}^{\theta(x_i^\mu)} e_2(z)dz + \int_{-1}^{\theta(x_i^\mu)} Q_1(x_i^\mu, z)e_0(z)dz + \sum_{j=0}^{3} E_j(x_i^\mu), \quad (4.3) \]

where

\[ E_0 := \int_{-1}^{\theta(x)} Q_1(x,z)\zeta(z)dz - \lambda_1(x), \quad x \in [-1, 1], \]

\[ E_1 := \int_{-1}^{\theta(x)} Q_2(x,z)\rho(z)dz - \lambda_2(x), \quad x \in [-1, 1], \]

\[ E_2 := \int_{-1}^{x} S_1(x,z)\zeta(z)dz - \beta_1(x), \quad x \in [-1, 1], \]

\[ E_3 := \int_{-1}^{x} S_2(x,z)\rho(z)dz - \beta_2(x), \quad x \in [-1, 1]. \]
Multiply $F_i^{η}(x)$ by the two ends of (4.3) and add $i=0$ to $N$. Then

$$\sum_{i=0}^{N} u''(x_i^{η}) F_i^{η}(x) - \sum_{i=0}^{N} q_i^{η} F_i^{η}(x) = \sum_{i=0}^{N} \left( A_1(x_i^{η}) \int_{-1}^{x_i^{η}} e_1(z)dz \right) F_i^{η}(x) + \sum_{i=0}^{N} \left( A_2(x_i^{η}) \int_{-1}^{x_i^{η}} e_2(z)dz \right) F_i^{η}(x)$$

$$+ \sum_{i=0}^{N} \left( B_1(x_i^{η}) \int_{-1}^{θ(x_i^{η})} e_1(z)dz \right) F_i^{η}(x) + \sum_{i=0}^{N} \left( B_2(x_i^{η}) \int_{-1}^{θ(x_i^{η})} e_2(z)dz \right) F_i^{η}(x)$$

$$+ \sum_{i=0}^{N} \left( \int_{-1}^{x_i^{η}} S_1(x_i^{η},z)e_0(z)dz \right) F_i^{η}(x) + \sum_{i=0}^{N} \left( \int_{-1}^{x_i^{η}} S_2(x_i^{η},z)e_1(z)dz \right) F_i^{η}(x)$$

$$+ \sum_{i=0}^{N} \left( \int_{-1}^{θ(x_i^{η})} Q_1(x_i^{η},z)e_0(z)dz \right) F_i^{η}(x) + \sum_{i=0}^{N} \left( \int_{-1}^{θ(x_i^{η})} Q_2(x_i^{η},z)e_1(z)dz \right) F_i^{η}(x)$$

$$+ \sum_{j=0}^{3} \sum_{N} E_j(x_i^{η}) F_j^{η}(x), \quad x \in [η_0, η_{N+1}]. \quad (4.4)$$

According to the definition of $I_N$ and $q(x)$, we can get

$$I_N u''(x) - q(x) = I_N \left( A_1(x) \int_{-1}^{x} e_1(z)dz \right) + I_N \left( A_2(x) \int_{-1}^{x} e_2(z)dz \right)$$

$$+ I_N \left( B_1(x) \int_{-1}^{θ(x)} e_1(z)dz \right) + I_N \left( B_2(x) \int_{-1}^{θ(x)} e_2(z)dz \right)$$

$$+ I_N \left( \int_{-1}^{x} S_1(x,z)e_0(z)dz \right) + I_N \left( \int_{-1}^{x} S_2(x,z)e_1(z)dz \right)$$

$$+ I_N \left( \int_{-1}^{θ(x)} Q_1(x,z)e_0(z)dz \right) + I_N \left( \int_{-1}^{θ(x)} Q_2(x,z)e_1(z)dz \right)$$

$$+ \sum_{j=0}^{3} I_N E_j(x), \quad x \in [-1,1]. \quad (4.5)$$

Through (4.5), we can get:

$$e_2(x) = \sum_{j=0}^{3} I_N E_j(x) + \sum_{j=4}^{12} E_j(x) + A_1(x) \int_{-1}^{x} e_1(z)dz + A_2(x) \int_{-1}^{x} e_2(z)dz$$

$$+ B_1(x) \int_{-1}^{θ(x)} e_1(z)dz + B_2(x) \int_{-1}^{θ(x)} e_2(z)dz + \int_{-1}^{x} S_1(x,z)e_0(z)dz$$

$$+ \int_{-1}^{x} S_2(x,z)e_1(z)dz + \int_{-1}^{θ(x)} Q_1(x,z)e_0(z)dz + \int_{-1}^{θ(x)} Q_2(x,z)e_1(z)dz, \quad (4.6)$$
Using Dirichlet formula

\[
E_4(x) := (I - I_N)u''(x), \quad E_5(x) := (I_N - I)A_1(x) \int_{-1}^{x} e_1(z)dz,
\]
\[
E_6(x) := (I_N - I)A_2(x) \int_{-1}^{x} e_2(z)dz, \quad E_7(x) := (I_N - I)B_1(x) \int_{-1}^{\theta(x)} e_1(z)dz,
\]
\[
E_8(x) := (I_N - I)B_2(x) \int_{-1}^{\theta(x)} e_2(z)dz, \quad E_9(x) := (I_N - I) \int_{-1}^{x} S_1(x,z)e_0(z)dz,
\]
\[
E_{10}(x) := (I_N - I) \int_{-1}^{x} S_2(x,z)e_1(z)dz, \quad E_{11}(x) := (I_N - I) \int_{-1}^{\theta(x)} Q_1(x,z)e_0(z)dz,
\]
\[
E_{12}(x) := (I_N - I) \int_{-1}^{\theta(x)} Q_2(x,z)e_1(z)dz.
\]

Using Dirichlet formula

\[
\int_{-1}^{x} \int_{-1}^{\tau} \Phi(\tau,s)dsd\tau = \int_{-1}^{x} \int_{s}^{x} \Phi(\tau,s)d\tau ds. \quad (4.7)
\]

Acting on the right end of (4.6) gives:

\[
A_1(x) \int_{-1}^{x} e_1(z)dz = \int_{-1}^{x} A_1(x)(x-\tau)e_2(\tau)d\tau,
\]
\[
B_1(x) \int_{-1}^{\theta(x)} e_1(z)dz = \int_{-1}^{\theta(x)} B_1(x)(x-\tau)e_2(\tau)d\tau,
\]
\[
\int_{-1}^{x} S_1(x,z)e_0(z)dz = \int_{-1}^{x} \left[ \int_{-1}^{x} S_1(x,z)(z-\tau)dz \right] e_2(\tau)d\tau,
\]
\[
\int_{-1}^{\theta(x)} Q_1(x,z)e_0(z)dz = \int_{-1}^{\theta(x)} \left[ \int_{-1}^{\theta(x)} Q_1(x,z)(z-\tau)dz \right] e_2(\tau)d\tau,
\]
\[
\int_{-1}^{x} S_2(x,z)e_1(z)dz = \int_{-1}^{x} \left[ \int_{-1}^{x} S_2(x,z)dz \right] e_2(\tau)d\tau,
\]
\[
\int_{-1}^{\theta(x)} Q_2(x,z)e_1(z)dz = \int_{-1}^{\theta(x)} \left[ \int_{-1}^{\theta(x)} Q_2(x,z)dz \right] e_2(\tau)d\tau.
\]

This leads to the existence of constants \(C_1, C_2, C > 0\) such that

\[
\left| A_1(x) \int_{-1}^{x} e_1(z)dz + A_2(x) \int_{-1}^{x} e_2(z)dz + B_1(x) \int_{-1}^{\theta(x)} e_1(z)dz + B_2(x) \int_{-1}^{\theta(x)} e_2(z)dz \\
+ \int_{-1}^{x} S_1(x,z)e_0(z)dz + \int_{-1}^{x} S_2(x,z)e_1(z)dz + \int_{-1}^{\theta(x)} Q_1(x,z)e_0(z)dz + \int_{-1}^{\theta(x)} Q_2(x,z)e_1(z)dz \right|
\]
From Lemma 3.1, we can estimate \( e_2(x) \) in (4.5), as follows

\[
\| e_2 \|_{L^\infty(-1,1)} \leq C \left( \sum_{j=0}^{2} \| I_N E_j \|_{L^\infty(-1,1)} + \sum_{j=4}^{12} \| E_j \|_{L^\infty(-1,1)} \right). \tag{4.9}
\]

Next, we estimate the right term of the inequality (4.9).

We first estimate \( \| I_N E_0 \|_{L^\infty(-1,1)} \). Get it from (3.1c)

\[
\| I_N E_0 \|_{L^\infty(-1,1)} \leq C \log(N+1) \| E_0 \|_{L^\infty(-1,1)}. \tag{4.10}
\]

By Lemma 3.3, we can get \((x \in \delta_0)\)

\[
\| E_0(x) \| \leq CN^{-m} \| \partial^m_z \left( Q_1(z(-1,\theta(x),\cdot)) \psi(z(-1,\theta(x),\cdot)) \right) \|_{L^2(-1,1)} \leq CN^{-m} \| \partial^m_z(Q_1(x,z)\psi(z)) \|_{L^2(-1,1)} \leq CN^{-m} \| \partial^m_z(Q_1(x,\cdot)\psi(\cdot)) \|_{L^2(\theta(x),-1)}. \tag{4.11}
\]

By Lemma 3.4, we have \((x \in \delta, \mu > 0)\)

\[
\| E_0 \|_{L^\infty(-1,1)} \leq CN^{-m} \| \partial_z^m(Q_1(x,\cdot)) \|_{L^2(-1,\theta(x))} \| u^N \|_{L^2(-1,1)} \leq CN^{-m} \| \partial_z^m(Q_1(x,\cdot)) \|_{L^2(-1,\theta(x))} \left( \| e_0 \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} \right). \tag{4.12}
\]

Then

\[
\| E_0 \|_{L^\infty(-1,1)} \leq CN^{-m} \overline{Q}_1 \left( \| e_0 \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} \right), \tag{4.13}
\]

where

\[
\overline{Q}_1 := \max \left\{ \max_{x \in \delta_0} \| \partial_z^m(Q_1(x,\cdot)\psi(\cdot)) \|_{L^2(\theta(x),-1)} , \max_{x \in [\eta,1]} \| \partial_z^m(Q_1(x,\cdot)) \|_{L^2(-1,\theta(x))} \right\}.
\]

Combining (4.10) and (4.13) gives

\[
\| I_N E_0 \|_{L^\infty(-1,1)} \leq CN^{-m} \log(N+1) \overline{Q}_1 \left( \| e_0 \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} \right). \tag{4.14}
\]
Using an analysis process similar to estimating $\|I_N E_0\|_{L^\infty(-1,1)}$, we get

\begin{align}
\|I_N E_1\|_{L^\infty(-1,1)} & \leq CN^{-m} \log(N+1) \tilde{Q}_2 \left( \|e_1\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right), \\
\|I_N E_2\|_{L^\infty(-1,1)} & \leq CN^{-m} \log(N+1) \tilde{S}_1 \left( \|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right), \\
\|I_N E_3\|_{L^\infty(-1,1)} & \leq CN^{-m} \log(N+1) \tilde{S}_2 \left( \|e_1\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \right),
\end{align}

where

\begin{align}
\tilde{Q}_2 & := \max \left\{ \max_{x \in \delta_0} \left\| \partial_x^m (Q_2(x,\cdot) \chi(\cdot)) \right\|_{L^2(\delta(x),-1)}, \max_{x \in [\eta,1]} \left\| \partial_x^m (Q_2(x,\cdot)) \right\|_{L^2(-1,\delta(x))} \right\}, \\
\tilde{S}_1 & := \max_{x \in [-1,1]} \left\| \partial_x^m (S_1(x,\cdot)) \right\|_{L^2(-1,\cdot)}, \\
\tilde{S}_2 & := \max_{x \in [-1,1]} \left\| \partial_x^m (S_2(x,\cdot)) \right\|_{L^2(-1,\cdot)}.
\end{align}

Next estimate is $\|E_j\|_{L^\infty(-1,1)}$, $j=4,5,\ldots,12$. Note $u|_{\delta_0}(x) \in H^{m+2}(\delta_0)$, $\mu=0,1,\ldots,M$. Apply (3.1b) to $u(x)$ gets

$$
\|E_4\|_{L^\infty(-1,1)} \leq CN^{1-m} \|u^{(m+2)}\|_{L^2(A_0)},
$$

Now estimate $\|E_{11}\|_{L^\infty(-1,1)}$. For the sake of brevity, let

$$
\gamma(x) := \int_{-1}^{\delta(x)} Q_1(x,z) e_0(z)dz.
$$

When $m = 1$, we can get from (3.1b):

$$
\|(I_N-I)\gamma\|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \|\partial_x^1 \gamma\|_{L^2(-1,1)}.
$$

Obviously, we have

$$
\left| \partial_x^1 \gamma(x) \right| = \left| Q_1(x,\delta(x)) e_0(\delta(x)) \theta'(x) + \int_{-1}^{\delta(x)} \partial_x Q_1(x,z) e_0(z)dz \right|
\leq \|e_0\|_{L^\infty(-1,1)} \left( \left| Q_1(x,\delta(x)) \theta'(x) \right| + \int_{-1}^{\delta(x)} \partial_x Q_1(x,z)dz \right)
\leq C \|e_0\|_{L^\infty(-1,1)}.
$$

Combining (4.18) and (4.19), we can get:

$$
\|E_{11}\|_{L^\infty(-1,1)} = \|(I_N-I)\gamma\|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \|e_0\|_{L^\infty(-1,1)}.
$$
Combining (4.14), (4.31), (4.16), (4.20) and (4.21), we have
\[ \| E_5 \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_1 \|_{L^\infty(-1,1)}, \]
\[ \| E_6 \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_2 \|_{L^\infty(-1,1)}, \]
\[ \| E_7 \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_1 \|_{L^\infty(-1,1)}, \]
\[ \| E_8 \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_2 \|_{L^\infty(-1,1)}, \]
\[ \| E_9 \|_{L^\infty(-1,1)} \leq \| e_0 \|_{L^\infty(-1,1)}, \]
\[ \| E_{10} \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_1 \|_{L^\infty(-1,1)}, \]
\[ \| E_{11} \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \| e_1 \|_{L^\infty(-1,1)}. \]

Similarly, we have
\[ \| e_2 \|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}} \left( \log(N+1) \right) R \left( \| u \|_{L^\infty(-1,1)} + \| u' \|_{L^\infty(-1,1)} \right) + N^{1/2} \| u^{(m+2)} \|_{L^2(A_h)} \]
\[ + CN^{-1/2} \left( \| e_0 \|_{L^\infty(-1,1)} + \| e_1 \|_{L^\infty(-1,1)} \right), \]
where
\[ R := \max \{ \tilde{Q}_1, \tilde{Q}_2, S_1, S_2 \} \]

Now we need two \( \| e_1 \|_{L^\infty(-1,1)}, \| e_0 \|_{L^\infty(-1,1)}, \| e_2 \|_{L^\infty(-1,1)} \) and \( \| e_1 \|_{L^\infty(-1,1)} \) relation.

Multiply both ends of the first term in (4.2) by \( F_i(x) \) and add from \( i = 0 \) to \( N \) to get
\[ e_0(x) = \tilde{E}(x) + (I_N - I) \left( \int_{-1}^{x} e_1(s)ds \right) + \int_{-1}^{x} e_1(s)ds, \]
where
\[ \tilde{E}(x) := (I - I_N)u(x). \]

We have:
\[ \| e_0 \|_{L^\infty(-1,1)} \leq C \left( \| \tilde{E}(x) \|_{L^\infty(-1,1)} + \| (I_N - I)( \int_{-1}^{x} e_1(s)ds \|_{L^\infty(-1,1)} + \| e_1 \|_{L^\infty(-1,1)} \right]. \]

Applying (3.1b) to \( \tilde{E}(x) \), and applying (3.1b) when \( m = 1 \) to the right middle term of the above inequality, we get
\[ \| e_0 \|_{L^\infty(-1,1)} \leq C N^{-m-\frac{1}{2}} \| u^{(m+2)} \|_{L^2(A_h)} + C \| e_1 \|_{L^\infty(-1,1)}. \]

Similarly, for the second term in (4.2) we can get
\[ e_1(x) = \hat{E}(x) + (I_N - I) \left( \int_{-1}^{x} e_2(s)ds \right) + \int_{-1}^{x} e_2(s)ds, \]
where
\[ \hat{E}(x) := (I - I_N)u'(x). \]
This will get
\[ \| e_1 \|_{L^\infty(-1,1)} \leq C N^{-m-\frac{1}{2}} \| u^{(m+2)} \|_{L^2(A_h)} + C \| e_2 \|_{L^\infty(-1,1)}. \]

We can then derive (4.1) by combining the results (4.22), (4.24) and (4.26). \( \square \)
Theorem 4.2. Let \( u(x) \) be the solution of Eq. (2.2). Use the numerical format (2.13), (2.16), (2.20), (2.18), and (2.23) The approximation of \( u(x) \) is \( \varepsilon(x) \), the approximation of its derivative is \( \rho(x) \), and the approximation of its second derivative is \( \varphi(x) \). Then for a sufficiently large \( N \geq m - 1 \) there is

\[
\|e_i\|_{L^2(-1,1)} \leq CN^{-m}(R+1)^2 \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right), \quad i = 0,1,2.
\]

Proof. Using Lemma 3.1, we get from (4.6) and (4.8)

\[
\|e_2\|_{L^2(-1,1)} \leq C \left( \sum_{j=0}^{3} \|I_N E_j\|_{L^2(-1,1)} + \sum_{j=4}^{12} \|E_j\|_{L^2(-1,1)} \right).
\]

Applying \( E_0(x) \) from (3.1d) to (4.28), we get

\[
\|I_N E_0\|_{L^2(-1,1)} \leq C \|E_0\|_{L^\infty(-1,1)}.
\]

By (4.13) and Theorem 4.1, we have

\[
\|I_N E_0\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right),
\]

Similarly, we have

\[
\|I_N E_1\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right),
\]

\[
\|I_N E_2\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right),
\]

\[
\|I_N E_3\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right).
\]

Notice \( u''|_{\delta_\mu}(x) \in H^m(\delta_\mu), \mu = 0,1, \cdots, M. \) Applying (3.1a) to \( u''(x) \), we have

\[
\|E_4\|_{L^2(-1,1)} \leq CN^{-m}\|u^{(m+2)}\|_{L^2(\Lambda_0)}.
\]

Similar to the analysis process of (4.18)-(4.20), applying (3.1a) at \( m = 1 \) to \( \gamma(x) \)

\[
\|E_1\|_{L^2(-1,1)} = \|(I - I_N)\gamma(x)\|_{L^2(-1,1)} \leq CN^{-1}\|\epsilon_0\|_{L^\infty(-1,1)}.
\]

Using the estimate of \( \epsilon_0 \) in Theorem 4.1, we can convert the above inequality to

\[
\|E_1\|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_0)} + 1 \right).
\]
Similarly, we have

\[
\| E_5 \|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \| u \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} + \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + 1 \right),
\]

(4.35a)

\[
\| E_7 \|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \| u \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} + \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + 1 \right),
\]

(4.35b)

\[
\| E_9 \|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \| u \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} + \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + 1 \right),
\]

(4.35c)

\[
\| E_{10} \|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \| u \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} + \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + 1 \right),
\]

(4.35d)

\[
\| E_{12} \|_{L^2(-1,1)} \leq CN^{-m-1/2} R \left( \| u \|_{L^\infty(-1,1)} + \| u \|_{L^\infty(-1,1)} + \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + 1 \right).
\]

(4.35e)

Similarly to the analysis process of (4.18)-(4.20), applying (3.1a) when \( m = 1 \) to get

\[
\| E_i \|_{L^2(-1,1)} \leq CN^{-1} \| e_2 \|_{L^\infty(-1,1)}, \quad i = 6, 8.
\]

(4.36)

Combining (4.28), (4.30), (4.31), (4.32), (4.34), (4.35) and (4.36) gives the estimate of \( e_2 \) in (4.27).

Now to estimate \( e_1 \). Get from (4.25)

\[
\| e_1 \|_{L^2(-1,1)} \leq C \left( \| \hat{E}(x) \|_{L^2(-1,1)} + \left( \| |N - I| \int_{-1}^{x} e_2(s) \, ds \|_{L^2(-1,1)} + \| e_2 \|_{L^2(-1,1)} \right) \right).
\]

Applying (3.1a) to \( \hat{E}(x) \), and applying (3.1a) at \( m = 1 \) to the middle term at the right end of the above inequality

\[
\| e_1 \|_{L^2(-1,1)} \leq CN^{-m-1} \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + C \| e_2 \|_{L^2(-1,1)}.
\]

(4.37)

Substituting the estimate of \( e_2 \) in (4.27) into (4.37) gives the estimate of \( e_1 \) in (4.27).

Similarly, we can get for (4.23)

\[
\| e_0 \|_{L^2(-1,1)} \leq CN^{-m-2} \| u \|^{(m+2)} \| L^2(\Lambda_h) \| + C \| e_1 \|_{L^2(-1,1)}.
\]

(4.38)

Substituting the estimate of \( e_1 \) in (4.27) into (4.38) gives the estimate of \( e_0 \) in (4.27). \( \square \)

5 Numerical results

Example 5.1. Take in Eq. (1.1)

\[
T = 4, \quad a_1(t) = b_1(t) = t, \quad a_2(t) = b_2(t) = t^2,
\]

\[
K_1(t,s) = R_1(t,s) = \sin(t + s), \quad K_2(t,s) = R_2(t,s) = \cos(t + s),
\]

\[
\phi(t) = e^t, \quad \psi(t) = e^t, \quad \theta(t) = t - 1,
\]

\[
g(t) = e^t - te^t - t^2 e^t - te^{t-1} - t^2 e^{t-1} + 2\sin(t) + e^{t} \sin(2t) + e^{t} \cos(2t - 1).
\]
The exact solution of the equation is $y(t) = e^t$, $t \in (0, T]$.

Fig. 1 shows the convergence of the error under the $L^\infty$ and $L^2$ norms at $4 \leq N \leq 20$. The specific error data is shown in Table 1. Error exponential convergence, which is the same as the convergence proof result.

**Example 5.2.** Take in Eq. (1.1)

\[
T = 3, \quad a_1(t) = a_2(t) = 0, \quad b_1(t) = b_2(t) = g(t) = e^t, \\
K_1(t,s) = K_2(t,s) = R_1(t,s) = 0, \quad R_2(t,s) = e^{(t+s)}, \\
\phi(t) = t, \quad \varphi(t) = 1, \quad \theta(t) = \frac{t-1}{2}.
\]

The exact solution of the equation is

\[
y(t) = \begin{cases} 
  te^t + \frac{2}{27}(3t-7)e^{(3t-1)/2} + \frac{5}{9}te^{-1/2} + \frac{14}{27}e^{-1/2}, & t \in (0,1], \\
y_1(t), & t \in (1,3],
\end{cases}
\]
where
\[ y_1(t) = \frac{1}{4} (2t-1)e^{2t} + \frac{1}{16} (t-1)e^{2t-1} + \frac{3}{4} e^t + \frac{4}{675} (15t-41)e^{(5t-1)/2} \]
\[ + \frac{16}{54675} (45t-271)e^{(9t-7)/4} + \frac{20}{81} e^{(2t-2)/2} + \frac{1}{675} (375t-214)e^{(2t-1)/2} \]
\[ - \frac{19}{27} e^2 + \frac{1}{36} e - \frac{250}{243} e^{1/2} + \frac{5}{9} e^{-1/2} + \frac{547}{900} e^2 - \frac{2}{27} e + \frac{764}{2025} e^{1/2} + \frac{31966}{54675} e^{-1/2}. \]

Fig. 2 depicts the error convergence in the sense of $L^\infty$ and $L^2$ norms at $6 \leq N \leq 24$. The specific data errors are shown in Table 2. The error converges exponentially, which is the same as the result of convergence proof.

### 6 Conclusions

This paper presents a numerical method of piecewise spectral-collocation for second-order Volterra integro-differential equations with nonvanishing delay based on Legendre polynomials. The convergence of the method is proved and the numerical experiments also verify the correctness of the proof results.
The focus of this work is to construct the derivative approximation in the sense of integration, which makes it possible to use the Dirichlet formula to prove the convergence of the method in the theoretical proof process. In addition, we make the solution at the main discontinuity point of the Volterra integral differential equation solution. The division of intervals in this method greatly overcomes the disturbance of the main discontinuity of the solution of the equation to the convergence of the numerical error.

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