Dynamic Loadings Induced Vibration of Third Order Shear Deformable FG-CNTRC Beams: Gram-Schmidt-Ritz Method

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Abstract. This research work deals with a study on dynamic behavior of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) beams under various types of dynamic loads. Carbon nanotubes (CNTs) are used as the reinforcing materials that distribute continuously across the beam thickness. By using third order shear deformable theory (TSDT) in this current study, the straightness and normality of the transverse normal after deformation are unconstrained. The equations of motion based on TSDT are solved by Gram-Schmidt-Ritz method in which the displacement functions are generated via Gram-Schmidt procedure. Additionally, the time-integration of Newmark is also employed to carry out dynamic response of the beams under dynamic loads. Several effects such as material distributions, types of dynamic loads, boundary conditions and so on are taken into account. According to numerical results, it can be revealed that adding small amount of CNTs can reduce considerably the dynamic amplitude of FG-CNTRC beams. Moreover, the dynamic analysis of beam-like structures plays an important role in structural design because mass inertia matrix of the beam being involved in the equations of motion, which yields much larger deflection than that predicted by simple static analysis.

AMS subject classifications: 65M10, 78A48

Key words: CNTRC beam, dynamic loads, moving load, gram-schmidt procedure, third order shear deformation theory.

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1 Introduction

One of currently promising materials is carbon nanotubes (CNTs) that exhibit exceptional physical, mechanical, and electrical properties. CNTs have been used in various applications of nanotechnology, electronics, optics and material sciences [1–3]. By using CNTs as reinforcing materials in polymeric matrix, we can receive an advanced class of composite materials which is much better than traditional composites. Recently, high performance structural composites with multiple properties are needed in engineering communities. With help of CNTs, the advanced composite materials are excellent in high stiffness and low density. For example, adding only 1.8 vol% of graphene in poly (vinyl alcohol) composites leads to a 150% enhancement in tensile strength and around 10 times increase in Young’s modulus [4]. However, the critical challenge of producing polymer/CNT composites is how to enhance dispersion and alignment of CNTs in a polymer matrix. Xie et al. [5] reviewed available techniques and recent progress on dispersion and alignment of CNTs in the polymer matrix using ex situ technique, force and magnetic fields, electrospinning and liquid crystalline phase induced methods. The earliest study was found in [6] that presented carbon nanotube-reinforced composites (CNTRCs) made from polymer reinforced by aligned CNT arrays. Since then, some investigations on material properties of CNTRCs were reported in [7–9] and the numerical analysis and experiment of CNTRCs were reviewed by Yengejeh et al. [10].

An idea of functionally graded (FG) materials is applied to create new type of CNTRCs in which the volume fraction of CNTs is changed gradually across the desirable direction. Therefore, the new type of composites is called functionally graded carbon nanotube reinforced composites (FG-CNTRCs). Recent research activities and some highlight topics relevant to FG-CNTRCs were reviewed by Liew et al. [11]. For structural analysis, there exists number of investigations on static bending, buckling and vibration of FG-CNTRC structures [12–15]. Moreover, Rafiee et al. [16] showed the large amplitude vibration of FG-CNTRC beams with piezoelectric layers. Based on their results, it is revealed that natural frequency and the nonlinear to linear frequency ratio increase as the increase of CNT volume fraction. The nonlinear analysis of FG-CNTRC structures has been of interest to many researchers since the past until now. They have investigated nonlinear bending, post-buckling and nonlinear vibration of FG-CNTRC beams, plates and shells [17–24]. By using the incremental harmonic balance (IHB) method, the super harmonic resonances of FG-CNTRC was derived in the study of Zhihua et al. [25]. Karami et al. [26] used second order shear deformation plate theory for analyzing static, stability and vibration of FG-CNTRC plates including size-dependent parameters. With the Donnell-type kinematic assumption, Qin et al. [27] applied Chebyshev polynomials to solve the free vibration problem of rotating FG-CNTRC cylindrical shells. Natural frequencies associated with their mode shapes of circular, annular and sector plates made of FG-CNTRC were presented by Zhong et al. [28] using the Ritz variational energy method. For small-scale structures, it has been proven that the size-dependent parameters play a vital role in mechanical characteristics of such small structures [29]. Finite element
method (FEM) known as powerful tool was employed to solve the mechanical problem of macro-to-nanostructures in several topics, elastodynamics problems [30], nano beams surrounded by an elastic matrix [31], torsional and longitudinal frequency and wave response of microtubules [32], transverse and longitudinal vibration of embedded carbon and silica carbide nanotubes [33], microtubules surrounded by an elastic matrix [34]. The Chebyshev-Ritz method was also applied to solve vibration analysis of nanostructures based on the modified couple stress model and refined beam theory [35].

For recent studies on dynamic analysis of composite structures, free and forced vibrations of functionally graded (FG) porous beams were solved in the study of Chen et al. [36]. They also extended their work to deal with the problems of FG graphene nanoplatelets [37]. Songsuwan et al. [38] presented the dynamic deflections of sandwich FG beams under harmonic moving load with various boundary conditions. Simsek and Al-Shujairi [39] analyzed static bending as well as free and forced vibration of FG sandwich beams under two successive moving harmonic loads. Also, the two masses moving on composite beams reinforced by graphenes were considered in the study of Wang et al. [40] to investigate the dynamic responses of such beams. In [40], different strain functions of shear stress were used to construct displacement field for new high order shear deformation theory. The frequency response curves of FG-CNTRC beams were presented in the study of nonlinear primary and super-harmonic resonances of the beams [41].

In recently mathematical techniques for science and engineering which can be modelled by using ordinary or partial differential equations, the concept and applications of deep collocation method based on Deep Neural Networks (DNNs) were proposed for solving such equations with a precise knowledge of the behavior of natural and engineered systems [42, 43]. This technique can be used to improve solutions in group of discretization including finite element, mesh-free and isogeometric methods. To solve engineering problems representing by energy minimization, it can be achieved by using continuous functions generated by Gram-Schmidt orthogonalization procedure. Therefore, the solutions can be approximated by numerically stable functions which lead to high spectral accuracy and fast convergence [44, 45].

According to above literature, to complete dynamic behavior of FG-CNTRC beams and to fulfill the gap of research, this study aims to investigate dynamic response of the beams under various types of dynamic loads. The equations of motion based on the third order shear deformable theory (TSDT) are derived from Lagrange’s equations. The Gram Schmidt procedure is employed to generate admissible displacement functions for various boundary conditions and the time-integration of Newmark is also adopted to carry out dynamic response of the beams under different types of dynamic loading as well as moving load.

2 FG-CNTRC beam
A straight FG-CNTRC beam made from a mixture of single-walled carbon nanotubes (SWCNTs) and an isotropic polymeric matrix is considered in Fig. 1. The beam has length \( L \) and thickness \( h \) which is subjected to dynamic loading \( P(t) \). As shown in the figure, there are four different patterns of reinforcement over the cross section that can be defined as uniform distribution (UD) and functionally graded distribution (FG-X, -O and –V).

The effective material properties of FG-CNTRC beams can be estimated using the rule of mixture. Therefore, the expressions of the effective Young’s modulus and shear modulus of FG-CNTRC beams are as follows:

\[
E_{11} = \eta_1 \bar{V}_{cnt} E_{11}^{cnt} + \bar{V}_p E^p, \tag{2.1a}
\]
\[
\frac{\eta_2}{E_{22}} = \frac{\bar{V}_{cnt} E_{22}^{cnt}}{E_{22}^p} + \frac{\bar{V}_p}{E^p}, \tag{2.1b}
\]
\[
\frac{\eta_3}{G_{12}} = \frac{\bar{V}_{cnt} G_{12}^{cnt}}{G^{p}} + \frac{\bar{V}_p}{G^p}. \tag{2.1c}
\]

\( E_{11}^{cnt} \), \( E_{22}^{cnt} \) and \( G_{12}^{cnt} \) are defined as the Young’s modulus and shear modulus of SWCNT, respectively, and \( E^p \) and \( G^p \) as the corresponding material properties of the polymer matrix. Also, \( \bar{V}_{cnt} \) and \( \bar{V}_p \) are the volume fractions for carbon nanotube and the polymer matrix, respectively, with the relation of

\[
\bar{V}_{cnt} + \bar{V}_p = 1, \tag{2.2}
\]

To consider the size-dependent material properties of SWCNT, the CNT efficiency parameters, \( \eta_i, \ (i=1,2,3) \), are introduced. They can be determined from matching the elastic
moduli of CNTRCs estimated by the molecular dynamic (MD) simulation with the numerical results estimated by the rule of mixture. By using the same rule, Poisson’s ratio ($\nu$) and mass density ($\rho$) of the CNTRC beams are written as:

\[
\nu = \bar{V}_{cnt}\nu^{cnt} + \bar{V}_p\nu^p, \quad (2.3a)
\]
\[
\rho = \bar{V}_{cnt}\rho^{cnt} + \bar{V}_p\rho^p, \quad (2.3b)
\]

where $\nu^{cnt}$, $\nu^p$ and $\rho^{cnt}$, $\rho^p$ are the Poisson’s ratios and densities of the SWCNT and polymeric matrix, respectively. For different patterns of CNT reinforcement distributed across the cross section, the continuous mathematical functions used for describing the distributions of material constituents are given below:

**UDBeam:**

\[
\bar{V}_{cnt} = V_{cnt}, \quad (2.4a)
\]

**FG-OBeam:**

\[
\bar{V}_{cnt} = 2 \left( 1 - 2 \frac{|z|}{h} \right) V_{cnt}, \quad (2.4b)
\]

**FG-XBeam:**

\[
\bar{V}_{cnt} = 4 \frac{|z|}{h} V_{cnt}, \quad (2.4c)
\]

**FG-VBeam:**

\[
\bar{V}_{cnt} = \left( 1 + \frac{2z}{h} \right) V_{cnt}, \quad (2.4d)
\]

where $V_{cnt}$ is the given volume fraction of CNTs, which can be obtained from the following equation [3, 4]:

\[
V_{cnt} = \frac{W_{cnt}}{W_{cnt} + \left( \frac{\rho^{cnt}}{\rho^p} \right) \left( 1 - W_{cnt} \right)}, \quad (2.5)
\]

where $W_{cnt}$ is the mass fraction of CNTs. From Eq. (2.4), it can be defined that FG-O-, X- and V-beams are functionally graded beams in which their material constituents are varied continuously across the cross section; while, the UD-beam has uniformly distributed carbon nanotube reinforcement. In this study, we use the CNT efficiency parameters ($\eta_i$) associated with the given volume fraction ($V_{cnt}$) from [46], which are: $\eta_1 = 1.2833$ and $\eta_2 = \eta_3 = 1.0556$ for the case of $V_{cnt} = 0.12$; $\eta_1 = 1.3414$ and $\eta_2 = \eta_3 = 1.7101$ for the case of $V_{cnt} = 0.17$; $\eta_1 = 1.3238$ and $\eta_2 = \eta_3 = 1.7380$ for the case of $V_{cnt} = 0.28$. These parameters are used to take into account load transfer between CNTs and polymer matrix covering surface, strain gradient, intermolecular coupling and other effects. Their values are determined from matching the Young’s modulus $E_{11}$ and $E_{22}$ obtained by the rule of mixture to the molecular dynamics simulation [47].

However, all material property models from Eqs. (2.1a)-(2.5) are based on the assumption that CNTs spread continuously within the polymer matrix according to Eq. (2.4) without considering any effect of non-uniformly dispersed or agglomerated CNTs [48] and fracture properties at the interphase zone between CNTs and polymer matrix [49].
3 Governing equations

In accordance with the TSDT of Reddy [50], the displacements of an arbitrary point of the beams along the \(x\) - and \(z\)-axes, which are denoted by \(u(x,z)\), \(w(x,z)\), respectively, can be expressed as follows:

\[
\begin{align*}
    u(x,z,t) &= u_0(x,t) + z\psi(x,t) - \frac{4}{3h^2}z^3 \left(\psi + \frac{\partial w_0}{\partial x}\right), \\
    w(x,z,t) &= w_0(x,t),
\end{align*}
\]

where \(u_0\) and \(w_0\) are the mid-plane displacements, \(\psi\) is the rotation of cross-section and \(t\) is time. Based on the displacements in Eq. (3.1), the normal strain \(\varepsilon_{xx}\) and shear strain \(\gamma_{xz}\) components are given by

\[
\begin{align*}
    \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^2\varepsilon_{xx}^{(3)}, \\
    \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{xz}^{(0)} + z\gamma_{xz}^{(2)},
\end{align*}
\]

in which

\[
\begin{align*}
    \left\{ \begin{array}{c}
    \varepsilon_{xx}^{(0)} \\
    \varepsilon_{xx}^{(1)} \\
    \varepsilon_{xx}^{(3)}
    \end{array} \right\} & = \left\{ \begin{array}{c}
    \frac{\partial u_0}{\partial x} \\
    \frac{\partial \psi}{\partial x} \\
    \frac{\partial^2 w_0}{\partial x^2}
    \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c}
    \gamma_{xz}^{(0)} \\
    \gamma_{xz}^{(2)}
    \end{array} \right\} = \left\{ \begin{array}{c}
    \psi + \frac{\partial w_0}{\partial x} \\
    \frac{\partial^2 w_0}{\partial x^2}
    \end{array} \right\}.
\end{align*}
\]

The corresponding normal stress \(\sigma_{xx}\) and shear stress \(\sigma_{xz}\) can be obtained from the elastic constitutive law as [46]:

\[
\sigma_{xx} = \frac{E(z)}{(1-\nu_{12}\nu_{21})}\varepsilon_{xx} \quad \text{and} \quad \sigma_{xz} = G_{12}(z)\gamma_{xz}.
\]

Note that \(\nu\) in Eq. (2.3a) is \(\nu_{12}\) and \(\nu_{21}\) is obtained from \(\nu_{21} = (E_{22}/E_{11})\nu_{21}\).

The strain energy of FG-CNTRC beams is defined as follows:

\[
U_s = \frac{b}{2} \int_{-L/2}^{L/2} \int_{-h/2}^{h/2} \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{xz} \gamma_{xz} \right) dz dx.
\]

Inserting the stress and strain components of Eqs. (3.2) and (3.4) into the strain energy
equation of (3.5), we have

\[
U_s = \frac{b}{2} \int_{-L/2}^{L/2} \left[ A_{11} \left( \frac{\partial u_0}{\partial x} \right)^2 + 2B_{11} \frac{\partial u_0}{\partial x} \frac{\partial \psi}{\partial x} + D_{11} \left( \frac{\partial \psi}{\partial x} \right)^2 - 2c_1 E_{11} \frac{\partial u_0}{\partial x} \frac{\partial \psi}{\partial x} 
- 2c_1 E_{11} \frac{\partial^2 w_0}{\partial x^2} - 2c_1 F_{11} \left( \frac{\partial \psi}{\partial x} \right)^2 - 2c_1 F_{11} \frac{\partial \psi}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + c_1^2 H_{11} \left( \frac{\partial \psi}{\partial x} \right)^2 
+ 2c_1^2 H_{11} \frac{\partial \psi}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + c_1^2 H_{11} \left( \frac{\partial^2 w_0}{\partial x^2} \right)^2 + A_{55} \left( \psi + \frac{\partial w_0}{\partial x} \right)^2 
- 2c_2 D_{55} \left( \psi + \frac{\partial w_0}{\partial x} \right)^2 + F_{55} c_2 \left( \psi + \frac{\partial w_0}{\partial x} \right)^2 \right] dx,
\]

where

\[
\{ A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11} \} = b \int_{-h/2}^{h/2} \frac{E(z)}{(1-v_1^2 v_2^2)} \{ 1, z, z^2, z^3, z^4, z^6 \} dz,
\]

\[
\{ A_{55}, D_{55}, F_{55} \} = b \int_{-h/2}^{h/2} G_{12}(z) \{ 1, z^2, z^4 \} dz,
\]

which are defined as the material stiffness components. The theoretical constants of TSDT are \( c_1 = 4/3 h^2 \) and \( c_2 = 4/h^2 \).

For vibration analysis, the kinetic energy of the beams is

\[
U_k = \frac{b}{2} \rho(z) \int_{-L/2}^{L/2} \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dz dx.
\]

Again inserting displacement field from Eq. (3.1) into the kinetic energy equation of (3.7), we have

\[
U_k = \frac{b}{2} \int_{-L/2}^{L/2} \left[ I_0 \left( \frac{\partial u_0}{\partial t} \right)^2 + \left( \frac{\partial w_0}{\partial t} \right)^2 \right] + 2I_1 \frac{\partial u_0}{\partial t} \frac{\partial \psi}{\partial t} + 2I_2 \left( \frac{\partial \psi}{\partial t} \right)^2 
- 2c_1 I_3 \frac{\partial u_0}{\partial t} \frac{\partial \psi}{\partial t} - 2c_1 I_3 \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial x \partial t} - 2c_1 I_4 \left( \frac{\partial \psi}{\partial t} \right)^2 
- 2c_1 I_4 \left( \frac{\partial \psi}{\partial x} \frac{\partial^2 w_0}{\partial t} + c_1^2 I_6 \left( \frac{\partial \psi}{\partial t} \right)^2 + 2c_1^2 I_6 \frac{\partial \psi}{\partial t} \frac{\partial^2 w_0}{\partial x \partial t} + c_1^2 I_6 \left( \frac{\partial^2 w_0}{\partial x \partial t} \right)^2 \right] dx,
\]

where

\[
\{ I_0, I_1, I_2, I_3, I_4, I_6 \} = b \int_{-h/2}^{h/2} \rho(z) \{ 1, z, z^2, z^3, z^4, z^6 \} dz
\]

are the inertia coefficients.
In addition, for forced vibration analysis, the work done ($U_{ex}$) by external fixed point load is $U_{ex} = P_0 w_0 (\tilde{x}_p)$, where $P_0$ is the load magnitude and $\tilde{x}_p$ indicates the position of the load on the top surface of the beams. And the work done by moving load is

$$U_{ex} = \int_{-L/2}^{L/2} P_0 \delta(x - x_p) w_0(x,t) dx,$$

in which $x_p = -L/2 + vt$, other domains are $-L/2 \leq x_p \leq L/2$ and $0 \leq t \leq L/v$, where $\delta(\cdot)$ is the Dirac delta function and $v$ is the velocity of the moving load.

From all energies described above, we can use them to create the total energy ($\Pi$) for the beam system. This total energy is obtained from the balance of elastic strain energies stored in deformed bodies, the kinetic energy and the work done by applied force. Therefore, the total energy can be expressed as follows:

$$\Pi = U_s - U_k - U_{ex}. \quad (3.9)$$

To solve the total energy in Eq. (3.9), we can use the Ritz method with admissible displacement functions generated by Gram-Schmidt procedure, which must satisfy at least the essential or geometric boundary conditions. In order to consider FG-CNTRC beams with different boundary conditions such as clamped (C) having geometric boundary conditions ($u_0 = w_0 = \psi = 0$) and hinged (H) having geometric boundary conditions ($u_0 = w_0 = 0, \psi \neq 0$) at any end of the beams, the displacement functions are

$$\begin{cases}
  u_0(x,t) = \sum_{j=1}^{J} A_j(t) N_u^u(x), \\
  w_0(x,t) = \sum_{j=1}^{J} B_j(t) N_u^w(x), \\
  \psi(x,t) = \sum_{j=1}^{J} C_j(t) N_u^\psi(x),
\end{cases} \quad (3.10)$$

where $N_u^u(x), N_u^w(x)$ and $N_u^\psi(x)$ are shape functions generated by the Gram-Schmidt procedure. Conveniently, $N_u^\alpha$ is used in which $\alpha = u, w, \psi$. The main reason for selecting Gram-Schmidt procedure to generate the displacement functions is that this procedure allows us to obtain numerically stable functions to be used in vibration analysis of such novel composite beams via the Ritz method. It is also proven that the Gram-Schmidt-Ritz method can provide a high spectral accuracy and fast convergence than other methods such as Galerkin, finite element and differential quadrature methods [51]. In general, for modern structural analysis including several prime inputs, sensitivity analysis (SA) approach presented by Vu-Bac et al. [52] may be recommended to improve and quantify the effects of correlated input parameters on model outputs.

By using the first term of three displacements ($N_u^a$) of Eq. (3.10) that satisfy the geometric boundary condition. The subsequent terms can be obtained from the following
Table 1: First terms of shape functions for different boundary conditions.

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>( N_2^0(x) )</th>
<th>( N_3^0(x) )</th>
<th>( N_4^0(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-C</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
</tr>
<tr>
<td>C-C</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
</tr>
<tr>
<td>C-H</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
</tr>
<tr>
<td>H-H</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>((\frac{1}{2} + \frac{x}{L}))</td>
<td>1</td>
</tr>
</tbody>
</table>

procedure:

\[
N_j^0(x) = [x - \tilde{\beta}_j] N_j^a(x), \quad (3.11a)
\]

\[
N_j^0(x) = [x - \tilde{\beta}_{j-1}] N_{j-1}^a(x) - \delta_{j-2} N_{j-2}^a(x), \quad j > 2, \quad (3.11b)
\]

where

\[
\tilde{\beta}_{j-1} = \frac{\int_{-L/2}^{L/2} x \left( N_{j-1}^a(x) \right)^2 dx}{\int_{-L/2}^{L/2} (N_{j-1}^a(x))^2 dx} \quad \text{and} \quad \delta_{j-2} = \frac{\int_{-L/2}^{L/2} x \left( N_{j-1}^a(x) N_{j-2}^a(x) \right) dx}{\int_{-L/2}^{L/2} (N_{j-1}^a(x))^2 dx}. \quad (3.12)
\]

For a beam problem with different boundary conditions, the first terms of all displacements are given in Table 1. The two letter notations represent boundary conditions at both ends of the beam, for example, C-C is the notation of beam clamped at both ends while H-H is the hinged beam.

These shape functions are expanded to suitable number of terms \( J \) which can find from convergence study. Inserting the admissible displacement functions written above into the total potential energy of Eq. (3.9) and then following the Lagrange equation method

\[
\frac{\partial \Pi}{\partial q_j} - \frac{d}{dt} \frac{\partial \Pi}{\partial \dot{q}_j} = 0, \quad (3.13)
\]

with \( q_j \) representing the time-dependent unknown parameters \( (A_j(t), B_j(t), C_j(t)) \), one can obtain the following equation of motion:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
+ \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{A} \\
\dot{B} \\
\dot{C}
\end{bmatrix}
= \begin{bmatrix}
0 \\
F \\
0
\end{bmatrix}. \quad (3.14)
\]

It is noted that the over-dot stands for the derivative with respective to time, \( K \) and \( M \) are the stiffness and mass matrices, respectively, in which their size is \((3J \times 3J)\). Additionally, \( F \) is the vector of dynamic force due to the dynamic or moving load. The matrix elements
in Eq. (3.14) are given by

\[
K_{jm}^{11} = A_{11} \int_{-L/2}^{L/2} \partial N_j^{\mu} \partial N_m^{\mu} dx, \quad K_{jm}^{12} = -c_1 E_{11} \int_{-L/2}^{L/2} \partial N_j^{\mu} \partial^2 N_m^{\mu} dx, \tag{3.15a}
\]

\[
K_{jm}^{13} = \int_{-L/2}^{L/2} \left( B_{11} \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} - c_1 E_{11} \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} \right) dx, \tag{3.15b}
\]

\[
K_{jm}^{21} = -c_1 E_{11} \int_{-L/2}^{L/2} \frac{\partial^2 N_j^{\mu}}{\partial x^2} \frac{\partial N_m^{\mu}}{\partial x} dx, \tag{3.15c}
\]

\[
K_{jm}^{22} = \int_{-L/2}^{L/2} \left( c_1^2 H_{11} \frac{\partial^2 N_j^{\mu}}{\partial x^2} \frac{\partial^2 N_m^{\mu}}{\partial x^2} + A_{55} \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} - 2c_2 D_{55} \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} \right) dx, \tag{3.15d}
\]

\[
K_{jm}^{23} = \int_{-L/2}^{L/2} \left( c_1^2 H_{11} \frac{\partial^2 N_j^{\mu}}{\partial x^2} \frac{\partial^2 N_m^{\mu}}{\partial x^2} - c_1 F_{11} \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} + A_{55} \frac{\partial N_j^{\mu}}{\partial x} N_m^{\phi} 
- 2c_2 D_{55} \frac{\partial N_j^{\mu}}{\partial x} N_m^{\phi} + c_2 F_{55} \frac{\partial N_j^{\mu}}{\partial x} N_m^{\phi} \right) dx, \tag{3.15e}
\]

\[
K_{jm}^{31} = \int_{-L/2}^{L/2} \left( B_{11} \frac{\partial N_j^{\phi}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} - c_1 E_{11} \frac{\partial N_j^{\phi}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} \right) dx, \tag{3.15f}
\]

\[
K_{jm}^{32} = \int_{-L/2}^{L/2} \left( c_1^2 H_{11} \frac{\partial^2 N_j^{\phi}}{\partial x^2} \frac{\partial^2 N_m^{\mu}}{\partial x^2} - c_1 F_{11} \frac{\partial N_j^{\phi}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} + A_{55} \frac{\partial N_j^{\phi}}{\partial x} N_m^{\phi} 
- 2c_2 D_{55} \frac{\partial N_j^{\phi}}{\partial x} N_m^{\phi} + c_2 F_{55} \frac{\partial N_j^{\phi}}{\partial x} N_m^{\phi} \right) dx, \tag{3.15g}
\]

\[
K_{jm}^{33} = \int_{-L/2}^{L/2} \left( D_{11} \frac{\partial N_j^{\mu}}{\partial x} + c_1 F_{11} \frac{\partial N_j^{\phi}}{\partial x} \frac{\partial N_m^{\phi}}{\partial x} + c_1^2 H_{11} \frac{\partial N_j^{\phi}}{\partial x} \frac{\partial N_m^{\phi}}{\partial x} \right) dx, \tag{3.15h}
\]

and

\[
M_{jm}^{11} = I_0 \int_{-L/2}^{L/2} N_j^{\mu} N_m^{\mu} dx, \tag{3.16a}
\]

\[
M_{jm}^{12} = -c_1 I_3 \int_{-L/2}^{L/2} N_j^{\mu} \frac{\partial N_m^{\mu}}{\partial x} dx, \tag{3.16b}
\]

\[
M_{jm}^{13} = I_1 \int_{-L/2}^{L/2} \left( I_1 N_j^{\mu} N_m^{\phi} - c_1 I_3 N_j^{\mu} N_m^{\phi} \right) dx, \tag{3.16c}
\]

\[
M_{jm}^{21} = -c_1 I_3 \int_{-L/2}^{L/2} \frac{\partial N_j^{\mu}}{\partial x} N_m^{\mu} dx, \tag{3.16d}
\]

\[
M_{jm}^{22} = I_0 N_j^{\mu} N_m^{\mu} + c_1^2 I_6 \frac{\partial N_j^{\mu}}{\partial x} \frac{\partial N_m^{\mu}}{\partial x} \right) dx, \tag{3.16e}
\]

\[
M_{jm}^{23} = -c_1 I_3 \int_{-L/2}^{L/2} \left( c_1^2 I_6 \frac{\partial N_j^{\mu}}{\partial x} N_m^{\phi} - c_1 I_4 \frac{\partial N_j^{\mu}}{\partial x} N_m^{\phi} \right) dx, \tag{3.16f}
\]
\[ M_{31}^{jm} = \int_{-L/2}^{L/2} \left( I_1 N_j^m N_m^j - c_1 I_2 N_j^m N_m^j \right) dx, \quad (3.16g) \]
\[ M_{32}^{jm} = \int_{-L/2}^{L/2} \left( c_2 I_6 N_j^m \frac{\partial N_m^j}{\partial x} - c_1 I_4 N_j^m \frac{\partial N_m^j}{\partial x} \right) dx, \quad (3.16h) \]
\[ M_{33}^{jm} = \int_{-L/2}^{L/2} \left( I_2 N_j^m N_m^j - 2c_1 I_4 N_j^m N_m^j + c_1^2 I_6 N_j^m N_m^j \right) dx, \quad (3.16i) \]

and
\[ F_j = P(t) N_j^m (\tilde{x}_p). \quad (3.17) \]

The equation of motion in Eq. (3.14) can be solved in time domain by using the average acceleration method of Newmark.

For free vibration analysis with harmonic phenomenon, it is assumed that the unknown vector \( [A_j(t), B_j(t), C_j(t)]^T \) for \( j = 1, 2, \ldots, J \) are expressed as
\[
\begin{bmatrix}
A_j(t) \\
B_j(t) \\
C_j(t)
\end{bmatrix} =
\begin{bmatrix}
A_j e^{i\omega t} \\
B_j e^{i\omega t} \\
C_j e^{i\omega t}
\end{bmatrix},
\] \( (3.18) \)
in which \( i = \sqrt{-1} \) and \( \omega \) is natural frequency. Substituting Eq. (3.18) into Eq. (3.14) without considering any force vector, we can obtain an eigenvalue equation for free vibration problem as
\[
\begin{bmatrix}
K^{11} & K^{12} & K^{13} \\
K^{21} & K^{22} & K^{23} \\
K^{31} & K^{32} & K^{33}
\end{bmatrix}
- \omega^2
\begin{bmatrix}
M^{11} & M^{12} & M^{13} \\
M^{21} & M^{22} & M^{23} \\
M^{31} & M^{32} & M^{33}
\end{bmatrix}
\begin{bmatrix}
\bar{A} \\
\bar{B} \\
\bar{C}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}. \quad (3.19) \]
Solving Eq. (3.19) as standard eigenvalue problem leads to a set of natural frequencies of FG-CNTRC beams in which the lowest eigenvalue is the fundamental frequency of the beams.

### 4 Numerical results

#### 4.1 Validation

By using Gram Schmidt to generate the shape functions in displacement field, it is important to carry out convergence study. Computations have been carried out for FG-CNTRC beams with breadth \( b = 0.5 \text{m} \) and thickness \( h = 1.0 \text{m} \) and the material properties of the beams are composed of CNTs having \( E_{\text{cnt}}^{11} = 5646.6 \text{GPa}, E_{\text{cnt}}^{22} = 7080 \text{GPa}, G_{\text{cnt}}^{12} = 1944.5 \text{GPa}, \)
\( \nu_{\text{cnt}}^{12} = 0.175 \) and \( \rho_{\text{cnt}} = 2100 \text{kg/m}^3 \) and polymeric matrix having \( E^{\text{p}} = 2.5 \text{GPa}, \nu^{\text{p}} = 0.3 \) and \( \rho^{\text{p}} = 1190 \text{kg/m}^3 \), which are used throughout this paper. The frequency results are presented in non-dimensional form of \( \tilde{\omega} = \omega L^2 \sqrt{\rho^{\text{p}} / (E^{\text{p}} h^2)} \). In Table 2, the convergent rates
Table 2: Convergence study on natural frequency ($\omega$) of FG-X beam hinged at both ends.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>17.8105</td>
<td>47.5052</td>
<td>265.9612</td>
<td>292.1336</td>
<td>446.6755</td>
</tr>
<tr>
<td>4</td>
<td>16.9093</td>
<td>42.9789</td>
<td>79.5648</td>
<td>162.4784</td>
<td>264.2235</td>
</tr>
<tr>
<td>6</td>
<td>16.9068</td>
<td>42.8828</td>
<td>70.0746</td>
<td>108.0695</td>
<td>186.2388</td>
</tr>
<tr>
<td>8</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7086</td>
<td>98.8314</td>
<td>133.6986</td>
</tr>
<tr>
<td>10</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7039</td>
<td>98.1692</td>
<td>129.2584</td>
</tr>
<tr>
<td>12</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7039</td>
<td>98.1514</td>
<td>129.0206</td>
</tr>
<tr>
<td>14</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7039</td>
<td>98.1512</td>
<td>129.0206</td>
</tr>
<tr>
<td>15</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7039</td>
<td>98.1512</td>
<td>129.0206</td>
</tr>
<tr>
<td>*</td>
<td>16.9068</td>
<td>42.8821</td>
<td>69.7038</td>
<td>98.1513</td>
<td>129.0220</td>
</tr>
</tbody>
</table>

*–Results of [46].

Table 3: Fundamental frequency ($\omega$) of UD and FG-X beams with different boundary conditions ($L/h=12$).

<table>
<thead>
<tr>
<th>B.C.</th>
<th>Source</th>
<th>UD-Beam</th>
<th>FG-X Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V=0.12$</td>
<td>$V=0.17$</td>
<td>$V=0.28$</td>
</tr>
<tr>
<td>[46]</td>
<td></td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

are shown for the 1st to 5th modes of natural frequencies of FG-X beams containing 0.17 of CNT volume fraction ($V_{cnt}=0.17$) and the beam thickness ratio ($L/h=12$) is considered in this table. As observed, $J=15$ can provide convergent results for all considered modes of vibration. It is clearly seen that using Gram Schmidt functions can achieve excellent results as compared to available results in [46].

In addition, the FG-CNTRC beams with uniform distribution of CNTs (UD-beams) and FG-X beams are chosen to be analyzed in Table 3. In this table, the fundamental frequency results of the beams are presented with different boundary conditions (H-H, C-C and C-H) and the values of CNT volume fraction are varied. For the cases of H-H and C-C boundary conditions, the frequency results are validated by comparing with the previous results of [46]. There exists excellent agreement between the results. New frequency results for the case of C-H boundary condition are also included in this table.

4.2 Dynamic response of CNTRC beam under different dynamic loads

After convergent study and validation of our modeling, the dynamic behavior of FG-CNTRC under different types of dynamic loadings is considered in this section. The time-integration technique of Newmark is used to solve the time-dependent unknown parameters and then the dynamic deflection can be computed. The amplitude of loadings
is set as $P_0 = 5\text{kN}$ throughout this paper and the dynamic deflection is presented in the form of normalized deflection \( \left( w(x,t)/w_s \right) \), where $w_s$ is the static deflection of simply supported isotropic beams made of polymeric material, $w_s = \frac{P_0 L^4}{48EI}$. Four types of dynamic loadings are applied to the beam, as shown in Fig. 2, such as heaviside step loading in Fig. 2(a), half-cycle sine loading in Fig. 2(b), transient loading with a linear decreasing in Fig. 2(c) and exponential decay loading Fig. 2(d).

In Fig. 3, the normalized dynamic deflection measured at the mid-span ($x = 0$) is presented for FG-CNTRC beams with different patterns of reinforcement. The beams are hinged at both ends (H-H). It is clearly seen that FG-O beam is the weakest one having very large deflection compared to that of FG-V, UD and FG-X beams, respectively.

Collecting data from Fig. 3 in time domain, one can obtain the amplitude in frequency domain by using Fast Fourier Transform (FFT). The representation in the frequency domain of the beams considered in Fig. 3 is shown in Fig. 4. It is observed that the peak of the amplitude in each small graph shown in Fig. 4 occurs at the frequency closed to nat-
Figure 3: Normalized dynamic deflection of FG-CNTRC beams under Haveside step loading: Effect of pattern of reinforcement \( V_{cnt} = 0.12, \frac{L}{h} = 10 \).

Figure 4: The representation in the frequency domain of the beams considered in Fig. 3.

Natural frequency of the beam. For example, the peak amplitude of FG-O beam is at 21.1Hz, which is very close to its natural frequency of the beam (21.4Hz).
Another extensive study presented in Fig. 5 is to consider the effect of boundary conditions for FG-O beams under heaviside step loading. As can be seen, the fully clamped beam (C-C) has the smallest dynamic deflection compared to that of others with different supports.

To consider the significant differences between static and dynamic deflections of FG-CNTRC beams under the same loading magnitude acting at the mid-span of the beams, Figs. 6 and 7 compare such deflections in relation to the effects of pattern of reinforcement and CNT volume fraction, respectively. The static deflections are computed by excluding
the mass inertia terms, setting $U_k = 0$, then the static bending problem is solved. For dynamic deflections, the results are collected from the maximum amplitude in dynamic analysis which are measured from different positions along the beam-span. According to these comparisons, the dynamic analysis is necessary for designing the beams under time dependent loading which cannot simplify the problem to be static analysis. Moreover, the patterns of reinforcement and CNT volume fraction are the key factors for reducing the gap between static and dynamic analyses. For instance, the minimum gap between the
analyses is found in the case of FG-X beam and increasing the CNT volume fraction leads to the considerable reduction of the gap.

Fig. 8 considers the dynamic deflection of hinged FG-CNTRC beams subjected to half-cycle sine loading. Similarly, the deflection of FG-O beam is high as compared to that of others. Within the duration of force action ($t = 0s - 0.5s$), there is a significant increase of dynamic deflection for every beam and, in free vibration zone of $t = 0.5s - 1.0s$, the slight change of the deflection is observed. In Fig. 9, the effect of boundary conditions is investigated for the case of FG-O beam under the half-cycle sine loading. The lowest deflection is obtained for the case of the beam clamped at both ends (C-C) throughout the considered time.

Again, collecting data from Fig. 9 to transform the time domain into the frequency domain, the results of the transformation using FFT are presented in Fig. 10 for the beams considered in Fig. 9.

Next, the transient loading with a finite decreasing is applied to FG-CNTRC beams in Figs. 11 and 12, in which the effects of pattern of reinforcement and boundary conditions are examined respectively. The dynamic deflections of FG-CNTRC beams with different patterns of reinforcement are plotted in Fig. 11 for the beams with H-H boundary condition. As can be seen, the deflections reduce linearly according to the linear decrease of loading at the forced vibration zone ($t=0s - 0.5s$). After that period of time, the beams are in the free vibration zone without any dynamic excitation of external force. Additionally, the significant effect of boundary conditions for FG-O beam under the transient loading is investigated and presented in Fig. 12.

The exponential loading is used to apply onto the FG-CNTRC beams in Figs. 13 and 14. The beams are hinged at both ends (H-H). The dynamic deflection reduces contin-
Figure 10: The representation in the frequency domain of the beams considered in Fig. 9.

Figure 11: Normalized dynamic deflection of FG-CNTRC beams under transient loading; Effect of pattern of reinforcement ($V_{cnt}=0.12$, $/h=20$).

uously which is dependent on value of exponential function of the external excitation in the forced vibration zone ($t=0s-0.5s$). The beams with different patterns of reinforcement are chosen to consider in Fig. 13 for investigating their dynamic behavior. The
influence of length to height ratio or beam thickness ratio is studied in Fig. 14 by varying the ratio values. The beam with low thickness ratio is thick beam which has less dynamic deflection through the time.

Fig. 15 compares dynamic deflections of FG-X beams supported by hinged boundary conditions (H-H) under four different types of external dynamic loadings. The beams are...
4.3 Dynamic response of CNTRC beam under moving load

In this section, we assume that the load \(P_0\) moves on the CNTRC beam hinged at both ends with constant velocity \(v\), as shown in Fig. 16. The significant effects of pattern and proportion of reinforced materials and velocity of the moving load are considered and used as the main parameters for this dynamic investigation of the beam.

To investigate the influence of patterns of reinforcement on dynamic behavior, the time history of the normalized deflection at the center of FG-CNTRC beams subjected to the moving load is illustrated in Fig. 17. This figure shows the maximum deflection time-history versus the normalized time \(t^* = vt/L\) for the case of FG-O beam and the minimum one occurring in the case of FG-X beam. Additionally, Fig. 18 shows the time history of the transverse deflection at the mid-span of hinged FG-X beams composed of different amount of CNTs. An increase in volume fraction of CNTs leads to dramatically decrease of dynamic deflection throughout the time-history. Therefore, it is evident that the dynamic deflection of FG-CNTRC beams is greatly influenced by amount of CNTs added inside the beams.

In Fig. 19, the constant velocity of the moving load is varied from 0m/s to 300m/s. The maximum dynamic deflections of hinged FG-CNTRC beams under moving loads with different velocities are plotted in this figure. As illustrated, the maximum deflection of FG-O beam is very high compared to that of other beams for every value of load veloc-
Figure 15: Normalized dynamic deflection of FG-CNTRC beams under different loadings (a) heaviside step loading, (b) half-cycle sine loading, (c) linear decreasing loading and (d) exponential loading ($L/h = 10$).

Figure 16: A FG-CNTRC beam under moving load with constant velocity.

The maximum dynamic deflections are fluctuated in the low range of velocity from zero to around 100m/s before increased dramatically beyond that range.
Figure 17: Normalized dynamic deflection of FG-CNTRC beams under moving load: Effect of pattern of reinforcement ($V_{cnt} = 0.12$, $L/h = 10$, $v = 40\text{m/s}$).

Figure 18: Normalized dynamic deflection of FG-CNTRC beams under moving load: Effect of CNT volume fraction ($L/h = 10$, $v = 40\text{m/s}$).

5 Conclusions

Dynamic behavior of FG-CNTRC beams subjected to different dynamic loads has been investigated using Gram-Schmidt-Ritz method in conjunction with the time-integration of Newmark. The equations of motion based on TSDT in which relaxing the assumption of straightness and normality of the transverse normal after deformation are implemented to describe dynamic behavior of the beams in current study. An accuracy of our
modeling is validated in free vibration analysis. Various effects such as material distributions, dynamic loading types, boundary conditions and etc. that have significant impact on dynamic response of the beams are taken into account.

Based on numerical experiments, it can give concluding that the dynamic amplitude of the beams can be reduced by adding CNTs as the reinforcing materials. The distribution of CNTs in form of FG-X makes the beams stronger than other distributions such as UD, FG-V and FG-O, respectively. In dynamic analysis, boundary condition is also one of significant effects on variation of time-history of dynamic deflection in which the beam clamped at both ends has less deflection throughout the time domain. Additionally, in case of moving load, increasing the velocity of the load leads to significant change in dynamic response of the beam. The maximum dynamic deflections are fluctuated in low range of velocity before going up after that range. Designing beam under dynamic loads cannot be simplified by using static analysis due to the lack of mass inertia matrix which may lead to significant errors.

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