A Numerical-Analytical Method for Time-Fractional Dual-Phase-Lag Models of Heat Transfer

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Received 2 August 2020; Accepted (in revised version) 22 December 2020

Abstract. The aim of this paper is to present the backward substitution method for solving a class of fractional dual-phase-lag models of heat transfer. The proposed method is based on the Fourier series expansion along the spatial coordinate over the orthonormal basis formed by the eigenfunctions of the corresponding Sturm-Liouville problem. This Fourier expansion of the solution transforms the original fractional partial differential equation into a sequence of multi-term fractional ordinary differential equations. These fractional equations are solved by the use of the backward substitution method. The numerical examples with temperature-jump boundary condition and parameters of the tissue confirm the high accuracy and efficiency of the proposed numerical scheme.

AMS subject classifications: 65N35, 80A20

Key words: Heat transfer, dual-phase-lag model, fractional partial differential equation, semi-analytical method.

1 Introduction

Fractional partial differential equations (FPDEs) are widely used in various areas of science and engineering. Their advantages become apparent in modeling electrical properties of real materials, the so-called anomalous transport phenomena, and in the theory of complex systems. The FPDEs describe important physical phenomena that arise in

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amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, dielectrics and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes [1–10].

Recently there has been a growing interest for fractional bioheat dual-phase-lag (DPL) heat transfer models [11–16, 18–23] in order to get a precise prediction of thermal data within living biological tissues in different thermal treatment processes.

In this paper we present a novel method for the class of FPDEs

\[ \mathcal{L}_t[v] = \mathcal{M}_t \left\{ \frac{\partial^2 v(x,t)}{\partial x^2} \right\} - p^2 \mathcal{P}_t \{ v(x,t) \} + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \]  

(1.1)

where

\[ \mathcal{L}_t = D^{(\mu)}_t + \sum_{k=l+1}^{K} \alpha_k D^{(\mu_k)}_t, \quad \mathcal{M}_t = \sum_{k=l+1}^{K} \alpha_k D^{(\mu_k)}_t, \quad \mathcal{P}_t = \sum_{k=l+1}^{K} \alpha_k D^{(\mu_k)}_t, \]  

(1.2)

are time fractional differential operators and \( \mu \in \{l−1,l\] \) is the order of the higher derivative of the differential operator \( \mathcal{L}_t \). The integer number \( l \) defines the maximal value of the fractional order \( \mu \) and so, the amount of the initial conditions needed for the FPDE. The values \( 0 \leq \mu_k < \mu \) are fractional or integer constant numbers, \( p \geq 0 \) and \( \alpha_k, k=1,\cdots,K \), are real numbers.

Eq. (1.1) is subjected to the initial conditions (ICs)

\[ v(x,0) = v_0(x), \quad \frac{\partial v(x,0)}{\partial t} = v_1(x), \quad \cdots, \quad \frac{\partial^l v(x,0)}{\partial t^l} = v_{l-1}(x). \]  

(1.3)

We utilize \( l=2 \) and \( l=3 \) for DPL models of the first and second order respectively.

We write the boundary conditions (BCs) along the spatial coordinate in the general form

\[ B_0[v]_{x=0} = g_0(t), \quad B_1[v]_{x=1} = g_1(t), \]  

(1.4)

where the operators \( B_0, B_1 \) conform Dirichlet’s, Neumann’s or Robin’s conditions.

The operator \( D^{(\nu)}_t \) denotes the Caputo fractional derivative defined by [26]

\[ D^{(\nu)}_t [T(x,t)] = \begin{cases} 
\frac{1}{\Gamma(n-\nu)} \int_0^x T^{(n)}(x,\tau)d\tau, & n-1<\nu<n, \\
T^{(n)}(x,t), & \nu=n,
\end{cases} \]  

(1.5)

where \( n \in \mathcal{N} = \{1,2,\cdots\} \) is the set of positive integers, and \( \Gamma(z) \) denotes the gamma function. In particular, for the power functions we get:

\[ D^{(\nu)}_t [t^z] = \begin{cases} 
0, & \text{if } z \in \mathcal{N}_0 \text{ and } z<n, \\
\frac{\Gamma(z+1)}{\Gamma(z+1-\nu)} t^{z-\nu}, & \text{if } z \in \mathcal{N}_0 \text{ and } z \geq n \text{ or } z \notin \mathcal{N}_0 \text{ and } z>n-1.
\end{cases} \]  

(1.6)
where $\mathbb{N}_0 = \{0,1,2,\cdots\}$ is the set of nonnegative integers. This formula is widely used throughout the paper.

Let us note that this class of FPDEs includes many different known equations as particular cases. For example:

- the time-fractional sub-diffusion equation [27]
  \[
  D_t^{(\nu)}[v(x,t)] = a \frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t), \quad 0 < \nu < 1, \quad a > 0, \tag{1.7}
  \]

- the time-fractional telegraph equation [28–30]
  \[
  D_t^{(\nu)}[v(x,t)] + \alpha_1 D_t^{(\nu-1)}[v(x,t)] + \alpha_2 v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t), \quad 0 < \nu < 1, \quad \alpha_1, \alpha_2 > 0, \tag{1.8}
  \]

- the multi-term time-fractional diffusion and diffusion-wave equations [31–33]
  \[
  D_t^{(\mu)}[v(x,t)] + \sum_{k=1}^{n} \alpha_k D_t^{(\mu_k)}[v(x,t)] = K e \frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t), \quad 0 < \nu_i < \mu < 2, \quad \alpha_k, K_e > 0, \tag{1.9}
  \]

- the time-fractional modified anomalous sub-diffusion equation [34–38]
  \[
  \frac{\partial v(x,t)}{\partial t} = \left[ \alpha_1 D_t^{(1-\nu_1)} + \alpha_2 D_t^{(1-\nu_2)} \right] \frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t), \quad 0 < \nu_1, \nu_2 < 1, \quad \alpha_1, \alpha_2 > 0. \tag{1.10}
  \]

The DPL models of the heat transfer are of particular interest in this field [39–41].

Thermal therapy encompasses all therapeutic treatments based on transfer of thermal energy into or out of the body by various ways (direct heating through the skin, laser radiation, ultrasound, microwaves etc.). Most models of these processes are based on the well-known Pennes’ bioheat equation which in 1D case can be written as follows [42]

\[
\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} + w_b c_b (T_b - T) + Q, \quad Q = Q_m + Q_{ext}. \tag{1.11}
\]

Here $q$ the heat flux, $T$ the tissue temperature, $\rho$ is the tissue mass density; $c_p$ the specific heat of tissue; $Q_m$ the metabolic heat generation; $w_b$ is the mass flow rate of blood per unit volume of tissue; $c_b$ is the blood specific heat; $\rho_b$ the mass density of blood; and $T_b$ the blood temperature. The heat generation due to the external sources, e.g., microwave or laser energy absorption is $Q_{ext}$. Let us note that one gets the similar equation in the problem on conduction of heat in rods when the rod loses heat by radiation to a surrounding medium (see, e.g., [43]). The Pennes bioheat equation was developed on the base of the classical Fourier’s law

\[ q = -K \nabla T, \]
where $K$ is the tissue thermal conductivity. This law assumes that the propagation speed of thermal disturbance is infinite i.e., infinitely fast propagation of thermal signal in the medium, which is contradictory to physical reality. More realistic DPL and fractional DPL(FDPL) models of the heat transfer in the biological tissue are derived in the next section. As it will be shown this type of equations also falls into the group the equations described by (1.1).

In this paper we use a combination of the method of separating variables and the Fourier expansion with the backward substitution method (BSM) [44,45], which belongs to the category of meshless methods [46–52]. In order to solve the problem by the Fourier expansion we first transform the nonhomogeneous boundary conditions along the spatial coordinate into homogeneous ones. Then, we seek solution of the homogeneous problem in the form of the expansion $\sim \sum w_n(t) \psi_n(x)$. Here $\psi_n(x)$ are eigenfunctions of the Sturm-Liouville problem

$$\frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad B_0[\psi(0)] = 0, \quad B_1[\psi(1)] = 0,$$

where the boundary operators $B_0, B_1$ are the same as in the original problem. The eigenfunctions $\psi_n(x)$ form an orthonormal basis in the Hilbert space $L^2([0,1])$. Due to the orthogonality of the basis functions $\psi_n(x)$ we get a sequence of the multi-term fractional ordinary differential equations (FODEs) for $w_n(t)$, which can be solved efficiently by the backward substitution method. The stability and accuracy of the proposed method has been studied by solving eight examples, the half of those concerns the bioheat transfer problems.

The paper is organized as follows. Some necessary preliminaries of the problems are discussed in Section 2. The main algorithm is described in Section 3. In Section 4, several numerical examples are provided to verify the accuracy and applicability of the proposed approach. The short conclusions are placed in Section 5.

## 2 Preliminary data

The DPL family of models is based on the introduction of two time lags into the classical Fourier law [53–55]

$$q(x,t+\tau_q) = -K \frac{\partial}{\partial x} T(x,t+\tau_T),$$

where $K = W/(m^\circ C)$ is the thermal conductivity; $\tau_q$ and $\tau_T$ stand for the heat flux and temperature-gradient phase lags respectively. Both are positive and intrinsic properties of the material.

Applying the generalized Taylor’s formula (see [56, Eqs. (3.12) and (4.1)]), we get the
Then we present an important particular case—the FPDL models of bioheat transfer. They are described in the next two subsections.

Taking two or three terms in the expansions (2.2a), (2.2b), one gets the FDPL models of the first or second orders. The equation of the fractional models of the first order should be considered together with the equation of the energy balance:

\[
\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x} q + S,
\]

where \(S, [S]=W/m^3\) represents the internal source of heat; \(\rho c_p \frac{\partial q}{\partial t} = \partial E/\partial t\) is the temporal change of the internal energy \(E = \rho c_p T, [E]=J/m^3\); \(\rho, [\rho]=kg/m^3\) denotes material density and \(c_p, [c_p]=1/(kg^\circ C)\) is the specific heat.

Combining (2.3) and (2.4), we get the FPDE of the FDPL\(_{\alpha,\beta}(1,1)\) model

\[
\rho c_p R_{1,t}^\alpha \left[ \frac{\partial T}{\partial t} \right] = K R_{1,t}^{\beta} \left[ \frac{\partial^2 T}{\partial x^2} \right] + R_{1,t}^\alpha [S]
\]

or in the expanded form:

\[
\rho c_p \left( \frac{\partial}{\partial t} + \frac{\tau_\alpha}{\Gamma(\alpha+1)} D_{t}^{(\alpha+1)} \right) [T] = K \left( \frac{\partial^2}{\partial x^2} \right) \left[ \frac{\partial^2 T}{\partial x^2} \right] + \left( \frac{\tau_\alpha}{\Gamma(\alpha+1)} D_{t}^{(\alpha)} \right) [S].
\]

Here and below we write \(D_{t}^{\beta}\) as \(D_{t}^{1+\alpha T}\) assuming that conditions of such transform are fulfilled (e.g., see Lemma 3.13 in [26]). Let us introduce the reference size of the domain \(L_0 [m] \left( x = L_0 x_s \right)\) and the reference time of the heating process \(t_0 [s] (t = t_0 t_s)\). Then dimensionless form of the FDPL\(_{\alpha,\beta}(1,1)\) equation can be written as follows

\[
R_{1,t}^\alpha \left[ \frac{\partial v}{\partial t_\star} \right] = F_0 R_{1,t}^{\beta} \left[ \frac{\partial^2 v}{\partial x_\star^2} \right] + R_{1,t}^\alpha [S_\star].
\]
In this case we combine the equation of the energy balance (2.4) with the following one

\[ R^{\alpha}_{1,t_*} = 1 + \left( \frac{\tau_q}{t_0} \right)^a \frac{D_t^{(a)}}{\Gamma(\alpha+1)}, \quad R^{\beta}_{1,t_*} = 1 + \left( \frac{\tau_T}{t_0} \right)^b \frac{D_t^{(b)}}{\Gamma(\beta+1)}. \]  

(2.7)

\( Fo = Kt_0 / (\rho c_p L_0^2) \) is the Fourier number; parameters \( \tau_q / t_0, \tau_T / t_0 \) characterize the lagging; \( T = T_0 \delta, T_0 \) is the reference value of the temperature, e.g., the initial temperature of the medium, \( S_* = St_0 / (\rho c_p T_0) \) denotes the dimensionless source function and we assume that it is a function of \( x \) and \( t \) only. The equation can be rewritten in the compact form (1.1), (1.2) with the following parameters:

\[ \mu = \alpha + 1, \quad \mathcal{L}_t = a_1 D_t^{(1)} + a_2 D_t^{(0)} + a_3 D_t^{(\beta)}, \quad p = 0, \]  

(2.9a)

\[ f(x,t) = \left( \frac{t_0}{\tau_q} \right)^a \Gamma(\alpha+1) R^{\alpha}_{1,t_*} [S], \]  

(2.9b)

where

\[ a_1 = \left( \frac{t_0}{\tau_q} \right)^a \Gamma(\alpha+1), \quad a_2 = \left( \frac{t_0}{\tau_q} \right)^a \Gamma(\alpha+1) Fo, \quad a_3 = a_2 \left( \frac{\tau_T}{t_0} \right)^\beta \Gamma(\beta+1). \]  

(2.10)

Here we omit the asterisk in notation of the dimensionless values.

**2.2 FDPL\(_{\alpha, \beta}(2,2)\) model**

In this case we combine the equation of the energy balance (2.4) with the following one

\[ R^{\alpha}_{2,t} [q] = -K R^{\beta}_{2,t} \left[ \frac{\partial T}{\partial x} \right], \]  

(2.11)

where

\[ R^{\alpha}_{2,t} \overset{\text{def}}{=} 1 + \frac{\tau_q^a}{\Gamma(\alpha+1)} D_t^{(a)} + \frac{\tau_q^{2a}}{\Gamma(2\alpha+1)} D_t^{(2a)}, \quad R^{\beta}_{2,t} \overset{\text{def}}{=} 1 + \frac{\tau_T^\beta}{\Gamma(\beta+1)} D_t^{(\beta)} + \frac{\tau_T^{2\beta}}{\Gamma(2\beta+1)} D_t^{(2\beta)}. \]

Using transforms similar to the ones described in the previous subsection, we get the FPDE of the DPL\(_{\alpha, \beta}(2,2)\) model

\[ \rho c_p R^{\alpha}_{2,t} \left[ \frac{\partial T}{\partial t} \right] = K R^{\beta}_{2,t} \left[ \frac{\partial^2 T}{\partial x^2} \right] + R^{\alpha}_{2,t} [S]. \]  

(2.12)
In the expanded dimensionless form the FPDE can be written as follows:

\[
D_t^{(2\alpha+1)}[\nu] + \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1) \frac{\partial^2 \nu}{\partial T^2} + \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1) \frac{\partial \nu}{\partial T} = F_0 \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1) \mathcal{R}_{2,f}[\mathcal{S}].
\]

(cf. (2.5)). The FPDE can be written in the compact form (1.1), (1.2) with the following parameters:

\[
f(x,t) = \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1) \mathcal{R}_{2,f}[\mathcal{S}],
\]

where

\[
a_1 = \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1), \quad a_2 = \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1), \quad a_3 = F_0 \left(\frac{t_0}{\tau_q}\right)^{2\alpha} \Gamma(2\alpha+1),
\]

\[
a_4 = a_5 = \frac{\tau_T}{t_0} \frac{1}{\Gamma(\beta+1)}.
\]

Here we omit the asterisk in notation of the dimensionless values.

### 2.3 FPDL models of the bioheat transfer

In the case of the Pennes model of the heat transfer the source term in the equation of the energy balance (2.4) takes the form:

\[
S = S(T,x,t) = w_b c_b (T_b - T) + S_m + S_c.
\]

Here \(w_b\) is the mass flow rate of blood per unit volume of tissue, \([w_b] = \text{kg}/(\text{sm}^3)\), \(c_b\) is the blood specific heat, \([c_b] = \text{J}/(\text{kg} \cdot \text{C})\), \(S_m\) is the metabolic heat generation and \(S_c\) is the heat generation due to external sources, e.g., due to the electromagnetic radiation absorbed. In this study we assume that: 1) \(T_b = \text{const}\); 2) \(S_c(x)\) is a function of \(x\) and \(t\) only; \(S_c = S_c(x,t)\) and we neglect the metabolic heat generation \(S_m\). Under these assumptions we get the FPDE of the FDPL_{\alpha,\beta}(1,1) model

\[
\rho c_p \mathcal{R}_{1,f} \frac{\partial T}{\partial t} = k \mathcal{R}_{1,f} \frac{\partial^2 T}{\partial x^2} - w_b c_b \mathcal{R}_{1,f} [T - T_b] + \mathcal{R}_{1,f} [S_c].
\]

(cf. (2.5)). The dimensionless form of the FPDE with the above-mentioned assumptions is as follows:

\[
\mathcal{R}_{1,f} \frac{\partial \nu}{\partial t} = F_0 \mathcal{R}_{1,f} \frac{\partial^2 \nu}{\partial x^2} - p^2 \mathcal{R}_{1,f} [\nu] + \mathcal{R}_{1,f} [S_{c,s}].
\]
Here
\[ v = (T - T_b) / T_b, \quad p^2 = \frac{w_b c_{b0} t_0}{\rho c_p}, \quad S_{c,v} = \frac{S_{c0} t_0}{\rho c_p T_b}, \]
(2.15)
and operators \( \mathcal{R}_{1,t}^{\alpha}, \mathcal{R}_{1,t}^{\beta} \) are given in (2.7). The FPDE can be written in the compact form (1.1), (1.2) with the same operators \( \mathcal{L}_t, \mathcal{M}_t \) as the ones given in (2.9), (2.10) and with
\[
\mathcal{P}_t = a_4 D_t^{(0)} + a_5 D_t^{(a)},
\]
(2.16)
where
\[
a_4 = \left( \frac{t_0}{\tau_q} \right)^a \Gamma (a+1), \quad a_5 = 1, \quad f (x,t) = \mathcal{R}_{1,t}^{\alpha} [S_c],
\]
(2.17)
and we omit the asterisk in notation of the dimensionless values.

It is easy to verify that in the case of FDPL \( \alpha, \beta (2,2) \) model we get the FPDE (1.1) with
\[
\mathcal{P}_t = a_6 D_t^{(0)} + a_7 D_t^{(a)} + a_8 D_t^{(2a)},
\]
(2.18a)
\[
a_6 = \left( \frac{t_0}{\tau_q} \right)^{2a} \Gamma (2a+1), \quad a_7 = \left( \frac{t_0}{\tau_q} \right)^a \frac{\Gamma (2a+1)}{\Gamma (a+1)}, \quad a_8 = 1, \quad f (x,t) = \mathcal{R}_{2,t}^{\alpha} [S_c],
\]
(2.18b)
and parameter \( p \) is given in (2.15).

3 Main algorithm

3.1 The problems with a single space harmonic

In this subsection we consider the particular case, when the source term in the FPDE (1.1) is the product \( \theta (t) \sin (n \pi x) \), and the solution satisfies the homogeneous Dirichlet conditions along \( x = 0 \) and \( x = 1 \). So, we consider the FPDE
\[
D_t^{(\mu)} [v] + \sum_{k=1}^l a_k D_t^{(\mu_k)} [v] = \left( \sum_{k=l+1}^K a_k D_t^{(\mu_k)} \right) \frac{\partial^2 v}{\partial x^2} - p^2 \left( \sum_{k=j+1}^L a_k D_t^{(\mu_k)} \right) v + \theta_n (t) \sin (n \pi x),
\]
(3.1)
with the boundary conditions
\[
v (0,t) = v (1,t) = 0.
\]
(3.2)
Suppose that ICs (1.3) also have the special form:
\[
v (x,0) = w_{n,0} \sin (n \pi x), \quad \frac{\partial v}{\partial t} (x,0) = w_{n,1} \sin (n \pi x), \ldots, \quad \frac{\partial^{l-1} v}{\partial t^{l-1}} (x,0) = w_{n,l-1} \sin (n \pi x).
\]
(3.3)
Using the substitution $v(x,t) = w_n(t) \sin(n \pi x)$, we transform Eq. (3.1) into the multi-term FODE

$$D_t^{(\mu)}[w_n] + \sum_{k=1}^{I} \alpha_k D_t^{(\mu_k)}[w_n] = -(n \pi)^2 \left( \sum_{k=I+1}^{J} \alpha_k D_t^{(\mu_k)}[w_n] \right) + \theta_n(t),$$

which can be written in a short form

$$D_t^{(\mu)}[w_n(t)] = \sum_{k=1}^{K} \beta_{k,n} D_t^{(\mu_k)}[w_n(t)] + \theta_n(t).$$

Here we denote

$$\beta_{k,n} = \begin{cases} -\alpha_k, & k \leq I, \\ -(n \pi)^2 \alpha_k, & I+1 \leq k \leq J, \\ -p^2 \alpha_k, & J+1 \leq k \leq K. \end{cases}$$

The equation is subjected to the ICs

$$w_n(0) = w_{n,0}, \quad \frac{dw_n}{dt}(0) = w_{n,1},$$

for the DPL-\(\alpha,\beta\) (1,1) model and

$$w_n(0) = w_{n,0}, \quad \frac{dw_n}{dt}(0) = w_{n,1}, \quad \frac{d^2w_n}{dt^2}(0) = w_{n,2},$$

for the DPL-\(\alpha,\beta\) (2,2) model when $2\alpha + 1 > 2$.

Let $\varphi_m(t)$ be some system of basis functions on $[0,T]$ which are chosen in such a way that the right hand side of (3.5) can be represented in the form of the series

$$\sum_{k=1}^{K} \beta_{k,n} D_t^{(\mu_k)}[w_n(t)] + \theta_n(t) = \sum_{m=1}^{\infty} q_m \varphi_m(t), \quad 0 \leq t \leq T.$$ 

Thus, the original equation (3.5) can be rewritten in the form

$$D_t^{(\mu)}[w_n(t)] = \sum_{m=1}^{\infty} q_m \varphi_m(t).$$

Besides, we assume that for each $\varphi_m(t)$ there exists $\varphi_m(t)$ given in the explicit analytic form, which satisfies the equation

$$D_t^{(\mu)}[\varphi_m(t)] = \varphi_m(t).$$
Throughout the paper we use the generalized power functions or the Müntz polynomials basis (MPB) \[57\]
\[
\phi_m(t) = t^\delta_m, \quad \delta_m = \sigma(m - 1), \quad 0 < \sigma \leq 1, \quad m = 1, \ldots
\]
as the basis functions. So, the solution is sought in the class of functions which can be approximated by the Müntz polynomials basis and for which there exist fractional derivatives of the original equation (3.5).

It is easy to show that the functions
\[
\phi_m(t) = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \mu + 1)} t^{\delta_m + \mu}
\]
satisfy Eq. (3.11) and satisfy zero ICs
\[
\phi_m^{(i)}(0) = 0, \quad i = 0, 1, \ldots, l - 1,
\]
where \(l = 2\) for FDPL\(_{\alpha,\beta}(1,1)\) and \(l = 2\) or 3 for FDPL\(_{\alpha,\beta}(2,2)\) models. So, any linear combination
\[
\sum_{m=1}^{\infty} q_{n,m} \phi_m(t)
\]
satisfies the modified equation (3.10) and zero ICs (3.14).

Let us denote
\[
w_{n,p}(t) = \sum_{i=0}^{l-1} \frac{1}{i!} w_{n,i} t^i,
\]
\[
 w_{n,p}(t) = w_{n,0} + w_{n,1} t, \quad \text{(3.15)}
\]
for the DPL\(_{\alpha,\beta}(1,1)\) model and
\[
w_p(t) = w_{n,0} + w_{n,1} t \quad (\text{for } 2\alpha + 1 \leq 2) \quad \text{or} \quad w_{n,0} + w_{n,1} t + 0.5 w_{n,2} t^2 \quad (\text{for } 2 < 2\alpha + 1 \leq 3)
\]
for the DPL\(_{\alpha,\beta}(2,2)\) model. The function \(w_p(t)\) satisfies the ICs (3.7)/(3.8) and the sum
\[
w_n(t, q) = w_{n,p}(t) + \sum_{m=1}^{\infty} q_{n,m} \phi_m(t)
\]
satisfies Eq. (3.10) and original ICs (3.7)/(3.8) with any choice of parameters \(q_{n,m}\).

Thus, the function \(w_n(t, q)\) is the solution of the problem (3.10), (3.7)/(3.8) with any choice of parameters \(q_{n,m}\). On the other hand, the modified equation (3.10) coincides with the original equation (3.5), if and only if, when the condition (3.9) is fulfilled. Then, the function \(w_n(t, q)\) should satisfy (3.9) to be the solution of the original equation (3.5).
Thus, the free parameters $q_{n,m}$ are determined by the backward substitution of $w_n(x,q)$ into (3.9):

$$\sum_{m=1}^{\infty} q_{n,m} \left[ \varphi_m(t) - \sum_{k=1}^{K} \beta_{k,n} \phi_m^{(\mu_k)}(t) \right] = \theta_n(t) + \sum_{k=1}^{K} \beta_{k,n} w_{n,p}^{(\mu_k)}(t) = F_n(t), \quad (3.17)$$

where we denote

$$\varphi_{m}^{(\mu_k)}(t) = D_t^{(\mu_k)}[\varphi_m(t)] = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \mu + 1)} D_t^{[\delta_m + \mu - \mu_k]} = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \mu + 1 - \mu_k)}, \quad (3.18a)$$

$$w_{n,p}^{(\mu_k)}(t) = \sum_{i=0}^{N_c-1} \frac{1}{i!} w_{n,i} D_t^{[i]} \left[ t^{\mu_k} \right] = \sum_{i=0}^{N_c-1} 1 \sum_{\mu_k \leq i} \frac{1}{i!} w_{n,i} D_t^{[i]} = \sum_{\mu_k \leq i} \frac{1}{i!} w_{n,i} t^{\mu_k - \mu_k} = \frac{1}{i!} w_{n,i} t^{\mu_k - \mu_k}. \quad (3.18b)$$

Eq. (3.17) should be fulfilled at each point of the interval $[0,T]$. It gives the exact solution of the original problem if it exists. To get the approximate solution we consider the truncated series

$$w_{n,M}(t,q) = w_{n,p}(t) + \sum_{m=1}^{M} q_{n,m} \varphi_m(t), \quad (3.19)$$

which satisfies the equation

$$D_t^{(\mu)}[w_{n,M}(t)] = \sum_{m=1}^{M} q_{n,m} \varphi_m(t). \quad (3.20)$$

The coefficients $q_{n,1}, \ldots, q_{n,M}$ are the solution of the truncated finite linear system

$$\sum_{m=1}^{M} q_{n,m} \left[ \varphi_m(t) - \sum_{k=1}^{K} \beta_{k,n} \varphi_m^{(\mu_k)}(t) \right] = F_n(t). \quad (3.21)$$

Here $F_n(t)$ is the same as in (3.17).

Applying the collocation procedure to Eqs. (3.21) at the Gauss-Chebyshev (GC) collocation points

$$t_i = \frac{T}{2} \left[ 1 + \cos \left( \frac{\pi(2i-1)}{2N_c} \right) \right] \in [0,T], \quad i = 1,2,\ldots, \quad N_c \geq M, \quad (3.22)$$

we get a system of linear algebraic equations which can be solved by standard methods.

### 3.2 General case

Let us consider the FPDE (1.1) with the Dirichlet boundary condition

$$v(0,t) = g_0(t), \quad v(1,t) = g_1(t). \quad$$
The substitution
\[ v(x,t) = v_g(x,t) + w(x,t), \tag{3.23} \]
where
\[ v_g(x,t) = g_0(t) + x(g_1(t) - g_0(t)) \]
transforms it into FPDE with a new source term
\[ D^{(\mu)}_t [w] + \sum_{k=1}^{K} a_k D^{(\mu_k)}_t [v_g] + \sum_{k=1}^{K} \beta_k D^{(\mu_k)}_t [v_g] - \theta_n(t) = f_1(x,t), \tag{3.24} \]
which is subjected to the homogeneous BCs
\[ w(0,t) = 0, \quad w(1,t) = 0. \]
Here
\[ f_1(x,t) = f(x,t) - D^{(\mu)}_t [v_g] - \sum_{k=1}^{K} a_k D^{(\mu_k)}_t [v_g] - \sum_{k=1}^{K} \beta_k D^{(\mu_k)}_t [v_g]. \]
The functions
\[ \psi_n(x) = \sqrt{2} \sin n\pi x, \quad n = 1, 2, \cdots, \tag{3.25} \]
are eigenfunctions of the Sturm-Liouville problem
\[ \frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad \psi(0) = 0, \quad \psi(1) = 0. \]
They form an orthonormal basis in the Hilbert space \( L_2([0,1]) \) with the condition of orthogonality
\[ \langle \psi_m, \psi_n \rangle_1 = \int_0^1 \psi_m(x) \psi_n(x) dx = \delta_{m,n}. \tag{3.26} \]
We seek the solution the FPDE (3.24) in the form of the expansion
\[ w(x,t) = \sum_{n=1}^{\infty} w_n(t) \psi_n(x). \tag{3.27} \]
Substituting \( w(x,t) \) in (3.24) and using the projecting \( \langle \cdots, \psi_n \rangle_1 \), we get the system of separated equations for each harmonic \( w_n(t) \):
\[ D^{(\mu)} [w_n(t)] = \sum_{k=1}^{K} \beta_{k,n} D^{(\mu_k)} [w_n(t)] + \theta_n(t), \quad \theta_n(t) = \int_0^1 f_1(x,t) \psi_n(x) dx, \tag{3.28} \]
(cf. (3.5)).
The initial values for the harmonic \( w_n(t) \) follow from (1.3), (3.23):

\[
w_n(0) = w_{n,0} = \int_0^1 \left[ v_0(x) - v_g(x,0) \right] \psi_n(x) \, dx,
\]

\[
\frac{d^i w_n(0)}{dt} = w_{n,i} \int_0^1 \left[ v_i(x) - \frac{\partial^i v_g(x,0)}{\partial t} \right] \psi_n(x) \, dx, \quad i = 1, 2, \ldots, l - 1.
\]

So, each initial value problem (3.28), (3.29) has the same form as the one described in previous subsection and can be solved by the algorithm presented above. The approximate solution \( w_{n,M}(t) \) of the \( n^{th} \) problem can be written in the form

\[
w_{n,M}(t) = w_{n,p}(t) + \sum_{m=1}^M q_{n,m} \phi_m(t)
\]

(cf. Eq. (3.19)).

Here \( w_{n,p}(t) \) are the particular solutions which satisfy the ICs (3.29) (cf. (3.15)). Finally, the approximate solution of the original problem can be written in the form:

\[
v_{N,M}(x,t) = v_g(x,t) + w_p(x,t) + \sum_{m=1}^M Q_m(x) \phi_m(t),
\]

where

\[
w_p(x,t) = \sum_{n=1}^N w_{n,p}(t) \psi_n(x), \quad Q_m(x) = \sum_{n=1}^N q_{n,m} \psi_n(x).
\]

In case of the boundary conditions

\[
B_0[v(0,t)] = g_0(t), \quad B_0[v(1,t)] = g_1(t),
\]

other than the Dirichlet ones we use the orthonormalized eigenfunctions of the Sturm-Liouville problem

\[
\frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad B_0[\psi(0)] = 0, \quad B_1[\psi(1)] = 0,
\]

as the basis system of the Fourier expansion (3.27).

4 Numerical illustration

Example 4.1. Let us consider FPDE of the DPL\(_{\alpha,\beta}(1,1)\) model [41]:

\[
\frac{\partial T}{\partial t} + \tau_q^\alpha D_{1}^{(1+\alpha)}[T] = \left(1 + \tau_q^\beta D_{1}^{(\beta)}\right) \frac{\partial^2 T}{\partial x^2} + f(x,t), \quad 0 \leq \alpha, \beta \leq 1, \quad 0 \leq x \leq 2, \quad 0 \leq t \leq 1.
\]
with boundary and initial conditions

\[
T(x,0) = \frac{\partial T}{\partial t}(x,0) = 0, \quad x \in [0,2], \quad (4.2a)
\]
\[
T(0,t) = T(2,t) = 0, \quad t \in [0,1]. \quad (4.2b)
\]

The BCs, ICs and the source term \( f(x,t) \) conform the exact solution

\[
T_{ex}(x,t) = t^{\alpha+\beta+1} \sin(2\pi x).
\]

The values of \( \tau_q \) and \( \tau_T \) are defined as given in [41], namely, \( \tau_q = 14\) and \( \tau_T = 0.056\). The substitution \( x = 2y \) transforms the original problem into the following one:

\[
\begin{align*}
\frac{\partial T}{\partial t} + \tau_q^a D_t^{(1+a)} [T] &= \frac{1}{4} \left( 1 + \tau_T^\beta D_t^{(\beta)} \right) \frac{\partial^2 T}{\partial y^2} + h(y,t), \quad 0 \leq y \leq 1, \quad 0 \leq t \leq 1, \quad (4.3a) \\
T(y,0) &= \frac{\partial T}{\partial t}(y,0) = 0, \quad y \in [0,1], \quad (4.3b) \\
T(0,t) &= T(1,t) = 0, \quad t \in [0,1], \quad (4.3c)
\end{align*}
\]

with the exact solution

\[
T_{ex}(y,t) = t^{\alpha+\beta+1} \sin(4\pi y).
\]

Finally, after substitution \( T(y,t) = w(t) \sin(4\pi y) \) we get the FODE

\[
D_t^{(1+a)} [w] + \frac{1}{\tau_q^a} \frac{dw}{dt} = -\frac{4\pi^2}{\tau_q^a} \left( 1 + \tau_T^\beta D_t^{(\beta)} \right) w + \theta(t), \quad 0 \leq t \leq 1, \quad (4.4)
\]

where \( \theta(t) \) conforms the exact solution \( w_{ex}(t) = t^{\alpha+\beta+1} \). The initial conditions

\[
w(0) = \frac{dw}{dt}(0) = 0 \quad (4.5)
\]

follow from (4.3b).

Applying the algorithm described in the previous section, we suppose that

\[
-\frac{1}{\tau_q^a} \frac{dw}{dt} - \frac{4\pi^2}{\tau_q^a} \left( 1 + \tau_T^\beta D_t^{(\beta)} \right) w + \theta(t) = \sum_{m=1}^{\infty} q_m \phi_m(t), \quad 0 \leq t \leq 1. \quad (4.6)
\]

Instead of the original equation (4.4) get the modified one

\[
D_t^{(1+a)} [w] = \sum_{m=1}^{\infty} q_m \phi_m(t). \quad (4.7)
\]

Here the functions

\[
\phi_m(t) = t^{\delta_m} = t^{\sigma(m-1)}
\]
are the same as the ones given in (3.12). This equation has the analytical solution
\[ w(t, q) = \sum_{m=1}^{\infty} q_m \phi_m(t), \quad \phi_m(t) = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \alpha + 2)} t^{\delta_m + 1 + \alpha}. \] (4.8)

If the coefficients \( q_m \) satisfy the relation
\[ \sum_{m=1}^{\infty} q_m \left[ \phi_m(t) + \frac{1}{\tau_q} \frac{d\phi_m(t)}{dt} + \frac{4\pi^2}{\tau_q^2} \left( \phi_m(t) + \tau_q^\beta D_1^{(\beta)}[\phi_m(t)] \right) \right] = \theta(t), \quad 0 \leq t \leq 1, \]
then the modified equation (4.7) is equivalent to the original FODE (4.4). We determine the coefficients \( q_m, m = 1, \ldots, M \) of the approximate solution
\[ w_M(t, q) = \sum_{m=1}^{M} q_m \phi_m(t) \]
as a solution of the collocation system
\[ \sum_{m=1}^{M} q_m \left[ \phi_m(t_j) + \frac{1}{\tau_q} \frac{d\phi_m(t_j)}{dt} + \frac{4\pi^2}{\tau_q^2} \left( \phi_m(t_j) + \tau_q^\beta D_1^{(\beta)}[\phi_m(t_j)] \right) \right] = \theta(t_j), \quad j = 1, \ldots, N_c \geq M. \]

Here \( t_j, j = 1, \ldots, N_c \) are the Gauss-Chebyshev (GC) collocation points
\[ t_j = \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi (2j-1)}{2N_c} \right) \right] \in [0,1]. \]

We get a system of linear algebraic equations which can be solved by standard methods. To test the accuracy of the proposed method we use the maximal absolute error
\[ E_{\text{max}}(M) = \max_{1 \leq j \leq N_t} \left| w_{\text{ex}}(t_j) - w_M(t_j, q) \right|, \]
and the error in the discrete \( H^1(\Omega) \) norm
\[ E_{H^1}(M) = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} \left( \left( \frac{\partial w_{\text{ex}}(t_j)}{\partial t} - \frac{\partial w_M(t_j, q)}{\partial t} \right)^2 \right)^2}, \]
which also includes the errors of the first derivative. We use \( N_t = 1000 \) test points \( t_j \) evenly distributed inside \([0,1]\).

To illustrate the efficiency of the method proposed we also calculate the convergence order
\[ \text{CO}(M) = \log_2 \left( \frac{E_{\text{max}}(M)}{E_{\text{max}}(2M)} \right). \]
Table 1: Example 4.1. The errors and convergence order with respect to the change of $M$. $\alpha = 0.2$, $\beta = 0.7$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$E_{max}$</th>
<th>$E_{H1}$</th>
<th>CO($M$)</th>
<th>$E_{max}$</th>
<th>$E_{H1}$</th>
<th>CO($M$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.10E-2</td>
<td>9.36E-2</td>
<td>7.28</td>
<td>1.42E-2</td>
<td>6.41E-2</td>
<td>9.54</td>
</tr>
<tr>
<td>4</td>
<td>1.36E-4</td>
<td>1.44E-3</td>
<td>16.0</td>
<td>1.90E-5</td>
<td>1.53E-4</td>
<td>8.78</td>
</tr>
<tr>
<td>8</td>
<td>2.08E-9</td>
<td>2.71E-8</td>
<td>9.53</td>
<td>4.33E-8</td>
<td>5.13E-7</td>
<td>6.23</td>
</tr>
<tr>
<td>16</td>
<td>2.83E-12</td>
<td>1.21E-11</td>
<td>1.87</td>
<td>5.77E-10</td>
<td>1.95E-9</td>
<td>3.38</td>
</tr>
<tr>
<td>32</td>
<td>7.72E-13</td>
<td>3.33E-12</td>
<td>3.13</td>
<td>5.56E-11</td>
<td>2.13E-10</td>
<td>2.94</td>
</tr>
<tr>
<td>64</td>
<td>8.83E-14</td>
<td>3.83E-13</td>
<td>-</td>
<td>7.25E-12</td>
<td>3.20E-11</td>
<td>-</td>
</tr>
</tbody>
</table>

see [41] Table 1 $E_{max}(t=1)=1.0601E^{-5}$

When the number of the free parameters increases from $M$ to $2M$, then the error decreases in $2^p$ times, if CO($M$) = $p$. Thus, this parameter characterizes the efficiency of the method. The data of Table 1 correspond to $\alpha = 0.2$, $\beta = 0.7$. The last row of the table contains the results obtained by applying the combination of the finite difference and the Legendre spectral method given in [41] with the same values of $\alpha$ and $\beta$.

The table illustrates that the errors decrease monotonically with the growth of number $M$ of the Müntz’s polynomials $\phi_m(t)$ in the approximate solution. The method possesses a high convergence order and provides a high precision of the approximate solution with a moderate number of the free parameters. The graphic a) in Fig. 1 shows dependence of the error on the Müntz parameter $\sigma$. In general, the error increases slightly with the growth of $\sigma$. However, this behaviour is not monotonic. The drastic change of the error is observed at $\sigma = 0.35$ and $\sigma = 0.7$. For these values of $\sigma$ the basis functions $\phi_m(t)$ up to the constant coincide with the exact solutions. Indeed, if we take $\sigma = 0.7$, then we get

$$\phi_2(t) \sim t^{0.7(2-1)+1+\alpha} = t^{1.9} = w_{\text{ex}}(t).$$

We observe the same behaviour of errors for $\sigma = 0.35$ ($\phi_3(t) \sim w_{\text{ex}}(t)$) and $\sigma = 0.14$ ($\phi_5(t) \sim w_{\text{ex}}(t)$).

![Figure 1: The maximal absolute error as a function of the parameter $\sigma$ of the Müntz polynomials basis. $M=32$.](image-url)
Table 2: Example 4.1. The behaviour of errors with the growth of $M$ for $\alpha = 0.2$, $\beta = 0.7$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.14$</td>
<td>1.49E-1</td>
<td>4.05E-1</td>
<td>1.49E-1</td>
<td>4.05E-1</td>
<td>1.49E-1</td>
<td>4.05E-1</td>
</tr>
<tr>
<td>$\sigma = 0.35$</td>
<td>2.15E-2</td>
<td>9.57E-2</td>
<td>1.21E-2</td>
<td>5.50E-2</td>
<td>6.66E-16</td>
<td>7.60E-16</td>
</tr>
<tr>
<td>$\sigma = 0.7$</td>
<td>1.62E-3</td>
<td>1.78E-3</td>
<td>4.44E-16</td>
<td>4.83E-16</td>
<td>5.55E-16</td>
<td>5.47E-16</td>
</tr>
</tbody>
</table>

Table 3: Example 4.1. The behaviour of errors with the growth of $M$ for $\alpha = 0.6$, $\beta = 0.4$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.10$</td>
<td>9.02E-2</td>
<td>1.94E-1</td>
<td>9.02E-2</td>
<td>1.94E-1</td>
<td>9.02E-2</td>
<td>1.94E-1</td>
</tr>
<tr>
<td>$\sigma = 0.20$</td>
<td>9.54E-3</td>
<td>2.31E-2</td>
<td>6.21E-3</td>
<td>1.50E-2</td>
<td>4.44E-16</td>
<td>5.25E-16</td>
</tr>
<tr>
<td>$\sigma = 0.4$</td>
<td>4.03E-4</td>
<td>1.69E-3</td>
<td>3.33E-16</td>
<td>2.79E-16</td>
<td>3.33E-16</td>
<td>2.17E-16</td>
</tr>
</tbody>
</table>

The data of Table 2 illustrate the behaviour of the errors for $\sigma = 0.14$, $\sigma = 0.35$ and $\sigma = 0.7$. This example demonstrates that in the framework of the present technique an additional information on the exact solution of the problem can be used to improve the approximate solution. For example, suppose, the solution is a smooth analytic function of $t$ and it can be well approximated by polynomials, i.e. by the functions $t^m$ with integer $m$. Taking

$$\delta_m = m - \mu, \quad m = 1, 2, \ldots,$$

in the definition (3.12) of the functions $\varphi_m(t)$, we gain (see (3.13)):

$$\delta_m = m - \mu \Rightarrow \varphi_m(t) = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \mu + 1)} t^{\delta_m + \mu} = \frac{\Gamma(\delta_m + 1)}{\Gamma(\delta_m + \mu + 1)} t^m.$$

As a result, we get integer degrees $\delta_m + \mu$ and the approximate solution $w_M(t, q)$ in the form of the polynomial of the degree $M$.

The graphic b) in Fig. 1 shows dependence of the error on the value of the Müntz’s parameter $\sigma$ for $\alpha = 0.6$, $\beta = 0.4$.

The errors decrease drastically at $\sigma = 0.10$, $\sigma = 0.20$ and $\sigma = 0.4$ when $w_{ex}(t) \sim \varphi_5(t)$, $\varphi_3(t)$ and $\varphi_2(t)$ correspondingly. The data of Table 3 illustrate this behaviour of the errors. The graphic c) in Fig. 1 and the data of Table 4 conform the case $\alpha = 0.7$, $\beta = 0.5$. Here $w_{ex}(t) \sim \varphi_6(t)$, $\varphi_3(t)$ and $\varphi_2(t)$ with $\sigma = 0.10$, $\sigma = 0.25$ and $\sigma = 0.5$. 

see [41] Table 1 $E_{\text{max}}(t=1)=3.2771E-5$
Table 4: Example 4.1. The behaviour of errors with the growth of $M$ for $\alpha = 0.7$, $\beta = 0.5$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H_1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H_1}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{H_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.22E-1</td>
<td>2.24E-1</td>
<td>1.22E-1</td>
<td>2.24E-1</td>
<td>1.22E-1</td>
<td>2.24E-1</td>
</tr>
<tr>
<td>2</td>
<td>2.10E-2</td>
<td>3.85E-2</td>
<td>1.29E-2</td>
<td>2.34E-2</td>
<td>4.44E-16</td>
<td>5.18E-16</td>
</tr>
<tr>
<td>3</td>
<td>1.18E-3</td>
<td>3.79E-3</td>
<td>3.33E-16</td>
<td>3.01E-16</td>
<td>4.44E-16</td>
<td>3.99E-16</td>
</tr>
<tr>
<td>4</td>
<td>9.43E-5</td>
<td>2.77E-4</td>
<td>1.29E-2</td>
<td>2.34E-2</td>
<td>4.44E-16</td>
<td>3.81E-16</td>
</tr>
<tr>
<td>5</td>
<td>4.96E-6</td>
<td>1.43E-5</td>
<td>3.33E-16</td>
<td>4.07E-16</td>
<td>2.22E-16</td>
<td>3.76E-16</td>
</tr>
<tr>
<td>6</td>
<td>3.33E-16</td>
<td>4.75E-16</td>
<td>2.22E-16</td>
<td>2.90E-16</td>
<td>2.22E-16</td>
<td>3.03E-16</td>
</tr>
</tbody>
</table>

Example 4.2. Let us consider the time-fractional DPL$_{\alpha,\beta}(1,1)$ equation

$$
\frac{\partial T}{\partial t} + \tau_\theta^\alpha D_{1+\alpha}^t [T] = \left(1 + \tau_\theta^\beta D_{1}^t \right) \frac{\partial^2 T}{\partial x^2} + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1, \tag{4.10a}
$$

$$
T(x,0) = \frac{\partial T}{\partial t}(x,0) = 0, \quad x \in [0,1], \tag{4.10b}
$$

$$
T(0,t) = T(1,t) = 0, \quad t \in [0,1], \tag{4.10c}
$$

where the source term $f(x,t)$ conforms the exact solution $T_{\text{ex}}(x,t) = \frac{256}{\tau_\theta} t^3 x^2 (1-x)^2$ and $\tau_\theta = 16$, $\tau_T = 0.043$ (the motivation for this choice see in [41, 58]).

We seek the approximate solution in the form of the truncated Fourier series

$$
T(x,t) = \sum_{n=1}^{N} w_n(t) \psi_n(x), \tag{4.11}
$$

over the orthonormal basis

$$
\psi_n(x) = \sqrt{2} \sin n\pi x, \quad n = 1,2,\cdots.
$$

Substituting (4.11) in (4.10a)-(4.10c) and using the projecting $\langle \cdot \rangle_1$ given in (3.26), we get the system of the separated equations similar to (4.4) for each harmonic $w_n(t)$:

$$
D_{1+\alpha}^t[w_n] + \frac{1}{\tau_\theta^\beta} \frac{d w_n}{d t} = - \frac{(n\pi)^2}{\tau_\theta^\beta} \left(1 + \tau_T^\beta D_{1}^t \right) w_n + \theta_n(t), \quad 0 \leq t \leq 1, \tag{4.12}
$$

where

$$
\theta_n(t) = \frac{1}{\tau_\theta^\beta} \int_0^1 f(x,t) \psi_n(x) \, dx.
$$

The equation is subjected to the ICs which follow from (4.10b):

$$
w_n(0) = \frac{\partial w_n(0)}{\partial t} = 0. \tag{4.13}$$
Table 5: Example 4.2. The errors $E_{\text{max}}$, $E_{\text{rel}}$ the convergence order and CPU time with respect to the change of $N$ at the time moments $t = 1$ and $t = 5$. $M = 14$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{\text{rel}}$</th>
<th>$\text{CO}(N)$</th>
<th>CPU, s</th>
<th>$E_{\text{max}}$</th>
<th>$E_{\text{rel}}$</th>
<th>$\text{CO}(N)$</th>
<th>CPU, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7.27E-1</td>
<td>3.76E-2</td>
<td>1.92</td>
<td>0.01</td>
<td>90.8</td>
<td>3.76E-2</td>
<td>1.92</td>
<td>0.01</td>
</tr>
<tr>
<td>8</td>
<td>1.93E-1</td>
<td>7.55E-3</td>
<td>1.98</td>
<td>0.03</td>
<td>26.1</td>
<td>7.55E-3</td>
<td>1.98</td>
<td>0.03</td>
</tr>
<tr>
<td>16</td>
<td>4.89E-2</td>
<td>1.39E-3</td>
<td>1.99</td>
<td>0.04</td>
<td>6.11</td>
<td>1.39E-3</td>
<td>1.99</td>
<td>0.04</td>
</tr>
<tr>
<td>32</td>
<td>1.23E-2</td>
<td>2.47E-4</td>
<td>2.00</td>
<td>0.09</td>
<td>1.53</td>
<td>2.47E-4</td>
<td>2.00</td>
<td>0.09</td>
</tr>
<tr>
<td>64</td>
<td>3.07E-3</td>
<td>4.38E-5</td>
<td>2.00</td>
<td>0.17</td>
<td>3.84E-1</td>
<td>4.38E-5</td>
<td>2.00</td>
<td>0.17</td>
</tr>
<tr>
<td>256</td>
<td>1.92E-4</td>
<td>1.37E-6</td>
<td>2.00</td>
<td>0.70</td>
<td>2.40E-2</td>
<td>1.37E-6</td>
<td>2.00</td>
<td>0.70</td>
</tr>
<tr>
<td>1024</td>
<td>1.20E-5</td>
<td>4.28E-8</td>
<td>2.00</td>
<td>2.79</td>
<td>1.50E-3</td>
<td>4.28E-8</td>
<td>2.00</td>
<td>2.79</td>
</tr>
<tr>
<td>4096</td>
<td>7.49E-7</td>
<td>1.34E-9</td>
<td>2.00</td>
<td>10.9</td>
<td>9.37E-5</td>
<td>1.34E-9</td>
<td>2.00</td>
<td>10.9</td>
</tr>
</tbody>
</table>

Thus, we get the sequence of the time-fractional problems each of which is identical to the problem (4.4), (4.5) and can be solved by the usage of the algorithm described above.

To study the influence of $N$ and $M$ on the error of the approximate solution $v_{N,M}(x,t)$ we introduce the convergence orders

$$
\text{CO}(N,t) = \log_2 \frac{E_{\text{max}}(N,M,t)}{E_{\text{max}}(2N,M,t)}, \quad \text{CO}(M,t) = \log_2 \frac{E_{\text{max}}(N,M,t)}{E_{\text{max}}(N,2M,t)}.
$$

Here and below to test the accuracy of the proposed method at each time moment we use the maximal absolute error

$$
E_{\text{max}}(t) = \max_{1 \leq i \leq N_t} |T_{ex}(x_i,t) - T_{N,M}(x_i,t)|,
$$

and the relative error

$$
E_{\text{rel}}(t) = \sqrt{\frac{\sum_{i=1}^{N_t} [T_{ex}(x_i,t) - T_{N,M}(x_i,t)]^2}{\Sigma_{i=1}^{N_t} [T_{ex}(x_i,t)]^2}}.
$$

The data shown in Table 5 correspond to $\alpha = 0.3$, $\beta = 0.8$. The value of the Müntz’s parameter is taken $\sigma = 0.15$. The data of the table show that the method has approximately the second order of the convergence with respect to the number of harmonics $N$. The errors $E_{\text{max}}$ rather differ for $t = 1$ and $t = 5$, but the relative errors $E_{\text{rel}}$ are the same. The behaviour of the $E_{\text{max}}$ as a function of $N$ with different fixed $M$ is shown with more details on the graphic in the left hand side of Fig. 2.

With the growth of $N$ the graphs $\log E_{\text{max}}$ move away from the original curves depending on the value of $M$. This occurs when the error due to the approximation in time becomes dominated and the growth of the number of the spatial harmonics $N$ does not improve the accuracy of the approximate solution. Table 6 and the graphic in the right hand side of Fig. 2 display the behaviour of the $E_{\text{max}}$ as a function of $M$ with fixed $N$. 
The graphics in Fig. 3 show the difference between the exact solution and the numerical approximations with $N = 4096$ spatial harmonics. The left hand side subfigure corresponds to the case when the solution is sought in the interval $0 \leq t \leq 1$ and the right hand side subfigure corresponds to the interval $0 \leq t \leq 5$. In both cases the number of the Müntz’s polynomials $\phi_m(t)$ in the approximate solutions is the same $M = 14$.

This problem with the same parameters was also considered by Zheng et al. in [41] by using the finite difference/Legendre spectral method. The most accurate results obtained there contain the error $3.0312E-4$ (Table 4).

**Example 4.3.** Let us consider FDPE of the fractional DPL model with the temperature-jump boundary condition which arises in the heat conduction of nanostructures [40]:

$$\frac{\partial u}{\partial t} + D_t^{(1+\alpha)} [u] = \frac{K_n^2}{3[\Gamma(1+\alpha)]^{1/\alpha}} \left(1 + B^a D_t^{(\alpha)}\right) \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < \alpha < 1,$$

(4.15)
where $K_n$ is the so-called Knudsen number, $B = \frac{\gamma}{\gamma - 1}$. The Robin BCs

$$u_x(0,t) - bu(0,t) = g_0(t), \quad u_x(1,t) + bu(1,t) = g_1(t), \quad b = \frac{1}{\gamma K_n}, \quad 0 \leq t \leq 1, \quad (4.16)$$

and the ICs

$$u(x,0) = u_t(x,0) = 0 \quad (4.17)$$

conform the analytical solution $u(x,t) = t^3 \sin(\pi x)$.

The function

$$u_g(x,t) = \frac{g_1(t) - g_0(t)(1+b) + xb(g_0(t) + g_1(t))}{2b + b^2} \quad (4.18)$$

satisfies BCs (4.16). Let us write the solution of the given problem as the sum

$$u(x,t) = u_g(x,t) + w(x,t). \quad (4.19)$$

Therefore, $w(x,t)$ is a solution of the equation

$$\frac{\partial w}{\partial t} + D^{(1+\alpha)}_t [w] = \frac{K_n^2}{3[\Gamma(1+\alpha)]^{1/\alpha}} \left( 1 + B^\alpha D^{(\alpha)}_t \right) \frac{\partial^2 w}{\partial x^2} + f_1(x,t) \quad (4.20)$$

subjected to the homogeneous BCs

$$w_x(0,t) - bw(0,t) = 0, \quad w_x(1,t) + bw(1,t) = 0, \quad (4.21)$$

and the ICs

$$w(x,0) = -u_g(x,0), \quad w_t(x,0) = -u_{gt}(x,0),$$
which follow from (4.17), (4.19). The new source term is

\[ f_1(x,t) = f(x,t) - \partial u_g / \partial t - D_t^{(1+\alpha)}[u_g]. \]

The functions

\[ \psi_n(x) = \sqrt{\frac{2\lambda_n^2}{\lambda_n^2 + 2b + b^2}} \left( \cos \lambda_n x + \frac{b}{\lambda_n} \sin \lambda_n x \right), \quad n = 1, 2, \ldots \]  \hspace{1cm} (4.22)

are eigenfunctions of the Sturm-Liouville problem

\[ \frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad \psi(0) - b \psi(0) = 0, \quad \psi(1) + b \psi(1) = 0, \]

and they form the orthonormal basis in the Hilbert space \( L_2([0,1]) \)

\[ \langle \psi_m, \psi_n \rangle_1 = \int_0^1 \psi_m(x) \psi_n(x) dx = \delta_{m,n}. \]

Here \( \lambda_n \) are the solutions of the transcendental equation

\[ 2b \lambda \cos \lambda = (\lambda^2 - b^2) \sin \lambda. \]

We solve the problem with the following parameters \([40, 59]\): \( K_n = \frac{4}{\pi}, \quad \gamma = 0.1, \quad B = \frac{4}{3}. \)

In this case \( b = 2.5\pi \) and the first three roots are: \( \lambda_1 = 2.520514685, \quad \lambda_2 = 5.126590445, \quad \lambda_3 = 7.853981634. \) The corresponding eigenfunctions are shown in Fig. 4.

We seek the solution of Eq. (4.20) in the form of the series over the basis (4.22):

\[ w(x,t) = \sum_{n=1}^{\infty} w_n(t) \psi_n(x) \]
Table 7: Example 4.3. The errors $E_{\text{max}}$, $E_{\text{rel}}$, the convergence order and CPU time with respect to the change of $N$ at the time moments $t=1$ and $t=5$. $M=12$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{\text{rel}}$</th>
<th>CO($N$)</th>
<th>CPU, s</th>
<th>$E_{\text{max}}$</th>
<th>$E_{\text{rel}}$</th>
<th>CO($N$)</th>
<th>CPU, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3.92E-3</td>
<td>3.35E-3</td>
<td>2.92</td>
<td>0.04</td>
<td>3.55E-3</td>
<td>2.92</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>8</td>
<td>5.17E-4</td>
<td>3.29E-4</td>
<td>3.10</td>
<td>0.07</td>
<td>6.47E-2</td>
<td>3.29E-4</td>
<td>3.10</td>
<td>0.09</td>
</tr>
<tr>
<td>16</td>
<td>6.02E-5</td>
<td>2.69E-5</td>
<td>3.09</td>
<td>0.16</td>
<td>7.53E-3</td>
<td>2.69E-5</td>
<td>3.09</td>
<td>0.21</td>
</tr>
<tr>
<td>32</td>
<td>7.05E-6</td>
<td>2.21E-6</td>
<td>3.06</td>
<td>0.28</td>
<td>8.81E-4</td>
<td>2.21E-6</td>
<td>3.06</td>
<td>0.40</td>
</tr>
<tr>
<td>64</td>
<td>8.47E-7</td>
<td>1.87E-7</td>
<td>3.03</td>
<td>0.52</td>
<td>1.06E-3</td>
<td>1.87E-7</td>
<td>3.03</td>
<td>0.74</td>
</tr>
<tr>
<td>256</td>
<td>1.28E-8</td>
<td>1.41E-9</td>
<td>3.01</td>
<td>1.86</td>
<td>1.60E-6</td>
<td>1.41E-9</td>
<td>3.01</td>
<td>2.35</td>
</tr>
<tr>
<td>1024</td>
<td>2.01E-10</td>
<td>8.53E-12</td>
<td>2.89</td>
<td>7.3</td>
<td>2.48E-8</td>
<td>8.53E-12</td>
<td>2.89</td>
<td>15.6</td>
</tr>
</tbody>
</table>

and transform the FPDE into a sequence of independent FODEs:

$$
\frac{dw_n}{dt} + D_1^{1+\alpha} \left[ w_n \right] = - \frac{\lambda_n^2 K_n^2}{\Gamma(1+\alpha)} \left( 1 + B^a D_1^{(a)} \right) \frac{d^2 w_n}{dx^2} + F_n(t),
$$

(4.23)

and

$$
\psi_n(x) = h_{0,n}, \quad \frac{dw_n(0)}{dt} = h_{1,n}.
$$

Here

$$
F_n(t) = \int_0^1 f_1(x,t) \psi_n(x) dx, h_{0,n} = - \int_0^1 u_g(x,0) \psi_n(x) dx, h_{1,n} = - \int_0^1 u_{g,t}(x,0) \psi_n(x) dx.
$$

Table 7 shows the errors, the convergence order and CPU time at the time moments $t=1$ and $t=5$. In Fig. 5 the absolute maximal error $E_{\text{max}}$ is shown at the time moment $t=1$ as a function of $N$ with fixed $M$ and as a function of $M$ with fixed $N$. The graphics in Fig. 6 show the absolute differences between the exact solution and the numerical approximations for both variants of the time interval $[0,1]$ and $[0,5]$ with the same number of the Müntz polynomials $M=20$.

Example 4.4. Let us consider equation of the DPL model [60]:

$$
\frac{\partial^3 u}{\partial t^3} + 2 \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2} + 8 \frac{\partial^3 u}{\partial x^2 \partial t} + 4 \frac{\partial^4 u}{\partial x^2 \partial t^2} + f(x,t) \quad \text{in} \quad (0,1) \times (0,1),
$$

(4.24a)

$$
u(0,t) = 0, \quad u(1,t) = e^t,
$$

(4.24b)

$$
u(x,0) = x^2, \quad \partial u_t(x,0) = x^2, \quad \partial u_{tt}(x,0) = x^2,
$$

(4.24c)

which conforms the analytical solution $u(x,t) = x^2 e^t$. So, this PDE corresponds to the DPL(2,2) model of the integer order. Using the substitution $u(x,t) = u_g(x,t) + w(x,t)$,
where \( u_g(x,t) = xu(1,t) = xe^t \), we transform the original problem into the one with the homogeneous BCs

\[
\frac{\partial^3 w}{\partial t^3} + 2\frac{\partial^2 w}{\partial t^2} + 2\frac{\partial w}{\partial t} = 4\frac{\partial^2 w}{\partial x^2} + 8\frac{\partial^3 w}{\partial x^2 \partial t} + 4\frac{\partial^4 w}{\partial x^2 \partial t^2} + f_1(x,t),
\]

\( w(0,t) = 0, \quad w(1,t) = 0, \quad (4.25a) \)

and with the ICs

\[
w(x,0) = x^2 - x, \quad \partial w_t(x,0) = x^2 - x, \quad \partial w_{tt}(x,0) = x^2 - x. \quad (4.26)
\]
Table 8: Example 4.4. The DPL_{\alpha,\beta}(2,2) model. The errors, the convergence order and CPU time with respect to the change of N. M = 14.

<table>
<thead>
<tr>
<th>N</th>
<th>E_{max}</th>
<th>E_{rel}</th>
<th>CO(N)</th>
<th>CPU, s</th>
<th>E_{max}</th>
<th>E_{rel}</th>
<th>CO(N)</th>
<th>CPU, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8.02E-3</td>
<td>3.25E-3</td>
<td>1.88</td>
<td>0.27</td>
<td>8.60E-4</td>
<td>7.08E-4</td>
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<td>0.15</td>
</tr>
<tr>
<td>8</td>
<td>2.06E-3</td>
<td>6.56E-4</td>
<td>1.99</td>
<td>0.57</td>
<td>2.42E-4</td>
<td>1.50E-4</td>
<td>1.92</td>
<td>0.32</td>
</tr>
<tr>
<td>16</td>
<td>5.20E-4</td>
<td>1.21E-4</td>
<td>2.00</td>
<td>2.34</td>
<td>1.65E-5</td>
<td>5.38E-5</td>
<td>1.98</td>
<td>1.33</td>
</tr>
<tr>
<td>32</td>
<td>1.30E-4</td>
<td>2.18E-5</td>
<td>2.00</td>
<td>4.75</td>
<td>4.19E-5</td>
<td>9.74E-5</td>
<td>1.98</td>
<td>1.33</td>
</tr>
<tr>
<td>64</td>
<td>3.26E-5</td>
<td>3.89E-6</td>
<td>2.00</td>
<td>9.50</td>
<td>1.65E-5</td>
<td>9.74E-5</td>
<td>1.99</td>
<td>1.33</td>
</tr>
<tr>
<td>128</td>
<td>2.04E-6</td>
<td>1.22E-7</td>
<td>2.00</td>
<td>19.4</td>
<td>2.64E-7</td>
<td>3.09E-8</td>
<td>2.00</td>
<td>10.7</td>
</tr>
<tr>
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<td>77</td>
<td>1.66E-8</td>
<td>9.80E-10</td>
<td>2.00</td>
<td>43</td>
</tr>
<tr>
<td>2048</td>
<td>7.96E-9</td>
<td>1.19E-10</td>
<td>2.00</td>
<td>303</td>
<td>1.04E-9</td>
<td>1.37E-10</td>
<td>2.00</td>
<td>175</td>
</tr>
</tbody>
</table>

The approximate solution of the problem (4.25a), (4.25b), (4.26) can be represented the form of the series over the orthonormal basis \( \psi_n(x) = \sqrt{2} \sin n\pi x \). As a result we get the sequence of the separate equations for each harmonic \( w_n(t) \):

\[
\frac{d^3 w_n}{dt^3} + 2 \frac{d^2 w_n}{dt^2} + 2 \frac{dw_n}{dt} = - (n\pi)^2 \left( 4w_n + 8 \frac{dw_n}{dt} + 4 \frac{d^2 w_n}{dt^2} \right) + F_n(t),
\]

(4.27)

which are subjected to the ICs

\[
w_n(0) = w_{n,t}(0) = w_{n,tt}(0) = h_n = \int_0^1 (x^2 - x) \psi_n(x) dx,
\]

\[
F_n(t) = \int_0^1 f_1(x,t) \psi_n(x) dx.
\]

According to the algorithm described above we look for the approximate solution as a sum

\[
w_{n,M}(t) = w_{n,p}(t) + \sum_{m=1}^{M} q_{n,m} \phi_m(t).
\]

Here \( \phi_m(t) \) is given in (3.13) and the particular solution is: \( w_{n,p}(t) = h_n \left( 1 + t + 0.5t^2 \right) \).

We get the approximate solution in the form

\[
u_{N,M}(x,t) = u_g(x,t) + w_p(x,t) + \sum_{m=1}^{M} Q_m(x) \phi_m(t),
\]

(4.28)

where

\[
w_p(x,t) = \sum_{n=1}^{N} w_{n,p}(t) \psi_n(x), \quad Q_m(x) = \sum_{n=1}^{N} q_{n,m} \psi_n(x).
\]

(4.29)

The behaviour of errors with the growth of N and the convergence order are shown in the left hand side of Table 8. The last row contains the error obtained in [60] by the usage of the combination of the finite element method with the implicit Euler scheme.
Let us consider the FPDE of the DPL $\alpha, \beta$ $(2,2)$ model similar to (4.24a), (4.24b), (4.24c):

\begin{equation*}
D_t^{(1+2\alpha)}[u] + 2D_t^{(1+\alpha)}[u] + 2\frac{\partial u}{\partial t} = \left( 4 + 8D_t^{(\beta)} + 4D_t^{(2\beta)} \right) \frac{\partial^2 u}{\partial x^2} + f(x,t),
\end{equation*}

$u(0,t) = 0, \quad u(1,t) = \sin 1 \sin t,$

$u(x,0) = 0, \quad \partial u_t(x,0) = \sin x, \quad \partial u_{tt}(x,0) = 0,$

with the analytical solution $u(x,t) = \sin x \sin t$. Using the substitution $u(x,t) = x \sin t + w(x,t)$ and representing the approximate solution in the form of the series over the orthonormal basis $\psi_n(x) = \sqrt{2} \sin n\pi x$, we get

\begin{equation*}
D_t^{(1+2\alpha)}[w_n] + 2D_t^{(1+\alpha)}[w_n] + 2\frac{dw_n}{dt} = -\left( n\pi \right)^2 \left( 4w_n + 8D_t^{(\beta)}[w_n] + 4D_t^{(2\beta)}[w_n] \right) + F_n(t),
\end{equation*}

$w_n(0) = 0, \quad w_{n,t}(0) = \int_0^1 \left( \sin x - x \right) \psi_n(x) dx, \quad w_{n,tt}(0) = 0.$

The data shown in the right hand side of Table 8 correspond to $\alpha = 0.8, \beta = 0.7$.

The graphics in Fig. 7 show the absolute differences between the exact solution and the numerical approximations for the DPL models of integer and fractional orders with the same numbers of the harmonics $N = 4096$ and of Müntz’s polynomials $M = 14$.

One of the most important applications of the time fractional DPL models is the heat transfer process in biological tissues. This concerns the therapeutic procedures as well as the temperature-based disease diagnostics. So, the next examples are placed to demonstrate the ability of the present method to handle with the fractional DPL models using realistic parameters of the bioheat equation.

**Example 4.5.** Let us consider FPDE (1.1) of the bioheat FDPL $\alpha, \beta$ $(1,1)$ model with coefficients which conform the parameters of the tissue: $K = 0.5 \text{ W/(m}\cdot\text{C}^0); \quad c_p = 4180 \text{ J/(kg}\cdot\text{C}^0); \quad \rho = 1000 \text{ kg/m}^3; \quad \tau_q = 16 \text{ s}; \quad \tau_T = 0.05 \text{ s}$. We use the reference values of the length and time:
Table 9: Example 4.5. The errors, CO and CPU time of approximate solution of the DPL_{\alpha,\beta}(1,1) FPDE. \( \alpha=0.6, \beta=0.7. \)

<table>
<thead>
<tr>
<th>N</th>
<th>( M=10 )</th>
<th>( M=20 )</th>
<th>( M=25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E_{\text{max}} )</td>
<td>( \text{CO}(N) )</td>
<td>( \text{CPU} )</td>
</tr>
<tr>
<td>4</td>
<td>2.21E-4</td>
<td>3.04</td>
<td>0.8</td>
</tr>
<tr>
<td>8</td>
<td>2.69E-5</td>
<td>3.07</td>
<td>0.8</td>
</tr>
<tr>
<td>16</td>
<td>3.19E-6</td>
<td>3.18</td>
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</tr>
<tr>
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<td>3.2</td>
</tr>
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<td>4.12E-7</td>
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<td>6.3</td>
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<tr>
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<tr>
<td>1024</td>
<td>1.09E-7</td>
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<td>13</td>
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</table>

\( L_0=0.01 \text{m}, \ t_0=10 \text{s}. \) These thermophysical parameters are typical for bioheat transfer problems [21–25]. The dimensionless blood perfusion term is taken \( p^2 = w_b c_L L_0^2 / K = 0.5. \) This matches the data of [21–23], where the analogous parameter \( \beta = \sqrt{p} \) changes from 0.1 to 1. The governing FPDE can be written in the compact form as follows

\[
D_t^{(a+1)} u + a_1 \frac{\partial u}{\partial t} = \left(a_2 + a_3 D_t^{(\beta)}\right) \frac{\partial^2 u}{\partial x^2} - p^2 \left(a_4 D_t^{(0)} + a_5 D_t^{(\alpha)}\right) u + f(x,t). \quad (4.30)
\]

The coefficients \( a_i \) are given in (2.10), (2.17).

The source term \( f(x,t) \), ICs \( u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x) \) and Robin’s boundary conditions

\[
\begin{align*}
  u_x(0,t) - c_1 u(0,t) &= g_0(t), & u_x(1,t) + c_2 u(1,t) &= g_1(t), & 0 \leq t \leq 1, \\
\end{align*}
\]

conform the analytical solution \( u(x,t) = \sin x \sin t. \)

The function

\[
\begin{align*}
  u_8(x,t) &= \frac{g_1(t) - g_0(t)(1+c_2) + x(c_2 g_0(t) + c_1 g_1(t))}{c_1 + c_2 + c_1 c_2},
\end{align*}
\]

satisfies BCs (4.31) except the case \( c_1 = c_2 = 0. \)

Let us write the solution of the given problem as the sum

\[
\begin{align*}
  u(x,t) &= u_8(x,t) + w(x,t). \\
\end{align*}
\]

Then we get the FPDE for \( w(x,t) \) subjected to the homogeneous boundary conditions

\[
\begin{align*}
  w_x(0,t) - c_1 w(0,t) &= 0, & w_x(1,t) + c_2 w(1,t) &= 0.
\end{align*}
\]

We seek the approximate solution of the FPDE in the form of the series

\[
\begin{align*}
  w(x,t) &= \sum_{n=1}^{N} w_n(t) \psi_n(x). \\
\end{align*}
\]
over the orthonormal basis system
\[ \psi_n(x) = \frac{1}{\sqrt{g_n}} \left( \cos \lambda_n x + \frac{1}{\lambda_n} \sin \lambda_n x \right), \quad n = 1, 2, \ldots. \]

Here
\[ g_n = \frac{1}{2} \left( 1 + \frac{c_1^2}{\lambda_n^2} \right) + \frac{c_1}{2\lambda_n^2} + \frac{c_2^2 + c_1^2}{2\lambda_n^2 \lambda_n^2 + c_2^2} \]
and \( \lambda_n \) are the roots of the equation
\[ (\lambda^2 - c_1 c_2) \sin \lambda = (c_1 + c_2) \lambda \cos \lambda. \]

The functions \( \psi_n(x) \) are the eigenfunctions of the Sturm-Liouville problem
\[ \frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad \psi(0) - c_1 \psi(0) = 0, \quad \psi(1) + c_2 \psi(1) = 0, \]
and they form the orthonormal basis in the Hilbert space \( L_2([0,1]) \)
\[ \langle \psi_m, \psi_n \rangle_1 = \int_0^1 \psi_m(x) \psi_n(x) dx = \delta_{m,n}. \]

In this example we take \( c_1 = 1, \ c_2 = 2 \) and the first three roots are: \( \lambda_1 = 1.509410345, \lambda_2 = 3.871244368, \lambda_3 = 6.720171109 \). As a result we get the sequence of the independent FODEs:
\[ D^{(a+1)}_t [w_n] + a_1 D^{(a+1)}_t [u] + a_2 \frac{\partial u}{\partial t} = -\lambda_n^2 \left( a_2 + a_3 D^{(b)}_t \right) w_n - p^2 \left( a_4 + a_5 D^{(a)}_t \right) w_n + F_n(t), \quad (4.34) \]
which are solved by the method described above. Table 9 shows the errors, convergence order and CPU time for \( \alpha = 0.6, \ beta = 0.7 \). The graphic in the left-hand-side of Fig. 8 shows the absolute differences between the exact solution and the numerical approximation for \( N = 1024 \) Fourier’s harmonics and \( M = 25 \) Müntz’s polynomials.

**Example 4.6.** In the similar way the FDPL \( \alpha, \beta (2,2) \) model can be written in the form
\[ D^{(2a+1)}_t [u] + a_1 D^{(a+1)}_t [u] + a_2 \frac{\partial u}{\partial t} = \left( a_3 + a_4 D^{(b)}_t + a_5 D^{(2b)}_t \right) \frac{\partial^2 u}{\partial x^2} - p^2 \left( a_6 D^{(0)}_t + a_7 D^{(a)}_t + a_8 D^{(2a)}_t \right) u + f(x,t), \quad (4.35) \]
with the coefficients \( a_i \) which conform the thermophysical parameters of the tissue and blood given in Example 4.5. The source term \( f(x,t) \), ICs, and Robin’s boundary conditions conform the analytical solution \( u(x,t) = \sin x \exp t \). After transform (4.32) we seek
Figure 8: Absolute errors with respect to the exact solution for the numerical approximations of $DPL_{\alpha,\beta}(1,1)$ (left) and $DPL_{\alpha,\beta}(2,2)$ (right) models. $\alpha = 0.6, \beta = 0.7, N = 1024, M = 25$.

Table 10: Example 4.6. The errors, CO and CPU time of approximate solution of the $DPL_{\alpha,\beta}(2,2)$ FPDE. $\alpha = 0.6, \beta = 0.7$.

<table>
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<th>CO(N)</th>
<th>CPU</th>
<th>$E_{\text{max}}$</th>
<th>CO(N)</th>
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</tbody>
</table>

The solution of the FPDEs in the form of the series (4.33). As a result we get the FODE for each harmonic $w_n$

$$
D_t^{(2\alpha+1)} [w_n] + a_1 D_t^{(\alpha+1)} [w_n] + a_2 \frac{dw_n}{dt} = -\lambda^2 \left( a_3 + a_4 D_t^{(\beta)} + a_5 D_t^{(2\beta)} \right) w_n - p^2 \left( a_6 + a_7 D_t^{(\beta)} + a_8 D_t^{(2\beta)} \right) w_n + F_n(t), \tag{4.36}
$$

Table 10 shows the errors, convergence order and CPU time for $\alpha = 0.6, \beta = 0.7$. The graphic in the right-hand-side of Fig. 8 shows the absolute differences between the exact solution and the numerical approximation for $N = 1024$ Fourier’s harmonics and $M = 20$ Müntz’s polynomials.

The all problems considered above have exact analytical solutions and have been used to verify the accuracy and convergence of the method presented. But the reason for using
fractional DPL models is that they allow greater degrees of freedom. So, in the next examples we demonstrate the difference between DPL models of integer and fractional types.

**Example 4.7.** Let us consider the FPDE of the FDPL \(\alpha, \beta\) \((1,1)\) model without the inner sources \(f(x,t) = 0\)

\[
D_t^{(\alpha+1)}[u] + a_1 \frac{\partial u}{\partial t} = \left(a_2 + a_3 D_t^{(\beta)} \right) \frac{\partial^2 u}{\partial x^2} - p^2 \left(a_4 + a_5 D_t^{(\alpha)} \right) u.
\]

The coefficients of the FPDE conform the thermophysical parameters of the tissue and blood given in Example 4.5.

The boundary conditions

\[
\frac{\partial u(0,t)}{\partial x} = q_0(t), \quad u(1,t) = u_0(t),
\]

correspond to the prescribed flux \(q_0(t)\) at the left hand side boundary and the prescribed temperature \(u_0(t)\) at the right hand side. We utilize zero initial conditions \(u(x,0) = \partial u(x,0)/\partial t = 0\).

We seek the solution of the FPDEs in the form of the series over the orthonormal basis system \(\psi_n(x) = \sqrt{2}\cos((n-1/2)\pi x), n = 1, 2, \ldots\). The data presented in Figs. 9, 10 are obtained by using \(N = 512\) harmonics and \(M = 25\) Müntz’s polynomials. The heat flux at the left hand side boundary is taken in the form \(q_0(t) = e^{5t}\) and the temperature at the right hand side boundary is fixed \(u(1,t) = 0\). Fig. 9 shows distributions of the dimensionless temperature at the time moments \(t = 0.4, 0.6, 0.8, 1.0\) for integer DPL\(_{1,1}\) \((1,1)\) and fractional DPL\(_{0.5,0.5}\) \((1,1)\) bioheat transfer models. Fig. 10 demonstrates evolution of the dimensionless temperature at the points \(x = 0.2, 0.4, 0.8\) with the growth of \(t\). The both figures show that fractional derivatives have a strong influence on the models of the heat transfer.

![Figure 9: The distribution of the temperature along the interval \([0,1]\) at the different time moments. Here I corresponds to the fractional model DPL\(_{0.5,0.5}\) \((1,1)\), II corresponds to the integer model DPL\(_{1,1}\) \((1,1)\).](image-url)
Example 4.8. Let us consider the FPDE of the FDPL$_{\alpha, \beta}(1,1)$ model with the external source

$$D_t^{(\alpha+1)} u + a_1 \frac{\partial u}{\partial t} = (a_2 + a_3 D_t^{(\beta)}) \frac{\partial^2 u}{\partial x^2} - p^2 \left( a_4 D_t^{(0)} + a_5 D_t^{(\alpha)} \right) u + \left( 1 + a_1 D_t^{(\alpha)} \right) [S_e(x,t)],$$

(4.37)

where the coefficients of the FPDE conform the thermophysical parameters of the tissue and blood given in Example 4.5. The source function has the form of the product

$$S_e(x,t) = FX(x,\zeta) FT(t),$$

where $FX(x,\zeta) = AS_6(x - \zeta, h)$ and $S_6(x,h)$ is the sextic B-spline [62] given by:

$$S_6(x,h) = \frac{1}{h^6} \begin{cases} 
(x+3h)^6, & x \in [-3h, -2h], \\
(x+3h)^6 - 7(x+2h)^6, & x \in [-2h, -h], \\
(x+3h)^6 - 7(x+2h)^6 + 21(x+2h)^6, & x \in [-h, 0], \\
(x+3h)^6 - 7(x+2h)^6 + 21(x+2h)^6 - 35(x)^6, & x \in [0,h], \\
(x-4h)^6 - 7(x-3h)^6 + 21(x-2h)^6, & x \in [h,2h], \\
(x-4h)^6 - 7(x-3h)^6, & x \in [2h,3h], \\
(x-4h)^6, & x \in [3h,4h], \\
0, & \text{otherwise.}
\end{cases}$$

This function belongs to the $C^{(5)}$ class of the smoothness and is a function with the finite support $[-3h,4h]$. The function depicted in Fig. 11 corresponds to the parameters: $A = 1$, $h = 0.1$, $\zeta = 0.5$.

We utilize zero initial conditions $u(x,0) = \partial u (x,0) / \partial t$ and the boundary condition of the Neumann type $\partial u(0,t) / \partial x = \partial u(1,t) / \partial x = 0$. Thus, we model problem when there is
no heat flux at the boundaries $x = 0$ and $x = 1$. It is well known that the spectrum of the Sturm-Liouville problem
\[ \frac{d^2 \psi}{dx^2} = -\lambda^2 \psi, \quad \psi_x(0) = 0, \quad \psi_x(1) = 0, \]
contains zero eigenvalue $\lambda_0$ corresponding to the eigenfunction $\psi_0(x) = 1$ which should be added to the other ones $\psi_n(x) = \sqrt{2}\cos(n\pi x)$, $n = 1, 2, \cdots$, to form the orthonormal system complete in a Hilbert space $L_2([0,1])$. Thus, we seek the solution of the FPDEs in the form of the series
\[ w(x,t) = w_0(t) + \sum_{n=1}^{N} w_n(t) \psi_n(x) \quad (4.38) \]
(cf. (4.33)). As a result we get the sequence of the FODEs
\[ D_t^{(\alpha+1)}[w_0] + a_1 \frac{dw_0}{dt} = -p^2 \left( a_4 + a_5 D_t^{(\alpha)} \right) w_0 + F_0(t), \quad (4.39a) \]
\[ D_t^{(\alpha+1)}[w_n] + a_1 \frac{dw_n}{dt} = -\lambda_n^2 \left( a_2 + a_3 D_t^{(\beta)} \right) w_n - p^2 \left( a_4 + a_5 D_t^{(\alpha)} \right) w_n + F_n(t), \quad (4.39b) \]
where
\[ F_n(t) = A_n(\xi) \left( 1 + a_1 D_t^{(\alpha)} \right) [FT(t)], \quad A_n(\xi) = \int_0^1 FX(x,\xi) \psi_n(x) dx. \]
The FODEs (4.39a), (4.39b) have been solved with zero ICs $w_n(0) = dw_n(0)/dt = 0$, $n = 0, 1, 2, \cdots$, utilizing the algorithm described above.
Figure 12: The evolution of $u(x,t)$ with the growth of $t$ at the points $x=0.2$, $x=0.3$, $x=0.4$. Here I corresponds to the fractional model DPL$_{0.5,0.5}(1,1)$, II corresponds to the integer model DPL$_{1,1}(1,1)$.

Figure 13: The distribution of the temperature along the interval $[0,1]$ at the different time moments. Here I corresponds to the fractional model DPL$_{0.5,0.5}(1,1)$, II corresponds to the integer model DPL$_{1,1}(1,1)$.

The data presented in Figs. 12, 13 correspond to the time source function $FT(t) = e^{5t}$ and are obtained by using $N = 300$ harmonics and $M = 25$ Müntz’s polynomials. The data show the strong influence of the fractional derivatives on the modeling of the heat transfer.

5 Conclusions

This paper presents a new numerical method for solving time fractional PDEs of the DPL models of the heat transfer. To transform the original FPDE into a sequence of independent fractional ordinary differential equations for each harmonic we use the basis system which is formed by the eigenfunctions of the corresponding Sturm-Liouville problems. Then, we apply an effective numerical technique for solving the resulting multi-term FODEs. The accuracy and effectiveness of the presented method is illustrated by solving time fractional DPL problems of the first and second orders. In particular, several
tests were performed to verify that the present method can handle with the fractional DPL models of the bioheat equations. The proposed method can also be used for high dimension problems with coefficients variable in time. In particular, it can be applied for solving time-fractional telegraph equations and the time-fractional modified anomalous sub-diffusion equations. These problems will be studied in the nearest future. It should be noted that the present method is ideal for the use of parallelizing algorithm.

Acknowledgements

The author thanks the editor and anonymous reviewers for their constructive comments on the manuscript. The work was supported by the Natural Science Foundation of China (No. 12072103), the Fundamental Research Funds for the Central Universities (No. B200202126), the Natural Science Foundation of Jiangsu Province (No. BK20190073), the State Key Laboratory of Acoustics, Chinese Academy of Sciences (No. SKLA202001), the State Key Laboratory of Mechanical Behavior and System Safety of Traffic Engineering Structures, Shijiazhuang Tiedao University (No. KF2020-22), the Key Laboratory of Intelligent Materials and Structural Mechanics of Hebei Province (No. KF2021-01) and the China Postdoctoral Science Foundation (Nos. 2017M611669, 2018T110430).

References


