A Two-Level Factored Crank-Nicolson Method for Two-Dimensional Nonstationary Advection-Diffusion Equation With Time Dependent Dispersion Coefficients and Source Terms

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Abstract. This paper deals with a two-level factored Crank-Nicolson method in an approximate solution of two-dimensional evolutionary advection-diffusion equation with time dependent dispersion coefficients and sink/source terms subjects to appropriate initial and boundary conditions. The procedure consists to reducing problems in many space variables into a sequence of one-dimensional subproblems and then find the solution of tridiagonal linear systems of equations. This considerably reduces the computational cost of the algorithm. Furthermore, the proposed approach is fast and efficient: unconditionally stable, temporal second order accurate and fourth order convergent in space and it improves a large class of numerical schemes widely studied in the literature for the considered problem. The stability of the new method is deeply analyzed using the $L^\infty(t_0,T_f;L^2)$-norm whereas the convergence rate of the scheme is numerically obtained in the $L^2$-norm. A broad range of numerical experiments are presented and critically discussed.

AMS subject classifications: 35K20, 65M06, 65M12

Key words: Two-dimensional advection-diffusion equation, time dependent dispersion coefficients, Crank-Nicolson approach, a two-level factored Crank-Nicolson method, stability and convergence rate.

1 Introduction and motivation

The advection-diffusion-reaction (ADR) equation still attracts research interests for its relevance to broad range of practical applications in environmental fluid mechanic, biology,
chemistry and applied mathematics among other fields. This model is usually solved in the literature under the assumption that the dispersion coefficients are time and space independent. Some laboratory scale experiments suggest that in porous media transport problems, the dispersive parameter may be time-dependent [2, 3]. The factors that can affect this transport of pollutants include the solute properties, fluid velocity field within the subsurface and microgeometry such as the shape, size and location of the solid part of the medium or the layout of the voids. The transport equation often models flow in porous media, thermal pollution in river systems, dispersion of dissolved salts in groundwater, water transfer in soils, dispersion of tracers in suburface, the spread of pollutants in rivers and streams, dispersion of dissolved material estuaries and coastal seas, the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the problems, the dispersive parameter may be time-dependent [2, 3]. The factors that can affect this transport of pollutants include the solute properties, fluid velocity field within the subsurface and microgeometry such as the shape, size and location of the solid part of the medium or the layout of the voids. The transport equation often models flow in porous media, thermal pollution in river systems, dispersion of dissolved salts in groundwater, water transfer in soils, dispersion of tracers in suburface, the spread of pollutants in rivers and streams, dispersion of dissolved material estuaries and coastal seas, the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the problems, the dispersive parameter may be time-dependent [2, 3]. The factors that can affect this transport of pollutants include the solute properties, fluid velocity field within the subsurface and microgeometry such as the shape, size and location of the solid part of the medium or the layout of the voids. The transport equation often models flow in porous media, thermal pollution in river systems, dispersion of dissolved salts in groundwater, water transfer in soils, dispersion of tracers in subsurface, the spread of pollutants in rivers and streams, dispersion of dissolved material estuaries and coastal seas, the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the sorption of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42].
scheme must be able to simulate transport phenomenon accurately by suppressing instabilities which can arise in the discretization. Standard temporal discretization, such as the second-order central difference scheme fails to approach the exact solution of the initial-boundary value problem (1.1)-(1.3), unless a large number of grid points are considered. In practical applications, it is often preferable to apply high-order schemes which provide less computations to get accurate solutions. Both compact finite difference procedures and multi-level time-split techniques feature high-order accuracy and small stencils. Most recently, several researchers have deeply analyzed implicit-explicit approaches, compact finite difference methods and multi-level time-split algorithms for the approximate solutions of mixed Stokes-Darcy’s model, evolutionary reaction-diffusion equation, systems of ordinary differential equations with time-variable coefficients, non-stationary convection-diffusion equation, time-dependent convection-diffusion-reaction problem, unsteady coupled Burgers’equations and Navier-Stokes equations [7, 11, 17, 19, 20, 22, 23, 25–33, 35, 40, 41].

In this paper, we are interested in a computed solution of the two-dimensional evolutionary advection-diffusion equations with time-dependent dispersion coefficients (1.1)-(1.3), using a two-level factored Crank-Nicolson approach. The proposed technique is fast, efficient and accurate: unconditional stability, second order convergent in time, fourth order accurate in space and less computational costs than a large class of numerical methods widely studied in the literature for solving Eqs. (1.1)-(1.3). For more detail the readers can consult [6, 8, 11, 15, 19, 22, 25, 36] and references therein. In fact, solving problem (1.1)-(1.3) by the use of the well-known Crank-Nicolson formulation leads to the inversion of a block tridiagonal matrix at each time step which requires a substantial amount of computations, while the proposed numerical scheme only needs to invert a tridiagonal matrix at each calculation step. This can be easily obtained by applying the tridiagonal Thomas algorithm in one-dimension. In comparing with the methods mentioned above, the two-level factored Crank-Nicolson yields a considerable savings in computing time. Moreover, implicit-explicit schemes often suffer from instability issues because require a time step restriction to ensure both stability and convergence of the algorithm. It is worth mentioning that the considered scheme is based on reducing problems in several space variables into collections of one-dimensional subproblems and then solve systems of linear equations with tridiagonal matrix coefficients.

The aim of this paper is to analyze the following three items:

a) a full description of the two-level factored Crank-Nicolson formulation for solving the parabolic partial differential equation (PDE) (1.1) subjects to initial and boundary conditions (1.2) and (1.3), respectively;

b) stability analysis of the proposed algorithm;

c) a large set of numerical examples which confirm the theoretical study are considered and discussed.
In the following we proceed as follows: Section 2 deals with a full description of a two-level factored Crank-Nicolson method for solving the system of Eqs. (1.1)-(1.3). In Section 3, using the Von Neumann stability approach we show that the scheme is unconditionally stable. A wide set of numerical experiments which confirms the theoretical analysis (stability and convergence rate) are considered and critically discussed in Section 4. Both general conclusion and future direction of works are presented in Section 5.

2 Overview of the three-level factored Crank-Nicolson method

This section sets notations and provides a detailed description of the two-level factored Crank-Nicolson formulation for solving the two-dimensional initial-boundary value problem (1.1)-(1.3).

Let \( K, M \) and \( N \) be three positive integers. Set
\[
k := \Delta t = \frac{T_f - t_0}{K}, \quad h_x := \Delta x = \frac{b_1 - a_1}{M} \quad \text{and} \quad h_y := \Delta y = \frac{b_2 - a_2}{N},
\]
be the time step and grid spacings, respectively. Putting \( t^n = t_0 + kn, \ n = 0, 1, \ldots, K; \ x_i = a_1 + ih_x, \ i = 0, 1, \ldots, M \) and \( y_j = a_2 + jh_y, \ 0, 1, \ldots, N \). In addition, suppose \( \Omega_k = \{t^n, 0 \leq n \leq K\}; \ \overline{\Omega}_h = \{(x_i, y_j), \ 0 \leq i \leq M \text{ and } 0 \leq j \leq N\}; \ \Omega_0 = \overline{\Omega}_h \cap \Omega \) and \( \partial \Omega_h = \overline{\Omega}_h \cap \partial \Omega \).

Let \( c_i^n = (c_{ij}^n, \ n = 0, 1, \ldots, K, \ 0 \leq i \leq M, \ 0 \leq j \leq N\), where \( c_{ij}^n = c(x_i, y_j, t^n) \), be the space of grid functions defined on \( \Omega_h \times \Omega_k \). We introduce the following operators
\[
\delta t^n c_{ij}^{n+1} = \frac{c_{ij}^{n+1} - c_{ij}^n}{k}, \quad \Delta c_{ij}^n = \frac{c_{ij}^{n+1} - c_{ij}^{n-1}}{2h_x}, \quad \nabla_{x} c_{ij}^n = \frac{c_{ij}^{n+1} - c_{ij}^{n-1}}{2h_x}, \quad \nabla_{y} c_{ij}^n = \frac{c_{ij}^{n+1} - c_{ij}^{n-1}}{2h_y}, \quad \delta_{x} c_{ij}^n = \frac{\Delta x c_{ij}^n - \nabla_{x} c_{ij}^n}{h_x}, \quad \delta_{y} c_{ij}^n = \frac{\Delta y c_{ij}^n - \nabla_{y} c_{ij}^n}{h_y}.
\]

Using Eq. (2.1), it is easy to see that
\[
\delta_{x} c_{ij}^n = \frac{1}{2} \left( \Delta x c_{ij}^n + \nabla_{x} c_{ij}^n \right) \quad \text{and} \quad \delta_{y} c_{ij}^n = \frac{1}{2} \left( \Delta y c_{ij}^n + \nabla_{y} c_{ij}^n \right).
\]

Let consider the following discrete norms
\[
\|c^n\|_{L^2(\Omega)} = \left( h_xh_y \sum_{i=1}^{M} \sum_{j=1}^{N} |c_{ij}^n|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|c^n\|_{L^\infty(t_0, T_f; L^2(\Omega))} = \max_{1 \leq n \leq K} \|c^n\|_{L^2(\Omega)},
\]
where \( |\cdot| \) denotes the C-norm. The spaces \( L^2(\Omega) \) and \( L^\infty(t_0, T_f; L^2(\Omega)) \) are equipped with the norm \( \|\cdot\|_{L^2(\Omega)} \) and \( \|\cdot\|_{L^\infty(t_0, T_f; L^2(\Omega))} \), respectively. We recall that a two-level factored
Crank-Nicolson procedure consists to reducing problems in many space variables into a sequence of one-dimensional subproblems and then find the solution of linear systems with associated tridiagonal matrix. This considerably reduces the computational cost of the scheme.

We should give an equivalent form of Eq. (1.1), which will be used in the following. Since the dispersion coefficients are time dependent and constant in space variables, that is, \( D_t = D_t(t) \) and \( D_t = D_t(t) \), multiplying both sides of (1.1) by 1/R and rearranging some terms to get

\[
\frac{\partial c}{\partial t} = D_t \frac{\partial^2 c}{\partial x^2} + D_t \frac{\partial^2 c}{\partial y^2} - u \frac{\partial c}{\partial x} - v \frac{\partial c}{\partial y} - \mu c + q,
\]

where

\[
D_t = \frac{\tilde{D}_t}{R}, \quad D_t = \frac{\tilde{D}_t}{R}, \quad u = \tilde{u}, \quad v = \tilde{v} \quad \text{and} \quad q = \tilde{q}.
\]

Now, applying the Taylor series for the function \( c \) about \((x_i, y_j, t^n)\) with time step \( k \) using forward and backward difference representations gives

\[
c_{ij}^{n+1} = c_{ij}^n + k c_{ij}^{n+1} + \frac{k^2}{2} c_{2i,2j}^{n} + \mathcal{O}(k^3) \quad \text{and} \quad c_{ij}^n = c_{ij}^{n+1} - k c_{ij}^{n+1} + \frac{k^2}{2} c_{2i,2j}^{n+1} + \mathcal{O}(k^3),
\]

where \( c_t = \frac{\partial c}{\partial t} \) and \( c_{2t} = \frac{\partial^2 c}{\partial t^2} \). Subtracting the second equation of (2.5) from the first one, it is not difficult to see that

\[
2(c_{ij}^{n+1} - c_{ij}^n) = k(c_{ij}^{n+1} + c_{ij}^n) + \frac{k^2}{2} (c_{2i,2j}^n - c_{2i,2j}^{n+1}) + \mathcal{O}(k^3).
\]

Expanding the Taylor series for \( c_{2t} \) about \((x_i, y_j, t^n)\) with time step \( k \) using forward difference formulation provides

\[
c_{2i,2j}^{n+1} = c_{2i,2j}^n + \mathcal{O}(k).
\]

A combination of Eqs. (2.7)-(2.6) results in

\[
\frac{c_{ij}^{n+1} - c_{ij}^n}{k} = \frac{1}{2} \left( c_{ij}^{n+1} + c_{ij}^n \right) + \mathcal{O}(k^2).
\]

Using Eq. (2.3), simple calculations yield

\[
c_{ij}^{n+1} = D_t^{n+1} c_{ij}^{n+1} + D_t^n c_{ij}^{n+1} - u_t^n c_{ij}^{n+1} - v_t^n c_{ij}^{n+1} - \mu_t^n c_{ij}^{n+1} + q_t^n,
\]

\[
c_{ij}^n = D_t^n c_{ij}^n + \mathcal{O}(k^2), \quad u_t^n = \mathcal{O}(k^2), \quad v_t^n = \mathcal{O}(k^2), \quad \mu_t^n = \mathcal{O}(k^2), \quad q_t^n = \mathcal{O}(k^2).
\]

The application of the Taylor series expansion for \( c \) about \((x_i, y_j, t^n)\) and \((x_i, y_j, t^{n+1})\) with mesh spaces \( h_x \) and \( h_y \), using central difference formulas gives

\[
c_{ij}^{n+1} = \delta_x c_{ij}^{n+1} + \mathcal{O}(h_x^2), \quad c_{ij}^n = \delta_x c_{ij}^n + \mathcal{O}(h_x^2), \quad (2.10a)
\]

\[
c_{ij}^{n+1} = \delta_y c_{ij}^{n+1} + \mathcal{O}(h_y^2), \quad c_{ij}^n = \delta_y c_{ij}^n + \mathcal{O}(h_y^2), \quad (2.10b)
\]
and

\[ c_{2x,ij}^{n+1} = \delta_x^2 c_{ij}^{n+1} + \mathcal{O}(h_x^2), \quad c_{2x,ij}^{n} = \delta_x^2 c_{ij}^{n} + \mathcal{O}(h_x^2), \quad (2.11a) \]
\[ c_{2y,ij}^{n+1} = \delta_y^2 c_{ij}^{n+1} + \mathcal{O}(h_y^2), \quad c_{2y,ij}^{n} = \delta_y^2 c_{ij}^{n} + \mathcal{O}(h_y^2). \quad (2.11b) \]

Substituting the second and fourth equations of (2.10) and (2.11) into relation (2.9b), it is not hard to observe that

\[ c_{i,j}^{n+1} = D_t^p \delta_x c_{ij}^{n+1} + D_{tx}^p \delta_y c_{ij}^{n+1} - u_{ij}^{n+1} \delta_x c_{ij}^{n+1} - v_{ij}^{n+1} \delta_y c_{ij}^{n+1} - \mu_{ij}^{n+1} c_{ij}^{n+1} + \mathcal{O}(h_x^2 + h_y^2), \quad (2.12a) \]
\[ c_{i,j}^{n} = D_t^p \delta_x c_{ij}^{n} + D_{tx}^p \delta_y c_{ij}^{n} - u_{ij}^{n} \delta_x c_{ij}^{n} - v_{ij}^{n} \delta_y c_{ij}^{n} - \mu_{ij}^{n} c_{ij}^{n} + \mathcal{O}(h_x^2 + h_y^2). \quad (2.12b) \]

Plugging Eqs. (2.8), (2.12a), (2.12b) and rearranging terms to get

\[ \frac{c_{ij}^{n+1} - c_{ij}^{n}}{k} = \frac{1}{2} \left\{ D_t^p \delta_x c_{ij}^{n+1} + D_{tx}^p \delta_y c_{ij}^{n+1} - u_{ij}^{n+1} \delta_x c_{ij}^{n+1} - v_{ij}^{n+1} \delta_y c_{ij}^{n+1} - \mu_{ij}^{n+1} c_{ij}^{n+1} + \mathcal{O}(k^2 + h_x^2 + h_y^2) \right\} c_{ij}^{n+1} \]

Solving this equation for \( c_{ij}^{n+1} \) provides

\[ \left\{ I - \frac{k}{2} \left[ D_t^p \delta_x^2 + D_{tx}^p \delta_y^2 - u_{ij}^{n+1} \delta_x - v_{ij}^{n+1} \delta_y - \mu_{ij}^{n+1} I \right] \right\} c_{ij}^{n+1} \]
\[ = \left\{ I + \frac{k}{2} \left[ D_t^p \delta_x^2 + D_{tx}^p \delta_y^2 - u_{ij}^{n} \delta_x - v_{ij}^{n} \delta_y - \mu_{ij}^{n} I \right] \right\} c_{ij}^{n} \]
\[ + \frac{k}{2} (q_{ij}^{n+1} + q_{ij}^{n}) + \mathcal{O}(k^3 + kh_x^2 + kh_y^2), \quad (2.13) \]

where \( I \) represents the identity operator. Since \((1-a)(1-b) = 1-a-b+ab\), for any real numbers \( a \) and \( b \), to obtain a factored expression, we should add the term

\[ \frac{k^2}{4} \left[ D_t^p \delta_x^2 - u_{ij}^{n+1} \delta_x - \frac{1}{2} \mu_{ij}^{n+1} I \right] \left[ D_{tx}^p \delta_y^2 - v_{ij}^{n+1} \delta_y - \frac{1}{2} \mu_{ij}^{n+1} I \right] c_{ij}^{n+1}, \]

in both sides of Eq. (2.13) and also manipulate the right hand side of this equation. This fact results in

\[ \left\{ I - \frac{k}{2} \left[ D_t^p \delta_x^2 - u_{ij}^{n+1} \delta_x - \frac{1}{2} \mu_{ij}^{n+1} I \right] \right\} \left\{ I - \frac{k}{2} \left[ D_{tx}^p \delta_y^2 - v_{ij}^{n+1} \delta_y - \frac{1}{2} \mu_{ij}^{n+1} I \right] \right\} c_{ij}^{n+1} \]
\[ = \left\{ I + \frac{k}{2} \left[ D_t^p \delta_x^2 - u_{ij}^{n} \delta_x - \frac{1}{2} \mu_{ij}^{n} I \right] \right\} \left\{ I + \frac{k}{2} \left[ D_{tx}^p \delta_y^2 - v_{ij}^{n} \delta_y - \frac{1}{2} \mu_{ij}^{n} I \right] \right\} c_{ij}^{n} \]
\[ + \frac{k}{2} (q_{ij}^{n+1} + q_{ij}^{n}) + \mathcal{O}(k^3), \quad (2.14) \]
where $\psi_{ij}$ is the error term which is given by

$$
\psi_{ij}^{n} = \frac{k^2}{4} \left\{ \left[ D_{t}^{n+1} \delta_{x}^2 - u_{ij}^{n+1} \delta_{x} - \frac{1}{2} \mu_{ij}^{n+1} I \right] \left[ D_{t}^{n+1} \delta_{y}^2 - v_{ij}^{n+1} \delta_{y} - \frac{1}{2} \mu_{ij}^{n+1} I \right] c_{ij}^{n+1} \right. \\
\left. - \left[ D_{x}^{n} \delta_{x}^2 - u_{ij}^{n} \delta_{x} - \frac{1}{2} \mu_{ij}^{n} I \right] \left[ D_{y}^{n} \delta_{y}^2 - v_{ij}^{n} \delta_{y} - \frac{1}{2} \mu_{ij}^{n} I \right] c_{ij}^{n} \right\} + O(k^3 + kh_x^2 + kh_y^2). \tag{2.15}
$$

Truncating the infinitesimal term $O(k^3 + kh_x^2 + kh_y^2)$ in Eq. (2.13) and replacing the exact solution $c_{ij}^{n}$ with its numerical approximation $C_{ij}^{n}$, we obtain a one-step linearized implicit scheme defined as

$$
\{ I - \frac{k}{2} \left[ D_{t}^{n+1} \delta_{x}^2 - D_{x}^{n} \delta_{x}^2 - u_{ij}^{n+1} \delta_{x} - u_{ij}^{n} \delta_{x} - \frac{1}{2} \mu_{ij}^{n+1} I - \mu_{ij}^{n} I \right] \} C_{ij}^{n+1} = \{ I + \frac{k}{2} \left[ D_{y}^{n} \delta_{y}^2 - v_{ij}^{n} \delta_{y} - \mu_{ij}^{n} I \right] \} C_{ij}^{n} + \frac{k}{2} (q_{ij}^{n+1} + q_{ij}^{n}). \tag{2.16}
$$

Furthermore, using relation (2.14) a two-step linearized equation can be constructed as follows

$$
\{ I - \frac{k}{2} \left[ D_{t}^{n+1} \delta_{x}^2 - D_{x}^{n} \delta_{x}^2 - u_{ij}^{n+1} \delta_{x} - \frac{1}{2} \mu_{ij}^{n+1} I \right] \} c_{ij}^{n} = \{ I + \frac{k}{2} \left[ D_{y}^{n} \delta_{y}^2 - v_{ij}^{n} \delta_{y} - \mu_{ij}^{n} I \right] \} c_{ij}^{n} + \frac{k}{2} (q_{ij}^{n+1} + q_{ij}^{n}) \tag{2.17a}
$$

$$
\left\{ I - \frac{k}{2} \left[ D_{t}^{n+1} \delta_{y}^2 - v_{ij}^{n+1} \delta_{y} - \frac{1}{2} \mu_{ij}^{n+1} I \right] \right\} c_{ij}^{n+1} = c_{ij}^{n}, \tag{2.17b}
$$

where the superscript asterisk denotes an intermediate value and $\psi_{ij}^{n}$ is defined by Eq. (2.15).

In the literature, many splitting schemes for solving the transport equations have been used to advance the solution in time. The most popular of these approaches are the three-level time-split MacCormack and the compact ADI methods described in details in [14, 32]. Implicit schemes may be constructed using a large number of techniques (for instance, see Eq. (2.16)). Most common of these procedures is the Euler implicit formulation or trapezoidal method. Although these schemes are unconditionally stable, they create the necessity of computing the flux jacobian and then, produce a large systems of linear equations to be solved as efficiently as possible. For two-dimensional problems, this becomes the major effort in calculating a solution with one-step implicit models. To overcome this drawback, a two-level factored Crank-Nicolson approach is proposed. This allows to solve one-dimensional implicit problems at each time step.
Now, omitting the term $\psi_{ij}^n$ in Eq. (2.17a), we get the desired numerical algorithm. For $n=0,1,\cdots,K-1$, $i=1,\cdots,M-1$, and $j=1,\cdots,N-1$,

$$\begin{align*}
\{ \mathcal{I} - \frac{k}{2} \left[ D_{ij}^{n+1} \delta_x^2 - u_{ij}^{n+1} \delta_x^2 - \frac{1}{2} \mu_{ij}^{n+1} \mathcal{I} \right] \} C^n_{ij} & = \{ \mathcal{I} + \frac{k}{2} \left[ D_{ij}^n \delta_x^2 - u_{ij}^n \delta_x^2 - \frac{1}{2} \mu_{ij}^n \mathcal{I} \right] \} \{ \mathcal{I} + \frac{k}{2} \left[ D_{ij}^n \delta_y^2 - v_{ij}^n \delta_y^2 - \frac{1}{2} \mu_{ij}^n \mathcal{I} \right] \} C^n_{ij} \\
& + \frac{k}{2} (q_{ij}^{n+1} + q_{ij}^n), \tag{2.18a}
\{ \mathcal{I} - \frac{k}{2} \left[ D_{ij}^{n+1} \delta_y^2 - v_{ij}^{n+1} \delta_y^2 - \frac{1}{2} \mu_{ij}^{n+1} \mathcal{I} \right] \} C^n_{ij} & = C^n_{ij}, \tag{2.18b}
\end{align*}$$

subject to initial and boundary conditions, for $n=1,\cdots,K-1$

$$\begin{align*}
C^n_{0j} & = 0, \quad C^n_{ij} = C^n_{ij} = 0, \quad C^*_{nj} = C^*_{nj} = 0, \quad C^*_{MN} = C^*_{MN} = 0, \tag{2.19a}
C^n_{j0} & = C^n_{ij} = 0, \quad C^*_{j0} = C^*_{ij} = 0, \quad C^*_{jN} = C^*_{ij} = 0, \tag{2.19b}
\end{align*}$$

for $i=0,1,\cdots,M$ and $j=0,1,\cdots,N$. Defining the following operators

$$\begin{align*}
L^-_x & = \mathcal{I} - \frac{k}{2} \left[ D_{ij}^{n+1} \delta_x^2 - u_{ij}^{n+1} \delta_x^2 - \frac{1}{2} \mu_{ij}^{n+1} \mathcal{I} \right], \tag{2.20a}
L^-_y & = \mathcal{I} - \frac{k}{2} \left[ D_{ij}^{n+1} \delta_y^2 - v_{ij}^{n+1} \delta_y^2 - \frac{1}{2} \mu_{ij}^{n+1} \mathcal{I} \right], \tag{2.20b}
L^+_x & = \mathcal{I} + \frac{k}{2} \left[ D_{ij}^n \delta_x^2 - u_{ij}^n \delta_x^2 - \frac{1}{2} \mu_{ij}^n \mathcal{I} \right], \tag{2.20c}
L^+_y & = \mathcal{I} + \frac{k}{2} \left[ D_{ij}^n \delta_y^2 - v_{ij}^n \delta_y^2 - \frac{1}{2} \mu_{ij}^n \mathcal{I} \right]. \tag{2.20d}
\end{align*}$$

It is worth noticing to recall that the two-level factored Crank-Nicolson approach constructed is composed of two phases, as specified in Eqs. (2.18a)-(2.18b). In each stage, both operators $L^-_x$ and $L^-_y$ compute implicitly. Thus, the growth of the error cannot cause any instability of the algorithm. Finally, it comes from Eq. (2.15) that the truncation error $\psi_{ij}^n$ satisfies: $\psi_{ij}^n = \mathcal{O}(k^2 + h_x^4 + h_y^4)$ (indeed, $kh_x^2 \leq k^2 + h_x^4$ and $kh_y^2 \leq k^2 + h_y^4$). Thus, the new approach is second order accurate in time and fourth order convergent in space.

In the rest of this work, we assume that the exact solution $c \in L^\infty(t_0,T_f;L^2(\Omega)) \cap H^1(t_0,T_f;L^2(\Omega)) \cap L^2(t_0,T_f;H^2(\Omega))$, that is, there exists a constant $\gamma > 0$, which does not depend on the time step $k$ and the mesh sizes $h_x$ and $h_y$, so that

$$\| c \|_{L^\infty(t_0,T_f;L^2(\Omega))} + \| c \|_{H^1(t_0,T_f;L^2(\Omega))} + \| c \|_{L^2(t_0,T_f;H^2(\Omega))} \leq \gamma. \tag{2.21}$$
3 Unconditional stability of the two-level factored Crank-Nicolson procedure

In this section we prove the unconditional stability of the numerical scheme (2.18a)-(2.19) for solving the two-dimensional advection-diffusion equation with time dependent dispersion coefficients and sink/source terms (1.1)-(1.3). Assuming that the boundary condition defined by Eq. (1.3) is accurate, we apply the Fourier method to the difference Eqs. (2.17a)-(2.17b), by computing the amplification factor to obtain an algebraic criterion (if it exists) for the stability analysis of the proposed technique. Following the Von Neumann criterion for necessary condition of stability, we suppose that both analytical and numerical solutions \( c^n_{ij} \) and \( C^n_{ij} \) can be expressed in the form of Fourier series

\[
c^n_{ij} = \alpha^n \exp(i\phi_x h_x + j\phi_y h_y), \quad C^n_{ij} = \beta^n \exp(i\phi_x h_x + j\phi_y h_y),
\]

and then, the error \( e^n_{ij} = c^n_{ij} - C^n_{ij} \) (where \( c^n_{ij} = c(x_i, y_j, t^n) \) and \( C^n_{ij} = C(x_i, y_j, t^n) \) are the exact solution of Eqs. (2.17a)-(2.17b) and (2.18a)-(2.18b), respectively) must satisfy

\[
e^n_{ij} = \tilde{c}^n \exp(i\phi_x h_x + j\phi_y h_y),
\]

where: \( \alpha^n, \beta^n \) and \( \tilde{c}^n = \alpha^n - \beta^n \), are the amplitudes at time level \( n \), \( i \) denotes the imaginary unit, \( \phi_x \) and \( \phi_y \) represent the wave numbers in the \( x \) and \( y \) directions, respectively. The products \( \phi_x h_x \) and \( \phi_y h_y \) are called the phase angles.

**Theorem 3.1.** Let \( c^n_{ij} \) and \( C^n_{ij} \) be the exact and numerical solutions provided by the algorithms (2.17a)-(2.17b) and (2.18a)-(2.18b), respectively. Assuming that the functions \( \tilde{D}_t(t), \tilde{D}_x(t), \tilde{u}(x,y,t), \tilde{v}(x,y,t) \) and \( \mu(x,y,t) \) are nonnegative and time variable increasing, the two-level factored Crank-Nicolson approach (2.18a)-(2.19) for solving the initial-boundary value problem (1.1)-(1.3) is unconditionally stable, that is

\[
\|C\|_{L^\infty(I_0, T; L^2(\Omega))} \leq \tilde{C}_1,
\]

where \( \tilde{C}_1 \) is a positive parameter independent of the time step \( \Delta t \) and grid spaces \( h_x \) and \( h_y \).

**Proof.** Firstly, we recall that \( C^n_{ij} \) is the solution provide by Eqs. (2.18a)-(2.18b) and \( e^n_{ij} \) is the exact solution of Eqs. (2.17a)-(2.17b).

Subtracting approximation (2.18a) from (2.17a) and approximation (2.18b) from (2.17b), we get

\[
\left\{ I - \frac{k}{2} \left[ D_t^{n+1} \delta^2_x - \mu_{ij}^{n+1} \delta^x_x - \frac{1}{2} \mu_{ij}^{n+1} l_x \right] \right\} e^n_{ij} = \left\{ I + \frac{k}{2} \left[ D_t^n \delta^2_x - \mu_{ij}^n \delta^x_x - \frac{1}{2} \mu_{ij}^n l_x \right] \right\} e^n_{ij} + \psi^n_{ij}, \quad \text{(3.4a)}
\]

\[
\left\{ I - \frac{k}{2} \left[ D_t^{n+1} \delta^2_y - \mu_{ij}^{n+1} \delta^y_y - \frac{1}{2} \mu_{ij}^{n+1} l_y \right] \right\} e^n_{ij} = \left\{ I + \frac{k}{2} \left[ D_t^n \delta^2_y - \mu_{ij}^n \delta^y_y - \frac{1}{2} \mu_{ij}^n l_y \right] \right\} e^n_{ij} + \psi^n_{ij} + \xi^n_{ij}, \quad \text{(3.4b)}
\]
where the intermediate error term $e_{ij}^n$ is defined as $e_{ij}^n = c_{ij}^n - C_{ij}^n$. Now, substituting Eq. (3.4b) into (3.4), it is not hard to see that

$$
\left\{ I - \frac{k}{2} \left[ D_{ij}^{n+1} \delta x^2 - u_{ij}^{n+1} \delta x - \frac{1}{2} \mu_{ij}^{n+1} I \right] \right\} \left\{ I - \frac{k}{2} \left[ D_{ij}^{n+1} \delta y^2 - v_{ij}^{n+1} \delta y - \frac{1}{2} \mu_{ij}^{n+1} I \right] \right\} e_{ij}^{n+1}
$$

$$= \left\{ I + \frac{k}{2} \left[ D_{ij}^{n} \delta x^2 - u_{ij}^{n} \delta x - \frac{1}{2} \mu_{ij}^{n} I \right] \right\} \left\{ I + \frac{k}{2} \left[ D_{ij}^{n} \delta y^2 - v_{ij}^{n} \delta y - \frac{1}{2} \mu_{ij}^{n} I \right] \right\} e_{ij}^{n} + \psi_{ij}^{n}.
$$

(3.5)

Using (2.20), Eq. (3.5) is equivalent to

$$
L_x^- L_y^- (e_{ij}^{n+1}) = L_x^+ L_y^+ (e_{ij}^{n}) + \psi_{ij}^{n},
$$

(3.6)

where $\psi_{ij}^{n}$ is given by relation (2.15). Utilizing Eq. (3.2), it is easy to observe that

$$
e_{ij}^{n} = \bar{\epsilon}^{n} \exp(I \phi_x h_x + j \phi_y h_y),
$$

$$
e_{ij}^{n+1} = \bar{\epsilon}^{n+1} \exp(I \phi_x h_x + j \phi_y h_y).
$$

This fact, together with relation (3.6) result in

$$
\bar{\epsilon}^{n+1} L_x^- (\exp(I \phi_x h_x)) L_y^- (\exp(I \phi_y h_y)) = \bar{\epsilon}^{n} L_x^+ (\exp(I \phi_x h_x)) L_y^+ (\exp(I \phi_y h_y)) + \psi_{ij}^{n}.
$$

Neglecting the error term $\psi_{ij}^{n}$, this can be approximate as

$$
\bar{\epsilon}^{n+1} L_x^- (\exp(I \phi_x h_x)) L_y^- (\exp(I \phi_y h_y)) = \bar{\epsilon}^{n} L_x^+ (\exp(I \phi_x h_x)) L_y^+ (\exp(I \phi_y h_y)),
$$

which is equivalent to

$$
\frac{\bar{\epsilon}^{n+1}}{\bar{\epsilon}^{n}} = \frac{L_x^+ (\exp(I \phi_x h_x)) L_y^+ (\exp(I \phi_y h_y))}{L_x^- (\exp(I \phi_x h_x)) L_y^- (\exp(I \phi_y h_x))}.
$$

(3.7)

It is worth noticing to mention that relation (3.7) defines the amplification factor provided by the numerical scheme (2.18a)-(2.19). In order to establish the unconditional stability of the two-level factored Crank-Nicolson method, we should show that the modulus of the amplification factor given by Eq. (3.7) is less than or equal 1.

Now, taking the squared modulus in both sides of (3.7) gives

$$
\left| \frac{\bar{\epsilon}^{n+1}}{\bar{\epsilon}^{n}} \right|^2 = \left| \frac{L_x^+ (\exp(I \phi_x h_x))}{L_x^- (\exp(I \phi_x h_x))} \right|^2 \left| \frac{L_y^+ (\exp(I \phi_y h_y))}{L_y^- (\exp(I \phi_y h_x))} \right|^2.
$$

(3.8)

For the convenient of writing, we should prove only the following estimate

$$
\left| \frac{L_x^+ (\exp(I \phi_x h_x))}{L_x^- (\exp(I \phi_x h_x))} \right|^2 \leq 1.
$$

(3.9)
The proof of inequality
\[
\left| L^+ \left( \exp\left(\hat{i}i\phi h_x\right) \right) \right|^2 \leq 1,
\]
(3.10)
is similar. It comes from the definition of the linear operators \( \delta^x \) and \( \delta^\phi \) (respectively, \( L^+_x \)) given by relation (2.1) (respectively, Eq. (2.20)) that
\[
L^+_x \left( \exp\left(\hat{i}i\phi h_x\right) \right) = \left\{ \exp\left(\hat{i}i\phi h_x\right) + \frac{k}{2} \left[ D_l^n \frac{\exp\left(i(i+1)\phi h_x\right) - 2\exp\left(\hat{i}i\phi h_x\right) + \exp\left(i(i-1)\phi h_x\right)}{h_x^2} \right. \\
- u_{ij}^n \frac{\exp\left(i(i+1)\phi h_x\right) - \exp\left(i(i-1)\phi h_x\right)}{2h_x} - \left. \frac{1}{2} h_{ij}^n \exp\left(\hat{i}i\phi h_x\right) \right] \right\} \exp\left(\hat{i}i\phi h_x\right).
\]
(3.11)

Since
\[
\exp\left(\hat{i}i\phi h_x\right) - 2 + \exp\left(-\hat{i}i\phi h_x\right) = 2\cos(\phi h_x) - 2 = -4\sin^2(\phi h_x/2),
\]
\[
\exp\left(\hat{i}i\phi h_x\right) - \exp\left(-\hat{i}i\phi h_x\right) = 2i\sin(\phi h_x),
\]
plugging this into Eq. (3.11) provides
\[
L^+_x \left( \exp\left(\hat{i}i\phi h_x\right) \right) = \left\{ 1 + \frac{k}{2} \left[ -4D_l^n \frac{\sin^2(\phi h_x/2)}{h_x^2} - iu_{ij}^n \frac{\sin(\phi h_x)}{h_x} - \frac{1}{2} h_{ij}^n \right] \right\} \exp\left(\hat{i}i\phi h_x\right).
\]

Taking the squared modulus of both sides to get
\[
\left| L^+_x \left( \exp\left(\hat{i}i\phi h_x\right) \right) \right|^2 = \left[ 1 - \frac{k}{2} \left( 4D_l^n \frac{\sin^2(\phi h_x/2)}{h_x^2} + \frac{1}{2} h_{ij}^n \right) \right]^2 + \frac{k^2}{4} \left[ u_{ij}^n \frac{\sin(\phi h_x)}{h_x} \right]^2.
\]
(3.12)

Furthermore, it is not difficult to observe that
\[
L^-_x \left( \exp\left(\hat{i}i\phi h_x\right) \right) = \left\{ \exp\left(\hat{i}i\phi h_x\right) - \frac{k}{2} \left[ D_l^{n+1} \frac{\exp\left(i(i+1)\phi h_x\right) - 2\exp\left(\hat{i}i\phi h_x\right) + \exp\left(i(i-1)\phi h_x\right)}{h_x^2} \right. \\
- u_{ij}^{n+1} \frac{\exp\left(i(i+1)\phi h_x\right) - \exp\left(i(i-1)\phi h_x\right)}{2h_x} - \left. \frac{1}{2} h_{ij}^{n+1} \exp\left(\hat{i}i\phi h_x\right) \right] \right\} \exp\left(\hat{i}i\phi h_x\right).
\]
The squared modulus results in
\[
|L_x(\exp(i\psi_x h_x))|^2 = \left[1 + \frac{k}{2} \left(4D_i^{n+1} \frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2} u_{ij}^{n+1} \right) \right]^2 + \frac{k^2}{4} \left[ u_{ij}^{n+1} \frac{\sin(\phi_x h_x)}{h_x} \right]^2.
\] (3.13)

Now, it follows from the assumption of Theorem 3.1 that the functions \(D_t, u, \mu\) are nonnegative and increasing in the time-variable \(t\). Using this fact, it is not hard to see that
\[
\left[1 + \frac{k}{2} \left(4D_i^{n+1} \frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2} u_{ij}^{n+1} \right) \right]^2 + \frac{k^2}{4} \left[ u_{ij}^{n+1} \frac{\sin(\phi_x h_x)}{h_x} \right]^2 \leq 1.
\]

Combining estimate (3.14) together with Eqs. (3.12) and (3.13), simple calculations give
\[
\left| \frac{L_x(\exp(i\psi_x h_x))} {L_x(\exp(i\phi_x h_x))} \right|^2 \leq \left[1 + \frac{k}{2} \left(4D_i^{n+1} \frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2} u_{ij}^{n+1} \right) \right]^2 + \frac{k^2}{4} \left[ u_{ij}^{n+1} \frac{\sin(\phi_x h_x)}{h_x} \right]^2 \leq 1.
\]

This completes the proof of estimate (3.9). In a similar manner, one proves estimate (3.10).

Now, a combination of inequalities (3.9), (3.10) and (3.8) results in
\[
\frac{|\hat{\varphi}^{n+1}|^2}{|\hat{\varphi}^n|^2} \leq 1,
\]
which is equivalent to
\[
|\hat{\varphi}^{n+1}| \leq |\hat{\varphi}^n| \quad \text{for} \quad n = 1, \cdots, K - 1.
\]

By induction, it is easy to observe that, for \(n = 1, \cdots, K\),
\[
|\hat{\varphi}^n| \leq |\hat{\varphi}^1|.
\] (3.15)

Utilizing the definition of \(L^2\)-norm given by (2.2) and Eq. (3.2), straightforward computations result in
\[
\|e^n\|_{L^2(\Omega)} = \left( h_x h_y \sum_{i=1}^{M} \sum_{j=1}^{N} |e_{ij}^n|^2 \right)^{\frac{1}{2}} = ((b_1 - a_1)(b_2 - a_2))^{\frac{1}{2}} |\hat{\varphi}^1|.
\] (3.16)

But, we have that
\[
\|C^n\|_{L^2(\Omega)} - \|e^n\|_{L^2(\Omega)} \leq \|C^n - e^n\|_{L^2(\Omega)} = \|e^n\|_{L^2(\Omega)}.
\]
This fact, together with relation (3.16), estimates (2.21) and (3.15) yield
\[ \|C^n\|_{L^2(\Omega)} \leq \left( (b_1-a_1)(b_2-a_2) \right)^{1/2} |\mathcal{E}| + \gamma \quad \text{for } n = 1, \cdots, K. \]

Indeed, \( c \) is the exact solution of the initial-boundary value problem (1.1)-(1.3). Taking the maximum over \( n \), this ends the proof of Theorem 3.1.

4 Numerical experiments and convergence rate

In this section, we perform numerical evidences to illustrate the validity and efficiency of the proposed unconditional stability and higher-order two-level factored scheme (2.18a)-(2.19) in an approximate solution of the initial-boundary value problem (1.1)-(1.3). We consider three examples described in [2] to demonstrate the effectiveness and utility of the new technique in two-dimensional case. We observe from each case satisfactory results. Thus, the considered approach provides better performances for multidimensional problems. The predicted convergence rate and unconditional stability from the theory are confirmed (see Section 2, p. 8 (paragraph below Eq. (2.20), last line) and Section 3, Theorem 3.1). While the graphs of the numerical solution (see Figs. 1-4) indicate that the new method is unconditionally stable, the convergence rate of the algorithm is obtained by listing in Tables 1-3 the errors between the computed solution and the analytical ones with values of the mesh grid \( h = h_x = h_y = 2^{-4} \) and time step \( k = 2^{-8} \). As indicated in [2], the examples we consider correspond to different cases of \( D_l(t) \) and \( D_r(t) \): linear dispersion coefficients
\[ D_l(t) = a^2 D_r(t) = \frac{D_0}{\theta} t + D_1, \]
asymptotic dispersion coefficients
\[ D_l(t) = a^2 D_r(t) = D_0 \frac{t}{t + \theta} + D_1 \]
and exponential dispersion coefficients
\[ D_l(t) = a^2 D_r(t) = D_0 \left[ 1 - \exp \left( -\frac{t}{\theta} \right) \right] + D_1. \]

In this study, we take \( R = 1, L = n_1 = 0.25, D_1 = 1, D_0 = 2, a = 1, \theta = 500, (t_0, T_f) = (1,3), \)
\( \Omega = (a_1, b_1) \times (a_2, b_2) = (2,4) \times (2,4). \) \( u(x,y,t), v(x,y,t) \) and \( \mu(x,y,t) \) are functions defined as:
\[ u(x,y,t) = x^2 + y + t, \quad v(x,y,t) = x + y^2 + t \quad \text{and} \quad \mu(x,y,t) = xy + t^2. \]
The mass injection rate \( q(x,y,t) \) is given by
\[ q(x,y,t) = \frac{L}{n_1} \delta(x) \delta(y) \delta(t). \]
while the initial condition is defined by \( f(x,y,t) = 0 \) and the boundary condition \( g(x,y,t) \) is determined by the analytical solution.

To verify the unconditional stability and convergence rate of the numerical scheme, we take the mesh size and time step in the range \( h \in \{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \} \) and \( k = 2^{-l}, l = 2, 4, \cdots, 8 \), respectively. For \( h = 2^{-4} \) and \( k = 2^{-8} \), we compute the norms of exact solution \( \|c\|_{L^2} \), approximate ones \( \|C\|_{L^2} \) and error estimates, \( \|E\|_{L^2} \) related to the two-level factored Crank-Nicolson procedure to see that the approach is unconditionally stable, temporal second order convergent and spatial fourth order accurate. In addition, for different values of \( k \) and \( h \), the \( L^\infty(t_0,T_f;L^2) \)-norm of the exact and computed solutions together with the error are plot versus \( n \). From this analysis, we observe that the new technique is fast, efficient and effective than a broad range of numerical methods [6, 8, 11, 15, 19, 25, 36] widely studied in literature for solving the two-dimensional unsteady advection-diffusion equation with sink/source terms. Finally, when \( k = 2^{-8} \) that is, for the value of \( K = 2^9 \), it comes from Tables 1-3 that the ”Ratio = \( E(2^n)/E(2^{n+1}) \)” of the approximation errors in two adjacent time-levels can be used to estimate the corresponding convergence rate with respect to \( k \).
Figure 2: Stability and convergence of a three-level factored Crank-Nicolson method. Asymptotic dispersion coefficients.

We recall that the norms of the approximate solution $C$, the exact one $c$ and the error $E$, are defined as follows

$$
\|C(n)\|_{L^2(\Omega)} = \left( h_x h_y \sum_{i=1}^{M} \sum_{j=1}^{N} |C_{ij}^n| \right)^{\frac{1}{2}}, \quad \|c(n)\|_{L^2(\Omega)} = \left( h_x h_y \sum_{i=1}^{M} \sum_{j=1}^{N} |c_{ij}^n| \right)^{\frac{1}{2}},
$$

$$
\|E(n)\|_{L^2(\Omega)} = \left( h_x h_y \sum_{i=1}^{M} \sum_{j=1}^{N} |C_{ij}^n - c_{ij}^n| \right)^{\frac{1}{2}} \quad \text{for } n = 0, 1, \ldots, K.
$$

**Problem 4.1** (Linear dispersion coefficients). The dispersion coefficients are defined by

$$
D_t(t) = \frac{D_0}{\theta} t + D_1 \quad \text{and} \quad D_\tau(t) = \frac{1}{a^2} \left( \frac{D_0}{\theta} t + D_1 \right)
$$

(for example, see [2]). The parameters $R$, $L$, $n_1$, $D_1$, $D_0$ and $a$ are given by $R = 1$, $L = n_1 = 0.25$, $D_1 = 1$, $D_0 = 2$, $a = 1$, $(t_0, T_f) = (1, 3)$ and $\Omega = (2, 4)^2$. Furthermore,

$$
u(x,y,t) = x^2 + y^2 + t, \quad \mu(x,y,t) = xy + t^2$$
and the function \( q(x,y,t) \) is defined by
\[
q(x,y,t) = \frac{L}{h_1} \delta(x) \delta(y) \delta(t),
\]
where \( \delta(\cdot) \) denotes the dirac function. The analytic solution \( c \) taken in [2] is given by
\[
c(x,y,t) = \frac{aL}{2\pi a^2((D_0/\theta)t+2D_1)t} \exp \left[ -\mu t - R \frac{(x-\frac{\alpha}{R}t)^2 + a^2(y-\frac{\alpha}{R}t)^2}{2((D_0/\theta)t+2D_1)t} \right],
\]
where \( \theta = 500 \). The initial condition \( f = 0 \) and the boundary condition \( g \) is obtained from the exact solution \( c \). We suppose that the grid spacing \( h \) and time step \( k \) satisfy \( k = h^2 \).

**Problem 4.2 (Asymptotic dispersion coefficients).** In this case, the dispersion coefficients \( D_l(t) \) and \( D_\tau(t) \) are given in [2] by
\[
D_l(t) = D_0 \frac{t}{t+\theta} + D_1 \quad \text{and} \quad D_\tau(t) = D_0 \frac{t}{a^2} \frac{t}{t+\theta} + D_1 \frac{1}{a^2}.
\]
Also in this test, the other quantities are those considered in Problem 4.1. The exact solution $c$ is given in [2] by

$$c(x,y,t) = \frac{aL}{4n_1 \pi \left( (D_0+D_1)t - D_0 \theta \ln \left( 1 + \frac{t}{\theta} \right) \right)} \exp \left[ -\mu t - \frac{R}{4 \left( (D_0+D_1)t - D_0 \theta \ln \left( 1 + \frac{t}{\theta} \right) \right)} \left( x - \frac{\mu}{R} t \right)^2 + a^2 \left( y - \frac{\nu}{R} t \right)^2 \right].$$
Table 1: Analytical solution “c”, numerical one “C”, error “E” and convergence rates “Ratio = E(2^n)/E(2^{n+1})” of the new algorithm with mesh size \( h = 2^{-4} \) and time step \( k = 2^{-8} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2^2 )</th>
<th>( 2^3 )</th>
<th>( 2^4 )</th>
<th>( 2^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( | c(n) |_{l_2} )</td>
<td>0.6505 \times 10^{-9}</td>
<td>0.4337 \times 10^{-9}</td>
<td>0.1849 \times 10^{-9}</td>
<td>2.8700 \times 10^{-10}</td>
</tr>
<tr>
<td>( | C(n) |_{l_2} )</td>
<td>0.6374 \times 10^{-9}</td>
<td>0.4118 \times 10^{-9}</td>
<td>0.1772 \times 10^{-9}</td>
<td>2.7900 \times 10^{-10}</td>
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<tr>
<td>( | E(n) |_{l_2} )</td>
<td>3.6660 \times 10^{-10}</td>
<td>1.8920 \times 10^{-10}</td>
<td>9.4500 \times 10^{-11}</td>
<td>4.3700 \times 10^{-11}</td>
</tr>
<tr>
<td>Ratio</td>
<td>---</td>
<td>1.9376</td>
<td>2.0021</td>
<td>2.1625</td>
</tr>
</tbody>
</table>

Table 2: Approximated solution “C”, exact solution “c”, error “E” and convergence rates “Ratio = E(2^n)/E(2^{n+1})” for the proposed approach with mesh grid \( h = 2^{-4} \) and step size \( k = 2^{-8} \).

<table>
<thead>
<tr>
<th>( n )</th>
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<th>( 2^3 )</th>
<th>( 2^4 )</th>
<th>( 2^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( | c(n) |_{l_2} )</td>
<td>0.1108 \times 10^{-9}</td>
<td>0.6360 \times 10^{-10}</td>
<td>0.2070 \times 10^{-10}</td>
<td>0.8100 \times 10^{-11}</td>
</tr>
<tr>
<td>( | C(n) |_{l_2} )</td>
<td>0.1079 \times 10^{-9}</td>
<td>0.6200 \times 10^{-10}</td>
<td>0.2030 \times 10^{-10}</td>
<td>0.8000 \times 10^{-11}</td>
</tr>
<tr>
<td>( | E(n) |_{l_2} )</td>
<td>0.2512 \times 10^{-10}</td>
<td>0.1280 \times 10^{-10}</td>
<td>0.6300 \times 10^{-11}</td>
<td>0.2700 \times 10^{-11}</td>
</tr>
<tr>
<td>Ratio</td>
<td>---</td>
<td>1.9625</td>
<td>2.0317</td>
<td>2.3334</td>
</tr>
</tbody>
</table>

Table 3: Analytical and computed solutions, error and convergence rates “Ratio = E(2^n)/E(2^{n+1})” of the new procedure with spacing \( h = 2^{-4} \) and time step \( k = 2^{-8} \).

<table>
<thead>
<tr>
<th>( n )</th>
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<th>( 2^3 )</th>
<th>( 2^4 )</th>
<th>( 2^5 )</th>
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<tbody>
<tr>
<td>( | c(n) |_{l_2} )</td>
<td>0.1108 \times 10^{-9}</td>
<td>0.6360 \times 10^{-10}</td>
<td>0.2070 \times 10^{-10}</td>
<td>0.8100 \times 10^{-11}</td>
</tr>
<tr>
<td>( | C(n) |_{l_2} )</td>
<td>0.1079 \times 10^{-9}</td>
<td>0.6200 \times 10^{-10}</td>
<td>0.2030 \times 10^{-10}</td>
<td>0.8000 \times 10^{-11}</td>
</tr>
<tr>
<td>( | E(n) |_{l_2} )</td>
<td>0.2512 \times 10^{-10}</td>
<td>0.1280 \times 10^{-10}</td>
<td>0.6300 \times 10^{-11}</td>
<td>0.2700 \times 10^{-11}</td>
</tr>
<tr>
<td>Ratio</td>
<td>---</td>
<td>1.9625</td>
<td>2.0317</td>
<td>2.3334</td>
</tr>
</tbody>
</table>

where \( \theta = 500 \). The function \( f = 0 \) and the function \( g \) is determined by the analytical solution \( c \). Similar to Problem 4.1 the time step \( k \) and mesh size \( h \) satisfy equality \( k = h^2 \).

Like in case 1, the time step and mesh grid are chosen such that: \( k = 2^{-l}, l = 2, 4, 6, 8 \) and \( h \in \{ 2^{-l}, l = 1, 2, 3, 4 \} \). For \( k = 2^{-8} \) that is, \( K = 2^9 \) we list in Table 2 the numerical solution “C”, exact solution “c” and error “E” related to a two-level factored Crank-Nicolson procedure to see that the method is second order convergent in time and fourth order accurate in space. Furthermore, we plot the analytical solution and approximate one together with the error versus \( n \) to see the efficiency and effectiveness of the considered method.

**Problem 4.3** (Exponential dispersion coefficients). The dispersion coefficients taken in [2] are defined as

\[ D_{1}(t) = D_{0} \left[ 1 - \exp \left( -\frac{t}{\theta} \right) \right] + D_{1} \quad \text{and} \quad D_{\tau}(t) = \frac{1}{\alpha^{2}} \left( D_{0} \left[ 1 - \exp \left( -\frac{t}{\theta} \right) \right] + D_{1} \right). \]

We recall that the physical parameters used in this test are given by \( R = 1, L = n_{1} = 0.25, D_{1} = 1, D_{0} = 2, a = 1, \theta = 500, [t_{0}, T_{f}] = [1,3], \Omega = (a_{1}, b_{1}) \times (a_{2}, b_{2}) = (2,4) \times (2,4). u(x,y,t), \)
\( v(x,y,t) \) and \( \mu(x,y,t) \) are functions defined as:

\[
\begin{align*}
    u(x,y,t) &= x^2 + y + t, \\
    v(x,y,t) &= x + y^2 + t, \\
    \mu(x,y,t) &= xy + t^2, \\
    q(x,y,t) &= \frac{L}{n_1} \delta(x) \delta(y) \delta(t).
\end{align*}
\]

The exact solution \( c \) is given by

\[
c(x,y,t) = \frac{aL}{4n_1 \pi} \left[ (D_0 + D_1)t + D_0 \theta \left( \exp \left( -\frac{t}{\theta} \right) - 1 \right) \right] \\
\times \exp \left[ -\mu t - R \frac{(x - \frac{u}{R}t)^2 + a^2 (y - \frac{v}{R}t)^2}{4 \left( (D_0 + D_1)t + D_0 \theta \left( \exp \left( -\frac{t}{\theta} \right) - 1 \right) \right)} \right].
\]

The initial condition \( f = 0 \) and the boundary condition is directly determined from the exact solution \( c \).

The theoretical analysis provided in Section 3, Theorem 3.1 and Section 2, pp. 6 (paragraph below Eq. (2.20), last line) has suggested that the new algorithm is unconditionally stable, second order convergent in time and fourth order accurate in space. We observe from Problems 4.1-4.3 that the expected results from the theory are confirmed. More precisely, Figs. 1-4 and Tables 1-3 indicate that the two-level factored technique is stable, temporal second order accurate and spatial fourth order convergent. Thus, the proposed method applied to initial-boundary value problem (1.1)-(1.3) is fast, efficient and effective.

5 General conclusions and future works

This paper has described in detail a two-level factored Crank-Nicolson scheme for solving the two-dimensional unsteady transport equation with time dependent dispersion coefficients and sink/source terms (1.1) subjects to initial-boundary conditions (1.2)-(1.3) and has provided the stability together with the convergence rate of the method. The theory has suggested that our method is unconditionally stable, second order accuracy in time and fourth order convergent in space. This theoretical analysis is confirmed by a large set of numerical examples (see both Figs. 1-4 and Tables 1-3). Numerical evidences also have shown that the new algorithm is: (a) more efficient and effective than a broad range of numerical methods \([6, 8, 11, 15, 19, 25, 36]\) applied to the initial-boundary value problem (1.1)-(1.3); (b) fast and robust tools for integrating systems of hyperbolic/parabolic PDEs. Moreover, the two-level factored formulation is a suitable approach for solving from low to high Reynolds number flows where the viscous region is too thin by providing less computations at each calculation step. This substantially reduces the computational cost of the algorithm. Furthermore, for two- and multi-dimensional problems, the method reduces to find the solution of a tridiagonal linear system of equations which can be easily obtained by applying the Thomas approach. Our
future works will consider the numerical solution of the two-dimension nonstationary convection-diffusion equation with spatial variable coefficients and sink/source terms using the new technique.

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