Subelliptic Harmonic Maps with Values in Metric Spaces of Nonpositive Curvature

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Abstract. We prove the Hölder continuity of a harmonic map from a domain of a sub-Riemannian manifold into a locally compact manifold with nonpositive curvature, and more generally into a non-positively curved metric space in the Alexandrov sense.

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1 Introduction

Sub-Riemannian manifolds naturally arise in many areas of mathematics. They are generalizations of Riemannian manifolds, where the quadratic form on the tangent bundle no longer is strictly positive definite, but where still all directions can be generated from commutators of positive ones. Thus, a sub-Riemannian manifold is a connected smooth manifold equipped with a positive definite quadratic form $Q$ defined on a smooth distribution $S$ with rank $m$ of the tangent bundle, and $S$ is assumed to satisfy the Hörmander condition. This means that the vector fields in $S$ together with their brackets up to some order generate the whole tangent bundle. Thus, while we no longer require positive definiteness, as discovered by Hörmander [6] this condition still implies some useful analysis inequalities, and with some additional effort, one can usually derive the same results as

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are known for Riemannian manifolds. To see the structure, let us first point out that for any point \( x \in M \), one can use the quadratic \( Q_x \) to define a unique linear mapping \( g_S(x): T^*_x \rightarrow T_x \) via

\[
Q_x(g_S(x)\xi, X) = \langle \xi, X \rangle, \quad \forall X \in S_x
\]

for \( \xi \in T^*_x \), where \( \langle \, \rangle \) is the pairing between \( T^*_x \) and \( T_x \). \( g_S \) varies smoothly on \( M \), and is semi positive definite.

There is one difficulty in sub-Riemannian geometry, however. Since the metric is not positive definite, there is no canonically defined volume measure. To overcome this problem, we choose an arbitrary Riemannian metric \( g \) and we equip the bundle complementary to the distribution \( S \) with the restriction of the metric \( g \) and we then declare it to be orthogonal to \( S \). We then obtain a measure and a volume form, although admittedly the choice of the metric \( g \) is arbitrary and not determined by the sub-Riemannian structure.

On the positive side, a distance function is naturally associated with the distribution. In fact, for any given two points, by a classical theorem of Chow and Rasshevsky, see [2, 15], because of the Hörmander condition, we can find a curve connecting them whose tangent vectors are contained in the distribution \( S \) and therefore have positive norm, and in particular, the curve then has positive length. We can then define the distance by taking the infimum of the lengths of such curves. Hence, the distance is finite, and we obtain a metric. In particular, we can find shortest curves, geodesics, connecting any two points.

Dong [3] showed the existence of harmonic maps from sub-Riemannian manifolds into Riemannian manifolds of non-positive curvature. Here, in order to define the energy functional, the above volume form is needed. In this contribution, we show the regularity of such harmonic maps. Our main theorem is

**Theorem 1.1.** Any harmonic map \( u: M \rightarrow N \) from a sub-Riemannian manifold \( M \) into a locally compact Riemannian manifold \( N \) with non-positive curvature is Hölder continuous.

In fact, our proofs make no use of the smooth structure of the target, hence our method applies to a more general setting, and we can prove

**Theorem 1.2.** Let \( u: M \rightarrow N \) be a harmonic map from a sub-Riemannian manifold \( M \) into a locally compact metric space \( N \) which is non-positively curved in the Alexandrov sense. Then \( u \) is Hölder continuous.

As mentioned, the existence of the harmonic map has been obtained in [3] when the target is smooth with non-positive curvature. The method can be extended to more general metric spaces of non-positive curvature, for instance by using the constructions of [8].
Our regularity proof uses the convexity of the squared distance functions on the non-positively curved target space in an essential way. More precisely, the squared distance function is strictly convex on the universal cover of a non-positively curved target space, but we may localize in the domain to lift a map into such a space to the universal cover. In particular, a harmonic map composed with a convex function is subharmonic, and so, from the distance functions, we obtain a family of subharmonic functions on local balls in the domain. We can then apply the De Giorgi-Nash-Moser iteration to derive a Harnack inequality. Finally, by an application of the map-valued Poincaré inequality and some geometric argument due to Lin in [13], see also [10], we can deduce the Hölder continuity of the harmonic map.

The paper is organized as follows. In the next section, we will introduce basic material about sub-Riemannian geometry and harmonic maps. The proof of two important ingredients, the doubling condition and the Poincaré inequality in sub-Riemannian manifolds are contained in Section 3. The proof of the regularity will then be given in the last section. In the sequel, we make an agreement that the positive constant $C$ may vary from line to line. And we will not distinguish subsequences.

2 Preliminaries

In this section, we first introduce some material from sub-Riemannian geometry and harmonic maps. Basic references are [11, 17]. Let $M$ be a smooth manifold and $S$ a distribution of the tangent bundle $TM$ with rank $m$. A triple $(M, Q_S, S)$ is called a sub-Riemannian manifold of rank $m$, if $Q_S$ is a Riemannian metric on $S$. Suppose $X_i$ is a local basis of $S$ defined on some open subset $U \subset M$. Denote

$$\Gamma(U, S) = \text{span}\{X_1, \cdots, X_m\},$$

and inductively define

$$\Gamma^{i+1}(U, S) = \Gamma^i(U, S) + \left[\Gamma(U, S), \Gamma^i(U, S)\right],$$

where $[\cdot, \cdot]$ is the Lie bracket of the vector fields.

**Definition 2.1.** $S$ is said to satisfy the Hörmander condition of order $r$ if

$$\Gamma^r(U, S)_x = T_x M, \quad \forall x \in U.$$
It is easy to check that the definition is independent of the particular choice of the local basis. As already explained in the introduction, for \( x \in M \), the quadratic \( Q_x \) yields a unique linear mapping \( g_S(x) : T^*_x \to T_x \) by

\[
Q_x(g_S(x)\xi, X) = \langle \xi, X \rangle, \quad \forall X \in S_x
\]

for \( \xi \in T^*_x \), where \( \langle \cdot, \cdot \rangle \) is the pairing between \( T^*_x \) and \( T_x \). Clearly, \( g_S \) varies smoothly on \( M \) as \( g_S(v,w) \) is a smooth function \( \forall v, w \in S \) and it is nonnegative since the distribution is semi positive definite. In order to define a Laplace operator, in addition, one needs a volume form. However, such a volume form is not canonically defined. One can introduce a compatible volume form as follows. Choose any background Riemannian metric \( g_0 \), consider the vertical distribution \( V \) of \( S \) with respect to the metric \( g_0 \), to obtain an orthogonal decomposition of the tangent bundle, \( TM = S \oplus V \). Then we define a metric \( g = g_S + g_0, \cdot \), i.e. we assert that the vectors in \( V \) and \( S \) be orthogonal and use the restriction metric on \( V \) and \( S \) respectively. In this way, we get an extension of the original sub-metric and we can define a volume form. Classical examples are given by the Carnot group, in particular the Heisenberg group, or contact manifolds, see [4].

We now introduce the fundamental distance function that is naturally associated with the distribution, called the Carnot-Carathéodory distance. We emphasize that this distance is not induced by the Riemannian metric \( g \), and hence it may be very different from the Riemannian distance.

We say a curve \( \gamma : I \to M \) is horizontal if \( \dot{\gamma}(t) \in S_{\gamma(t)}, \forall t \in I \). \( (\gamma(t), \xi(t)) \) is called a lift of the curve \( \gamma(t) \) if \( g_S(\gamma(t))\xi(t) = \dot{\gamma}(t) \). Define the length of the curve \( \gamma \) by

\[
l(\gamma) = \int_a^b \sqrt{\langle g_S(\xi(t)), \xi(t) \rangle} \, dt,
\]

where \( \langle \cdot, \cdot \rangle \) is the pair between \( TM \) and \( T^*M \). For any two points \( p, q \in M \), the distance between them is then defined as the infimum of all horizontal curves connecting the two points \( p, q \). More precisely, we define

\[
d(p, q) = \inf_\gamma \{ l(\gamma) : \gamma \text{ is a horizontal curve} \}.
\]

By a classical theorem of Chow and Rashevsky, there always exists a horizontal curve on a connected manifold, hence the distance is finite. We refer to the book [1] for a proof of the fact that \( d \) is indeed a metric on \( M \). As indicated in [9], the basic ingredients to prove the regularity of a harmonic map into non-positively curved target are the Poincaré inequality and the doubling condition of the domain manifold.
3 Proof of the doubling and Poincaré inequalities

In this section, we shall treat these two properties in a sub-Riemannian manifold.

**Proposition 3.1.** On a sub-Riemannian manifold $M$, there exist two positive constants $s, C$ such that, for any point $p \in M$, and any $0 < r < R \leq \frac{1}{2} \text{diam} M$, we have

$$
\mu(B_R(p)) \leq C \left( \frac{R}{r} \right)^s \mu(B_r(p)),
$$

where $\mu = \mu_g$ is the volume measure with respect to the Riemannian metric $g$.

The proof of this proposition is already indicated in the paper [14], see also [3]. Note that the Proposition 3.1 immediately implies that the measure is doubling which is what we need.

We shall now give the proof of the Poincaré inequality. A smooth vector field is called horizontal if $X_p \in S_p$ for any point $p \in M$. We may define a horizontal gradient vector field associated with a smooth function, namely, we define $\nabla_S f$ as the unique vector field satisfying the condition $Q(\nabla_S f, X) = df(X)$, for any horizontal vector field $X$. We can now state the following Poincaré inequality.

**Proposition 3.2.** There exists a positive constant $C$ such that

$$
\int_{B_r(p)} |f - f_{B_r(p)}|^2 d\mu \leq Cr^2 \int_{B_r(p)} |\nabla_S f|^2 d\mu,
$$

where $f$ is a smooth function on $M$, and $B_r(p)$ is a metric ball. $f_{B_r(r)}$ is the average integral of $f$ over the ball $B_r(p)$.

**Proof.** We will apply the classical Poincaré inequality for vector fields satisfying the Hörmander condition on a Euclidean domain, see [7]. Let $(U_i, \phi_i)$ be local coordinate charts that cover the manifold $M$. Then we get a finite family of open subsets $\Omega_i = \phi_i(U_i)$ of $\mathbb{R}^n$. Let $X^a_i$ be a local basis of the sub-bundle $S$ restricted on $U_i$. By our assumption, $X^a_i$ together with their communicators generate the tangent bundle on $U_i$. Set $\tilde{X}^a_i = (d\phi_i) X^a_i$, then $\tilde{X}^a_i$ still satisfies the Hörmander condition since $\phi_i$ is a diffeomorphism and

$$
\left[ (d\phi_i) X^a_i, (d\phi_i) X^b_j \right] = (d\phi_i) \left[ X^a_i, X^b_j \right].
$$

Set $g_i = (\phi_i^{-1})^* g$. Since $M$ is compact, there exist positive constants $C_i$ such that

$$
C_i^{-1} g_i \leq g \leq C_i g_i \quad \text{in} \quad U_i,
$$

and the local coordinates $\phi_i$ are $C^\infty$-smooth.
where $g_0$ is the standard Euclidean metric. Hence, the volume measure is also comparable. In other words, we have
\[
C_i^{-1} d\mu_g \leq d\mathcal{L}^n \leq C_i d\mu_g, \tag{3.1}
\]
where $d\mathcal{L}^n$ is the standard Lebesgue measure on Euclidean space. In particular, for any subset $A \subset \mathbb{R}^n$,
\[
C_i^{-1} d\mu_g (\phi_i^{-1}(A)) \leq d\mathcal{L}^n (A) \leq C_i d\mu_g (\phi_i^{-1}(A)). \tag{3.2}
\]

Now we need to show that the corresponding non-isotropic distance is also comparable.

**Lemma 3.1.** Let $d$ denote the distance function on $M$, then there exists a positive constant $C$ such that
\[
C^{-1} d \leq d_0 \leq C d, \tag{3.3}
\]
where $d_0$ is the non-isotropic distance on $\mathbb{R}^n$ induced by the vector fields $\bar{X}^a_i$.

**Proof.** To see this, fix $p, q \in M$, there is a horizontal curve $\gamma$ minimizing the distance $d(p, q)$ with $\dot{\gamma} \in S_{\gamma(t)}$ and parameterized by a multiple of arc length, see [17]. Let $U_{i_k}$ be finitely many charts covering the curve $\gamma$. By our particular choice, the length of the curve $\gamma$ is then given by
\[
d(p, q) = l(\gamma) = \int_0^1 \sqrt{\langle g_S (\xi(t), \xi(t)) \rangle} dt := \delta.
\]

On $U_{i_k}$, we may suppose $\dot{\gamma}(t) = \sum_{k=1}^{m} a_k^i(t) X^a_{i_k}(t)$, then $\sum_k \sum_{\alpha, \beta} g^a_{S} \partial^a_{\alpha} a_k^i \partial^a_{\beta} = \delta^2$. Let $\lambda^{i_k}_{\min}$ be the smallest positive eigenvalue of $g^k_s$ on $S$ restricted on $U_{i_k}$. Set $\lambda_{\min} = \min_k \{\lambda^{i_k}_{\min}\}$. Thus
\[
\lambda_{\min} \sum_k (a_k^i)^2 \leq \delta^2. \tag{3.4}
\]

Consider the curve
\[
\gamma_i = \phi_i \circ \gamma, \quad t \in (0,1).
\]
We may connect the curves $\gamma_i$ to get a piece-wise smooth curve $\tilde{\gamma}$. On each component $\gamma_i$, direct computation gives
\[
\dot{\gamma}_i = (d\phi_i) (\dot{\gamma}) = \sum_{i=1}^{m} a_k^i \bar{X}^a_{i_k}.
\]
That is, $\bar{\gamma}$ is an admissible curve. Thus we conclude that
\[ d_0(\phi_i(p), \phi_i(q)) \leq \frac{1}{\lambda_{\text{min}}}. \]

Since $M$ is compact, $\lambda_{\text{min}}^i$ has a positive lower bound. Hence, by definition, we have
\[ d_0(\phi_{i_k}(p), \phi_{i_k}(q)) \leq \lambda_{\text{min}}^i. \]
So that
\[ \sum_k d_0(\phi_{i_k}(p), \phi_{i_k}(q)) \leq C d(p, q). \]

On the other hand, for any $x, y \in \phi_i(U_i)$, we can still choose a minimizing curve $\eta$ connecting $x, y$ such that
\[ \dot{\eta}(t) = \sum b^l_\beta(t) \bar{X}^\beta_{j_l}(t), \quad t \in (0, 1), \quad d_0(x, y) = \sum_l (b^l_\beta)^2. \]

Consider the curve $\eta_i = \phi_i^{-1} \circ \eta$, then we have $\dot{\eta}_i = \sum b^l_\beta X^\beta_{j_l}$ and
\[ d(\phi_i^{-1}(x), \phi_i^{-1}(y)) \leq \int_0^1 \sqrt{\langle g_S(\zeta(t)), \zeta(t) \rangle} \, dt \leq \lambda_{\text{max}}^i \sum_l (b^l_\beta)^2 \leq C_i d_0(x, y), \]
where $\lambda_{\text{max}}^i$ is the maximal positive eigenvalue of the metric $g_S$ restricted on $U_i$ and $(\beta(t), \zeta(t))$ is the lift of the curve $\beta(t)$. We have shown that the two non-isotropic distances are comparable. \[ \square \]

Now let $B_r(p)$ be a metric ball with radius $r$ sufficiently small such that $B_r(p)$ is contained in a coordinate chart $U_i$. By Lemma 3.1, the ball $B_i = B_{C_i^{-1} r}(\phi_i(p))$ is contained in the ball $\phi_i(B_r(p))$. Now applying the Poincaré inequality on $\mathbb{R}^n$, we have for any smooth function $f \in C^\infty(M)$
\[ \int_{B_i} \left| f \circ \phi_i^{-1} - (f \circ \phi_i^{-1})_{B_i} \right|^2 d\mathcal{L}^n \leq C r^2 \int_{B_i} \sum_{k=1}^m \left| \bar{X}_k (f \circ \phi_i^{-1}) \right|^2 d\mathcal{L}^n. \quad (3.5) \]

We now prove the following simple claims, where we use $\lesssim$ to indicate that the two related quantities are comparable, i.e. for some positive constant $C$
\[ A \lesssim B \iff C^{-1} A \leq B \leq CA. \]

Here, a slightly different definition of the non-isotropic distance in Euclidean space is used.
Claim 3.1. The average integral of the two functions \( f \) and \( f \circ \phi_i^{-1} \) is comparable, that is,

\[
f_{\phi_i^{-1}(B_i)} \lesssim (f \circ \phi_i^{-1})_{B_i}
\]

Proof. Combining the inequalities (3.1) and (3.2), we have

\[
f_{\phi_i^{-1}(B_i)} = \frac{1}{\mu_s(\phi_i^{-1}(B_i))} \int_{\phi_i^{-1}(B_i)} f \, d\mu
\leq \frac{C_i}{\mathcal{L}^n(B_i)} \int_{B_i} f \circ \phi_i^{-1} J_{\phi_i} \, d\mathcal{L}^n
\leq \frac{C_i}{\mathcal{L}^n(B_i)} \int_{B_i} f \circ \phi_i^{-1} \, d\mathcal{L}^n,
\]

where \( J_{\phi_i} \) is the Jacobian of the transformation \( \phi_i \) satisfying \( 0 < J_{\phi_i} < C_i \).

Similarly we have

Claim 3.2.

\[
\int_{B_i} \left| f \circ \phi_i^{-1} - (f \circ \phi_i^{-1})_{B_i} \right|^2 \, d\mathcal{L}^n \lesssim \int_{\phi_i^{-1}(B_i)} \left| f - f_{\phi_i^{-1}(B_i)} \right|^2 \, d\mu.
\]

We can now prove that

Claim 3.3.

\[
\int_{B_i} \sum_{k=1}^m |X_k(f \circ \phi_i^{-1})|^2 \, d\mathcal{L}^n \lesssim \int_{\phi_i^{-1}(B_i)} \sum_{k=1}^m |X_k f|^2 \, d\mu.
\]

Proof. Note that for any \( y \in B_i \)

\[
X_k(f \circ \phi_i^{-1})(y) = (df_i)(X_k)(f \circ \phi_i^{-1}) = X_k f|_{\phi_i^{-1}(y)}.
\]

The desired result follows from (3.6) and (3.1).

In view of (3.6), and noting the inequality (3.5), we obtain

\[
\int_{\phi_i^{-1}(B_i)} |f - f_{\phi_i^{-1}(B_i)}| \, d\mu \leq C \int_{\phi_i^{-1}(B_i)} \sum_{k=1}^m |X_k f|^2 \, d\mu.
\]

We need to show that the inequality (3.7) is independent of the particular choice of the local basis \( X_k \). To see this, we may assume that the horizontal gradient of the function is of the form \( \nabla_S f = x_l X_l \). By our definition, we have

\[
Q(\nabla_S f, X_k) = df(X_k) \quad \text{or} \quad x_l q_{kl} = X_k f,
\]
where $q_{kl} = Q(X_k, X_l)$ are the coefficients of the quadratic form $Q$. We write the formula in matrix form

\[ q \cdot x = Xf, \]

where $x = (x_1, \cdots, x_m)^T$ and $X = (X_1, \cdots, X_m)$. Since $Q$ is positive definite on the subbundle $S$, we can solve the equation to obtain

\[ x = q^{-1} \cdot Xf. \]

Hence, the norm of the horizontal gradient is given by

\[ |\nabla_S f|^2 = Q(\nabla S f, \nabla S f) = x^T \cdot qx = (Xf)^T \cdot (q^{-1})^T \cdot (Xf). \]

But the quadratic form $Q$ is positive definite on $S$, and so is the matrix $q$, we then conclude that

\[ |\nabla_S f|^2 \lesssim |Xf|^2. \]

Finally, we claim that the subset $\phi_i^{-1}(B_i)$ contains a metric ball of a radius that is a multiple of $r$, the radius of the ball $B_r(p)$. Recalling $B_i = B_{C^{-1}r}(\phi_i(p))$ and the relation of (3.3) in Lemma 3.1 we claim that

Claim 3.4. The ball centered at $p$ of radius $C^{-2}r$ is contained in $\phi_i^{-1}(B_i)$, that is

\[ B = B_{C^{-2}r}(p) \subset \phi_i^{-1}(B_i). \]

Proof. This is because for $x \in B$, we have $d(x, p) \leq C^{-2}r$. Thus,

\[ d_0(\phi_i(p), \phi_i(x)) \leq Cd(x, p) \leq C^{-1}r. \]

We then finish the proof.

We now want to show that the Poincaré inequality indeed holds in a more general setting, i.e. we allow the smooth map to take values in a manifold. More precisely, we can prove the following proposition.

Proposition 3.3. Let $u$ be a smooth map from a sub-Riemannian manifold $M$ into a smooth nonpositively curved manifold $N$. With the notation in Proposition 3.2, there exists a positive constant $C$ with

\[ \int_{B_r(p)} d_N(u(x), u_B)^2 d\mu \leq C r^2 \int_{B_r(p)} e_S(u) d\mu, \] (3.8)

where $u_B$ is the center of the mass of $u$, that is,

\[ \int_B d_N^2(u(x), u_B) d\mu = \inf_{p \in N} \frac{1}{\mu(B)} \int_B d_N^2(u(x), p) d\mu, \]

$d_N$ is the Riemannian distance on $N$, and $e_S(u)$ is the horizontal energy density of the map $u, B = B_r(p)$, see below the definition.
Proof. The proof is an application of Proposition 3.2. Namely, we first embed the manifold isometrically into some Euclidean space, and denote this embedding by \( F: N \hookrightarrow \mathbb{R}^K \). By the Nash theorem, this is possible. We may simply compose a smooth map \( u \) with the embedding to obtain a smooth function \( \bar{u} = F \circ u \) and apply the Poincaré inequality to the function \( \bar{u} \). We have

\[
\int_B |\bar{u} - \bar{u}_B|^2 \, d\mu \leq C r^2 \int_B |\nabla S \bar{u}|^2 \, d\mu. \tag{3.9}
\]

Since \( u_B \) is the center of the mass of \( u \), we immediately get

\[
\int_{B_r(p)} d_N^2(u_B, u(x)) \leq \int_{B_r(p)} d_N^2(u(x), u(q)), \quad \forall q \in N, \ x \in B.
\]

But \( F \) is an isometric embedding, hence a distance-preserving map from \( N \) to \( F(N) \), where \( F(N) \) is equipped with the induced metric from the ambient space \( \mathbb{R}^K \). We have for any \( q \in F(N) \), with \( p = F^{-1}(q) \),

\[
\int_B d_N^2(u(x), u_B) \, d\mu \leq \int_B d_N^2(u(x), q) \, d\mu
\]
\[
= \int_B |F(u(x)) - F(u(p))|^2 \, d\mu
\]
\[
\leq 2 \int_B |\bar{u}(x) - \bar{u}_B|^2 \, d\mu. \tag{3.10}
\]

On the other hand, \( |\nabla S \bar{u}| \leq |dF| \sqrt{e_S(u)} \). Since \( N \) is compact, \( |dF| \) is bounded. Hence, we conclude that

\[
\int_B |\nabla S \bar{u}|^2 \, d\mu \leq C \int_B e_S(u) \, d\mu.
\]

Combining (3.9) and (3.10), we have

\[
\int_B d_N^2(u(x), u_B) \, d\mu \leq C r^2 \int_B e_S(u) \, d\mu,
\]

which is the desired result. \( \Box \)

We now introduce the harmonic map starting from a domain in a sub-Riemannian manifold to another Riemannian manifold, see [3, 11]. A map

\[
u: M \rightarrow N
\]
is called a subelliptic harmonic map if \( u \) is a critical point of the following horizontal energy:

\[
E_S(v) = \frac{1}{2} \int_M e_S(v) \, d\mu,
\]

where \( e_S(v) \) is the horizontal energy. Under a local orthonormal basis, \( \{e_A\} \), we have, where we make an agreement that \( \{e_i\}_{i=1}^m \) span the sub-bundle \( S \), while \( \{e_{\alpha}\}_{\alpha=m+1}^n \) span the vertical bundle

\[
e_S(v) = \sum_{i=1}^m g_N(d\nu(e_i), d\nu(e_i)).
\]

As before, we may embed the manifold \( N \) into some Euclidean space \( \mathbb{R}^K \). It is not hard to see that the Euler-Lagrange equation for the subelliptic harmonic map \( u \) is

\[
\tau_S(u) = \Delta_S u - \nabla^2 \Pi(\nabla_S u, \nabla_S u),
\]

where \( \Delta_S = \text{div}(\nabla_S) \) is the horizontal Laplacian operator. And here we have viewed \( u \) as a map into the Euclidean space \( \mathbb{R}^K \), and \( \Pi \) is the nearest point projection from a neighbourhood of \( N \) in \( \mathbb{R}^K \) into \( N \). We may also derive a Leibniz formula for the composition of two maps. Namely, we have

\[
\tau_S(k \circ u) = dk(\tau_S(u)) + tr_g \nabla^2_N k(\nabla_S u, \nabla_S u),
\]

where \( \nabla_N \) is the Levi-Civita connection on \( N \), \( k: N \to L \) is a smooth map.

### 4 Proof of the main theorems

In this section, we shall give a proof of the regularity property of the subelliptic harmonic map whose existence is obtained in [3]. The proof is divided into several steps, see [10]. As before we use the simplified notation \( B = B_r(x_0) \).

**Proof of Theorem 1.1.** **Step 1:** We will first employ the general fact that a subelliptic harmonic map composed with convex function is sub-harmonic. Namely, \( \Delta_S f \geq 0 \), where \( f = k \circ u \). This directly follows from the composition formula (3.11) by taking \( L = \mathbb{R} \) and noticing the convexity of \( k \).

**Step 2:** We prove a Harnack inequality. The following Sobolev inequality is proved in [11]:

\[
\left( \frac{1}{\mu(B)} \int_B f^{2q} \, d\mu \right)^{\frac{q-2}{q}} \leq C_S \frac{r(B)^2}{\mu(B)^{\frac{q}{2}}} \int_B |\nabla_S f|^2 \, d\mu, \quad \forall f \in C_0^\infty(B).
\]
Remark 4.1. The Sobolev inequalities are related to the volume growth of a metric ball, so that the exponent is no longer the dimension of the manifold. Here we see the difference between the non-isotropic distance and the Riemannian distance, see [3, 16].

By using the De Giorgi-Nash-Moser iteration, see [18], we can derive the following Harnack inequality:

$$\sup_{B_r(x_0)} f \leq C \left( \frac{1}{\mu(B_{2r}(x_0))} \int_{B_{2r}(x_0)} f^q d\mu \right)^{\frac{1}{q}}, \quad \forall q > 1. \quad (4.1)$$

Alternatively, we can use another version of the Harnack inequality, which is a direct consequence of the inequality (4.1) and the John-Nirenberg inequality, see [9]

$$\inf_{B_r(x_0)} f \geq C \frac{1}{\mu(B_r(x_0))} \int_{B_{2r}(x_0)} f d\mu. \quad (4.2)$$

**Step 3:** By using some geometric argument and local compactness of the target, we can show that the function $$f = d^2_N(u, q)$$ for any $$q \in N$$ is Hölder continuous, see [10, 13]. We begin with the following lemma.

**Lemma 4.1.** Let $$u : M \to N$$ be a subelliptic harmonic map and $$f = d^2_N(u, q)$$ be a smooth function satisfying $$\Delta_S f \geq 0$$ weakly for any $$q \in N$$, where $$B_{4r}(x_0)$$ is a geodesic ball such that $$0 < 4r \leq \frac{1}{2} \text{diam}(M)$$. Let $$0 < \kappa_1 \leq \kappa \leq \kappa_0 < \infty$$, suppose that

$$\text{diam} \left( u(B_{2r}(x_0)) \right) := \sup \left\{ d_N(u(x), u(y)) \mid x, y \in B_{2r}(x_0) \right\} = \kappa.$$

There exists an $$\epsilon_0 > 0$$ depending on the geometry of $$M$$ and $$N$$ and on $$\kappa_0$$ and $$\kappa_1$$ with the property that if $$0 < \epsilon \leq \epsilon_0$$ and $$u(B_{2r}(x_0))$$ is covered by $$k$$ balls $$B_1, \ldots, B_k$$ of radius $$\epsilon$$, then $$u(B_r(x_0))$$ can be covered by $$k - 1$$ of those balls.

**Proof.** The proof uses the argument of [13]. Without loss of generality, we assume $$\kappa = 1$$. Suppose

$$B_i \subset B_{2\epsilon}(p_i) \quad \text{with} \quad p_i = u(x_i), \quad i = 1, \ldots, k.$$

Since $$N$$ is locally compact and $$u(B_{2r}(x_0))$$ is bounded, there exists a positive integer $$k_0$$ which depends on the geometry of the target $$N$$ such that any ball of radius 2 with center $$p_i$$ contains at most $$k_0$$ points with mutual distance at least $$\frac{1}{5}$$. In other words, we may assume that the balls $$B_{1/4}, i = 1, \ldots, k_0$$ cover the set $$u(B_{2r}(x_0))$$. Hence, there exists at least one point, say $$p_1$$, such that

$$\mu \left( u^{-1}(B_{1/4}(p_1)) \cap B_r(x_0) \right) \geq \frac{1}{k_0} \mu(B_r(x_0)) \geq \frac{\eta_0}{k_0} r^Q, \quad (4.3)$$
where $\eta_0$ depends on the geometry of the domain $M$. We now consider the function

$$g(x) = d^2_N(u(x), p_1).$$

And define

$$M_0 = \sup_{B_{2r}(x_0)} g(x).$$

Obviously, $\frac{1}{2} \leq M_0 \leq \text{diam}(u_{2r}(x_0)) = 1$. Note that

$$g \leq \frac{1}{16} \text{ on } u^{-1}(B_{\frac{1}{4}}(p_1)).$$

Consider the function

$$h(x) = M_0 - g(x) \geq 0 \text{ on } B_{2r}(x_0).$$

Then

$$h \geq \frac{1}{8} \text{ on } u^{-1}(B_{\frac{1}{4}}(p_1)). \quad (4.4)$$

By our assumption, we have

$$\Delta_S h \leq 0 \text{ weakly in } B_{2r}(x_0).$$

We can now apply the Harnack inequality (4.2) to the function $h$ and combine the inequalities (4.3), (4.4), and we find that

$$\inf_{B_r(x_0)} h(x) \geq \delta_0 \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} h d\mu \geq \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0) \cap u^{-1}(B_{1/4}(p_1))} h d\mu \geq \eta \quad (4.5)$$

for some $\eta > 0$. We now claim that for sufficiently small $\varepsilon > 0$, there cannot hold

$$u(B_r(x_0)) \cap (B_{2\varepsilon}(p_i)) \neq \emptyset. \quad (4.6)$$

Indeed, suppose $M_0 = d^2_N(u(y), p_1)$ for some point $y \in B_{2r}(x_0)$. Since $u(B_{2r}(x_0))$ is covered by $\bigcup_{j=1}^K B_{2\varepsilon}(p_j)$, we can find some $p_i$ such that $u(y) \in B_{2\varepsilon}(p_i)$. Now if (4.6) does hold, we can find some point $y_1 \in B_r(x_0)$ such that $d_N(u(y_1), p_i) \leq 2\varepsilon$. Hence,

$$d_N(u(y_1), u(y)) \leq d_N(u(y_1), p_i) + d_N(u(y), p_i) \leq 4\varepsilon.$$
We conclude that
\[
\inf_{B_r(x_0)} h \leq d^2_N(u(y),p_1) - d^2_N(u(y_1),p_1) \\
\leq d_N(u(y),u(y_1))(d_N(u(y),p_1) + d_N(u(y_1),p_1)) \leq 8\epsilon.
\]

Therefore, if we choose \( \varepsilon_0 = \frac{\delta}{16} \), we get a contradiction.

To finish the proof, we first apply the inequality (4.1) to deduce that \( |f|_{L^\infty} \) is bounded by its \( L^q \)-norm \( \forall q > 2 \). Then using the Poincaré inequality Proposition 4.1 and the assumption that the map \( u \) has finite energy, we conclude that \( f \) is bounded, hence \( u \) is itself bounded. The Hölder continuity will follow from the next lemma.

**Lemma 4.2.** Under the assumptions of Lemma 4.1, for any \( \varepsilon > 0 \), there exists \( \rho > 0 \) such that \( \text{diam}(u(B_{\rho}(x))) \leq \varepsilon, \forall x \in B_r(x_0) \).

**Proof.** Since \( u \) is bounded, we may assume \( \text{diam}(u(B_2(x_0))) = 1 \). Letting \( \varepsilon > 0 \) and \( x \in B_r(x_0) \), we need to find some \( \rho > 0 \) with \( \text{diam}(u(B_{\rho}(x))) \leq \varepsilon \). Suppose \( u(B_r(x)) \) is covered by \( k \) balls of radius \( \varepsilon \) with \( k \) uniformly bounded independent of \( \varepsilon \). By Lemma 4.1, we can cover the subset \( u(B_{r/2}(x)) \) by \( k-1 \) balls of radius \( \varepsilon \). Iterating this procedure, we can find some positive integer \( l \) such that \( u(B_{2^{-l}r}(x)) \) is covered by a single ball of radius \( \varepsilon \), hence \( \text{diam}(u(B_{2^{-l}r}(x))) \leq \varepsilon \).

**Remark 4.2.** The Hölder continuity of a map \( f : \Omega \to N \) can be characterised by the following property, there is an \( r_0 > 0 \) and \( \alpha \in (0,1), C > 0 \) such that \( \forall x \in \Omega \), there holds
\[
\omega(f,x,r) \leq Cr^\alpha, \quad 0 < r \leq r_0,
\]
where \( \omega(f,x,r) \) is the oscillation of the map \( f \) on the metric ball \( B_r(x) \).

**Remark 4.3.** The method presented above can be applied to handle the case where the target is singular, since we only use the doubling property of the domain manifold and a map-valued Poincaré inequality, see Proposition 4.1 for a singular target case.

We now need to generalize the map-valued Poincaré inequality to a singular target that is non-positively curved in the Alexandrov sense.

**Proposition 4.1.** Let \( M \) be a Riemannian manifold, \( N \) a locally compact metric space with non-positive curvature in the Alexandrov sense. Then there holds the local Poincaré
inequality. More precisely, there is \( r_0 > 0 \) and a positive constant \( P \) such that for any Lipschitz continuous map \( f : B_r(x_0) \to N \) and \( 0 < r \leq r_0 \), we have

\[
\int_{B_r(x_0)} d_N^2(f_B, f(x)) d\mu \leq Pr^2 \int_{B_{2r}(x_0)} \text{Lip}_f(x)^2 d\mu, \tag{4.7}
\]

where \( \text{Lip}_f \) is the point-wise Lipschitz constant, i.e. for \( x \) is not an isolated point,

\[
\text{Lip}_f(x) = \limsup_{y \to x} \frac{d_N(f(y), f(x))}{d(x, y)},
\]

and 0 otherwise, \( f_B \) is the center of the mass on \( B_r(x_0) \), that is,

\[
\int_{B_r(x_0)} d_N^2(f(x), f_B) d\mu = \inf_{p \in N} \int_{B_r(x_0)} d_N^2(f(x), p) d\mu.
\]

We use the argument of Rellich compactness, see [10]. There are two properties that we need to prove when applying the methods to a non-smooth setting. Firstly, we want to show

**Lemma 4.3.** Let \( g : B_r(x_0) \to N \) be a Lipschitz map, where as before, \( N \) is assumed to be non-positively curved. Suppose \( g_B \) is the unique center of mass of the map \( g \). Let \( g_t(x) \) be a geodesic connecting \( g_B \) and the point \( g(x) \). Then \( g_B \) remains to be the center of mass along the geodesic, namely, \( g_{t,B} = g_B \).

**Remark 4.4.** The center of mass uniquely exists in a non-positively curved metric space, see [10].

**Proof of Lemma 4.3.** We want to show that

\[
\int_B d_N^2(g_t(x), g_B) d\mu \leq \inf_{q \in N} \int_B d_N^2(g_t(x), q) d\mu. \tag{4.8}
\]

For simplicity, we omit the volume element. For \( q \in N \), we compute

\[
\int_B d_N^2(g(x), g_B) \leq \int_B d_N^2(g(x), q) \\
\leq \int_B (d_N(g(x), g_t(x)) + d_N(g_t(x), q))^2 \\
= (1-t)^2 \int_B d_N^2(g(x), g_B) + \int_B d_N^2(g_t(x), q) \\
+ 2(1-t) \int_B d_N(g(x), g_B) d_N(g_t(x), q). \tag{4.9}
\]
Arranging the inequality (4.9), we obtain
\[
(2t-t^2) \int_B d_N^2(g(x),g_B) \leq 2(1-t) \int_B d_N(g(x),g_B)d_N(g_I(x),q) + \int_B d_N^2(g_I(x),q),
\]
which is equivalent to the following inequality:
\[
2(1-t) \int_B \left( t d_N^2(g(x),g_B) - d_N(g(x),g_B) d_N(g_I(x),q) \right) + t^2 \int_B d_N^2(g(x),g_B) \leq \int_B d_N^2(g_I(x),q).
\]
Thus, to show inequality (4.8), it suffices to show that for \( q = g_I, \)
\[
\int_B t^2d_N^2(g(x),g_B) \geq \int_B d_N^2(g(x),g_B)d_N(g_I(x),q). \tag{4.11}
\]
However, from the Cauchy-Schwarz inequality combined with the definition of the center of mass, we have
\[
2 \int_B d_N^2(g_I(x),g_B) \geq \int_B d_N^2(g_I(x),g_B) + \int_B d_N^2(g_I(x),g_{I,B}) \geq 2 \int_B d_N(g_I(x),g_B)d_N(g_I(x),g_{I,B}). \tag{4.12}
\]
These two inequalities (4.11) and (4.12) are equivalent because \( t d_N(g(x),g_B) = d_N^2(g_I(x),g_B), \) and we are done. \( \square \)

The other property that we want to generalize is the following Rellich type compactness.

**Proposition 4.2.** Let \( f_i: B_r(x_0) \to N \) be a sequence of Lipschitz maps satisfying
\[
\sup_i \int_{B_r(x_0)} \text{Lip}^2 f_i d\mu < \infty,
\]
\[
\int_{B_r(x_0)} d_N^2 (f_i(x), f_{I,B}) d\mu = 1.
\]
Then we have that \( f_i \to f \) strongly in \( L^2 \) and
\[
\int_{B_r(x_0)} d_N^2 (f(x), f_B) d\mu = 1.
\]
Proof: The proof is based on the generalized Arzela-Ascoli compactness theorem, for values in a metric space see [12]. This theorem tells us that a sequence of continuous maps $g_i : M \to N$ has a convergent subsequence if the $g_i$ satisfy the following two conditions: for any $a \in B_r(x_0)$

- the closure of the set $F_a = \{ g_i(a), i = 1, 2, \ldots \}$ is compact,

- $\forall \epsilon > 0$, there exists some neighbourhood $U_a$ of $a$ such that $d_N(g_i(x), g_i(y)) < \epsilon$, $\forall x, y \in U_a$, $\forall i$.

Now concerning the $f_i$, if they are all Lipschitz continuous with uniformly bounded Lipschitz constant, they automatically satisfy the two conditions. Hence, there exists a subsequence, still denoted by $f_i$, such that $f_i$ converges to some Lipschitz map $f$ uniformly. Then the Fatou lemma implies that $\int_B Lip f^2 d\mu < \infty$ and

$$\int_{B_r(x_0)} d^2_N(f(x), f_B) d\mu = 1.$$ 

Otherwise, we shall first mollify the family of maps $f_i$ to get another sequence of maps $f_{i,h}$ with uniformly bounded Lipschitz constant and then extract a subsequence. We then prove that the $f_i$ lie in an $\epsilon$-net of the sequential compact maps $f_{i,h}$. Together with the completeness of the function space $L^2(M, N)$, we conclude that $f_i$ converges to some Lipschitz map $f$ with $\int_{B_r(x_0)} d^2_N(f(x), f_B) d\mu = 1$. □

Remark 4.5. Note here that we need to mollify the maps, hence a construction of mollifiers is necessary. We can indeed construct such mollifiers in an NPC space, see [10].

Proof of Proposition 4.1. The proof proceeds by a contradiction argument. Suppose the inequality (4.7) does not hold. We may find a sequence of maps $f_i$ with

$$\int_{B_r(x_0)} d^2_N(f_i, f_{i,h}(x)) d\mu \geq \int_{B_{2r}(x_0)} Lip f^2_i(x) d\mu. \quad (4.13)$$

And by scaling, we may assume $r = 1$. At present, we do not know whether the left hand side of the inequality (4.13) stays finite. If not, we will replace the map $f_i$ by its modification $f_{i,t}$, where $f_{i,t}$ is the map as in Lemma 4.3. This is because the integral $F(t) = \int_B d^2_N(g_i(x), g_{i,B})$ is continuous with $t$, and we note that by Lemma 4.3

$$F(0) = 0, \quad F(1) = \int_B d^2_N(g(x), g_{1,B}) d\mu = \int_B d^2_N(g(x), g_B) d\mu.$$
Hence, we may find some $t \in (0,1)$ such that $F(t) = 1$. Therefore, without loss of generality we can assume that

$$
\int_{B_r(x_0)} d^2_N(f_i,B,f_i(x))d\mu = 1
$$

and $\int_B Lip f^2 d\mu \to 0$. Proposition 4.2 implies that we can extract a subsequence $f_i$ converging to some map $f$ with $\int_B d^2_N(f(x),f_B)d\mu = 1$, but $Lip f = 0$. We conclude that $f$ is indeed a constant map, but then $\int_B d^2_N(f(x),f_B)d\mu = 0$, a contradiction.

Now for any $\Omega \subset M$, let $H^1(\Omega,N)$ be the completion of the space of Lipschitz maps from $M$ to $N$ under the norm $\int_{\Omega}(f^2+Lip f^2)d\mu$. If $f \in H^1(\Omega,N)$, we choose $f_i \in Lip(\Omega,N)$ such that

$$
\int_{\Omega} d^2_N(f_i(x),f(x)) + |Lip f_i - Lip f|^2 d\mu \to 0 \quad \text{as} \quad i \to \infty.
$$

We may use $f_i$ to replace the map $f$ in (4.7) and then we obtain the desired result after taking limit.

\[\square\]

**Remark 4.6.** The fact that a subelliptic harmonic map composed with a convex function is actually subharmonic can be found in [9], where the author used the Dirichlet form. As our domain manifold supports the Poincaré inequality and ball-doubling property, the energy functional is actually associated with a Dirichlet form, see [5].

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**References**


