Discrete Duality Finite Volume Discretization of the Thermal-$P_N$ Radiative Transfer Equations on General Meshes

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Abstract. The discrete duality finite volume method has proven to be a practical tool for discretizing partial differential equations coming from a wide variety of areas of physics on nearly arbitrary meshes. The main ingredients of the method are: (1) use of three meshes, (2) use of the Gauss-Green theorem for the approximation of derivatives, (3) discrete integration by parts. In this article we propose to extend this method to the coupled grey thermal-$P_N$ radiative transfer equations in Cartesian and cylindrical coordinates in order to be able to deal with two-dimensional Lagrangian approximations of the interaction of matter with radiation. The stability under a Courant-Friedrichs-Lewy condition and the preservation of the diffusion asymptotic limit are proved while the experimental second-order accuracy is observed with manufactured solutions. Several numerical experiments are reported which show the good behavior of the method.

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1 Introduction

For discretizing the Boltzmann equation under its multigroup $P_N$ approximate form we proposed in [1] a DDFV (Discrete Duality Finite Volume) type method which benefits from several attractive features, namely: use of general meshes, second-order experimental accuracy on a broad variety of meshes, stability under a Courant-Friedrichs-Lewy (CFL) condition, preservation of the diffusion asymptotic limit and generalization from

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rectangle meshes to nearly arbitrary meshes of either legacy methods as the MAC [2] and Yee’s schemes [3] or the more recent STARMAP method [4].

This was considered as a first step before tackling the discretization of the coupling of the hydrodynamics and Boltzmann equations, thus allowing the Lagrangian type methods to be used for the hydrodynamics without the possible mesh distortions due to the matter motion being an inconvenient for the approximation of the Boltzmann equation. Here we propose to address a second step namely the discretization of the coupled thermal-Boltzmann equations on general meshes (without heat conduction to simplify) while the background material is assumed to be static. Furthermore we deal with both Cartesian and cylindrical coordinates. Of course the third (and last) step will be the discretization of the whole coupled hydrodynamics-Boltzmann equations that is to say the coupling not only of the thermal equation but also the mass and momentum conservation equations with the Boltzmann equation. Such a step will be the subject of ongoing works.

To our knowledge designing numerical methods combining all the features cited above is a rather challenging task: see [5, 6] where a Godunov-type method is proposed for solving the hyperbolic heat equation and, more generally, the abstract Friedrichs systems of equations on unstructured meshes. See also [7] where a discontinuous Galerkin finite element method is proposed for the coupled thermal-Boltzmann equations in Cartesian coordinates.

We will focus here on the Boltzmann equation for photons, namely the radiative transfer equation, but of course other type of particles (as neutrons for example) could be dealt with as well.

The organization of the paper is as follows. In Section 2 we recall what are the thermal-radiative transfer equations and their grey approximate form and what are the grey coupled thermal-$PN$ radiative transfer equations in both Cartesian and cylindrical coordinates. The finite volume method we propose is described in Section 3 in the two-dimensional cylindrical coordinates framework (we restrict ourselves to the 2D case for simplicity but the ideas behind the discretization can be generalized to the 3D case although the notation is inherently more intricate). Section 3.1 is devoted to the DDFV discretization of partial derivatives on general meshes and to some of its mimetic properties. Sections 3.2 and 3.3 detail respectively the space and time discretizations of the $PN$ radiative transfer equations. In Section 3.4 the $L^2$ stability of the scheme is studied (Theorems 3.1 to 3.3) while we prove that it is asymptotic preserving (Theorem 3.4). Section 4 details the space and time discretizations of the grey thermal-$PN$ radiative transfer equations. Finally three numerical experiments are presented in Section 5, the first one allowing us to assess the experimental order of convergence with a manufactured solution and the others being reference benchmarks inspired from the literature [8, 11, 12]. Concluding remarks are proposed in Section 6.

In all what follows the vectors and the matrices will be noted with bold letters while $x = (x, y, z)$ or $x = (r, \phi, z)$, $R$, $S^2$, $t$, $\delta^n$, $I_n$ and $i$ will denote respectively the position vector in Cartesian or cylindrical coordinates, the set of real numbers, the unit sphere in $R^3$, the
time, the delta Kronecker function, the $n$ by $n$ identity matrix and the imaginary unit number. Finally we will denote by $\Delta = [0, t_f]$ a time span and by $\Omega$ a bounded open polygonal domain, boundary $\partial \Omega$, unit outside vector $n = (n_x, n_y, n_z)$.

2 The thermal-radiative transfer equations

Let $c$ be the speed of light and $\nu$, $\omega$ the frequency and unit direction of the photons. We use the standard spherical parametrization of $\omega$ (see Fig. 1) in Cartesian coordinates:

$$\omega = \cos \psi \sqrt{1 - \mu^2} e_x + \sin \psi \sqrt{1 - \mu^2} e_y + \mu e_z,$$

or, in cylindrical coordinates:

$$\omega = \cos \psi \sqrt{1 - \mu^2} e_r + \sin \psi \sqrt{1 - \mu^2} e_\varphi + \mu e_z,$$

where $\psi$ is the azimuthal angle ($0 \leq \psi < 2\pi$) and $\mu = \cos \beta$, $\beta$ being the polar angle ($0 \leq \beta \leq \pi$).

We look for the material temperature $T = T(x, t)$ and the distribution function $\bar{f} = \bar{f}(x, \psi, \mu, \nu, t)$ of photons in the phase space, solutions to the thermal-radiative transfer equations in the laboratory frame:

$$\begin{align*}
\rho \frac{\partial \bar{f}}{\partial t} + c \int_0^{+\infty} \int_{S^2} \bar{\sigma}_a b v dv d\mu d\nu &= ch \int_0^{+\infty} \int_{S^2} \bar{\sigma}_a \bar{f} v dv d\psi d\mu, \quad \text{in } \Omega \times \Delta, \\
-\frac{1}{c} \frac{\partial \bar{f}}{\partial t} + \nabla \cdot (\omega \bar{f}) + (\bar{\sigma}_a + \bar{\sigma}_s) \bar{f} &= \bar{\sigma}_a b + \frac{1}{4\pi} \bar{\sigma}_s \int_{S^2} \bar{f} d\psi d\mu + \frac{1}{c} \bar{q}, \quad \text{in } \Omega \times \Delta, \\
\bar{f} &= \bar{f}_b \quad \text{if } \omega \cdot n \leq 0, \quad \text{on } \partial \Omega \times \Delta, \\
\bar{f}(0) &= \bar{f}_b(0), \quad \text{in } \Omega, \\
\bar{f}_b(0) &= \bar{f}_b(0), \quad \text{in } \Omega, \\
T(0) &= T_0, \quad \text{in } \Omega. 
\end{align*}$$

(2.1)

† That is the number of photons by unity of volume, solid angle and frequency. Instead of $\bar{f}$ the radiative intensity $I = c ch \nu \bar{f}$ is often used in the literature.
where \( \rho = \rho(x,t) \) is the mass density, \( \varepsilon = \varepsilon(\rho,T) \) the specific internal energy, \( \bar{\sigma}_a = \bar{\sigma}_a(\nu,\rho,T) \) and \( \bar{\sigma}_s = \bar{\sigma}_s(\nu,\rho,T) \) the absorption and the scattering opacities (supposed to be isotropic, that is independent of \( \psi \) and \( \mu \)) of the background material, \( \bar{q} = \bar{q}(x,\psi,\mu,\nu,t) \) the source of photons, \( b \) the function defined by\(^1\):

\[
b = b(\nu,T) = 2\frac{v^2}{c^3} \left( \exp\left( \frac{\hbar \nu}{kT} \right) - 1 \right)^{-1},
\]

\( T^0 = T^0(x) \) the initial material temperature, \( \bar{f}^0 = \bar{f}^0(x,\psi,\nu) \) the initial distribution function and \( \bar{f}_b = \bar{f}_b(x,\psi,\mu,\nu,t) \) a given boundary distribution function.

In cylindrical coordinates the second (radiative transfer) equation of (2.1) reads (see [13]):

\[
\frac{1}{c} \frac{\partial \bar{f}}{\partial t} + \cos \psi \sqrt{1 - \mu^2} \frac{\partial \bar{f}}{\partial r} + \frac{1}{r} \sin \psi \sqrt{1 - \mu^2} \left( \frac{\partial \bar{f}}{\partial \psi} - \frac{\partial \bar{f}}{\partial \phi} \right) + \mu \frac{\partial \bar{f}}{\partial \z} + \left( \bar{\sigma}_a + \bar{\sigma}_s \right) \bar{f} = \bar{\sigma}_a b + \frac{1}{4\pi} \bar{\sigma}_s \int_{S^2} \bar{f} d\psi d\mu + \frac{1}{c} \bar{q},
\]

that is, after multiplying by \( r \), under a conservative form:

\[
\frac{1}{c} \frac{\partial \bar{f}}{\partial t} + r \nabla (\omega \bar{f}) - \frac{\partial}{\partial \phi} (\sin \psi \sqrt{1 - \mu^2} \bar{f}) + r(\bar{\sigma}_a + \bar{\sigma}_s) \bar{f} = r \bar{\sigma}_a b + \frac{1}{4\pi} r \bar{\sigma}_s \int_{S^2} \bar{f} d\psi d\mu + \frac{1}{c} r \bar{q}. \tag{2.2}
\]

By definition of the direction \( \omega \) and of the divergence operator in cylindrical coordinates note that:

\[
r \nabla (\omega \bar{f}) = \frac{\partial}{\partial r} (r \cos \psi \sqrt{1 - \mu^2} \bar{f}) + \frac{\partial}{\partial \phi} (\sin \psi \sqrt{1 - \mu^2} \bar{f}) + \frac{\partial \bar{f}}{\partial \z} (r \mu \bar{f}).
\]

Besides the material temperature, the quantities of primary interest are the total radiation energy density \( E \), the radiation vector flux \( F \) and the radiation pressure tensor \( P \) which are defined from the distribution function \( \bar{f} \) by:

\[
E = h \int_0^{+\infty} \int_{S^2} \bar{f} \nu d\psi d\mu dv, \quad F = ch \int_0^{+\infty} \int_{S^2} \omega \bar{f} \nu d\psi d\mu dv, \quad P = h \int_0^{+\infty} \int_{S^2} \omega \otimes \omega \bar{f} \nu d\psi d\mu dv.
\]

If \( \bar{f} = b \) notice that:

\[
E = a T^4, \quad F = 0, \quad P = \frac{1}{3} a T^4 I_3,
\]

where \( a \) is the radiation constant:

\[
a = \frac{8\pi^5 \kappa^4}{15c^3 \hbar^3}
\]

\(^1\)We denote by \( h \) and \( \kappa \) the Planck and Boltzmann constants. Instead of \( b \) the Planck distribution function \( B = chv^2b \) is traditionally used in the literature.
thanks to the relation:

\[
\int_0^{+\infty} v^3 \left( \exp \left( \frac{hv}{kT} \right) - 1 \right)^{-1} dv = \frac{\pi^4 k^4 T^4}{15 h^4}.
\]

By analogy with this particular case the radiative temperature \(T_r\) is defined from the radiation energy \(E\) by the relation:

\[
E = a T_r^4.
\]

To simplify we will suppose from now on that the material obeys the perfect gas law, that is:

\[
\varepsilon = C_v T,
\]

where \(C_v = C_v(x)\) is the material specific heat.

Let denote:

\[
f = f(x, \psi, \mu, t) = h \int_0^{\infty} \bar{f} \nu d\nu, \quad q = q(x, \psi, \mu, t) = h \int_0^{\infty} \bar{q} \nu d\nu
\]

and suppose that:

\[
\bar{\sigma}_a(\nu, \rho, T) \simeq \sigma_a(\rho, T), \quad \bar{\sigma}_s(\nu, \rho, T) \simeq \sigma_s(\rho, T),
\]

where \(\sigma_a\) (resp. \(\sigma_s\)) is some kind of mean absorption (resp. scattering) opacity averaged over the frequency. The calculation of \(\sigma_a\) and \(\sigma_s\) in the general case is somewhat difficult. According to either an optically thin limit or an optically thick limit (close to the thermal equilibrium), we choose respectively Planck means (see [13] for example):

\[
\sigma_a = \int_0^{\infty} \bar{\sigma}_a b v d\nu \int_0^{\infty} b v d\nu, \quad \sigma_s = \int_0^{\infty} \bar{\sigma}_s b v d\nu \int_0^{\infty} b v d\nu
\]

or Rosseland means:

\[
\sigma_a = \int_0^{\infty} \frac{\partial b}{\partial T} v d\nu \int_0^{\infty} \frac{1}{\sigma_a} \frac{\partial b}{\partial T} v d\nu, \quad \sigma_s = \int_0^{\infty} \frac{\partial b}{\partial T} v d\nu \int_0^{\infty} \frac{1}{\sigma_s} \frac{\partial b}{\partial T} v d\nu.
\]

Integrate the second (radiative transfer) equation of system (2.1) or Eq. (2.2) multiplied by the energy \(h\nu\) of the photons over the frequency fan. By using the definitions (2.3) and (2.4) we obtain the so-called grey thermal-radiative transfer equations, in Cartesian coordinates:

\[
\begin{aligned}
\rho C_v \frac{\partial T}{\partial t} + ac \sigma_a T^4 &= c \sigma_a E, \\
\frac{1}{c} \frac{\partial f}{\partial t} + \nabla \cdot (\omega f) + (\sigma_a + \sigma_s) f &= \frac{1}{4\pi} \sigma_a T^4 + \frac{1}{4\pi} \sigma_s \int_{S^2} f d\psi d\mu + \frac{1}{c} q,
\end{aligned}
\]
or, in cylindrical coordinates:
\[
\begin{align*}
\rho c_v \frac{\partial T}{\partial t} + acr_T^4 &= c c_T E, \\
\frac{1}{c} \frac{\partial f}{\partial t} + r \nabla \cdot (\mathbf{w} f) - \frac{\partial}{\partial \psi} (\sin \psi \sqrt{1 - \mu^2} f) + r (c_T + c_a) f &= 0 \\
&= \frac{1}{4\pi} r c_T T^4 + \frac{1}{4\pi} r c_a j \int_{S^2} f d\psi d\mu + \frac{1}{c} r q.
\end{align*}
\]

The unknowns of these systems are the matter temperature \( T \) and the grey distribution function \( f \), that is the number of photons by unity of volume and solid angle.

### 2.1 The \( P_N \) approximation of the radiative equation

Although they are well-known by the community the basic ideas of the \( P_N \) model are introduced in this section, in particular in order to fix the notation and to recall the properties that need to be preserved at the discrete level.

Consider the last (radiative) equation of systems (2.5) and (2.6). Given \( k, m \) integers such that \( 0 \leq |m| \leq k \), consider the expansion of the distribution function \( f \) in the angular basis of real-valued spherical harmonics functions\(^\S\) \( X_k^m \):

\[
f = f(x, \psi, \mu, t) = \sum_{k=-k}^{k} f_k^m(x, t) X_k^m(\psi, \mu),
\]

where the expansion coefficients \( f_k^m \):

\[
f_k^m = f_k^m(x, t) = \frac{1}{4\pi} \int_{S^2} f(x, \psi, \mu, t) X_k^m(\psi, \mu) d\psi d\mu
\]

are called the moments of order \( k \) of \( f \). Let \( N \geq 1 \) be an arbitrary integer, the \( P_N \) approximation consists in neglecting the moments \( f_k^m \) such that \( k > N \).

From the recursion relations (A.4) we deduce that, for all \( k, m \) (with the convention \( f_k^m = 0 \) if \( k < 0 \) or \( |m| > k \)):

\[
\frac{1}{4\pi} \int_{S^2} \mathbf{w} f X_k^m = \left( \begin{array}{c}
\varepsilon_k^m (A_k^m f_{k+1}^m - B_k^m f_{k-1}^m) + \varepsilon_k^m (C_k^m f_{k+1}^m - D_k^m f_{k-1}^m) \\
\eta_k^m (A_k^m f_{k+1}^m - B_k^m f_{k-1}^m) + \eta_k^m (C_k^m f_{k+1}^m - D_k^m f_{k-1}^m) \\
E_k^m f_{k+1}^m + F_k^m f_{k-1}^m
\end{array} \right),
\]

where \( A_k^m, B_k^m, C_k^m, D_k^m, E_k^m, F_k^m \) are coefficients defined by (A.2) which depend only on \( k \), \( m \) and \( \varepsilon_k^m, \eta_k^m, \theta_k^m \) are parameters defined in Table 1 (see Appendix A).

For the continuation of the exposition it is convenient to denote by:

\(^\S\)The definition of the spherical harmonics functions and some of their properties are recalled in Appendix A for the reader convenience.
1. \( \mathbf{g} \) the column vector made up of the \( N_e \) so-called even moments \( f_{2l}^m \) (\( 0 \leq |m| \leq 2l \leq N \)) arranged as follows:

\[
\mathbf{g} = (f_{21}^m) = (f_0^1, f_2^{-2}, f_2^{-1}, f_2^0, f_2^1, f_2^2, \cdots)^t;
\]

2. \( \mathbf{h} \) the column vector made up of the \( N_o \) so-called odd moments \( f_{2l+1}^m \) (\( 0 \leq |m| \leq 2l + 1 \leq N \)) arranged as follows:

\[
\mathbf{h} = (f_{2l+1}^m) = (f_1^{-1}, f_1^0, f_3^{-3}, f_3^{-2}, f_3^1, f_3^0, f_3^2, f_3^3, \cdots)^t;
\]

3. \( \mathbf{X} \) the column vector made up of the \( N_e + N_o \) real-valued spherical harmonics \( X_k^m \) arranged as follows:

\[
\begin{align*}
k \text{ even} & : X = (X_0^0, X_2^{-2}, X_2^{-1}, X_2^0, X_2^1, X_2^2, \cdots, X_1^{-1}, X_1^0, X_1^1, X_1^2, X_3^{-3}, X_3^{-2}, X_3^{-1}, X_3^0, X_3^1, X_3^2, X_3^3, \cdots)^t, \\
k \text{ odd} & : X = (X_0^0, X_2^{-2}, X_2^{-1}, X_2^0, X_2^1, X_2^2, \cdots, X_1^{-1}, X_1^0, X_1^1, X_1^2, X_3^{-3}, X_3^{-2}, X_3^{-1}, X_3^0, X_3^1, X_3^2, X_3^3, \cdots)^t.
\end{align*}
\]

Thanks to the identities: \( A_k^m = D^{m+1}_{k+1}, B_k^m = C^{m+1}_{k-1}, A_k^m = B_{k+1}^{-m-1}, D_k^m = C_{k-1}^{-m+1}, E_k^m = F_{k+1}^m \), the relations (2.8) can then be rewritten in the more compact matrix form:

\[
\begin{align*}
&\left( \frac{1}{4\pi} \int_{S^2} \cos \theta \sqrt{1 - \mu^2} fX_k^m \right) (Ah,A^t)^t, \\
&\left( \frac{1}{4\pi} \int_{S^2} \sin \theta \sqrt{1 - \mu^2} fX_k^m \right) (Bh,B^t)^t, \\
&\left( \frac{1}{4\pi} \int_{S^2} \mu fX_k^m \right) (Ch,C^t)^t,
\end{align*}
\]

(2.9)

where \( A, B, C \) are \( N_e \times N_o \) constant sparse matrices that are given in Appendix B for the particular cases \( N = 1, 2, 3 \).

In the following (including the numerical experiments) we will limit ourselves to the case of reflecting boundary conditions: \( f_b = f(x, \omega - 2\omega.nn, v, t) \).

2.1.1 Cartesian coordinates

For all \( k, m \) such that \( 0 \leq |m| \leq k \leq N \), multiply the second equation of (2.5) by \( X_k^m \) and integrate over the unit sphere \( S^2 \). We get the \( P_N \) approximation of the radiative equation in Cartesian coordinates:

\[
\begin{align*}
\frac{1}{c} \frac{\partial}{\partial t} \left( \int_{S^2} fX_k^m \right) + \frac{\partial}{\partial x} \left( \int_{S^2} \cos \theta \sqrt{1 - \mu^2} fX_k^m \right) + \frac{\partial}{\partial y} \left( \int_{S^2} \sin \theta \sqrt{1 - \mu^2} fX_k^m \right) \\
+ \frac{\partial}{\partial z} \left( \int_{S^2} \mu fX_k^m \right) + (\sigma_a + \sigma_s) \int_{S^2} fX_k^m \\
= \sigma_a \int_{S^2} X_k^m + \frac{1}{4\pi} \sigma_s \int_{S^2} \left( \int_{S^2} f \right) X_k^m + \frac{1}{c} \int_{S^2} qX_k^m,
\end{align*}
\]

\( \sigma_a \) and \( \sigma_s \) are the so-called so-called \( \sigma_a \) and \( \sigma_s \) are the so-called

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\( ^3 \)In the framework of \( P_N \) approximations the inflow boundary condition is more difficult to deal with as soon as a Godunov’s type method is not used: see [14], paragraph 16-4, for example.
that is, after dividing by $4\pi$ and using (2.9):

\[
\begin{align*}
\frac{1}{c} \frac{\partial g}{\partial t} + \frac{\partial}{\partial x} (Ah) + \frac{\partial}{\partial y} (Bh) + \frac{\partial}{\partial z} (Ch) + Mg &= a, \\
\frac{1}{c} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (A'g) + \frac{\partial}{\partial y} (B'g) + \frac{\partial}{\partial z} (C'g) + Nh &= b,
\end{align*}
\]

(2.10)

where $M$ and $N$ are the diagonal $N_c \times N_c$ and $N_o \times N_o$ matrices defined by:

\[
M = \begin{pmatrix}
\sigma_a & 0 \\
0 & (\sigma_a + \sigma_s)I_{N_o-1}
\end{pmatrix}, \quad N = (\sigma_a + \sigma_s)I_{N_o}
\]

and $a$ and $b$ are the column vectors of order $N_c$ and $N_o$ defined by\footnote{The $q^m_k$ being the moments of the source $q$ defined like the moments $f^m_k$ of $f$ in (2.7).}:

\[
a = \frac{1}{c} \left( 1 + a \sigma_a T^4 + q_0^0 \rho_0^{-2} \rho_2^{-1} \rho_2^0 \rho_4^2 \rho_2^2 \cdots \right)^t, \quad b = \frac{1}{c} (q_1^{-1}, q_1^1, q_3^{-3}, q_3^{-3} q_3^0 q_3^0 q_3^2 q_3^2, \cdots)^t.
\]

2.1.2 Cylindrical coordinates

As in the Cartesian case we multiply the second equation of (2.6) by $X^m_k$ and we integrate over the unit sphere $S^2$, thus obtaining the $P_N$ approximation of the radiative equation in cylindrical coordinates:

\[
\begin{align*}
\frac{r}{c} \frac{\partial}{\partial t} \left( \int_{S^2} f X^m_k \right) + \frac{\partial}{\partial r} \left( r \int_{S^2} \cos \psi \sqrt{1 - \mu^2} f X^m_k \right) + \frac{\partial}{\partial \psi} \left( \int_{S^2} \sin \psi \sqrt{1 - \mu^2} f X^m_k \right) \\
+ r \frac{\partial}{\partial z} \left( \int_{S^2} \mu f X^m_k \right) - \int_{S^2} \frac{\partial}{\partial \psi} \left( \sin \psi \sqrt{1 - \mu^2} f X^m_k \right) + r \frac{\partial}{\partial z} \left( \int_{S^2} \mu f X^m_k \right)
\end{align*}
\]

(2.11)

This equation is similar to the Cartesian one except the fifth term of the left hand side which can be rewritten as:

\[\int_{S^2} \frac{\partial}{\partial \psi} (\sin \psi \sqrt{1 - \mu^2} f X^m_k) = \int_{S^2} \sin \psi \sqrt{1 - \mu^2} \frac{\partial X^m_k}{\partial \psi} = -m \int_{S^2} \sin \psi \sqrt{1 - \mu^2} f X^{-m}_k \]

because of the identity:

\[\frac{\partial X^m_k}{\partial \psi} = -m X^{-m}_k.\]

Thanks to the second line relation of (2.8) we have:

\[
\frac{1}{4\pi} \int_{S^2} \sin \psi \sqrt{1 - \mu^2} f X^{-m}_k = \eta^{-m} (A_k m f^m_{k+1} - B_k m f^m_{k-1}) + \theta^{-m} (C_k m f^m_{k+1} - D_k m f^m_{k-1}).
\]
As, for all $m \neq 0$, we have: $A^{-m}_k = C^m_k$, $B^{-m}_k = D^m_k$, $\eta^{-m} = -\zeta^m$, $\theta^{-m} = -\epsilon^m$, we obtain:

$$
\frac{1}{4\pi} \int_{S^2} \sin \psi \sqrt{1-\mu^2} f X^{-m}_k = -\epsilon^m (A^{m+1}_k f_{k+1} - B^m_{k-1}) - \zeta^m (C^m_{k+1} f_{k-1} - D^m_{k+1}).
$$

From this relation we deduce:

$$
-\frac{1}{4\pi} \int_{S^2} \frac{\partial}{\partial \psi} (\sin \psi \sqrt{1-\mu^2} f) X^m_k = -\frac{1}{4\pi} m \int_{S^2} \sin \psi \sqrt{1-\mu^2} f X^{-m}_k,
$$

that is, under a more compact matrix form:

$$
-\frac{1}{4\pi} \int_{S^2} \frac{\partial}{\partial \psi} (\sin \psi \sqrt{1-\mu^2} f) X = (Uh, -V_t^g)^t,
$$

(2.12)

where $U$ and $V$ are $N_e \times N_o$ constant matrices such that:

$$
U = V - A.
$$

(2.13)

These matrices are given in Appendix B for the particular cases $N=1,2,3$. After dividing Eq. (2.11) by $4\pi$ and using (2.9), (2.12) and (2.13) we obtain**:

$$
\begin{cases}
\frac{1}{c} \frac{\partial g}{\partial t} + \frac{\partial}{\partial r} (rAh) + \frac{\partial}{\partial \phi} (Bh) + \frac{\partial}{\partial z} (rCh) + Uh + rMg = ra,
\\
\frac{1}{c} \frac{\partial h}{\partial t} + r \frac{\partial}{\partial r} (A_t^g) + \frac{\partial}{\partial \phi} (B_t^g) + \frac{\partial}{\partial z} (C_t^g) - U_t^g + rNh = rb.
\end{cases}
$$

(2.14)

2.1.3 Some properties

In this section we state some results which have been proved, for example, in [1] in the Cartesian coordinates framework.

To avoid complicated calculations we will suppose that the boundary $\partial \Omega$ is made up of planes orthogonal to the axis. The reflecting boundary condition combined with the definition (2.7) of the moments of $f$ then lead to the following conditions:

1. if $n = \pm (1,0,0)$, $f_k^m = 0$ for $m$ odd $> 0$ or $m$ even $< 0$,

2. if $n = \pm (0,1,0)$, $f_k^m = 0$ for $m < 0$,

3. if $n = \pm (0,0,1)$, $f_k^m = 0$ for $k + m$ odd,

**Note that the Cartesian case can be retrieve formally from the cylindrical case by replacing respectively $r$, $U$, $\partial r$, $\partial \phi$ by $1$, $0$, $\partial x$, $\partial y$.**
so that, for all $x \in \partial \Omega$:

$$\left( n_x A + n_y B + n_z C \right) h = 0. \quad (2.15)$$

Multiply the first (resp. second) equation of (2.14) by $g$ (resp. $h$), integrate over the whole domain $\Omega$, add these two equations and denote:

$$S = S(t) = \frac{1}{2} \sum_{k=0}^{K} \sum_{m=-M}^{M} \int_{\Omega} r | f_k^m |^2 = \frac{1}{2} \int_{\Omega} r (\|g\|^2 + \|h\|^2).$$

Thanks to the boundary condition (2.15) we get the conservation equation in the cylindrical coordinates:

$$\frac{1}{c} \frac{\partial S}{\partial t} + \int_{\Omega} r (g^t M g + h^t N h) = \int_{\Omega} r (a^t g + b^t h). \quad (2.16)$$

If $\sigma_a = \sigma_s = 0$ and $q = 0$ (that is $M = 0$, $N = 0$, $a = 0$, $b = 0$) we observe that $S$ is preserved over the time.

Finally one recalls the following theorem whose proof is similar to that given in [1] in the framework of Cartesian coordinates.

**Theorem 2.1.** Let $l_r, t_r$ be reference length and time such that:

$$\frac{t_r}{l_r} = \frac{1}{\sigma_a l_r}, \quad \frac{1}{\sigma_s l_r} = \frac{1}{\epsilon}.$$

If $q = 0$ and $\epsilon$ tends toward 0, the $P_N$ radiative equation (2.14) provides the following asymptotic isotropic diffusion equation:

$$\frac{1}{c} \frac{\partial f_0^0}{\partial t} - \nabla \cdot \left( \frac{1}{3 \sigma_s} \nabla f_0^0 \right) = 0. \quad (2.17)$$

### 2.2 The grey approximation of the thermal-$P_N$ radiative transfer equations

Under their $P_N$ approximation form the coupled grey thermal-radiative transfer equations (2.5) and (2.6) with reflective boundary conditions then read (Cartesian coordinates):

$$\begin{cases} 
\rho C_v \frac{\partial T}{\partial t} + acc T^4 = c \sigma_a E, \\
\frac{1}{c} \frac{\partial g}{\partial t} + \frac{\partial}{\partial x} (A h) + \frac{\partial}{\partial y} (B h) + \frac{\partial}{\partial z} (C h) + M g = a, \\
\frac{1}{c} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (A^t g) + \frac{\partial}{\partial y} (B^t g) + \frac{\partial}{\partial z} (C^t g) + N h = b, \\
(n_x A + n_y B + n_z C) h = 0 
\end{cases} \quad (2.18)$$
and (cylindrical coordinates):

\[
\begin{align*}
\rho C_v \frac{\partial T}{\partial t} + ac\sigma_a T^4 &= ac\sigma_a E, \\
\frac{1}{c} \frac{\partial g}{\partial t} + \frac{\partial}{\partial r} (r A h) + \frac{\partial}{\partial \phi} (B h) + \frac{\partial}{\partial z} (r Ch) + U h + rM g &= ra, \\
\frac{1}{c} \frac{\partial h}{\partial t} + r \frac{\partial}{\partial r} (A^l g) + \frac{\partial}{\partial \phi} (B^l g) + r \frac{\partial}{\partial z} (C^l g) - U^l g + rN h &= rb, \\
(n_r A + n_\phi B + n_z C) h &= 0.
\end{align*}
\]

(2.19)

Thanks to the recursion relations (A.4), recalled in Appendix A, the expression of \(E\), \(F\) and \(P\) in function of the moments \(f_m^k\) are given by:

\[E = 4\pi f_0^0, \quad F = \frac{4\pi}{\sqrt{3}} c (f_1^1, f_1^{-1}, f_1^0)^t, \quad P = \frac{4\pi}{3} f_0^0 I_3 + \Pi,\]

where \(\Pi\) is the deviatoric tensor:

\[
\Pi = \frac{4\pi}{\sqrt{15}} \begin{pmatrix}
-\frac{1}{\sqrt{3}} f_2^0 + f_2^2 & f_2^{-2} & f_2^1 \\
& -\frac{1}{\sqrt{3}} f_2^0 - f_2^2 & f_2^{-1} \\
& & f_2^1 & f_2^{-1} & \frac{2}{\sqrt{3}} f_2^0
\end{pmatrix}.
\]

Arrived at this point it is worth noting that the grey thermal-\(P_N\) radiative transfer equations regarding the first component \(f_0^0\) of \(g\) and the three first components \(f_1^{-1}, f_1^0, f_1^1\) of \(h\), can be rewritten under the form \(††\):

\[
\begin{align*}
\rho C_v \frac{\partial T}{\partial t} + ac\sigma_a T^4 &= ac\sigma_a E, \\
\frac{1}{c} \frac{\partial E}{\partial t} + \frac{1}{c} \nabla \cdot F + \sigma_a E &= 4\pi \sigma_a b + \frac{1}{c} q, \\
\frac{1}{c} \frac{\partial F}{\partial t} + \frac{c}{3} \nabla E + c \nabla \cdot \Pi + (\sigma_a + \sigma_s) F &= \frac{1}{c} q.
\end{align*}
\]

(2.20)

The unknowns \(E, F, \Pi\) are then substituted for \(f\) and a closure relation for obtaining \(\Pi\) from \(E\) and \(F\) is then mandatory while the reflecting boundary condition boils down to \(††\):

\[
q = 4\pi q_0^0, \quad \tilde{q} = \frac{4\pi}{\sqrt{3}} (q_1^1, q_1^{-1}, q_1^0)^t.
\]

\(††\)With:
the relation: $F \cdot n = 0$. The simplest closure relation$^{\dagger \dagger}$:

$$\Pi = 0$$  \hspace{1cm} (2.21)

allows to retrieve the grey thermal-$P_1$ radiative transfer equations.

Suppose now that $q = 0$ and $q = 0$. If we suppose that $\sigma_s$ tends to 0 and $c, \sigma_a$ tend to $+\infty$ the system (2.20)-(2.21) tends to the non-equilibrium diffusion limit model:

$$\begin{cases} 
\rho C_v \frac{\partial T}{\partial t} + ac\sigma_a T^4 = c\sigma_a E, \\
\frac{1}{c} \frac{\partial E}{\partial t} - \nabla \cdot \left( \frac{1}{3(\sigma_a + \sigma_s)} \nabla E \right) + \sigma_a E = a\sigma_a T^4
\end{cases}$$  \hspace{1cm} (2.22)

while the reflecting boundary condition then boils down to the homogeneous Neumann’s one: $\nabla E \cdot n = 0$. To avoid the infinite propagation speed of this model it may be judicious to replace the diffusion coefficient:

$$\lambda = \frac{1}{3(\sigma_a + \sigma_s)}$$

by the flux-limited one:

$$\lambda = \lambda(E) = \frac{1}{\left( 3^m (\sigma_a + \sigma_s)^m + \frac{\|\nabla E\|^m}{E^m} \right)^{\frac{1}{m}}}$$  \hspace{1cm} (2.23)

where $m$ is a strictly positive integer (the limiter) to be chosen (see [8, 16]).

If we suppose that $\sigma_s$ tends to 0 and $c, \sigma_a$ tend to $+\infty$ then $E = aT^4$ and the system (2.20)-(2.21) tends to the equilibrium diffusion limit model:

$$\begin{cases} 
\frac{1}{c} \frac{\partial}{\partial t} \left( \rho C_v T + aT^4 \right) - \nabla \cdot \left( \frac{a}{3(\sigma_a + \sigma_s)} \nabla T^4 \right) = 0.
\end{cases}$$  \hspace{1cm} (2.24)

Thus we obtain a hierarchy of physical models for the approximation of the coupled grey thermal-radiative transfer equations (2.1):

1. the coupled grey thermal-$P_N$ radiative transfer equations (2.18) or (2.19);
2. the coupled grey thermal-$P_1$ radiative transfer equations (2.20)-(2.21);
3. the grey non-equilibrium diffusion limit (2.22);

$^{\dagger \dagger}$Note that in this case (Eddington approximation):

$$f = f_0^0 x_0^0 + f_1^{-1} x_1^{-1} + f_1^0 x_1^0 + f_1^1 x_1^1 = \frac{1}{4\pi} (E + \frac{3}{c} \omega \cdot F).$$

Many possible other (non-linear) closure relations can be used: see for example [15].
4. the grey equilibrium diffusion limit (2.24).

In paragraph 5.3 we will present and discuss numerical comparisons between the non-equilibrium diffusion limit model with various limiters and the coupled thermal-radiative transfer model about a benchmark proposed in the literature.

3 Discretization of the grey $P_N$ radiative transfer equation

We tackle the discretization of the two-dimensional version of Eq. (2.10) for which $g$, $h$ are supposed to be independent of $z$, that is, with reflective boundary conditions:

\begin{equation}
\begin{aligned}
\frac{1}{c} \frac{\partial g}{\partial t} + \frac{\partial}{\partial x} (Ah) + \frac{\partial}{\partial y} (Bh) + Mg = a, \quad \text{in } \Omega \times \Delta, \\
\frac{1}{c} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (A'g) + \frac{\partial}{\partial y} (B'g) + Nh = b, \quad \text{in } \Omega \times \Delta, \\
(n_x A + n_y B) h = 0, \quad \text{on } \partial \Omega \times \Delta.
\end{aligned}
\end{equation}

The distribution function $f$ is then necessarily an even function in the variable $\mu$. Consequently $f_{k}^m = 0$ if $k+m$ is odd, thanks to the definition of the Legendre functions.

In the cylindrical coordinates we have to discretize the two-dimensional version of Eq. (2.14) for which $g$, $h$ are supposed to be independent of $\phi$, that is, with reflective boundary conditions:

\begin{equation}
\begin{aligned}
\frac{1}{r} \frac{\partial g}{\partial t} + \frac{\partial}{\partial r} (rAh) + \frac{\partial}{\partial z} (rCh) + Uh + rMg = ra, \quad \text{in } \Omega \times \Delta, \\
\frac{1}{r} \frac{\partial h}{\partial t} + r \frac{\partial}{\partial r} (A'g) + r \frac{\partial}{\partial z} (C'g) - U'l + rNh = rb, \quad \text{in } \Omega \times \Delta, \\
(n_r A + n_z B) h = 0, \quad \text{on } \partial \Omega \times \Delta.
\end{aligned}
\end{equation}

The distribution function $f$ is then necessarily an even function in the variable $\psi$. Consequently $f_{k}^m = 0$ if $m < 0$, thanks to the definition of the Legendre functions.

In these both cases the number of equations reduces from $(N+1)^2$ to $\frac{1}{2}(N+1)(N+2)$.

3.1 Approximation of partial derivatives in the DDFV framework

Although previously defined in [1] the approximation of partial derivatives in the two-dimensional DDFV framework is recalled in this section for the sake of self-completeness.

Given $G$ a geometrical variable (segment or polygon), we will denote by $|G|$ its measure (length or area) and by $\partial G$ its boundary. For visualizing the geometrical definitions we are going to introduce now, the reader is referred to Figs. 2, 3 and 4.
1. We suppose that the polygonal domain $\Omega$ is covered with a given *primal* mesh made up of (non-necessarily convex) arbitrary sided polygons denoted by $P_p$ ($1 \leq p \leq N_P$). With every *primal* polygon $P_p$ we associate one *primal* point $x_p$: the centroid is a qualified candidate but other points could be chosen provided that $P_p$ is *star-shaped* with respect to $x_p$.

---

*A polygon $P$ is star-shaped with respect to a point $x \in P$ if, for all point $y \in P$, $xy \subset P$. For a non-convex*
2. By connecting the primal points and the midpoints $x_i$ of the sides $C_i$ of the primal polygons we obtain a dual mesh which covers $\Omega$ with (non-necessarily convex) dual polygons denoted by $D_d$ ($1 \leq d \leq N_D$). This mesh is often called median, barycentric or Donald’s dual mesh in the literature. With every dual polygon $D_d$ we can associate one vertex of the primal mesh, denoted by $x_d$, and such that $x_d \in D_d$: note that $D_d$ is star-shaped with respect to $x_d$. We denote $\Gamma_i = \partial D_d \cap \partial\Omega$. Note that:

$$\partial P_p = \bigcup_{i \in p} C_i, \quad \partial D_d = \left( \bigcup_{i \in d} \Gamma_i \right) \cup (D_d \cap \partial\Omega).$$

3. Besides the primal and dual meshes, we will need to define a third so-called intermediary mesh that is made up of (non-necessarily convex) quadrilaterals. Note that this mesh, often called diamond mesh in the literature owing to the shape of its elements, is sometimes called dual (which can cause confusion). For every interior (resp. boundary) primal side $C_i$ one defines the intermediary quadrilateral $I_i = x_p x_d x_e x_i$ (resp. degenerated quadrilateral $I_i = x_p x_d x_e$). By definition, the set of quadrilaterals $I_i$ ($1 \leq i \leq N_I$) is the intermediary mesh. We will finally denote by $I_{ip} = I_i \cap P_p = x_p x_d x_e x_i$ and $I_{iq} = I_i \cap P_q = x_q x_e x_i x_d$ what we will call the two intermediary (degenerated) sub-quadrilaterals of $I_i$.

In what follows, to ease the description, the midpoints $x_i$ of the primal sides will be called intermediary points while the vertices $x_d$ of the primal cells will be called dual points.
We will need to denote:

\[ n_i = (n_{xi}, n_{yi}) = \int_{C_i} n = x_i x_d^\perp, \quad v_i = (v_{xi}, v_{yi}) = \int_{\Gamma_i} n = x_p x_q^\perp \]

and, for \( x_d \in \partial \Omega \) (see Fig. 4):

\[ n_d = (n_{xd}, n_{yd}) = \int_{D_d \cap \partial \Omega} n = \frac{1}{2} (n_i + n_j). \]

Given some function \( p = p(x) \), the approximation of the partial derivatives of \( p \) will be done either in the intermediary cells by using the values of \( p \) at the primal and dual points or in the primal and dual cells by using the values of \( p \) at the intermediary points. We will denote systematically \( p_p = p(x_p), \ p_d = p(x_d), \ p_i = p(x_i) \), etc.

### 3.1.1 Approximation of partial derivatives in the intermediary cells

Given \( x_a, x_b, x_c \) three non-colinear points and \( T = x_a x_b x_c \) the associated triangle we will use the following approximation of \( \nabla p \) in \( T^\perp \):

\[
(\nabla p)_T \approx \frac{1}{|T|} \int_T \nabla p = \frac{1}{|T|} \int_{\partial T} p n.
\]

If \( p \) is supposed to be linear in \( T \) we obtain (favouring the point \( x_a \) for example):

\[
(\nabla p)_T \approx \frac{1}{2 |T|} ((p_a - p_b) n_{ca} + (p_a - p_c) n_{ab}),
\]

where \( n_{ab}, n_{ca} \) are the vectors defined by \( n_{ab} = (y_b - y_a, x_b - x_a) \), \( n_{ca} = (y_a - y_c, x_a - x_c) \) (see Fig. 5).

Given \( P \) a polygon let \( \mathcal{T} \) be a triangulation of \( P \). Thanks to (3.3) one can define an approximation of \( \nabla p \) in \( P \) (see [17], [19] for extensions to the 3D case):

\[
(\nabla p)_P \approx \frac{1}{|P|} \sum_{T \in \mathcal{T}} |T| (\nabla p)_T.
\]

Note that this approximation is independent of the choice of the triangulation \( \mathcal{T} \) of \( P \). For the particular case \( P = I_i = x_p x_d x_q x_e \) (see Fig. 3), we obtain the following approximation of \( \nabla p \) in the intermediary cell \( I_i \):

\[
\begin{align*}
(\nabla p)_{I_i} & \approx \frac{1}{2 |I_i|} ((p_q - p_p)n_i + (p_d - p_e)v_i), \quad x_i \notin \partial \Omega, \\
(\nabla p)_{I_i} & \approx \frac{1}{2 |I_i|} ((p_i - p_p)n_i + (p_e - p_d)v_i), \quad x_i \in \partial \Omega.
\end{align*}
\]

\(^\dagger\)Thanks to the Gauss-Green (divergence) theorem (\( n \) being the unit normal outside vector).
3.1.2 Approximation of partial derivatives in the primal and dual cells

Given $P_p$ (resp. $D_d$) a primal (resp. dual) cell, we use the following approximation of $\nabla p$ in $P_p$ and $D_d$:

$$(\nabla p)_p \simeq \frac{1}{|P_p|} \int_{P_p} \nabla p = \frac{1}{|P_p|} \int_{\partial P_p} p n_i,$$

$$(\nabla p)_d \simeq \frac{1}{|D_d|} \int_{D_d} \nabla p = \frac{1}{|D_d|} \int_{\partial D_d} p \nu.$$

If we suppose that $p$ is constant in the intermediary cell $I_i$ and that, for $x_d \in \partial \Omega$:

$$\int_{\partial D_d \cap \partial \Omega} \nabla p \simeq p_d n_d,$$

we obtain\(^\dagger\):

$$\begin{cases}
(\nabla p)_p \simeq \frac{1}{|P_p|} \sum_{i \in p} p_i n_i, \\
(\nabla p)_d \simeq \frac{1}{|D_d|} \sum_{i \in d} p_i \nu_i, & x_d \notin \partial \Omega, \\
(\nabla p)_d \simeq \frac{1}{|D_d|} \left( \sum_{i \in d} p_i \nu_i + p_d n_d \right), & x_d \in \partial \Omega.
\end{cases}$$

3.1.3 Some properties

When they are combined the approximations (3.4) and (3.5) benefit from several attractive features. Given $p$, $\nu$ some sufficiently smooth scalar and vector functions, suppose that

\(^\dagger\)The compact notation $i \in p$ ($i \in d$) will now mean $C_i \in \partial P_p$ ($\Gamma_i \in \partial D_d$).
the partial derivatives of $p$ are discretized in the intermediary cells by (3.4) and that the partial derivatives of $v$ are discretized in the primal and dual cells by (3.5). Then we obtain notably discrete counterparts of the identities: $\nabla \cdot (\nabla \times p) = \nabla \times (\nabla p) = 0$ and of the Green’s formula:

$$\int_{\Omega} p \nabla \cdot v + \int_{\partial \Omega} \nabla p \cdot v = \int_{\partial \Omega} pv \cdot n.$$ 

For more details see [1, 23, 24]. The approximation (3.4) is at the root of both diamond finite volume [18] and DDFV [20–22] methods for the discretization of general elliptic problems while the approximations (3.4) or/and (3.5) are at the root of DDFV methods for the discretization of various other partial derivatives equations as wave, Stokes, convection-diffusion or non-linear hyperbolic ones: see the more detailed, though non exhaustive, bibliography in [23] then [24].

3.2 Space discretization of the radiative equation

In this section we will use the gradient discretizations (3.4) and (3.5) for approximating the $P_N$ radiative equations (3.2). Doing the integration of the first equation on the primal cells $P_p$ and dual cells $D_d$ and integration of the second equation on the intermediary cells $I_i$ provides:

$$\frac{1}{c} \frac{\partial}{\partial t} \left( \int_{P_p} rg \right) + \int_{P_p} \left( \frac{\partial}{\partial r} (rAh) + \frac{\partial}{\partial z} (rCh) \right) + \int_{P_p} Uh + \int_{P_p} rMg = \int_{P_p} ra,$$

$$\frac{1}{c} \frac{\partial}{\partial t} \left( \int_{D_d} rg \right) + \int_{D_d} \left( \frac{\partial}{\partial r} (rAh) + \frac{\partial}{\partial z} (rCh) \right) + \int_{D_d} Uh + \int_{D_d} rMg = \int_{D_d} ra,$$

$$\frac{1}{c} \frac{\partial}{\partial t} \left( \int_{I_i} rh \right) + \int_{I_i} r \left( \frac{\partial}{\partial r} (A'g) + \frac{\partial}{\partial z} (C'g) \right) - \int_{I_i} U^t g + \int_{I_i} rNh = \int_{I_i} rb.$$

Do the approximations:

$$\int_{P_p} rg \simeq r_p |P_p| g_p, \quad \int_{P_p} rMg \simeq r_p |P_p| M_p g_p, \quad \int_{P_p} ra \simeq r_p |P_p| a_p,$$

$$\int_{D_d} rg \simeq r_d |D_d| g_d, \quad \int_{D_d} rMg \simeq r_d |D_d| M_d g_d, \quad \int_{D_d} ra \simeq r_d |D_d| a_d,$$

$$\int_{I_i} rh \simeq r_i |I_i| h_i, \quad \int_{I_i} rNh \simeq r_i |I_i| N_i h_i, \quad \int_{I_i} rb \simeq r_i |I_i| b_i,$$

$$\int_{P_p} Uh \simeq \sum_{i \in p} |P_p \cap I_i| Uh_i, \quad \int_{D_d} Uh \simeq \sum_{i \in d} |D_d \cap I_i| Uh_i,$$

$$\int_{I_i} U^t g \simeq \frac{1}{2} U^t (|P_p \cap I_i| g_p + |P_q \cap I_i| g_q + |D_d \cap I_i| g_d + |D_e \cap I_i| g_e) \quad (x_i \not\in \partial \Omega),$$

$$\int_{I_i} U^t g \simeq \frac{1}{2} U^t (|P_p \cap I_i| g_p + |D_d \cap I_i| g_d + |D_e \cap I_i| g_e) \quad (x_i \in \partial \Omega),$$
where $\bar{r}_d, \bar{r}_i$ are defined by\footnote{With these definitions note that $\bar{r}_d \neq 0$ and $\bar{r}_i \neq 0$ even if $r_d = 0$ and $r_i = 0.$}:

\[
\begin{align*}
\bar{r}_d &= \frac{1}{|D_d|} \sum_p |P_p \cap D_d|r_p, \\
\bar{r}_i &= \frac{1}{2} \left( r_p |P_p \cap I_i| + r_q |P_q \cap I_i| + \bar{r}_d |D_d \cap I_i| + \bar{r}_e |D_e \cap I_i| \right), \quad x_i \notin \partial \Omega, \\
\bar{r}_i &= \frac{1}{2} \left( r_p |P_p \cap I_i| + \bar{r}_d |D_d \cap I_i| + \bar{r}_e |D_e \cap I_i| \right), \quad x_i \in \partial \Omega,
\end{align*}
\]

and $a_d$ is defined by:

\[
a_d = \frac{\sum_p r_p |P_p \cap D_d| a_p}{\sum_p r_p |P_p \cap D_d|},
\]

so that:

\[
\sum_p r_p |P_p| = |D_d| = \sum_i |I_i|, \quad \sum_p r_p |P_p| a_p = \sum_d |D_d| a_d.
\]

For defining $M_p$ we use the natural definition of the primal opacities $\sigma_{ap} = \sigma_a(\rho_p, T_p)$ and $\sigma_{sp} = \sigma_s(\rho_p, T_p)$ while for defining $M_d$ and $N_i$ we use the following definitions of the dual and intermediary opacities $\sigma_{ad}, \sigma_{sd}$ and $\sigma_{ai}, \sigma_{si}$ at the points $x_d$ and $x_i$ from the primal opacities:

\[
\begin{align*}
\sigma_{ad} &= \frac{\sum_p |P_p \cap D_d| \sigma_{ap}}{\sum_p |P_p \cap D_d|}, \quad \sigma_{sd} = \frac{\sum_p |P_p \cap D_d| \sigma_{sp}}{\sum_p |P_p \cap D_d|}, \\
\sigma_{ai} &= \frac{|I_{ip}| \sigma_{ap} + |I_{iq}| \sigma_{aq}}{|I_{ip}| + |I_{iq}|}, \quad \sigma_{si} = \frac{|I_{ip}| \sigma_{sp} + |I_{iq}| \sigma_{sq}}{|I_{ip}| + |I_{iq}|}.
\end{align*}
\]

Finally, by using the approximations (3.4) for the derivatives of the components of $g$ in the intermediary cells $I_i$, the approximations (3.5) for the derivatives of the components of $rh$ in the primal and dual cells $P_p$ and $D_d$ and by denoting $X_i = r_t A + n_{zd} C, X_d = r_d A + n_{zd} C$ and $Y_i = v_i A + v_{zd} C$, we arrive to the space discretization of (3.2), that is, after dividing
The degrees of freedom of the method will be the vector of primal even moments \( g_p \), the vector of dual even moments \( g_d \) and the vector of intermediary odd moments \( h_i \). Besides these degrees of freedom the discretized system (3.7) involve the supplementary boundary degrees of freedom \( g_d \) (values of \( g \) at the boundary intermediary points \( x_i \)) and \( h_d \) (values of \( h \) at the boundary dual points \( x_d \)) which can be eliminated thanks to the boundary conditions. For example the reflecting boundary condition entails that either \( g_i = 0 \) or \( X_i'g_i = 0 \) (if \( x_i \in \partial \Omega \)) and either \( h_d = 0 \) or \( X_d'h_d = 0 \) (if \( x_d \in \partial \Omega \)).

Therefore the method has \( N_p (N_p + N_D) + N_o N_i \) degrees of freedom\(^4\). The location of the degrees of freedom in a pentagon, for instance, are displayed on Fig. 6: instead of a pentagon any (even non-convex) polygon could be chosen as well, provided that the degrees of freedom of \( g \) (resp. \( h \) ) are located at the centers and vertices (resp. middles of the sides).

As all the DDFV type methods our method provides accurate approximations of the solution on general meshes: see paragraph 5.1. Moreover, compared to more standard methods, it provides also accurate approximations of the derivatives of the solution: see [9, 10]. The price to pay is the doubling of degrees of freedom of temperature and even moments of the distribution function \( f \). Note, however, that despite this doubling the number of degrees of freedom of this method is lower than that, for example, of the commonly used discontinuous Galerkin method of order one.

\(^4\)Recall that \( N_p \), \( N_D \), \( N_I \), \( N_c \) and \( N_o \) denote respectively the number of primal cells, the number of dual cells (equal to the number of vertices), the number of intermediary cells (equal to the number of sides), the number of even moments and the number of odd moments.
3.3 Time discretization of the radiative equation

The streaming terms (that is terms involving the matrices $A$, $C$, $U$) are discretized by the standard, explicit, second-order accurate, leap-frog scheme. This scheme, which is by far the most commonly used to solve wave-like equations since the early Yee’s work [3], is stable under a Courant-Friedrichs-Lewy (CFL) condition: see for example [25].

On the other hand the absorption terms (that is terms involving the matrices $M$, $N$ which involve itself the opacities $\sigma_a$ and $\sigma_s$ depending on the material density and temperature) are discretized by the implicit, unconditionally stable, $\theta$-scheme ($0.5 \leq \theta \leq 1$). In what follows we have chosen the first-order accurate backward Euler scheme ($\theta = 1$). By doing so, notice that we have not to invert non-diagonal linear systems.

Thus we obtain the full space and time discretization of the radiative equations (3.2) in cylindrical coordinates:

$$
\begin{align*}
\frac{1}{c} \frac{g^{n+1}_p - g^n_p}{\Delta t} &+ \frac{1}{r_p |P_p|} \sum_{I_p \cap I_l} (f_{I_p} X_{I_p} h^{n+\frac{1}{2}}_l + |P_p \cap I_l| U h^{n+\frac{1}{2}}_l ) + M^n_p g^{n+1}_p = a^{n+\frac{1}{2}}_p, \\
\frac{1}{c} \frac{g^{n+1}_d - g^n_d}{\Delta t} &+ \frac{1}{r_d |D_d|} \sum_{I_d \cap I_l} (f_{I_d} Y_{I_d} h^{n+\frac{1}{2}}_l + |D_d \cap I_l| U h^{n+\frac{1}{2}}_l ) + M^n_d g^{n+1}_d = a^{n+\frac{1}{2}}_d, \\
\frac{1}{c} \frac{h^{n+\frac{1}{2}}_i - h^{n-\frac{1}{2}}_i}{\Delta t} &+ \frac{1}{|I_i|} (X_{I_i} (g^n_i - g^0_i) + Y_{I_i} (g^n_i - g^0_i)) \\
&- \frac{1}{2} \frac{1}{r_i |I_i|} U^i (|P_p \cap I_i| g^n_p + |P_q \cap I_i| g^n_q + |D_d \cap I_i| g^n_d + |D_c \cap I_i| g^n_c) \\
&+ N^{h}_{i} h^{n+\frac{1}{2}}_i = b^n_i, \quad x_i \notin \partial \Omega, \\
\frac{1}{c} \frac{h^{n+\frac{1}{2}}_i - h^{n-\frac{1}{2}}_i}{\Delta t} &+ \frac{1}{2} \frac{1}{|I_i|} (X_{I_i} (g^n_i - g^0_i) + Y_{I_i} (g^n_i - g^0_i)) \\
&- \frac{1}{2} \frac{1}{r_i |I_i|} U^i (|P_p \cap I_i| g^n_p + |D_d \cap I_i| g^n_d + |D_c \cap I_i| g^n_c) + N^{h}_{i} h^{n+\frac{1}{2}}_i = b^n_i, \quad x_i \in \partial \Omega,
\end{align*}
$$

where $g$ and $h$ are the space and time discretization of the radiative equations (3.2).
with:

\[
M_p^n = \begin{pmatrix} a_{np}^n & 0 \\ 0 & (a_{np}^n + a_{sp}^n I)_{M - 1} \end{pmatrix},
M_d^n = \begin{pmatrix} a_{nd}^n & 0 \\ 0 & (a_{nd}^n + a_{sd}^n I)_{M - 1} \end{pmatrix},
\]

\[
N_i^n = (a_{ni}^n + a_{si}^n) I_{N_i - 1},
\]

\[
a_{p}^{n+\frac{1}{2}} = \frac{1}{c} (a_{np}^n b_p^{n+\frac{1}{2}} + (q_p^{n+\frac{1}{2}} - q_p^n) (q_p^n - q_p^{n-1}) (q_p^{n-1} - q_p^{n-2}) (q_p^{n-2} - q_p^{n-3}) \cdots ),
\]

\[
a_{d}^{n+\frac{1}{2}} = \frac{1}{c} (a_{nd}^n b_d^{n+\frac{1}{2}} + (q_d^{n+\frac{1}{2}} - q_d^n) (q_d^n - q_d^{n-1}) (q_d^{n-1} - q_d^{n-2}) (q_d^{n-2} - q_d^{n-3}) \cdots ),
\]

\[
b_i^n = \frac{1}{c} ((q_i^n)_1^{-1} (q_i^n)_2^{-1} (q_i^n)_3^{-1} (q_i^n)_4^{-1} (q_i^n)_5^{-1} (q_i^n)_6^{-1} (q_i^n)_7^{-1} (q_i^n)_8^{-1} \cdots ).
\]

The initialization consists in setting (for \( x_i \notin \partial \Omega \)):

\[
\frac{2 h_i^2 - h_0^2}{c} \frac{1}{\Delta t} + \frac{1}{2 |I_i|} (X_i^0 (g_d^0 - g_p^0) + Y_i^0 (g_e^0 - g_d^0)) + \frac{1}{2 |I_i|} U_i^0 (|P_p \cap I_i| g_p^0 + |P_d \cap I_i| g_d^0 + |D_e \cap I_i| g_e^0) + N_i^0 h_i^2 = b_i^0,
\]

and (for \( x_i \in \partial \Omega \)):

\[
\frac{2 h_i^2 - h_0^2}{c} \frac{1}{\Delta t} + \frac{1}{2 |I_i|} (X_i^0 (g_d^0 - g_p^0) + Y_i^0 (g_e^0 - g_d^0)) + \frac{1}{2 |I_i|} U_i^0 (|P_p \cap I_i| g_p^0 + |D_d \cap I_i| g_d^0 + |D_e \cap I_i| g_e^0) + N_i^0 h_i^2 = b_i^0,
\]

with \( g_d^0 \) defined by:

\[
g_d^0 = \sum_p \frac{r_p |P_p \cap D_d| g_p^0}{\sum_p r_p |P_p \cap D_d|},
\]

so that:

\[
\sum_p r_p |P_p| g_p^0 = \sum_d |D_d| g_d^0.
\]
3.4 Properties

If we do the quite natural approximations:

\[
S^{n+\frac{1}{2}} \simeq \frac{1}{4} \sum p |P_p| g_p^{n+1} \cdot g^n_p + \frac{1}{4} \sum d |D_d| g_d^{n+1} \cdot g^n_d + \frac{1}{2} \sum i |I_i| ||h^n_i + \frac{1}{2}||^2,
\]

\[
\int_{\Omega} r(\mathbf{g}' M \mathbf{g} + \mathbf{h}' N \mathbf{h})^n \simeq \frac{1}{4} \sum p |P_p| (g^n_p)' (M^n_p g^{n+1}_p + M^{n-1}_p g^n_p)
\]

\[
+ \frac{1}{4} \sum d |D_d| (g^n_d)' (M^n_d g^{n+1}_d + M^{n-1}_d g^n_d)
\]

\[
+ \frac{1}{2} \sum i |I_i| (h^n_i + \frac{1}{2} + h^n_i - \frac{1}{2}) N_i h^n_i + \frac{1}{2} \sum i |I_i| (h^n_i + \frac{1}{2} + h^n_i - \frac{1}{2}),
\]

we obtain Theorem 3.1.

**Theorem 3.1.** Given the space and time discretization (3.8) of the PN radiative equation, the conservation equation (2.16) is discretized by:

\[
\frac{1}{c} (S^{n+\frac{1}{2}} - S^{n-\frac{1}{2}}) \Delta t + \int_{\Omega} r(\mathbf{g}' M \mathbf{g} + \mathbf{h}' N \mathbf{h})^n = \int_{\Omega} r(\mathbf{a} \cdot \mathbf{g} + \mathbf{b} \cdot \mathbf{h})^n. \tag{3.10}
\]

**Proof.** Multiply the first (resp. second) equation of (3.8) written at times \((n+\frac{1}{2})\Delta t\) and \((n-\frac{1}{2})\Delta t\) by:

\[
\frac{1}{4} |P_p| g^n_p \quad \text{(resp. } \frac{1}{4} |D_d| g^n_d \text{)}
\]

and multiply the third and fourth equations of (3.8) at time \(n\Delta t\) by:

\[
\frac{1}{2} |I_i| h^n_i + \frac{1}{2} + h^n_i - \frac{1}{2}.
\]
Then adding these equations on all the primal, dual and intermediary cells provides:

\[
\begin{align*}
\frac{1}{4} \frac{1}{\Delta t} \sum_p r_p |P_p| (g^{n+1}_p - g^n_p) \cdot g^n_p + & \frac{1}{4} \sum_p \sum_i r_i (g^n_i)^4 X_i h_i^{n+\frac{1}{2}} \\
+ \frac{1}{4} \sum_p \sum_i |P_p \cap I_i| (g^n_i)^4 U_i h_i^{n+\frac{1}{2}} + & \frac{1}{4} \sum_p r_p |P_p| (g^n_p)^4 M_p g^{n+1}_p \\
+ \frac{1}{4} \frac{1}{\Delta t} \sum_d \sum_p |P_p \cap I_d| (g^n_d - g^{n-1}_d) \cdot g^n_d + & \frac{1}{4} \sum_d \sum_i r_i (g^n_i)^4 Y_i h_i^{n+\frac{1}{2}} \\
+ \frac{1}{4} \frac{1}{\Delta t} \sum_d \sum_i D_d \cap I_d (g^n_d - g^{n-1}_d) \cdot g^n_d + & \frac{1}{4} \sum_d \sum_i r_i (g^n_i)^4 Y_i h_i^{n+\frac{1}{2}} \\
+ \frac{1}{4} \frac{1}{\Delta t} \sum_i r_i (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}})^4 (X_i (g^n_i - g^n_p) + Y_i (g^n_i - g^n_d)) \\
+ \frac{1}{4} \sum_{i \in d \Omega} r_i (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}})^4 (X_i (g^n_i - g^n_p) + Y_i (g^n_i - g^n_d)) \\
- \frac{1}{4} \sum_{i \in d \Omega} (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}})^4 U_i (|P_p \cap I_d| g^n_p + |P_q \cap I_d| g^n_q + |D_d \cap I_d| g^n_d + |D_e \cap I_d| g^n_e) \\
- \frac{1}{4} \sum_{i \in d \Omega} (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}})^4 U_i (|P_p \cap I_d| g^n_p + |D_d \cap I_d| g^n_d + |D_e \cap I_d| g^n_e) \\
+ \frac{1}{2} \sum_i |I_i| (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}})^2 N_i h_i^{n+\frac{1}{2}} & = \frac{1}{4} \sum_p r_p |P_p| (a^{n+\frac{1}{2}}_p + a^{n-\frac{1}{2}}_p) \cdot g^n_p + \frac{1}{4} \sum_d r_d |D_d| (a^{n+\frac{1}{2}}_d + a^{n-\frac{1}{2}}_d) \cdot g^n_d + \frac{1}{2} \sum_i b^n_i (h_i^{n+\frac{1}{2}} + h_i^{n-\frac{1}{2}}).
\end{align*}
\]

Thanks to the reflective boundary conditions we have, for all \( x, y, z \in \partial \Omega, X_i h_i^{n+\frac{1}{2}} = X_i h_i^{n-\frac{1}{2}} = \)
and sufficient conditions for this value to be positive are, for all $g, h$:

$$
\begin{align*}
S_0^+ = \cdots = S_{n+\frac{1}{2}} &= \frac{1}{4} \sum_p r_p |P_p| g^{n+1}_p \cdot g^n_p + \frac{1}{4} \sum_d \bar{r}_d |D_d| g^{n+1}_d \cdot g^n_d + \frac{1}{2} \sum_i \bar{r}_i |I_i| \|h_i^{n+\frac{1}{2}}\|^2 \\
&= \frac{1}{4} \sum_p r_p |P_p| g^{n+1}_p \cdot g^n_p + \frac{1}{4} \sum_d \bar{r}_d |D_d| g^{n+1}_d \cdot g^n_d + \frac{1}{2} \sum_i \bar{r}_i |I_i| \|h_i^{n+\frac{1}{2}}\|^2.
\end{align*}
$$

Proof. From Theorem 2.1 we deduce that for all $n$:

$$
S_0^+ = \cdots = S_{n+\frac{1}{2}}
= \frac{1}{4} \sum_p r_p |P_p| g^{n+1}_p \cdot g^n_p + \frac{1}{4} \sum_d \bar{r}_d |D_d| g^{n+1}_d \cdot g^n_d + \frac{1}{2} \sum_i \bar{r}_i |I_i| \|h_i^{n+\frac{1}{2}}\|^2.
$$

Thanks to the first two equations of (3.8), we have:

$$
\begin{align*}
g^{n+1}_p &= g^n_p - \frac{\Delta t}{r_p |P_p|} \sum_{i \in p} (\bar{r}_i |I_i| h_i^{n+\frac{1}{2}} + |P_p \cap I_i| U h_i^{n+\frac{1}{2}}), \\
g^{n+1}_d &= g^n_d - \frac{\Delta t}{\bar{r}_d |D_d|} \sum_{i \in d} (\bar{r}_i |I_i| h_i^{n+\frac{1}{2}} + |D_d \cap I_i| U h_i^{n+\frac{1}{2}}).
\end{align*}
$$
Inserting these values into (3.9) provides:

\[ S^{n+\frac{1}{2}} = \frac{1}{4} \sum_p \left( r_p |P_p||g_p^n| - \Delta t \sum_i (\bar{r}_i X_i h_i^n + \bar{r}_i |U_i h_i^n|) \cdot g_p^n 
\]
\[ + \frac{1}{4} \sum_d (\bar{r}_d |D_d| g_d^n - \Delta t \sum_i (\bar{r}_i Y_i h_i^n + |D_d| I_i |U_i h_i^n|) \cdot g_d^n 
\]
\[ + \frac{1}{2} \sum_i |I_i||h_i^n + \frac{1}{2}|^2 \right. \]

Thanks to the reflective boundary conditions and to the obvious geometrical identities\(\|\):

\[ |P_p| = \sum_i |I_{ip}|, \quad |D_d| = \frac{1}{2} \sum_i |I_i|, \]

we obtain the relation:

\[ S^{n+\frac{1}{2}} = \sum_i S_i^{n+\frac{1}{2}}, \]

where the values \(S_i^{n+\frac{1}{2}}\), defined by:

\[ S_i^{n+\frac{1}{2}} = \frac{1}{4} |I_{ip}| \left( r_p |g_p^n|^2 - \frac{\Delta t}{|I_{ip}|} \bar{r}_i (g_q^n)^t X_i h_i^n - \Delta t (g_q^n)^t U_i h_i^n + \bar{r}_i |h_i^n + \frac{1}{2}|^2 \right) 
\]
\[ + \frac{1}{4} |I_{iq}| \left( r_q |g_q^n|^2 + \frac{\Delta t}{|I_{iq}|} \bar{r}_i (g_q^n)^t X_i h_i^n - \Delta t (g_q^n)^t U_i h_i^n + \bar{r}_i |h_i^n + \frac{1}{2}|^2 \right) 
\]
\[ + \frac{1}{2} \sum_i |I_i| \left( \bar{r}_d |g_d^n|^2 - 2 \frac{\Delta t}{|I_i|} \bar{r}_i (g_d^n)^t Y_i h_i^n - \Delta t (g_d^n)^t U_i h_i^n + \bar{r}_i |h_i^n + \frac{1}{2}|^2 \right) 
\]
\[ + \frac{1}{2} \sum_i |I_i| \left( \bar{r}_c |g_c^n|^2 + 2 \frac{\Delta t}{|I_i|} \bar{r}_i (g_c^n)^t Y_i h_i^n - \Delta t (g_c^n)^t U_i h_i^n + \bar{r}_i |h_i^n + \frac{1}{2}|^2 \right) \]

appear to be approximations of \(S\) in the intermediary cells \(I_i\) at time \((n + \frac{1}{2}\Delta t)\). Then sufficient conditions for the \(S_i^{n+\frac{1}{2}}\) to be positive are therefore those expressed in a more compact form by (3.11). This ends the proof. □

Denote by \(L_{ip}\) (\(L_{iq}\)) the distance from the point \(x_p\) (\(x_q\)) to the side \(C_i = x_p x_q\) and denote by \(\theta_i\) the angle between \(n_i\) and \(\nu_i\) (see Fig. 7). From Theorem 3.2 we deduce Theorem 3.3.

**Theorem 3.3.** In the \(P_1\) case a sufficient condition for relations (3.11) to be satisfied is:

\[ \frac{1}{\sqrt{3}} c \Delta t \leq \min_i \left( L_{ip} \sqrt{r_p} / \sqrt{r_i}, L_{iq} \sqrt{r_q} / \sqrt{r_i}, \frac{1}{2} |C_i| \cos \theta_i \sqrt{r_d} / \sqrt{r_i}, \frac{1}{2} |C_i| \cos \theta_i \sqrt{r_c} / \sqrt{r_i} \right). \]

(3.12)

\(^1\)Recall that \(I_i = I_{ip} \cup I_{iq}\) (with \(I_{iq} = \emptyset\) if \(x_i \in \partial \Omega\)) and \(|I_i| = |I_{ip}| + |I_{iq}|\).
Proof. In the $P_1$ case recall that $\mathbf{U} = 0$. If we denote $g = f_0$ and $h = (f_1^*, f_1^{*})$ the first of the conditions (3.11), for example, reads:

$$
rp \frac{g^2}{\sqrt{3}} \frac{\bar{r}_i}{|I_{ip}|} \mathbf{g} \cdot \mathbf{n}_i + \bar{r}_i \| \mathbf{h} \|^2
$$

$$
= rp \left( \left( \frac{n_{zi}}{|C_i|} g - \frac{c \Delta t}{2 \sqrt{3} r_p} \frac{\bar{r}_i}{|I_{ip}|} |C_i| h_r \right)^2 + \left( \frac{n_{zi}}{|C_i|} g - \frac{c \Delta t}{2 \sqrt{3} r_p} \frac{\bar{r}_i}{|I_{ip}|} |C_i| h_z \right)^2 \right)
$$

$$
+ \left( \frac{\bar{r}_i}{r_p} - \frac{c^2 \Delta t^2 \bar{r}_i}{12 \frac{r_p^2}{|I_{ip}|^2}} \| \mathbf{h} \|^2 \right) \geq 0.
$$

A sufficient condition for this relation to be satisfied is**:

$$
\frac{1}{\sqrt{3}} c \Delta t \leq L_{ip} \frac{\sqrt{r_p}}{\sqrt{r_i}}.
$$

**Thanks to the identity (see Fig. 7):

$$
|I_{ip}| = \frac{1}{2} |C_i| L_{ip}.
$$
Similarly the third of the conditions (3.11), for example, reads:

\[
\tilde{r}_d s^2 - \frac{2 c \Delta t}{\sqrt{3} |I_i|} \tilde{r}_i \cdot \nu_i + \tilde{r}_i \|h\|^2 = \tilde{r}_d \left( \left( \frac{v_{ri}}{|x_p x_q|} g - c \Delta t \frac{r_i}{\|h\|} \right)^2 + \left( \frac{v_{\tilde{r}_i}}{|x_p x_q|} g - c \Delta t \frac{r_i}{\|h\|} \right)^2 + \left( \frac{r_i}{\|h\|} \right) \right) \geq 0.
\]

A sufficient condition for this relation to be satisfied is\[††:\]

\[
\frac{1}{\sqrt{3}} c \Delta t \leq \frac{1}{2} |C_i| \cos \theta_i \sqrt{\frac{r_i}{r_d}}.
\]

By doing similar calculations for the second and fourth condition of (3.11), we observe that (3.12) is a sufficient condition for all the relations (3.11) to be satisfied. This ends the proof.

Unlike the $P_1$ case the relations (3.11) do not seem to be directly exploitable for exhibiting a stability condition for the $P_N$ general case ($N > 1$). To circumvent this difficulty we have used the rule of thumb:

\[
c \Delta t \leq \min_p (L_p),
\]

where $L_p$ is the length of the shortest side of the primal cell $P_p$. This rough condition turns out to be effective for the numerical experiments.

Starting from the DDFV approximation of partial derivatives, it can be proved the following discrete counterpart of Theorem 2.1.

**Theorem 3.4.** Let $l_r$, $t_r$ be reference length and time such that:

\[
c \frac{t_r}{l_r} = \frac{1}{\sigma_d l_r} = \sigma_s l_r = \frac{1}{\epsilon}.
\]

If $q = 0$ and $\epsilon$ tends toward 0, the discretized radiative system of equations (3.8) provides a DDFV discretization of the diffusion equation (2.17).
Proof. It suffices to consider Eq. (3.8) and to apply a Hilbert expansion to both the vectors \( g \) and \( h \). Then, following the steps of the proof given in [1] for the continuous case, we obtain the equations:

\[
\begin{align*}
\frac{1}{c} & r_p | P_p | \frac{g^{n+1}_p - g^n_p}{\Delta t} - \frac{1}{6} \sum_{i\in P} \frac{r_i}{|I_i|} \left( (g^n_c - g^n_p) ||n_i||^2 + (g^n_c - g^n_p) v_i \cdot n_i \right) = 0, \\
\frac{1}{c} & r_d | D_d | \frac{g^{n+1}_d - g^n_d}{\Delta t} - \frac{1}{6} \sum_{i\in D} \frac{r_i}{|I_i|} \left( (g^n_c - g^n_p) n_i \cdot v_i + (g^n_c - g^n_p) ||v_i||^2 \right) = 0.
\end{align*}
\]

These equations coincide with the DDFV discretization of a diffusion equation in cylindrical coordinates similar to that proposed in [20, 21] in Cartesian coordinates. This ends the proof.

4 Discretization of the grey thermal-\( P_N \) radiative transfer equations

In this section we detail the discretization of the 2D grey thermal-\( P_N \) radiative transfer equations (2.19) in cylindrical coordinates. Because of the numerous variables the notation is rather complicated since certain values can have several indices. For example we have:

\[
(f^n_p)_k^m = \frac{1}{4\pi} \int_{S^2} f(x_p, \psi, \mu, n\Delta t) X_k^m(\psi, \mu) d\psi d\mu,
\]

\[
b^n_p = b(T^n_p) = \frac{2\pi h}{c^3} \int_0^\infty v^3 \left( \exp\left( \frac{\nu}{\kappa T^n_p} \right) - 1 \right)^{-1} dv,
\]

\[
\sigma^n_{ap} = \sigma_a(\rho_p, T^n_p), \quad \sigma^n_{sp} = \sigma_s(\rho_p, T^n_p)
\]

and so on.\ldots Recall that \( k \) is a lower index regarding the (polar) \( \mu \) discretization, \( m \) is an upper index regarding the (azimuthal) \( \psi \) discretization, \( p \) (resp. \( d, i \)) are lower indices regarding the primal (resp. dual, intermediary) spatial discretization and \( n \) is an upper index regarding the time discretization.

4.1 Space discretization

Integrate the first (thermal) equation of the system (2.19) on primal \textit{and} dual cells and do the approximations:

\[
\int_{P_p} r_p \rho = r_p | P_p | \rho_p, \quad \int_{P_p} r_p \rho C_v = r_p | P_p | \rho_p C_{vp},
\]

\[
\int_{D_d} r_p = r_d | D_d | \rho_d, \quad \int_{D_d} r_p \rho C_v = r_d | D_d | \rho_d C_{vd},
\]
where the dual mass density $\rho_d$ and the dual specific heat $C_{vd}$ are defined from their primal counterparts by:

$$
\rho_d = \frac{\sum_p r_p |P_p \cap D_d| P_p}{\sum_p r_p |P_p \cap D_d|}, \quad C_{vd} = \frac{\sum_p r_p |P_p \cap D_d| P_p C_{vp}}{\sum_p r_p |P_p \cap D_d|},
$$

so as to have:

$$
\sum_p r_p |P_p| \rho_p = \sum_d r_d |D_d| \rho_d, \quad \sum_p r_p |P_p| \rho_p C_{vp} = \sum_d r_d |D_d| \rho_d C_{vd}.
$$

After dividing by $r_p |P_p|$ or $r_d |D_d|$ and taking into account the space discretized $P_N$ grey radiative transfer equation (3.7), we obtain the space discretization of the grey thermal-$P_N$ radiative transfer equations in cylindrical coordinates:

$$
\begin{cases}
\rho_p C_{vp} \frac{\partial T_p}{\partial t} + ac \sigma_{ap} T_p^4 = 4\pi ac \sigma_{ap} (f_p)_0^0, \\
\rho_d C_{vd} \frac{\partial T_d}{\partial t} + ac \sigma_{ad} T_d^4 = 4\pi ac \sigma_{ad} (f_d)_0^0, \\
\frac{1}{c} \frac{\partial g_p}{\partial t} + \frac{1}{r_p |P_p|} \sum_i (r_i |X_i h_i + |P_p \cap I_i| U h_i|) + M_p g_p = a_p, \\
\frac{1}{c} \frac{\partial g_d}{\partial t} + \frac{1}{r_d |D_d|} \sum_i (r_i |Y_i h_i + |D_d \cap I_i| U h_i|) + M_d g_d = a_d, \\
\frac{1}{c} \frac{\partial h_i}{\partial t} + \frac{1}{2 |I_i|} (X^{(i)} (g_q - g_p) + Y^{(i)} (g_e - g_d)) \\
- \frac{1}{2 r_i |I_i|} U^{(i)} (|P_p \cap I_i| g_p + |P_q \cap I_i| g_q + |D_d \cap I_i| g_d + |D_e \cap I_i| g_e) \\
+ N_i h_i = b_i, \quad x_i \in \partial \Omega,
\end{cases}
$$

4.2 Time discretization

The thermal equation is discretized in time by the backward Euler scheme so that we obtain the following space-time discretization of grey thermal-$P_N$ radiative transfer equa-
tions in cylindrical coordinates:

\[
\begin{align*}
\frac{T_p^{n+1} - T_p^n}{\Delta t} + a c_\alpha \sigma^a_{\alpha p}(T_p^{n+1})^4 &= 4 \pi c \sigma^a_{\alpha p}(f_p^{n+1})_0, \\
\frac{T_d^{n+1} - T_d^n}{\Delta t} + a c_\alpha \sigma^a_{\alpha d}(T_d^{n+1})^4 &= 4 \pi c \sigma^a_{\alpha d}(f_d^{n+1})_0, \\
\frac{1}{c} \sum_{i \in p} (\vec{r}_i \cdot \mathbf{h}_i^{n+\frac{1}{2}} + |P_p \cap I_i| \mathbf{U} \mathbf{h}_i^{n+\frac{1}{2}}) + M_p^n g_p^{n+1} &= a_p^{n+\frac{1}{2}}, \\
\frac{1}{c} \sum_{i \in d} (\vec{r}_i \cdot \mathbf{h}_i^{n+\frac{1}{2}} + |D_d \cap I_i| \mathbf{U} \mathbf{h}_i^{n+\frac{1}{2}}) + M_d^n g_d^{n+1} &= a_d^{n+\frac{1}{2}}, \\
\frac{1}{c} \left| I_i \right| \left( \mathbf{X}_i^n (\mathbf{g}_q^n - \mathbf{g}_q^n) + \mathbf{Y}_i^n (\mathbf{g}_c^n - \mathbf{g}_d^n) \right) - \frac{1}{2} \frac{1}{\left| I_i \right|} \mathbf{U}^T \left( |P_p \cap I_i| \mathbf{g}_p^n + |P_q \cap I_i| \mathbf{g}_q^n + |D_d \cap I_i| \mathbf{g}_d^n + |D_c \cap I_i| \mathbf{g}_c^n \right) \\
&+ N_h^n h_i^{n+\frac{1}{2}} = b_i^n, \quad x_i \notin \partial \Omega, \\
\frac{1}{c} \left| I_i \right| \left( \mathbf{X}_i^n (\mathbf{g}_q^n - \mathbf{g}_q^n) + \mathbf{Y}_i^n (\mathbf{g}_c^n - \mathbf{g}_d^n) \right) - \frac{1}{2} \frac{1}{\left| I_i \right|} \mathbf{U}^T \left( |P_p \cap I_i| \mathbf{g}_p^n + |D_d \cap I_i| \mathbf{g}_d^n + |D_c \cap I_i| \mathbf{g}_c^n \right) + N_h^n h_i^{n+\frac{1}{2}} &= b_i^n, \quad x_i \in \partial \Omega.
\end{align*}
\]

Thanks to the definition of the first line of matrices \( \mathbf{A}, \mathbf{C}, \mathbf{U} \), the primal and dual equations regarding the first component of vector \( \mathbf{g} \) read\( ^{\text{\dagger\dagger}} \):

\[
\begin{align*}
\frac{1}{c} \sum_{i \in p} (\vec{r}_i \cdot \mathbf{F}_i^{n+\frac{1}{2}}) \cdot \mathbf{m}_i + c_\alpha \sigma^a_{\alpha p} E_p^{n+1} = a \sigma^a_{\alpha p} (T_p^{n+1})^4 + \frac{4 \pi}{c} \sigma^a_{\alpha p} (q_p^{n+\frac{1}{2}})_0, \\
\frac{1}{c} \sum_{i \in d} (\vec{r}_i \cdot \mathbf{F}_i^{n+\frac{1}{2}}) \cdot \mathbf{m}_i + c_\alpha \sigma^a_{\alpha d} E_d^{n+1} = a \sigma^a_{\alpha d} (T_d^{n+1})^4 + \frac{4 \pi}{c} \sigma^a_{\alpha d} (q_d^{n+\frac{1}{2}})_0.
\end{align*}
\]

\( ^{\text{\dagger\dagger}} \text{Recall that:} \\
E = 4 \pi f_0^0, \quad F = \frac{4 \pi}{\sqrt{3}} (f_1^1 f_1^{-1})^t.
\)
By eliminating
\[ E_{p}^{n+1} = 4\pi (f_{p}^{n+1})_{0}^{0} \quad \text{(resp.} \quad E_{d}^{n+1} = 4\pi (f_{d}^{n+1})_{0}^{0}) \]
between the first (resp. second) equation of (4.2) and (4.3), we obtain two similar non-linear equations on the unknowns \( T_{p}^{n+1} \) and \( T_{d}^{n+1} \):

\[
\begin{align*}
\rho_{p}C_{vp}T_{p}^{n+1} + \frac{ac\Delta t\sigma_{ap}^{n}}{1+c\Delta t\sigma_{ap}^{n}} (T_{p}^{n+1})^{4} &= \rho_{p}C_{vp}T_{p}^{n} + \frac{c\Delta t\sigma_{ap}^{n}}{1+c\Delta t\sigma_{ap}^{n}} \left( E_{p}^{n} - \frac{c\Delta t}{r_{p}|P_{p}|} \sum_{i\in p} r_{i} F_{i} \cdot n_{i} - 4\pi\Delta t(q_{p}^{n+\frac{1}{2}})_{0} \right), \\
\rho_{d}C_{vd}T_{d}^{n+1} + \frac{ac\Delta t\sigma_{ad}^{n}}{1+c\Delta t\sigma_{ad}^{n}} (T_{d}^{n+1})^{4} &= \rho_{d}C_{vd}T_{d}^{n} + \frac{c\Delta t\sigma_{ad}^{n}}{1+c\Delta t\sigma_{ad}^{n}} \left( E_{d}^{n} - \frac{c\Delta t}{r_{d}|D_{d}|} \left( \sum_{i\in d} r_{i} F_{i} \cdot n_{i} + 4\pi\Delta t(q_{d}^{n+\frac{1}{2}})_{0} \right) \right).
\end{align*}
\]

(4.4)

The initial dual temperature \( T_{d}^{0} \) is defined by:

\[
T_{d}^{0} = \frac{\sum_{p} r_{p}|P_{p}\cap D_{d}| \rho_{p}C_{vp}T_{p}^{0}}{\sum_{p} r_{p}|P_{p}\cap D_{d}| \rho_{p}C_{vp}}.
\]

Thanks to (3.6) and (4.1) note that the discretized primal, initial, thermal energy in the domain \( \Omega \) is then equal to its dual counterpart:

\[
\sum_{p} r_{p}|P_{p}| \rho_{p}C_{vp}T_{p}^{0} = \sum_{d} r_{d}|D_{d}| \rho_{d}C_{vd}T_{d}^{0}.
\]

The twin non-linear equations (4.4) can be rewritten as:

\[ AX^{\gamma} + BX^{\delta} = C, \]

where \( X = T_{p}^{n+1} \) or \( X = T_{d}^{n+1} \) is the unknown while \( \gamma = 1, \delta = 4 \) and:

\[
A = \rho_{p}C_{vp}, \quad B = \frac{ac\Delta t\sigma_{ap}^{n}}{1+c\Delta t\sigma_{ap}^{n}}, \quad C = \rho_{p}C_{vp}T_{p}^{n} + \frac{c\Delta t\sigma_{ap}^{n}}{1+c\Delta t\sigma_{ap}^{n}} \left( E_{p}^{n} - \frac{\Delta t}{r_{p}|P_{p}|} \sum_{i\in p} r_{i} F_{i}^{n+\frac{1}{2}} \cdot n_{i} + 4\pi\Delta t(q_{p}^{n+\frac{1}{2}})_{0} \right)
\]
or:

\[ A = \rho_d C vd, \quad B = \frac{ac\Delta t e^n_{ad}}{1 + c\Delta t e^n_{ad}}, \]

\[ C = \rho_d C vd T^n_{d} + \frac{c\Delta t e^n_{ad}}{1 + c\Delta t e^n_{ad}} \left( E^n_d - \frac{\Delta t}{\bar{T}_d[D_d]} \sum_{i \in D} \bar{T}_i F_{n+1/2,n_i} + 4\pi \Delta I q^n_{d} \right). \]

If \( A, B, C \) are positive coefficients, such non-linear equations can be solved by the dichotomy method (with \( \varepsilon = 10^{-12} \) for example):

1. \( i = 1, \)

2. \( X_{\min} = 0, X_{\max} = \min \left( \frac{(C_A)}{A}, \frac{(C_B)}{B} \right), \)

3. \( i = i + 1, \)

4. \( X_i = \frac{1}{2} (X_{\min} + X_{\max}), \)

5. \( C_i = AX^i + BX^i, \)

6. if \( |C_i - C| > \varepsilon C \) then:

   (a) do \( X_{\max} = X_i \) if \( C_i \geq C, \)

   (b) do \( X_{\min} = X_i \) if \( C_i < C, \)

   (c) go to step 3,

7. \( X = X_i. \)

For some strongly non-linear cases it may happen that the primal and dual temperatures \( T_p, T_d \) and the primal and dual radiation density energies \( E_p, E_d \) are decoupled over the time. One can remedy this problem by solving the following corrected first four equations
of (4.2): 
$$\begin{align*}
\rho_p C_{vp} T_p^{n+1} + \frac{ac\Delta t \sigma_{ap}}{1+c\Delta t \sigma_{ap}} (T_p^n)^4 & = \rho_p C_{vp} T_p^n + \frac{c\Delta t \sigma_{ap}}{1+c\Delta t \sigma_{ap}} \left( E_p^n - \frac{\Delta t}{r_p |P_p|} \sum_{i \in p} \bar{r}_i F_i^{n+\frac{1}{2}} \cdot n_i + 4\pi \Delta t q_p^{n+\frac{1}{2}} \right) \\
& + \alpha \Delta t \left( \frac{1}{|P_p|} \sum_{d} |D_d \cap P_p| \rho_d C_{vd} T_d^n - \rho_p C_{vp} T_p^n \right), \\
\rho_d C_{vd} T_d^{n+1} + \frac{ac\Delta t \sigma_{ad}}{1+c\Delta t \sigma_{ad}} (T_d^n)^4 & = \rho_d C_{vd} T_d^n + \frac{c\Delta t \sigma_{ad}}{1+c\Delta t \sigma_{ad}} \left( E_d^n - \frac{\Delta t}{r_d |D_d|} \sum_{i \in d} \bar{r}_i F_i^{n+\frac{1}{2}} \cdot n_i + 4\pi \Delta t q_d^{n+\frac{1}{2}} \right) \\
& + \alpha \Delta t \left( \frac{1}{|D_d|} \sum_{p} |D_d \cap P_p| \rho_p C_{vp} T_p^n - \rho_d C_{vd} T_d^n \right), \\
1 \frac{E_p^{n+1} - E_p^n}{\Delta t} + \frac{1}{c} \frac{1}{r_P |P_p|} \sum_{i \in p} \bar{r}_i F_i^{n+\frac{1}{2}} \cdot n_i + \sigma_{ap} E_p^{n+1} & = a \sigma_{ap} (T_p^{n+1})^4 + \frac{4\pi}{c} q_p^{n+\frac{1}{2}} + \alpha \left( \frac{1}{|P_p|} \sum_{d} |D_d \cap P_p| E_d^n - E_p^n \right), \\
1 \frac{E_d^{n+1} - E_d^n}{\Delta t} + \frac{1}{c} \frac{1}{r_D |D_d|} \left( \sum_{i \in d} \bar{r}_i F_i^{n+\frac{1}{2}} \cdot n_i \right) + \sigma_{ad} E_d^{n+1} & = a \sigma_{ad} (T_d^{n+1})^4 + \frac{4\pi}{c} q_d^{n+\frac{1}{2}} + \alpha \left( \frac{1}{|D_d|} \sum_{p} |D_d \cap P_p| E_p^n - E_d^n \right),
\end{align*}$$

where $\alpha$ is a positive number to be chosen according to the benchmark. Note that this type of problem may occur mainly for $N \geq 3$.

## 5 Numerical experiments

Let define the (double) discrete $L^2$-norm of some scalar function $p = p(x)$ by:

$$\|p\|_2 = \left( \frac{1}{2} \left( \sum_{p=1}^{N_p} |P_p| p_p^2 + \sum_{d=1}^{N_D} |D_d| p_d^2 \right) \right)^{\frac{1}{2}}$$

and let $l$ be the characteristic size of the mesh of the domain $\Omega$, defined by:

$$l = \left( \frac{|\Omega|}{N_p} \right)^{\frac{1}{2}}.$$

*Recall that $N_p$ ($N_D$) is the number of primal (dual) cells.*
The relative error between the exact solution \( p_e \) and the approximated one \( p_l \) is defined by:

\[
\epsilon_l = \frac{\|p_l - p_e\|_2}{\|p_e\|_2}.
\]

The order of the method is given by:

\[
\text{order} = \frac{\log(\epsilon_{2^l}) - \log(\epsilon_l)}{\log 2}.
\]

We deal with three benchmarks. The first one uses the method of manufactured solution in order to assess the experimental order of convergence of the method in the framework of cylindrical coordinates. The next one, inspired from [11], deals with the radiation of a line source in Cartesian and cylindrical coordinates while the last one deals with a coupled grey thermal-radiative transfer problem in Cartesian coordinates inspired from [8].

For all these benchmarks the time step is chosen either following the stability conditions (3.12) (if \( N = 1 \)) or (3.13) (if \( N > 1 \)).

To take into account the diversity of situations one can be confronted we have tested twelve groups of successively refined more or less distorted meshes of a square. The level of refinement of these meshes is the integer \( n \) such that the four sides of \( \Omega \) are divided into \( 2^{n-1} \times 10 \) segments. The coarsest \((n=1)\) of the quad (resp. Delaunay’s triangles, polygons) meshes has \( 10^2 \) (resp. 270, 156) elements while the finest \((n=9)\) for the quad meshes and \(n=7\) for the Delaunay’s triangles and polygons meshes) has \( 2560^2 \) (resp. 1174270, 588416) elements. The meshes with a 1-level of refinement are displayed on Fig. 8.

1. The cells of the meshes of the first group are squares.

2. The cells of the meshes of the second group are right triangles, constructed by dividing each square of the first group into four triangles.

3. The meshes of the third group are standard unstructured Delaunay’s meshes. Recall that a Delaunay’s mesh is a mesh whose cells (triangles or tetrahedra in general) can be inscribed in an open ball that does not contain any vertex of the mesh: see [27, 28] where an algorithm to generate such meshes for arbitrary polygons (polyhedra) has been first proposed.

4. The cells of the meshes of the fourth group are polygons generated by joining the centers of gravity of the triangles of the third group.

5. The cells of the meshes of the fifth group are continuously distorted quads which are made up by transforming the squares of the first group by the mapping: \( X = x + 0.1\sin(2\pi x)\sin(2\pi y), Y = y + 0.1\sin(2\pi x)\sin(2\pi y) \).
6. The cells of the meshes of the sixth group are either standard or non-conformal rectangles. By non-conformal rectangles we mean degenerated polygons whose several successive sides can be aligned: such cells are used in the framework of arbitrary mesh refinement (AMR) methods in order to vary easily the cell size.

7. The meshes of the seventh are inspired from those used in [29] for testing a discretization of diffusion equations.

8. The meshes of the eight group are constructed by dividing each square of the seventh group into four triangles.
9. The cells of the meshes of the ninth group are randomly distorted quads which are made up by transforming the interior vertices of the meshes of the first group by the mapping: $X = x + 0.4 r_x \Delta x$, $Y = y + 0.4 r_y \Delta x$, where $\Delta x$ is the length of the square sides and $r_x, r_y$ are random numbers between $-1$ and $1$. Note that such meshes can include non-convex quads.

10. The cells of the meshes of the tenth group are randomly distorted quads which are made up by transforming the interior vertices of the meshes of the first group by the mapping: $X = x + 0.4 \text{sign}(r_x) \Delta x$, $Y = y + 0.4 \text{sign}(r_y) \Delta x$, where $\Delta x$ is the length of the square sides and $r_x, r_y$ are random numbers between $-1$ and $1$. Note that such meshes can include non-convex quads.

11. The cells of the meshes of the eleventh group are randomly distorted triangles which are made up by transforming the interior vertices of the meshes of the second group by the mapping: $X = x + 0.4 r_x l$, $Y = y + 0.4 r_y l$, where $l$ is the minimum of the length of sides from vertex $(x, y)$ and $r_x, r_y$ are random numbers between $-1$ and $1$.

12. The twelfth group of meshes are made up of strongly non-convex quads, constructed from the first group of meshes in such a way that the center $(x, y)$ of each motif of four squares is replaced by the point $(x + \Delta x(1+\cos \theta), y + \Delta x(1+\sin \theta))$ with $\theta = 0.25 \pi$.

5.1 A simple manufactured solution

Suppose that the moments $q^m_k$ of source $q$ are chosen such that, for all $k, m$ with $0 \leq |m| \leq k \leq N$, the functions:

$$f^m_k = f^m_k(x, t) = \frac{1}{\pi r_0^2} \exp \left( - \left( \frac{t}{t_0} + \frac{\|x - x_0\|^2}{r_0^2} \right) \right)$$

are solutions to the grey $P_N$ radiative equations in cylindrical coordinates (3.2) with $\Omega = [0,0.5]^2$, $t_f = \text{Log5}$, $c = 1$, $\sigma_a = \sigma_s = 0$ and:

$$t_0 = 1, \quad x_0 = \left( \frac{1}{4}, \frac{1}{4} \right), \quad r_0 = \frac{1}{\sqrt{20\pi}}.$$

For all groups of meshes of Fig. 8 the method is second-order accurate except for the meshes of polygons and non-conformal rectangles (resp. randomly distorted quads) for which the accuracy drops to one and a half (resp. one): see Fig. 9 on which are reported the mesh size $l$ (abscissa) and the relative errors in the $L^2$-norm between the exact moment $f^0_k$ and the approximated one (ordinate) for the cases $P_1$ (stars) and $P_3$ (circles). For the moment we do not know why this difference arises. However, for the approximation (3.5) of $\nabla p$, recall that we have supposed that $p$ is only a piecewise constant function on the intermediary mesh. Notice that the errors are similar for the other moments. Similar observations have been reported for the same problem in the Cartesian case: see [1].
Figure 9: Discrete $L^2$ errors for $f(x)$ (stars: $P_1$, circles: $P_3$) for the twelve meshes of Fig. 8 (from left to right and top to bottom) in cylindrical coordinates.
5.2 The line source

This benchmark is inspired from [11]. Consider the grey $P_N$ radiative equations in Cartesian and cylindrical coordinates (3.1) and (3.2) with $\Omega = [0.6,0.6]^2$, $t_f = 0.5$, $c = 1$, $\sigma_a = 0$, $\sigma_s = 1$, $q = 0$ and define the initial distribution function as:

$$f^0 = f^0(x) = \frac{1}{\pi r_o^2} \exp \left(-\frac{\|x-x_o\|^2}{r_o^2}\right)$$

with $x_o = (0.15,0.30)$, $r_o = 3.57771 \times 10^{-2}$. The moments of $f^0$ are therefore $(f^0)_0 = f^0$ and $(f^0)_k = 0$ if $0 < |m| \leq k$.

Fig. 10 (resp. Fig. 11) show the isovalues of $f^0_0$ computed with the twelve groups of meshes of Fig. 8 in Cartesian (resp. cylindrical) coordinates: one can see that the approximation of the moment $f^0_0$ is almost independent of the type of mesh as well as in Cartesian as cylindrical coordinates.

![Figure 10: Isovalues of f_0^0 at time 0.5 for the twelve meshes of Fig. 8 (from left to right and top to bottom) in Cartesian coordinates.](image-url)
5.3 A coupled grey thermal-$P_N$ radiative transfer problem

This benchmark is inspired from [8, 12]. We look for the temperature $T$ and the radiative energy density $E$ solutions to both the non-equilibrium diffusion limit (2.22) and $P_N$ (3.1) approximations of the coupled grey thermal-radiative transfer equations in Cartesian coordinates, thus allowing numerical comparisons between these two models. The discretization of the non-equilibrium diffusion limit equation is detailed in Appendix C.

We have chosen $\Omega = [0,1]^2$, $t_f = 5$, $\rho = 1$, $C_v = 1$, $a = 1$, $c = 1$, $\sigma_s = 0$ and:

$$
\begin{align*}
\sigma_a(x,T) &= 10^3 T^{-3}, & \text{if } x \in \left( \left[ \frac{3}{16}, \frac{7}{16} \right] \times \left[ \frac{9}{16}, \frac{13}{16} \right] \right) \cup \left( \left[ \frac{9}{16}, \frac{13}{16} \right] \times \left[ \frac{3}{16}, \frac{7}{16} \right] \right), \\
\sigma_a(x,T) &= T^{-3}, & \text{if not.}
\end{align*}
$$

Figure 11: Isovalues of $f_0^0$ at time 0.5 for the twelve meshes of Fig. 8 (from left to right and top to bottom) in cylindrical coordinates.
The initial conditions are:

\[
\begin{cases}
T(0) = 1, & E(0) = 1, \\
T(x) = (10^{-3})^4, & E(x) = 10^{-3},
\end{cases}
\]

if \( \|x\| \leq \frac{4}{10} \),

if \( \|x\| > \frac{4}{10} \).

We use seven more and more refined meshes made up of squares or randomly distorted quads. Both types of meshes have \( 16^2 \) (see Fig. 12), \( 32^2 \), \( 64^2 \), \ldots, \( 1024^2 \) cells aligned with the possible discontinuity lines of \( \sigma_a \).

Fig. 13 (top-left) shows the time evolution of the temperature at point \((0.55,0.55)\), calculated with the non-equilibrium diffusion approximation with the limiter \( m = 2 \), for squares and randomly distorted quads meshes having \( 64^2 \) (black lines), \( 128^2 \) (blue lines), \( 256^2 \) (green lines), \( 512^2 \) (red lines) and \( 1024^2 \) (mauve lines) cells while Fig. 13 (top-right) shows the same values calculated with the \( P_9 \) approximation. One note that the convergence take place as soon as the number of cells reaches \( 256^2 \) while the type of mesh (squares or randomly distorted quads) is almost irrelevant.

For the \( 256^2 \) squares mesh, Fig. 13 (bottom-left) shows the time evolution of the temperature at points \((0.5,0.5)\) (black), \((0.55,0.55)\) (red) and \((0.6,0.6)\) (green) for:

1. the non-equilibrium diffusion approximation without limiter (stars);
2. the non-equilibrium diffusion approximation with limiter \( m = 1 \) (circles);
3. the non-equilibrium diffusion approximation with limiter \( m = 2 \) (triangles);
4. the \( P_9 \) approximation (full lines).

We can see that the non-equilibrium approximation with the limiter \( m = 2 \) (triangles) is the closest to the approximation \( P_9 \) (full lines) which serves as a reference. We have chosen

---

\( \dagger \)See definition (2.23).
Figure 13: Time evolution of temperature — top-left: convergence of the non-equilibrium diffusion approximation with limiter $m = 2$ at point $(0.55,0.55)$ for squares and distorted quads meshes having $64^2$ (black lines), $128^2$ (blue lines), $256^2$ (green lines), $512^2$ (red lines) and $1024^2$ (mauve lines) cells — top right: convergence of the $P_0$ approximation at point $(0.55,0.55)$ for the same meshes — bottom-left: convergence at points $(0.5,0.5)$ (black), $(0.55,0.55)$ (red), $(0.6,0.6)$ (green) for the $256^2$ squares mesh (stars: non-equilibrium diffusion without limiter, circles: non-equilibrium diffusion approximation with limiter $m = 1$, triangles: non-equilibrium diffusion approximation with limiter $m = 2$, full lines: $P_0$) — bottom-right: convergence at the point $(0.55,0.55)$ for the $P_3$ (black), $P_5$ (blue), $P_9$ (red), $P_{39}$ (mauve) approximations and for the non-equilibrium diffusion approximation with the limiter $m = 2$ (orange) for the same mesh.

$N = 9$ but one can see on Fig. 13 (bottom-right) that it is useless, for this benchmark, to increase $N$ beyond 5 since the lines corresponding to $N = 5$ (blue), $N = 9$ (green), $N = 19$ (red) and $N = 39$ (mauve) are almost superimposed.
6 Concluding remarks

We have extended the DDFV type method for discretizing the $P_N$-radiative equation to the coupled grey thermal-$P_N$ radiative transfer equations as well in Cartesian as cylindrical coordinates. Three benchmarks borrowed from the literature have been dealt with successfully.

As these numerical experiments have shown, the attractiveness of our method, compared with more standard others, comes from the fact that it is able to provide an accurate discrete solution whatever the type (polygonal, distorted, non-conformal, non-convex) of mesh or coordinates system (Cartesian, cylindrical) and this, only at the price of a doubling of the degrees of freedom of the material temperature and even moments of the distribution function of photons. As the method is explicit and easily parallelizable this doubling only results in a minimal increase of the computational cost.

It is worth noting that the method lends itself very well to the discretization of numerous other well-known systems of wave-type equations as, for example, acoustics, hyperbolic heat, linearized shallow water with Coriolis and friction terms, Maxwell, elastodynamics, Stokes or Dirac equations which can all be put in an abstract form similar to (2.10) and therefore fall under the approximation (3.8): see [24].

The important topic of coupling with the Lagrangian material equations will be the subject of a next paper while the extension of the method to the three-dimensional case should be the subject of future research.

Appendices

A Legendre functions and spherical harmonics

For the reader convenience we recall here some definitions and properties of the Legendre functions and of the complex and real-valued spherical harmonics functions.

A.1 Legendre functions

Given integers $k$, $m$ such that $0 \leq |m| \leq k$, the Legendre functions $P^m_k$ of degree $k$ and order $m$ are defined by:

$$
\begin{align*}
P^m_k (\mu) &= \frac{1}{2^k k!} (1-\mu^2)^{k/2} \frac{d^{k+m}}{d\mu^{k+m}} ((\mu^2-1)^k), \quad m \geq 0, \\
P^m_k (\mu) &= (-1)^m \frac{(k-|m|)!}{(k+|m|)!} P^{|m|}_k (\mu), \quad m < 0
\end{align*}
$$

and satisfy the orthogonality relation:

$$
\frac{1}{2} \int_{-1}^{1} P^m_k P^m_l d\mu = \frac{1}{(a^m_k)^2} \delta^m_k,
$$
where $a_k^m$ is a normalization coefficient defined by:

$$a_k^m = \sqrt{(2k+1)(k-m)!(k+m)!}.$$ 

They also satisfy the following recursion relations (with the convention $P_k^m = 0$ if $k < 0$ or $|m| > k$) which are the cornerstone of the $P_N$ approximation:

$$\begin{cases}
1 - \mu^2 P_k^m = \frac{1}{2k+1} (P_{k+1}^{m+1} - P_{k-1}^{m+1}), \\
1 - \mu^2 P_k^m = \frac{1}{2k+1} (-(k-m+1)(k-m+2)P_{k+1}^{m-1} + (k+m-1)(k+m)P_{k-1}^{m-1}), \\
\mu P_k^m = \frac{1}{2k+1} ((k-m+1)P_{k+1}^m + (k+m)P_{k-1}^m).
\end{cases} \tag{A.1}$$

### A.2 Complex-valued spherical harmonics

Given $k$, $m$ integers such that $0 \leq |m| \leq k$, the complex-valued spherical harmonics $Y_k^m$ are defined by:

$$Y_k^m = Y_k^m(\psi, \mu) = (-1)^m a_k^m e^{im\psi} P_k^m(\mu).$$

The $Y_k^m$ satisfy the relations:

$$\frac{1}{4\pi} \int_{S^2} Y_k^m \bar{Y}_l^m d\psi d\mu = \delta_{k,1} \delta_{m,0}, \quad \bar{Y}_k^m = (-1)^m Y_{-m}^m, \quad \frac{\partial Y_k^m}{\partial \psi} = im Y_k^m.$$

From the recursion relations (A.1) we deduce (see [26] with a slightly different notation):

$$\begin{cases}
a_k^m \sqrt{1-\mu^2} P_k^m = A_k^m a_{k+1}^{m+1} P_{k+1}^{m+1} - B_k^m a_{k-1}^{m+1} P_{k-1}^{m+1}, \\
d_k^m \sqrt{1-\mu^2} P_k^m = -C_k^m a_{k+1}^{m-1} P_{k+1}^{m-1} + D_k^m a_{k-1}^{m-1} P_{k-1}^{m-1}, \\
d_k^m \mu P_k^m = E_k^m a_{k+1}^m P_{k+1}^m + F_k^m a_{k-1}^m P_{k-1}^m
\end{cases} \tag{A.2}$$

with:

$$\begin{align*}
A_k^m &= \sqrt{\frac{(k+m+1)(k+m+2)}{(2k+1)(2k+3)}}, & B_k^m &= \sqrt{\frac{(k-m-1)(k-m)}{(2k-1)(2k+1)}}, \\
C_k^m &= \sqrt{\frac{(k-m+1)(k-m+2)}{(2k+1)(2k+3)}}, & D_k^m &= \sqrt{\frac{(k+m-1)(k+m)}{(2k-1)(2k+1)}}, \\
E_k^m &= \sqrt{\frac{(k-m+1)(k+m+1)}{(2k+1)(2k+3)}}, & F_k^m &= \sqrt{\frac{(k-m)(k+m)}{(2k-1)(2k+1)}},
\end{align*}$$
Therefore we obtain:

\[
\begin{align*}
\exp i\phi \sqrt{1 - \mu^2} Y^m_k &= -A^m_k Y^m_{k+1} + B^m_k Y^m_{k-1}, \\
\exp -i\phi \sqrt{1 - \mu^2} Y^m_k &= C^m_k Y^m_{k+1} - D^m_k Y^m_{k-1}, \\
\mu Y^m_k &= E^m_k Y^m_{k+1} + F^m_k Y^m_{k-1}.
\end{align*}
\] (A.3)

A.3 Real-valued spherical harmonics

The real-valued spherical harmonics \( X^m_k \) are defined from the complex-valued spherical harmonics \( Y^m_k \) by:

\[
\begin{align*}
X^m_k &= Y^m_k(\mu) = a^m_k P^m_k, & m = 0, \\
X^m_k &= \frac{(-1)^m}{\sqrt{2}} (Y^m_k(\psi,\mu) + \overline{Y^m_k(\psi,\mu)}) = a^m_k \sqrt{2} \cos(m\psi) P^m_k(\mu), & 0 < m \leq k, \\
X^m_k &= \frac{i}{\sqrt{2}} (Y^m_k(\psi,\mu) - \overline{Y^m_k(\psi,\mu)}) = a^{|m|}_k \sqrt{2} \sin(|m|\psi) P^{|m|}_k(\mu), & -k \leq m < 0
\end{align*}
\]

and satisfy the relations:

\[
\frac{1}{4\pi} \int_{S^2} X^m_k X^m_l d\psi d\mu = \delta^m_k \delta^m_l, \quad \frac{\partial X^m_k}{\partial \psi} = -m X^{-m}_k.
\]

From the recursion relations (A.3) we obtain after rather tedious calculations:

\[
\begin{align*}
\cos \psi \sqrt{1 - \mu^2} X^m_k &= \epsilon^m (A^m_k X^m_{k+1} - B^m_k X^m_{k-1}) - \zeta^m (C^m_k X^m_{k+1} - D^m_k X^m_{k-1}), \\
\sin \psi \sqrt{1 - \mu^2} X^m_k &= \eta^m (A^m_k X^{-m}_{k+1} - B^m_k X^{-m}_{k-1}) + \theta^m (C^m_k X^{-m}_{k+1} - D^m_k X^{-m}_{k-1}), \\
\mu X^m_k &= E^m_k X^m_{k+1} + F^m_k X^m_{k-1},
\end{align*}
\] (A.4)

where \( \epsilon^m, \zeta^m, \eta^m \) and \( \theta^m \) are parameters defined (according to the values of \( m \)) in Table 1.

| Table 1: Parameters involved in the relations (A.4). |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | \( m < -1 \)    | \( m = -1 \)    | \( m = 0 \)     | \( m = 1 \)     | \( m > 1 \)     |
| \( \epsilon^m \) | -\frac{1}{2}   | 0               | \frac{1}{\sqrt{2}} | \frac{1}{2}    | \frac{1}{2}    |
| \( \zeta^m \)   | -\frac{1}{2}   | -\frac{1}{2}   | 0               | \frac{1}{\sqrt{2}} | \frac{1}{2}    |
| \( \eta^m \)   | -\frac{1}{2}   | -\frac{1}{\sqrt{2}} | \frac{1}{\sqrt{2}} | \frac{1}{2}    | \frac{1}{2}    |
| \( \theta^m \) | -\frac{1}{2}   | -\frac{1}{2}   | 0               | 0              | \frac{1}{2}    |
B Matrices $A, B, C, U, V$ for the $P_1, P_2$ and $P_3$ cases

If $N=1$, the matrices $A, B, C, U, V$ read:

$$A = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{15}} \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{15}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{2}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{5}} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{15}} \\ 0 & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

while for $N=2$ we have:

$$A, B, C, U, V$$ read:
and for $N = 3$:

$$A = \begin{pmatrix}
0 & 0 & 1 \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$B = \begin{pmatrix}
\frac{1}{\sqrt{5}} & 0 & 0 & \frac{\sqrt{3}}{\sqrt{14}} & 0 & -\frac{1}{\sqrt{70}} & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
C Discretization of the non-equilibrium diffusion equation

With the changing of variable $U = aT^4$ the system (2.22) can be rewritten as:

\[
\frac{\rho C_v}{a^4} \frac{\partial}{\partial t} (U^4) \frac{\partial U}{\partial t} + c \sigma_a U^4 = c \sigma_a E,
\]

\[
\frac{1}{c} \frac{\partial E}{\partial t} - \frac{1}{3} \nabla \cdot \left( \frac{1}{\sigma_a + \sigma_s} \nabla E \right) + \sigma_a E = \sigma_a U^4
\]
and can be solved by *Picard* iterations ($r$ stands for the index of iterations):

$$
\begin{aligned}
\rho c v c (U^{n+1,r}, U^n) \frac{U^{n+1,r+1} - U^n}{\Delta t} &+ c \sigma_a (U^{n+1,r}) U^{n+1,r+1} = c \sigma_a E^{n+1,r+1}, \\
\frac{1}{c} E^{n+1,r+1} - E^n \frac{1}{\Delta t} - \frac{1}{3} \nabla \cdot \left( \frac{1}{\sigma_a(U^{n+1,r}) + \sigma_s(U^{n+1,r})} \nabla E^{n+1,r+1} \right) &+ \sigma_a (U^{n+1,r}) E^{n+1,r+1} \\
&= \sigma_a (U^{n+1,r}) U^{n+1,r+1}
\end{aligned}
$$

with:

$$
\alpha(U^{n+1,r}, U^n) = \frac{1}{(U^{n+1,r})^\frac{1}{2} + (U^n)^\frac{1}{2} + (U^{n+1,r})^\frac{1}{2} + (U^n)^\frac{1}{2}} \approx \frac{\partial}{\partial U} (U^\frac{1}{2}).
$$

The full space and time discretization of (2.22) then reads:

$$
\begin{aligned}
\rho p c v c (U^{n+1,r}, U^n) \frac{U^{n+1,r+1} - U^n}{\Delta t} &+ c \sigma_{ap} (U^{n+1,r}) U^{n+1,r+1} = c \sigma_{ap} E^{n+1,r+1}, \\
\rho p c v c (U^{n+1,r}, U^n) \frac{U^{n+1,r+1} - U^n}{\Delta t} &+ c \sigma_{ap} (U^{n+1,r}) U^{n+1,r+1} = c \sigma_{ap} E^{n+1,r+1}, \\
\frac{1}{c} E^{n+1,r+1} - E^n \frac{1}{\Delta t} &- \frac{1}{6} \sum_{i \in p \cap \Omega} \sigma_{ai}(U^{n+1,r}) + \sigma_{si}(U^{n+1,r}) |l_i| \left( (E^{n+1,r+1} - E^n + 1, r+1) \| n_i \|^2 + (E^{n+1,r+1} - E^n + 1, r+1) n_i \cdot n_i \right) \\
&+ \sigma_{ap} (U^{n+1,r}) E^{n+1,r+1} = \sigma_{ap} (U^{n+1,r}) U^{n+1,r+1}, \\
\frac{1}{c} E^{n+1,r+1} - E^n \frac{1}{\Delta t} &- \frac{1}{6} \sum_{i \in d \cap \Omega} \sigma_{ai}(U^{n+1,r}) + \sigma_{si}(U^{n+1,r}) |l_i| \left( (E^{n+1,r+1} - E^n + 1, r+1) n_i + (E^{n+1,r+1} - E^n + 1, r+1) \| n_i \|^2 \right) \\
&+ \sigma_{ad} (U^{n+1,r}) E^{n+1,r+1} = \sigma_{ad} (U^{n+1,r}) U^{n+1,r+1}.
\end{aligned}
$$

References


