Central Discontinuous Galerkin Methods for the Generalized Korteweg-de Vries Equation

Mengjiao Jiao\(^1\), Yingda Cheng\(^2\), Yong Liu\(^1\) and Mengping Zhang\(^1,\ast\)

\(^1\) School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.
\(^2\) Department of Mathematics, Department of Computational Mathematics, Science and Engineering, Michigan State University, East Lansing, MI 48824, USA.

Received 11 June 2019; Accepted (in revised version) 2 January 2020

Abstract. In this paper, we develop central discontinuous Galerkin (CDG) finite element methods for solving the generalized Korteweg-de Vries (KdV) equations in one dimension. Unlike traditional discontinuous Galerkin (DG) method, the CDG methods evolve two approximate solutions defined on overlapping cells and thus do not need numerical fluxes on the cell interfaces. Several CDG schemes are constructed, including the dissipative and non-dissipative versions. \(L^2\) error estimates are established for the linear and nonlinear equation using several projections for different parameter choices. Although we can not provide optimal \textit{a priori} error estimate, numerical examples show that our scheme attains the optimal \((k+1)\)-th order of accuracy when using piecewise \(k\)-th degree polynomials for many cases.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Kortewegde Vries equation, central DG method, stability, error estimates.

1 Introduction

In this paper, we develop central discontinuous Galerkin (CDG) finite element methods for solving the generalized Korteweg-de Vries (KdV) [23] equation

\[ u_t + f(u)_x + \sigma u_{xxx} = 0, \quad (1.1) \]

where \(\sigma\) is a given constant and \(f\) is a smooth function. We will focus on one-dimensional case in this paper, however, the numerical methodologies can be generalized to multi-dimensional case.

\ast Corresponding author. Email addresses: jmj123@mail.ustc.edu.cn (M. Jiao), ycheng@msu.edu (Y. Cheng), yong123@mail.ustc.edu.cn (Y. Liu), mpzhang@ustc.edu.cn (M. Zhang)

The KdV type equations and similar models that feature nonlinearity and dispersion arise as mathematical models for the propagation of physical waves in a wide variety of situations, e.g. [1, 2, 6, 7, 10, 20, 39]. They play an important role in applications, such as fluid mechanics [11, 37], nonlinear optics [3, 18], acoustics [38, 44], and plasma physics [41, 43]. They also have an enormous impact on the development of nonlinear mathematical science and theoretical physics. Due to their importance in applications and theoretical studies, various numerical methods have been proposed and used in practice to solve this type of equation, e.g. [8, 9, 21]. In many situations, such as in the quantum hydrodynamic models of semiconductor device simulations [17] and in the dispersive limit of conservation laws [25], the third order derivative terms might have small or even zero coefficients in some parts of the domain, which means the equation is “convection dominated”. The design of stable, efficient and high order methods, especially those for the convection dominated cases is a big challenge. Hence the discontinuous Galerkin (DG) method is a good choice because of its advantages in dealing with convection terms. The DG methods for KdV type equations have developed significantly in recent years. The first DG method, which is the local discontinuous Galerkin (LDG) method, for the KdV equation was introduced by Yan and Shu in [49]. The main idea of the LDG method is to introduce new auxiliary variables and rewrite the original equation into several first order equations. Later the LDG method was analyzed in [19, 29, 45, 46] and recently energy conserving LDG methods for wave equations are popular [12, 50]. In [14], a DG method, the so-called ultra-weak DG method [40] for the KdV equation was devised by using the ultra-weak formulation, which avoids the introduction of auxiliary variables and equations. Recently, several conservative DG methods [5, 13, 16, 22] were developed for KdV type equations to preserve quantities such as the mass and the $L^2$-norm of the solutions.

On the other hand, the central scheme of Nessyahu and Tadmor [36] computes hyperbolic conservation laws on a staggered mesh and avoids the Riemann solver. In [24], Kurganov and Tadmor introduce a new kind of central scheme without the large dissipation error related to the small time step size by using a variable control volume whose size depends on time step size. Liu [31] uses another coupling technique to avoid the excessive numerical dissipation for small time steps. The overlapping cells evolves two independent overlapping subcell averages for each given cell, which opens up many new possibilities. The advantages of overlapping cells motivates the combination of the central scheme and the DG method, which results in the CDG methods [30, 32, 33]. The CDG method evolves two copies of approximating solutions defined on staggered meshes and avoids using numerical fluxes which can be complicated and costly [26]. Like some previous central schemes, the CDG method also avoids the excessive numerical dissipation for small time steps by a suitable choice of the numerical dissipation term. Besides, the central method carry many features of standard DG methods, such as compact stencil, easy parallel implementation, etc. It is generally understood that the CDG method allows for a larger CFL number compared to the regular DG method [33]. Later in [34], the central local discontinuous Galerkin (CLDG) method was introduced to solve dif-
fusion equations, which is formulated based on the LDG scheme on overlapping cells. Recently, the CDG method has been used to solve systems of conservation laws in many applications [27, 28, 47, 48, 52].

The advantages of CDG method motivates us to apply it to generalized KdV equations. We use the ultra-weak formulation instead of the LDG formulation, which avoids auxiliary variables to save computational complexity and cost. Due to the use of overlapping cells, the CDG method in this paper does not need numerical fluxes. We are interested in the construction, analysis and numerical performance of the resulting methods. In particular, we study the stability and error estimates of the solution with several parameter choices related to numerical dissipation term. The $L^2$ error estimates are particularly challenging due to the interfacial terms involving derivatives of the solutions. We invoke the projection techniques introduced in [35] for the CDG method for hyperbolic problems, and another projection is suggested for even degree polynomials to further improve convergence order.

The rest of the paper is organized as follows. In Section 2, we introduce our central DG method for generalized KdV equations. In Section 3, we give the analysis of stability and error estimates for the scheme. We provide numerical examples to demonstrate the performance of the proposed method in Section 4. In Section 5, we give a few concluding remarks and perspectives for future work. Finally, in the appendix we provide proofs for some of the more technical results of the error estimates.

## 2 Numerical scheme

In this section, we describe CDG methods for the one-dimensional generalized KdV equation,

$$\begin{cases}
u_t + f(u)_x + \sigma u_{xxx} = 0, & x \in [a,b], \ t > 0, \\
u(x,0) = u_0(x), & x \in [a,b],
\end{cases}$$

(2.1)

with periodic boundary conditions or other types of boundary conditions on $[a,b]$, where $\sigma$ is a non-zero parameter. For a given interval $[a,b]$, we divide it into $N$ cells as follows:

$$a = x_0 < x_1 < \cdots < x_N = b.$$  

(2.2)

We denote

$$x_{j+1/2} = \frac{x_j + x_{j+1}}{2}, \quad I_{j+1/2} = (x_j, x_{j+1}), \quad h_{j+1/2} = x_{j+1} - x_j, \quad j = 0, \cdots, N-1,$$

(2.3)

and similarly for the dual mesh

$$I_j = (x_{j-1/2}, x_{j+1/2}), \quad h_j = x_{j+1/2} - x_{j-1/2}, \quad j = 1, \cdots, N-1.$$

(2.4)

For periodic boundary condition, the dual mesh can be extended to $I_0 = (x_{-1/2}, x_{1/2})$. For other boundary conditions, we will use two half cell at the boundaries $I_0 = (x_0, x_{1/2}), I_N = (x_{N-1/2}, x_N)$. 


(x_{N-1/2},x_N). We let h_{max}/h_{j+1/2} and assume the mesh is regular, i.e., max_{j}h_{j+1/2}/\min_{j}h_{j+1/2} is upper-bounded by a fixed constant during mesh refinements. We let \( V^k_h \) to be the set of piecewise polynomials of degree \( k \) over the subintervals \( \{ I_j \} \) with no continuity assumed across the subinterval boundaries. Likewise, \( W^k_h \) is the set of piecewise polynomials of degree \( k \) over the subintervals \( \{ I_{j+1/2} \} \) with no continuity assumed across the subinterval boundaries. For a function \( \varphi_h \in V^k_h \), we use \( (\varphi_h)_{j+1/2}^- \) or \( (\varphi_h)_{j+1/2}^+ \) to refer to the value of \( \varphi_h \) at \( x_{j+1/2} \) from the left cell \( I_j \) and the right cell \( I_{j+1} \). And for \( \psi_h \in W^k_h \), \( (\psi_h)_j^- \) or \( (\psi_h)_j^+ \) have similar meanings. \([\varphi_h]\) or \([\psi_h]\) are used to denote \( \varphi_h^+ - \varphi_h^- \) or \( \psi_h^+ - \psi_h^- \), i.e. the jump of \( \varphi_h \) or \( \psi_h \) at cell interfaces. For notations of different constants we follow [51] and [45]. We denote by \( C_a \) a positive constant independent of \( h \), which may depend on the solution of the problem and other parameters. To emphasize the nonlinearity of the flux \( f(u) \), we use \( C_a \) to denote a non-negative constant depending on the maximum of \( |f''| \). We remark \( C_a = 0 \) for linear fluxes \( f(u) = cu \) with a constant \( c \).

The first version of central DG method. The CDG method is defined on overlapping cells and uses both spaces \( V^k_h \) and \( W^k_h \). In addition, we require \( k \geq 2 \). This requirement is to guarantee consistency similar as the ultra-weak DG method in [14] for KdV equations, and is verified by our numerical experiments. We propose the following semi-discrete CDG scheme for periodic boundary condition based on the ultra-weak DG scheme [14]: find \( u_h(\cdot,t) \in V^k_h \) and \( v_h(\cdot,t) \in W^k_h \), such that for any \( \varphi_h \in V^k_h \) and \( \psi_h \in W^k_h \),

\[
\int_{I_j} (u_h)_t \varphi_h dx = \frac{1}{\tau_{max}} \int_{I_j} (v_h - u_h) \varphi_h dx + \int_{I_j} f(v_h)(\varphi_h)_x dx + \sigma \int_{I_j} v_h (\varphi_h)_{xx} dx \\
- (f'(v_h)(\varphi_h^+))_{j+1/2} + (f'(v_h)(\varphi_h^-))_{j-1/2} - \sigma(\varphi_h)_{j+1/2}^+ - \sigma(\varphi_h)_{j-1/2}^- + \sigma(\varphi_h)_{j+1/2}^- - \sigma(\varphi_h)_{j-1/2}^+ \\
- \sigma(\varphi_h)_{j+1/2}^- + \sigma(\varphi_h)_{j-1/2}^+, \tag{2.5a}
\]

\[
\int_{I_{j+1/2}} (v_h)_t \psi_h dx = \frac{1}{\tau_{max}} \int_{I_{j+1/2}} (u_h - v_h) \psi_h dx + \int_{I_{j+1/2}} f(u_h)(\psi_h)_x dx + \sigma \int_{I_{j+1/2}} u_h (\psi_h)_{xx} dx \\
- (f'(u_h)(\psi_h^+))_{j+1} + (f'(u_h)(\psi_h^-))_{j+1} - \sigma(u_h)_{j+1}^- - \sigma(u_h)_{j+1}^- + \sigma(u_h)_{j+1}^- - \sigma(u_h)_{j+1}^- \\
- \sigma(u_h)_{j+1}^- + \sigma(u_h)_{j+1}^+, \tag{2.5b}
\]

where \( \tau_{max} \) is a parameter to be discussed later. The formulation (2.5a) and (2.5b) can be derived by multiplying Eq. (1.1) with test functions and integrating it by parts repeatedly. The numerical dissipation term which involves the difference between the two duplicative solutions on different cells is firstly introduced in the central DG method in [31]. For non-periodic problems, the scheme can be defined in a similar spirit by replacing the values of the numerical solutions at the left and right end points \( x_0, x_N \) using the boundary
condition when available. An example which involves non-periodic boundary conditions will be shown in later numerical tests and we will see that the choice of $\tau_{\text{max}}$ is crucial for the stability of our numerical scheme.

In this paper, we investigate several choices of $\tau_{\text{max}}$. One can refer to [31, 32] for the origin of the parameter $\tau_{\text{max}}$. In the CDG scheme for conservation law [33] and the CLDG scheme for diffusion equations [34], $\tau_{\text{max}}$ was chosen to be $ch$ and $ch^2$, respectively, where $c$ is a constant independent of mesh size $h$. Here in scheme (2.5), we investigate $\tau_{\text{max}} = ch$, $ch^2$, $ch^3$, corresponding different magnitudes of numerical dissipation. Later in the numerical examples we will see that optimal $(k+1)$-th order can not be achieved if $\tau_{\text{max}} = ch$ or $ch^2$ in some cases, which means the choice of $\tau_{\text{max}}$ is crucial for the scheme.

The second version of central DG method. We also consider a non-dissipative version of the method by letting $\tau_{\text{max}} \to \infty$. This yields

$$\int_{I_j} (u_h)_t \phi_h dx = \int_{I_j} f(v_h) (\phi_h)_x dx + \sigma \int_{I_j} v_h (\phi_h)_{xxx} dx$$

$$- f(v_h) \phi_h^+ j+\frac{1}{2} + f(v_h) \phi_h^- j-\frac{1}{2} - \sigma v_h (\phi_h)_{xx} j+\frac{1}{2}$$

$$+ \sigma (v_h (\phi_h)_{xx}) j+\frac{1}{2} + \sigma (v_h (\phi_h)_x)_j+\frac{1}{2} - \sigma (v_h (\phi_h)_x ) j-\frac{1}{2}$$

$$- \sigma ((v_h)_x \phi_h^+) j+\frac{1}{2} + \sigma ((v_h)_x \phi_h^-) j-\frac{1}{2}.$$  \hspace{1cm} (2.6a)

$$\int_{I_{j+\frac{1}{2}}} (v_h)_t \psi_h dx = \int_{I_{j+\frac{1}{2}}} f(u_h) (\psi_h)_x dx + \sigma \int_{I_{j+\frac{1}{2}}} u_h (\psi_h)_{xxx} dx$$

$$- f(u_h) \psi_h^- j+1 + f(u_h) \psi_h^+ j - \sigma (u_h (\psi_h)_{xx}) j+1$$

$$+ \sigma (u_h (\psi_h)_{xx}) j+1 + \sigma ((u_h)_x (\psi_h)_x ) j+1 - \sigma ((u_h)_x (\psi_h)_x ) j+1$$

$$- \sigma ((u_h)_x \psi_h^+) j+1 + \sigma ((u_h)_x \psi_h^-) j.$$  \hspace{1cm} (2.6b)

Similar formulation without additional numerical dissipation has been considered in the CLDG method in [34], and conservative ultra-weak DG methods in [5] demonstrate some advantages for long time simulations. We will see that scheme (2.6) conserves the $L^2$ energy of the system, but it may results in sub-optimal convergence rate in numerical test. Both versions of the CDG methods on overlapping cells do not need a numerical flux to define the interface values of the solution, since the evaluation of the solution at the interface is in the middle of the staggered mesh, hence in the continuous region of the numerical solution. The semi-discrete schemes then are solved by chosen numerical ODE solvers and numerical initial condition, which is taken as the $L^2$ projection of the initial condition of the PDE onto the corresponding mesh in this paper.

3 Stability and error estimates

In this section, we will discuss $L^2$ stability of the semi-discrete schemes (2.5)-(2.6) for linear problems and error estimates for nonlinear problems. It is worth noting that the $L^2$
stability for CDG scheme for nonlinear problem is generally not available [33]. But under some assumptions we can still get the error estimate of the nonlinear case.

3.1 $L^2$ stability

Without loss of generality, we take $f(u) = u$ and $\sigma = 1$. Hence, we have

$$u_t + u_x + u_{xxx} = 0, \quad (x,t) \in [a,b] \times [0,T]$$

(3.1)

with periodic boundary condition. The analysis can be easily extended to compactly supported boundary condition.

**Theorem 3.1.** The numerical solutions $u_h$ and $v_h$ of the CDG scheme (2.5) for Eq. (3.1) have the following $L^2$ stability property

$$\frac{1}{2} \frac{d}{dt} \int_a^b (u_h^2 + v_h^2) dx = -\frac{1}{\tau_{\text{max}}} \int_a^b (v_h - u_h)^2 dx \leq 0. \quad (3.2)$$

The non-dissipative scheme (2.6) is energy conserving, i.e.

$$\frac{1}{2} \frac{d}{dt} \int_a^b (u_h^2 + v_h^2) dx = 0. \quad (3.3)$$

**Proof.** We only need to prove (3.2), and (3.3) follows immediately. Taking the test function $\varphi_h = u_h$ and $\psi_h = v_h$ in (2.5) respectively, summing up over $j$, observing $f(u) = u$ and the periodic (or compactly supported) boundary condition, we have

$$\frac{1}{2} \frac{d}{dt} \int_a^b ((u_h)^2 + (v_h)^2) dx$$

$$= \frac{1}{\tau_{\text{max}}} \int_a^b (v_h u_h - u_h^2 + u_h v_h - v_h^2) dx + \sum_j \left[ \int l_j (v_h (u_h)_x) dx + \int l_{j+\frac{1}{2}} (u_h (v_h)_x) dx 

+ \int l_j (v_h (u_h)_{xxx}) dx + \int l_{j+\frac{1}{2}} (u_h (v_h)_{xxx}) dx - (v_h u_h^+)_j + (v_h u_h^-)_{j+1} + (v_h v_h^+)_j - (v_h v_h^-)_{j+1} 

+ (u_h v_h^+)_j - (v_h (u_h)_{xx})_j + (v_h (u_h)_{xx})_{j+\frac{1}{2}} - (u_h (v_h)_{xx})_{j+\frac{1}{2}} + (u_h (v_h)_{xx})_{j+1} + (u_h (v_h)_{xx})_j 

+ ((v_h)_x (u_h)_{xx})_{j+\frac{1}{2}} - ((v_h)_x (u_h)_{xx})_{j+1} - ((v_h)_x (v_h)_{xx})_{j+\frac{1}{2}} - ((v_h)_x (v_h)_{xx})_{j+1} - ((v_h)_x (v_h)_{xx})_j 

- ((v_h)_xx (u_h)_x)_{j+\frac{1}{2}} - ((v_h)_xx (u_h)_x)_{j+1} + ((u_h)_xx (v_h)_x)_j + ((u_h)_xx (v_h)_x)_{j+1} + ((u_h)_xx (v_h)_x)_{j+2} \right]$$
we define and we are done.

\[ \int_{x_{j-1/2}}^{x_j} \left( v_h u_h^+ - \frac{1}{2} \phi_h \right) dx + \int_{x_{j+1/2}}^{x_j} \left( v_h u_h^- + \frac{1}{2} \phi_h \right) dx \]

\[ = -\frac{1}{\tau_{\text{max}}} \left[ I_j \right] u_h - v_h \right) \leq 0, \]

and we are done. \[ \square \]

### 3.2 \( L^2 \) error estimates

In this subsection, we show a priori error estimates of the scheme (2.5) and (2.6) for the nonlinear equation (2.1). Without loss of generality, we assume \( \sigma = 1 \) in following analysis. Due to the fact that the CDG scheme uses staggered meshes and different choices of \( \tau_{\text{max}} \), it’s challenging to obtain optimal error estimates for all cases. In the following theorem, we use three kind of projections with the aim of providing better error estimates corresponding to different parameter choices. Here and below, we use \( \| \cdot \| \) to denote the standard \( L^2 \) norm. For the proof, we recall the classical inverse and trace inequalities [15]. For any \( w_h \in V_h \) or \( u_h \in W_h \), there exists a positive constant \( C \) independent of \( w_h \) and \( h \), such that

\[ \| \partial_x w_h \| \leq Ch^{-1} \| w_h \|, \quad \| w_h \|_r \leq Ch^{-\frac{r}{2}} \| w_h \|, \quad \| w_h \|_{\infty} \leq Ch^{-\frac{1}{2}} \| w_h \|, \quad (3.4) \]

where \( \Gamma \) is the set of boundary points of all elements \( I_j \) or \( I_{j+\frac{1}{2}} \).

First we introduce some notations. For the numerical solutions \( u_h \) and \( v_h \) of the CDG scheme (2.5) for Eq. (2.1), we define

\[ \tilde{B}_j(u_h, v_h; \phi_h) = \frac{1}{\tau_{\text{max}}} \int_{I_j} (v_h - u_h) \phi_h dx + \int_{I_j} v_h (\phi_h)_{x=0} dx \]

\[ = -\frac{1}{\tau_{\text{max}}} \left[ I_j \right] u_h - v_h \right) \leq 0, \]

and we are done. \[ \square \]

### 3.2 \( L^2 \) error estimates

In this subsection, we show a priori error estimates of the scheme (2.5) and (2.6) for the nonlinear equation (2.1). Without loss of generality, we assume \( \sigma = 1 \) in following analysis. Due to the fact that the CDG scheme uses staggered meshes and different choices of \( \tau_{\text{max}} \), it’s challenging to obtain optimal error estimates for all cases. In the following theorem, we use three kind of projections with the aim of providing better error estimates corresponding to different parameter choices. Here and below, we use \( \| \cdot \| \) to denote the standard \( L^2 \) norm. For the proof, we recall the classical inverse and trace inequalities [15]. For any \( w_h \in V_h \) or \( u_h \in W_h \), there exists a positive constant \( C \) independent of \( w_h \) and \( h \), such that

\[ \| \partial_x w_h \| \leq Ch^{-1} \| w_h \|, \quad \| w_h \|_r \leq Ch^{-\frac{r}{2}} \| w_h \|, \quad \| w_h \|_{\infty} \leq Ch^{-\frac{1}{2}} \| w_h \|, \quad (3.4) \]

where \( \Gamma \) is the set of boundary points of all elements \( I_j \) or \( I_{j+\frac{1}{2}} \).

First we introduce some notations. For the numerical solutions \( u_h \) and \( v_h \) of the CDG scheme (2.5) for Eq. (2.1), we define

\[ \tilde{B}_j(u_h, v_h; \phi_h) = \frac{1}{\tau_{\text{max}}} \int_{I_j} (v_h - u_h) \phi_h dx + \int_{I_j} v_h (\phi_h)_{x=0} dx \]

\[ = -\frac{1}{\tau_{\text{max}}} \left[ I_j \right] u_h - v_h \right) \leq 0, \]

and we are done. \[ \square \]
Define\[\hat{B}_{j+rac{1}{2}}(u_h,v_h;\psi_h) = \frac{1}{\tau_{\text{max}}} \int_{l_{j+rac{1}{2}}} u_h(v_h - u_h)\psi_h dx + \int_{l_{j+rac{1}{2}}} u_h(\psi_h)_{xx} dx \]
\[- (u_h(\psi_h)_{xx})_{j+1} + (u_h(\psi_h)_{xx})_{j} + (u_h(x)\psi_h)_x)_{j+1} \]
\[- ((u_h)_x(x)\hat{\psi}^{-})_{j} - ((u_h)_{xx}\hat{\psi}^{-})_{j+1} + ((u_h)_{xx}\hat{\psi}^{+})_{j}, \quad (3.6)\]
and\[B_j(u_h,v_h;\varphi_h,\psi_h) = \int_{l_{j}} u_h(v_h)\varphi_h dx + \int_{l_{j+rac{1}{2}}} (v_h)_{j}\psi_h dx \]
\[- \hat{B}_j(u_h,v_h;\varphi_h) - \hat{B}_{j+rac{1}{2}}(u_h,v_h;\psi_h). \quad (3.7)\]
Obviously, we have\[B_j(u_h,v_h;\varphi_h,\psi_h) = \int_{l_{j}} f(v_h)\varphi_h dx + \int_{l_{j+rac{1}{2}}} f(u_h)\varphi_h dx - (f(v_h)\varphi_{h}^{-})_{j+\frac{1}{2}} \]
\[+ (f(v_h)\varphi_{h}^{+})_{j-\frac{1}{2}} - (f(u_h)\varphi_{h}^{-})_{j+1} + (f(u_h)\varphi_{h}^{+})_{j}, \quad \forall \varphi_h \in V^k_{h}, \; \psi_h \in W^k_{h}. \quad (3.8)\]
It is also clear that the exact solution \(u\) of (2.1) satisfies\[B_j(u,u;\varphi_h,\psi_h) = \int_{l_{j}} f(u)\varphi_h dx + \int_{l_{j+rac{1}{2}}} f(u)\varphi_h dx - (f(u)\varphi_{h}^{-})_{j+\frac{1}{2}} \]
\[+ (f(u)\varphi_{h}^{+})_{j-\frac{1}{2}} - (f(u)\varphi_{h}^{-})_{j+1} + (f(u)\varphi_{h}^{+})_{j}, \quad \forall \varphi_h \in V^k_{h}, \; \psi_h \in W^k_{h}. \quad (3.9)\]
Subtracting (3.8) from (3.9), we obtain the error equation\[B_j(u-u_h,u-v_h;\varphi_h,\psi_h) = \int_{l_{j}} (f(u) - f(v_h))\varphi_h dx + \int_{l_{j+rac{1}{2}}} (f(u) - f(u_h))\varphi_h dx \]
\[- (f(u) - f(v_h))\varphi_{h}^{-})_{j+\frac{1}{2}} + ((f(u) - f(v_h))\varphi_{h}^{+})_{j-\frac{1}{2}} \]
\[- ((f(u) - f(u_h))\varphi_{h}^{-})_{j+1} + ((f(u) - f(u_h))\varphi_{h}^{+})_{j}, \quad \forall \varphi_h \in V^k_{h}, \; \psi_h \in W^k_{h}. \quad (3.10)\]
Define\[H_j(f,u,u_h,v_h;\varphi_h,\psi_h) = \int_{l_{j}} (f(u) - f(v_h))\varphi_h dx + \int_{l_{j+rac{1}{2}}} (f(u) - f(u_h))\varphi_h dx \]
\[- ((f(u) - f(v_h))\varphi_{h}^{-})_{j+\frac{1}{2}} + ((f(u) - f(v_h))\varphi_{h}^{+})_{j-\frac{1}{2}} \]
\[- ((f(u) - f(u_h))\varphi_{h}^{-})_{j+1} + ((f(u) - f(u_h))\varphi_{h}^{+})_{j}. \quad (3.11)\]
Summing over all \(j\), the error equation becomes\[\sum_j B_j(u-u_h, u-v_h; \varphi_h, \psi_h) = \sum_j H_j(f;u,u_h,v_h;\varphi_h,\psi_h), \quad \forall \varphi_h \in V^k_{h}, \; \psi_h \in W^k_{h}. \quad (3.12)\]
3.2.1 Projection operators

To get error estimates for all cases, we will use the following three different types of projections.

**Type 1.** We introduce two special projections $P_h^*$ and $Q_h^*$ as follows. When $k \geq 3$, for a given function $w(x)$, we define $P_h^* w \in V_h^k$, such that $\forall j$,

$$
\int_{I_j} P_h^* w \varphi_h dx = \int_{I_j} w \varphi_h dx, \quad \forall \varphi_h \in P_h^{k-3}(I_j),
$$

$$
P_h^* w(x_j) = w(x_j),
$$

$$
(\tilde{P}_h w)_x(x_j) = w_x(x_j),
$$

$$
(P_h^* w)_{xx}(x_j) = w_{xx}(x_j).
$$

Similarly, we define $Q_h^* w \in W_h^k$, such that $\forall j$,

$$
\int_{I_{j+1/2}} Q_h^* w \psi_h dx = \int_{I_{j+1/2}} w \psi_h dx, \quad \forall \psi_h \in P_h^{k-3}(I_{j+1/2}),
$$

$$
Q_h^* w(x_{j+1/2}) = w(x_{j+1/2}),
$$

$$
(Q_h^* w)_x(x_{j+1/2}) = w_x(x_{j+1/2}),
$$

$$
(Q_h^* w)_{xx}(x_{j+1/2}) = w_{xx}(x_{j+1/2}).
$$

Here $P_h^{k-3}(I_j)$ and $P_h^{k-3}(I_{j+1/2})$ denote the spaces of polynomials of degree up to $k-3$ in the cell $I_j$ and $I_{j+1/2}$ respectively. Below, we discuss properties of the projections.

**Lemma 3.1.** When $k$ is an even number and $k \geq 3$, the projections $P_h^*$, $Q_h^*$ exist and are unique for any smooth function $w$, and the following estimates hold

$$
\| P_h^* w - w \| + h \| P_h^* w - w \|_\infty + h^{1/2} \| P_h^* w - w \|_\Gamma \leq C h^{k+1},
$$

$$
\| Q_h^* w - w \| + h \| Q_h^* w - w \|_\infty + h^{1/2} \| Q_h^* w - w \|_\Gamma \leq C h^{k+1},
$$

where $\Gamma$ denotes the set of boundary points of all elements $I_j$ or $I_{j+1/2}$ and $C$ is a constant that depends on $k$, $\| w \|_{H^{k+1}}$, but is independent of the mesh size $h$.

**Proof.** The proof of this lemma is contained in Appendix A.1. \qed

**Type 2.** Similar to [35], we define $P_h^*$ and $Q_h^*$ as the following projections onto $V_h^k$ and $W_h^k$ respectively on uniform meshes. That is, for a given function $w(x)$, we define $P_h^* w \in V_h^k$, such that $\forall j$,

$$
\int_{I_j} P_h^* w dx = \int_{I_j} w dx,
$$

$$
\tilde{P}_h (P_h^* w; \varphi_h)_j = \tilde{P}_h (w; \varphi_h)_j, \quad \forall \varphi_h \in P_h^k(I_j),
$$

$$
\tilde{P}_h (\tilde{P}_h w; \varphi_h)_j = \tilde{P}_h (w; \varphi_h)_j, \quad \forall \varphi_h \in P_h^k(I_j).
$$

The proof of this lemma is contained in Appendix A.1.
where $\tilde{P}_h(w;\varphi_h)_j$ is defined as follows
\[
\tilde{P}_h(w;\varphi_h)_j = \frac{1}{\tau_{\text{max}}} \left( \int_{x_j}^{x_{j+\frac{1}{2}}} w \left( x + \frac{h}{2} \right) \varphi_h dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w \left( x - \frac{h}{2} \right) \varphi_h dx - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w(x) \varphi_h dx \right)
+ \int_{x_j}^{x_{j+\frac{1}{2}}} w \left( x + \frac{h}{2} \right) (\varphi_h)_{xxx} dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w \left( x - \frac{h}{2} \right) (\varphi_h)_{xxx} dx
- w(x_j)((\varphi_h)_{xx}(x_{j+\frac{1}{2}}) - (\varphi_h)_{xx}(x_{j-\frac{1}{2} }))
+ w_x(x_j)((\varphi_h)_x(x_{j+\frac{1}{2}}) - (\varphi_h)_x(x_{j-\frac{1}{2} }))
- w_{xx}(x_j)((\varphi_h)_{x}(x_{j+\frac{1}{2}}) - (\varphi_h)_{x}(x_{j-\frac{1}{2} })).
\]

Similarly, we define $Q^*_h w \in W^k_h$, such that $\forall j$,
\[
\int_{I_{j+\frac{1}{2}}} Q^*_h w dx = \int_{I_{j+\frac{1}{2}}} w dx,
\]
\[
\tilde{Q}_h(Q^*_h w;\varphi_h)_{j+\frac{1}{2}} = \tilde{Q}_h(w;\varphi_h)_{j+\frac{1}{2}}, \quad \forall \psi_h \in P^k(I_{j+\frac{1}{2}}),
\]
where $\tilde{Q}_h(w;\varphi_h)_{j+\frac{1}{2}}$ is defined as follows
\[
\tilde{Q}_h(w;\varphi_h)_{j+\frac{1}{2}} = \frac{1}{\tau_{\text{max}}} \left( \int_{x_j}^{x_{j+\frac{1}{2}}} w \left( x + \frac{h}{2} \right) \varphi_h dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w \left( x - \frac{h}{2} \right) \varphi_h dx - \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w(x) \varphi_h dx \right)
+ \int_{x_j}^{x_{j+\frac{1}{2}}} w \left( x + \frac{h}{2} \right) (\varphi_h)_{xxx} dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w \left( x - \frac{h}{2} \right) (\varphi_h)_{xxx} dx
- w(x_{j+\frac{1}{2}})((\varphi_h)_{xx}(x_{j+1}) - (\varphi_h)_{xx}(x_{j-1}))
+ w_x(x_{j+\frac{1}{2}})((\varphi_h)_x(x_{j+1}) - (\varphi_h)_x(x_{j-1}))
- w_{xx}(x_{j+\frac{1}{2}})((\varphi_h)_{x}(x_{j+1}) - (\varphi_h)_{x}(x_{j-1})).
\]

Here $P^k(I_j)$ and $P^k(I_{j+\frac{1}{2}})$ denote the spaces of polynomials of degree up to $k$ in the cell $I_j$ and $I_{j+\frac{1}{2}}$ respectively. Note that $\tilde{P}_h(w;\varphi_h)_j = \tilde{Q}_h(w;\varphi_h)_{j+\frac{1}{2}} = 0$, $\forall w$ when $\varphi_h$ and $\psi_h$ are constants, so (3.16b) or (3.18b) alone misses one condition which is provided by (3.16a) or (3.18a). Next, we will discuss about the properties of the projections $P^*_h$ and $Q^*_h$.

**Lemma 3.2.** When $\tau_{\text{max}} = ch^3$, the projections $P^*_h$, $Q^*_h$ exist and are unique for any smooth function $w(x)$, and the following error estimates hold
\[
\|P^*_h w - w\| + h\|P^*_h w - w\|_{\infty} + h^{\frac{1}{2}}\|P^*_h w - w\|_{\Gamma} \leq C h^{k+1},
\]
\[
\|Q^*_h w - w\| + h\|Q^*_h w - w\|_{\infty} + h^{\frac{1}{2}}\|Q^*_h w - w\|_{\Gamma} \leq C h^{k+1},
\]
for all $k$, where $\Gamma$ denotes the set of boundary points of all elements $I_j$ or $I_{j+\frac{1}{2}}$ and $C$ is a constant that depends on $k$, $\|w\|_{H^{k+1}}$, but is independent of the mesh size $h$. 
Proof. The proof of this lemma is in Appendix A.2.

We note that the projections \( P_h^k \) and \( Q_h^k \) are not well defined for the dissipation free case \( \tau_{\text{max}} \to \infty \). For \( \tau_{\text{max}} = ch, ch^2 \), estimates (3.20) no longer hold because the scaling relation is broken.

**Proposition 3.1.** Assume that \( u \) is a \((k+1)\)-th degree polynomial function in \( P^{k+1}([a,b]) \). For a uniform partition on the interval \([a,b] \), set \( u_l = P_h^k u \in V_h^k \) and \( v_l = Q_h^k u \in W_h^k \), where \( P_h^k \) and \( Q_h^k \) are defined by (3.16) and (3.18). Then we have

\[
\tilde{B}_j(u_l,v_l;\varphi_h) = \tilde{B}_j(u,u;\varphi_h), \quad \forall \varphi_h \in P^k(I_j),
\]

\[
\tilde{B}_{j+\frac{1}{2}}(u_l,v_l;\psi_h) = \tilde{B}_{j+\frac{1}{2}}(u,u;\psi_h), \quad \forall \psi_h \in P^k(I_{j+\frac{1}{2}}).
\]

(3.21)

Proof. The proof of this proposition is in the Appendix A.3.

**Type 3.** Define \( P \) and \( Q \) as the standard \( L^2 \) projection of a function onto \( V_h^k \) and \( W_h^k \) respectively.


\[
\|Pw - w\| + h\|Pw - w\|_\infty + h^{\frac{k}{2}}\|Pw - w\|_\Gamma \leq Ch^{k+1},
\]

(3.22)

\[
\|Qw - w\| + h\|Qw - w\|_\infty + h^{\frac{k}{2}}\|Qw - w\|_\Gamma \leq Ch^{k+1},
\]

(3.23)

where \( \Gamma \) denotes the set of boundary points of all elements \( I_j \) or \( I_{j+\frac{1}{2}} \) and \( C \) is a constant that depends on \( k, \|w\|_{L^{k+1}} \), but is independent of the mesh size \( h \).

### 3.2.2 Error estimates of the CDG schemes

**Theorem 3.2.** Let \( u(\cdot,t) \) be the exact solution of Eq. (2.1), which is sufficiently smooth with bounded derivatives, and assume \( f \in C^2 \). The numerical solutions \( u_h \) and \( v_h \) of the CDG scheme (2.5) with \( \tau_{\text{max}} = ch^\mu, \mu = 1,2,3 \) and the scheme (2.6) for Eq. (2.1) satisfies the following \( L^2 \) error estimate

\[
\|u(\cdot,T) - u_h(\cdot,T)\|^2 + \|u(\cdot,T) - v_h(\cdot,T)\|^2 \leq C h^{2k+\alpha},
\]

(3.24)

where \( k \) is the polynomial degree in the finite element spaces \( V_h^k \) and \( W_h^k \) with \( k \geq 2 \), and the constant \( C \) depends on \( k \), the final time \( T \), the exact solution \( u \) and the bounds on the derivatives \( |f^{(m)}| \), \( m = 1,2 \), but is independent of the mesh size \( h \) and the numerical solution \( u_h, v_h \). For different choices of \( k \) and \( \tau_{\text{max}} \), we have

**Case 1.** If \( k \geq 3 \) and \( k \) is even, for scheme (2.5) with all choices of \( \tau_{\text{max}} \), (3.24) holds with \( \alpha = \mu - 4, \mu = 1,2,3 \) on nonuniform meshes.

**Case 2.** If \( k \geq 3 \) and \( k \) is odd, or \( k = 2 \), for scheme (2.5) with \( \tau_{\text{max}} = ch^3 \), then (3.24) holds with \( \alpha = -2 \) on uniform meshes. Here for \( k = 2 \), we require the convection term to be linear, i.e. \( f(u) = cu \).
Case 3. For all other choices of \( k \) and \( \tau_{\text{max}} \), (3.24) holds for scheme (2.5) and (2.6) with \( \alpha = -4 \) on nonuniform meshes. Here for \( k = 2, 3 \), we also require the convection term to be linear, i.e. \( f(u) = cu \).

Before the proof of Theorem 3.2, we need first prove a lemma to estimate the nonlinear part.

Let \( e_u = u - u_h, e_v = u - v_h \) be the error between the numerical and exact solutions. By taking
\[
\varphi_h = P_h u - u_h, \quad \psi_h = Q_h u - v_h, \quad \varphi^e = P_h u - u, \quad \psi^e = Q_h u - u,
\]
(3.25)
where \( P_h = P_h^*, P \) and \( Q_h = Q_h^*, Q \), we obtain the energy equality
\[
\sum_j B_j (\varphi_h - \varphi^e, \psi_h - \psi^e; \varphi_h, \psi_h) = \sum_j H_j (f; u, u_h, v_h; \varphi_h, \psi_h).
\]
(3.26)

We can rewrite the right hand side of (3.26) into the following form:
\[
\sum_j H_j (f; u, u_h, v_h; \varphi_h, \psi_h) \\
= \sum_j \int_{I_j} (f(u) - f(v_h))(\varphi_h)_{x} dx + \sum_j ((f(u) - f(v_h))(\varphi_h))_{j+\frac{1}{2}} \\
+ \sum_j \int_{I_j, \frac{1}{2}} (f(u) - f(u_h))(\psi_h)_{x} dx + \sum_j ((f(u) - f(u_h))(\psi_h))_{j}.
\]
(3.27)

And we have the following lemma:

Lemma 3.3. Suppose \( \varphi_h, \psi_h, \varphi^e, \psi^e \) are taken as forms in (3.25), then following inequality holds:
\[
\sum_j H_j (f; u, u_h, v_h; \varphi_h, \psi_h) \leq (C + C_* (h^{-1} \| e_u \|_\infty + h^{-1} \| e_v \|_\infty)) (\| \varphi_h \|^2 + \| \psi_h \|^2) \\
+ (C + C_* (h \| e_u \|_\infty + h \| e_v \|_\infty)) h^{2k},
\]
(3.28)
where \( C, C_* \) are constants which are independent of the mesh size \( h \) and the numerical solution \( u_h, v_h \).

Proof. The proof of this lemma is in Appendix A.4.

We are now ready to prove Theorem 3.2.

Proof. To deal with the nonlinearity of \( f(u) \), we would like first make a priori assumption that, for small enough \( h \), we have
\[
\| u - u_h \| \leq h^\frac{1}{2}, \quad \| u - v_h \| \leq h^\frac{1}{2},
\]
(3.29)
and then by interpolation property,
\[ \|e_u\|_\infty \leq Ch, \quad \|P_h u - u_h\|_\infty \leq Ch, \]
\[ \|e_v\|_\infty \leq Ch, \quad \|Q_h u - u_h\|_\infty \leq Ch, \]
(3.30)

where \( P_h = P_h^*, P, Q_h = Q_h^*, Q \). This assumption is not necessary for linear \( f \).
Later, we will justify this assumption for Case 1 with \( k \geq 3 \) and \( k \) is even, for Case 2 with \( k \geq 3 \) and \( k \) is odd, and for Case 3 with \( k \geq 4 \).

**Case 1.** Now we take
\[ \varphi_h = P_h^* u - u_h, \quad \psi_h = Q_h^* u - v_h \]
(3.31)
and
\[ \varphi^e = P_h^* u - u, \quad \psi^e = Q_h^* u - u. \]
(3.32)

From the definition of \( B_j \), we can obtain
\[ B_j(\varphi_h - \varphi^e, \psi_h - \psi^e; \varphi_h, \psi_h) = B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) - B_j(\varphi^e, \psi^e; \varphi_h, \psi_h). \]
(3.33)

By a similar argument as that in the stability proof, we have that for scheme (2.5),
\[ \sum_j B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2} \frac{d}{dt} \int_a^b (\varphi_h^2 + \varphi_h^2)dx + \frac{1}{\tau_{\text{max}}} \int_a^b (\psi_h - \psi_h)^2dx. \]
(3.34)

By using the definitions of the special projections (3.13) and (3.14), we have
\[ B_j(\varphi^e, \psi^e; \varphi_h, \psi_h) = \int_{l_j} (\varphi^e)_j \varphi_h dx + \int_{l_j + \frac{1}{2}} (\varphi^e)_{\psi_h} dx 
- \frac{1}{\tau_{\text{max}}} \int_{l_j} (\psi^e - \varphi^e) \varphi_h dx - \frac{1}{\tau_{\text{max}}} \int_{l_j + \frac{1}{2}} (\varphi^e - \psi^e) \psi_h dx
- \int_{l_j} \psi^e(\varphi_h)_{xxx} dx - \int_{l_j + \frac{1}{2}} \varphi^e(\psi_h)_{xxx} dx
\]
\[ \pm B_j^1 + B_j^2 + B_j^3, \]
(3.35)

where
\[ B_j^1 = \int_{l_j} (\varphi^e)_j \varphi_h dx + \int_{l_j + \frac{1}{2}} (\varphi^e)_{\psi_h} dx, \]
\[ B_j^2 = -\frac{1}{\tau_{\text{max}}} \int_{l_j} (\psi^e - \varphi^e) \varphi_h dx - \frac{1}{\tau_{\text{max}}} \int_{l_j + \frac{1}{2}} (\varphi^e - \psi^e) \psi_h dx, \]
(3.36)
\[ B_j^3 = -\int_{l_j} \psi^e(\varphi_h)_{xxx} dx - \int_{l_j + \frac{1}{2}} \varphi^e(\psi_h)_{xxx} dx. \]

By (3.15), we have
\[ \sum_j B_j^1 \leq \int_a^b (\varphi_h)^2dx + \int_a^b (\psi_h)^2dx + Ch^{2k+2}, \]
(3.37)
where $C$ is a constant that depends on $\|u\|_{H^{k+4}}$. Since
\begin{equation}
B_j^3 = -\int_{t_j}^{t_{j+\frac{1}{2}}} (\psi^e - \psi^f) (\varphi_h)_{xxx} dx - \int_{t_{j+\frac{1}{2}}}^{t_j} (\psi^e - \psi^f) (\psi_h)_{xxx} dx,
\end{equation}
(3.38)
therefore,
\begin{equation}
\sum_j \left( B_j^2 + B_j^3 \right) \leq \frac{C}{\tau_{\text{max}}} \|\psi^e - \psi^f\| \|\varphi_h - \psi_h\| + \frac{C}{h^3} \|\psi^e - \psi^f\| \|\varphi_h - \psi_h\|
\end{equation}
\begin{equation}
\leq C \tau^{-2} \|\varphi_h - \psi_h\|.
\end{equation}
(3.39)
Thus we have
\begin{equation}
\sum_j \left( B_j^1 + B_j^2 + B_j^3 \right) \leq \int_a^b (\varphi_h^2 + \psi_h^2) dx + Ch^{2k+2} + Ch^{-k-2} \|\varphi_h - \psi_h\|.
\end{equation}
(3.40)
Now, combining (3.26), (3.33), (3.34), (3.40) and (3.28), we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_a^b (\varphi_h^2 + \psi_h^2) dx + \frac{1}{\tau_{\text{max}}} \int_a^b (\varphi_h - \psi_h)^2 dx
\end{equation}
\begin{equation}
\leq (C + C_s (h^{-1} e_u \|e_u\|_{\infty} + h^{-1} e_v \|e_v\|_{\infty})) (\|\varphi_h\|^2 + \|\psi_h\|^2)
\end{equation}
\begin{equation}
+ (C + C_s (h \|e_u\|_{\infty} + h \|e_v\|_{\infty})) h^{2k} + Ch^{2k+2} + Ch^{-k-2} \|\varphi_h - \psi_h\|.
\end{equation}
(3.41)
Since $\tau_{\text{max}} = ch^\mu$, $\mu = 1, 2, 3$, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_a^b (\varphi_h^2 + \psi_h^2) dx + \frac{1}{h^\mu} \left( \|\varphi_h - \psi_h\| - \frac{C}{2} h^{k-2+\mu} \right)^2
\end{equation}
\begin{equation}
\leq (C + C_s (h^{-1} e_u \|e_u\|_{\infty} + h^{-1} e_v \|e_v\|_{\infty})) (\|\varphi_h\|^2 + \|\psi_h\|^2)
\end{equation}
\begin{equation}
+ (C + C_s (h \|e_u\|_{\infty} + h \|e_v\|_{\infty})) h^{2k} + Ch^{2k+2} + \frac{C^2}{4} h^{2k-4+\mu}.
\end{equation}
(3.42)
Finally, by using a priori assumption (3.29), (3.30) and Gronwall’s inequality we can get
\begin{equation}
\int_a^b (\varphi_h^2 + \psi_h^2) dx \leq Ch^{2k-4+\mu}.
\end{equation}
(3.43)
Combined with the approximation result (3.15), we reach the desired error estimate.

**Case 2.** The proof follows similar lines by taking
\begin{equation}
\varphi_h = P_h^s u - u_h, \quad \psi_h = Q_h^s u - v_h, \quad \varphi^e = P_h^s u - u, \quad \psi^e = Q_h^s u - u,
\end{equation}
(3.44)
and considering
\begin{equation}
B_j(\varphi_h - \psi^e; \psi_h) = B_j(\varphi_h; \psi_h) - B_j(\varphi^e; \psi_h).
\end{equation}
(3.45)
Similarly, we have
\[ \sum_j B_j(f_h, g_h; f_h, g_h) = \frac{1}{2} \frac{d}{dt} \int_a^b (q_h^2 + r^2_h) \, dx + \frac{1}{\tau_{\text{max}}} \int_a^b (q_h - \psi_h)^2 \, dx. \]  
(3.46)

The second term in the right-hand side of (3.45) can be written as a sum of three terms
\[ B_j(f^e, g^e; f_h, g_h) = B^1_j + B^2_j + B^3_j. \]  
(3.47)

where
\[ B^1_j = -\tilde{B}_j(f^e, g^e; f_h), \]
\[ B^2_j = -\tilde{B}_{j+\frac{1}{2}}(f^e, g^e; f_h), \]
\[ B^3_j = \int_{j-\frac{1}{2}}^j \partial_t f^e \, \psi_h \, dx + \int_{j+1}^j \partial_t g^e \, \psi_h \, dx. \]

From (3.20), we have
\[ \sum_j B_j^3 \leq \int_a^b (q_h)^2 \, dx + \int_a^b (\psi_h)^2 \, dx + C h^{2k+2}, \]  
(3.48)

where C is a constant that depends on $\|u\|_{H^{k+1}}$. For $B^1_j$, let $\tilde{u}_j$ be the Taylor polynomial of order $k+1$ of $u$ near $x_j$, i.e., $\tilde{u}_j = \sum_{i=0}^{k+1} \frac{1}{i!} u^{(i)}(x_j) (x - x_j)^i$, $x \in (x_{j-1}, x_{j+1})$. Let $a_j$ denote the residual term i.e. $a_j = u - \tilde{u}_j$. Recalling the Bramble-Hilbert lemma [15], we have
\[ ||r^i_j||_{L^\infty(I_j)} \leq C h^{k+\frac{3}{2}} |\partial^i u|_{H^{k+2}(I_j)}. \]  
(3.49)

Then we rewrite $f^e$ and $g^e$
\[ f^e = \mathcal{P}^*_h u - u = \mathcal{P}^*_h \tilde{u}_j - \tilde{u}_j + \mathcal{P}^*_h a_j - a_j, \]
\[ g^e = \mathcal{Q}^*_h u - u = \mathcal{Q}^*_h \tilde{u}_j - \tilde{u}_j + \mathcal{Q}^*_h a_j - a_j. \]  
(3.50)

Hence, using Proposition 3.1, we have
\[ B^1_j = -\tilde{B}_j(f^e, g^e; f_h) \]
\[ = -\tilde{B}_j(\mathcal{P}^*_h \tilde{u}_j - \tilde{u}_j + \mathcal{P}^*_h a_j - a_j; \mathcal{P}^*_h \tilde{u}_j - \tilde{u}_j + \mathcal{P}^*_h a_j - a_j; f_h) \]
\[ = -\tilde{B}_j(\mathcal{P}^*_h \tilde{u}_j - \tilde{u}_j, \mathcal{Q}^*_h \tilde{u}_j - \tilde{u}_j; f_h) - \tilde{B}_j(\mathcal{P}^*_h a_j - a_j, \mathcal{Q}^*_h a_j - a_j; f_h) \]
\[ = -\tilde{B}_j(\mathcal{P}^*_h a_j - a_j, \mathcal{Q}^*_h a_j - a_j; f_h). \]  
(3.51)

Therefore, by (3.20), (3.49) and Young’s inequality, we have
\[ \sum_j B^1_j \leq C h^{2k-2} + C \int_a^b \phi^2_h \, dx. \]  
(3.52)
Similarly, for \( B_j^2 \) we have
\[
\sum_j B_j^2 \leq Ch^{2k-2} + C \int_a^b \psi_h^2 dx. \tag{3.53}
\]

When \( k \geq 3 \), combining (3.28), (3.45), (3.46), (3.47), (3.48), (3.52) and (3.53), we obtain from (3.26)
\[
\frac{1}{2} \frac{d}{dt} \int_I (\phi_h^2 + \psi_h^2) dx + \frac{1}{t_{\max}} \int_I (\phi_h - \psi_h)^2 dx \\
\leq (C + C_s(h^{-1} ||e_u||_{\infty} + h^{-1} ||e_v||_{\infty}))(||\phi_h||^2 + ||\psi_h||^2) \\
+ (C + C_s(h ||e_u||_{\infty} + h ||e_v||_{\infty}))h^{2k} + Ch^{2k-2}. \tag{3.54}
\]

Finally, by using \textit{a priori} assumption (3.29), (3.30) and Gronwall’s inequality we can get
\[
\int_a^b (\phi_h^2 + \psi_h^2) dx \leq Ch^{2k-2}. \tag{3.55}
\]

For the case of \( k = 2 \), we assume that the convection term is linear, namely \( f(u) = cu \). This is to avoid the need of the \textit{a priori} assumption (3.29) which is no longer justifiable since our \( L^2 \) error estimate is only of order \( \mathcal{O}(h) \) in this case. The proof is similar to that for \( k \geq 3 \) case given above. By similar lines of proof, we have
\[
\frac{1}{2} \frac{d}{dt} \int_I (\phi_h^2 + \psi_h^2) dx + \frac{1}{t_{\max}} \int_I (\phi_h - \psi_h)^2 dx \\
\leq C(||\phi_h||^2 + ||\psi_h||^2) + Ch^2. \tag{3.56}
\]

And by Gronwall’s inequality we can get
\[
\int_a^b (\phi_h^2 + \psi_h^2) dx \leq Ch^2. \tag{3.57}
\]

Hence, (3.55) and (3.57) together with the approximation results (3.20), implies the desired error estimate.

**Case 3.** Following the same lines of proof, we take
\[
\phi_h = Pu - u_h, \quad \psi_h = Qu - v_h, \tag{3.58}
\]
in the error equation (3.26), and denote
\[
\phi^e = Pu - u, \quad \psi^e = Qu - u \tag{3.59}
\]
to obtain
\[
B_j(\phi_h - \phi^e, \psi_h - \psi^e; \phi_h, \psi_h) = B_j(\phi_h, \psi_h; \phi_h, \psi_h) - B_j(\phi^e, \psi^e; \phi_h, \psi_h). \tag{3.60}
\]
Similarly we have
\[ \sum_j B_j (\varphi_h; \varphi_h; \varphi_h; \varphi_h) = \frac{1}{2} \frac{d}{dt} \int_a^b (\varphi_h^2 + \varphi_h^2) dx + \frac{1}{t_{\text{max}}} \int_a^b (\varphi_h - \varphi_h)^2 dx \]  
(3.61)
and
\[ \sum_j B_j (\varphi^2; \varphi^2; \varphi_h; \varphi_h) \leq \int_I (\varphi_h) + (\varphi_h) dx + Ch^{2k-4}. \]  
(3.62)

Therefore, when \( k \geq 4 \) combine (3.28), (3.60), (3.61) and (3.62), we obtain from (3.26)
\[ \frac{1}{2} \frac{d}{dt} \int_I (\varphi_h^2 + \varphi_h^2) dx + \frac{1}{t_{\text{max}}} \int_I (\varphi_h - \varphi_h)^2 dx \]
\[ \leq (C + C_s(h^{-1} \| e_u \|_\infty + h^{-1} \| e_v \|_\infty)) (\| \varphi_h \|^2 + \| \varphi_h \|^2) \]
\[ + (C + C_s(h\| e_u \|_\infty + h\| e_v \|_\infty)) h^{2k} + Ch^{2k-4}. \]  
(3.63)

Finally, by using an a priori assumption (3.29), (3.30) and Gronwall’s inequality we can get
\[ \int_I (\varphi_h^2 + \varphi_h^2) dx \leq Ch^{2k-4}. \]  
(3.64)

For the case of \( k = 2, 3 \), we also assume that the convection term is linear, namely \( f(u) = cu \). This is to avoid the need of the a priori assumption (3.29) which is no longer justifiable since our \( L^2 \) error estimate is only of order \( O(1) \) or \( O(h) \) in this case. The proof is similar to that for \( k \geq 4 \) case given above. By similar lines of proof, we have
\[ \frac{1}{2} \frac{d}{dt} \int_I (\varphi_h^2 + \varphi_h^2) dx + \frac{1}{t_{\text{max}}} \int_I (\varphi_h - \varphi_h)^2 dx \]
\[ \leq C(\| \varphi_h \|^2 + \| \varphi_h \|^2) + Ch^{2k-4}, \quad k = 2, 3. \]  
(3.65)

And by Gronwall’s inequality we can get
\[ \int_a^b (\varphi_h^2 + \varphi_h^2) dx \leq Ch^{2k-4}, \quad k = 2, 3. \]  
(3.66)

By combining (3.64), (3.66) and the approximation result (3.22)-(3.23), we can get the desired error estimate. Also it is easy to check that the derivation above works for the dissipation free scheme (2.6).

To complete the proof of this theorem for nonlinear flux \( f(u) \), we need to verify the a priori assumption (3.29) for Case 1 with \( k \geq 3 \) and \( k \) is even, for Case 2 with \( k \geq 3 \) and \( k \) is odd, and for Case 3 with \( k \geq 4 \). Similar to [51] and [14], we can verify this by a proof by contradiction. By (3.24), we can consider \( h \) small enough so that \( Ch^{k+a} < \frac{1}{2} h^2 \), where \( C \) is the constant in (3.24) determined by the final time \( T \). Define \( t^* = \sup \{ t : \| u(\cdot, t) - u_h(\cdot, t) \| + \| u(\cdot, t) - v_h(\cdot, t) \| \leq h^2 \} \), then we have \( \| u(\cdot, t^*) - u_h(\cdot, t^*) \| + \| u(\cdot, t^*) - v_h(\cdot, t^*) \| = h^2 \) by continuity if \( t^* \) is finite. Clearly, (3.24) holds for \( t \leq t^* \), in particular, \( \| u(\cdot, t^*) - u_h(\cdot, t^*) \| + \| u(\cdot, t^*) - v_h(\cdot, t^*) \| \leq Ch^{k+a} < \frac{1}{2} h^2 \). This a contradiction if \( t^* < T \). Hence \( t^* \geq T \) and our a priori assumption is justified. \( \square \)
Remark 3.1. When $k < 2$, numerical experiments in the following subsection show that our scheme is not consistent for some $\tau_{\text{max}}$.

4 Numerical examples

In this section, we present numerical examples to demonstrate the performance of our scheme. Uniform meshes are used in all examples unless otherwise stated.

Example 4.1. We solve the linear third order equation given by

$$
\begin{align*}
    u_t + u_{xxx} &= 0, \\
    u(x,0) &= \sin(x), \\
    u(0,t) &= u(2\pi,t). 
\end{align*}
$$

(4.1)

Table 1: Errors and numerical orders of accuracy for Example 4.1 when using $p^k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\text{max}} = \text{CFL} \cdot h$ and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1.89E-000</td>
<td>-</td>
<td>8.37E-001</td>
<td>-</td>
<td>4.79E-001</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>4.01E-001</td>
<td>2.24</td>
<td>1.76E-001</td>
<td>2.24</td>
<td>1.04E-001</td>
<td>2.21</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.36E-001</td>
<td>0.77</td>
<td>1.05E-002</td>
<td>0.77</td>
<td>6.11E-002</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.90E-001</td>
<td>-0.73</td>
<td>1.73E-001</td>
<td>-0.72</td>
<td>1.00E-001</td>
<td>-0.72</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>5.68E-003</td>
<td>-</td>
<td>2.59E-003</td>
<td>-</td>
<td>3.18E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5.38E-004</td>
<td>3.40</td>
<td>2.96E-004</td>
<td>3.13</td>
<td>5.62E-004</td>
<td>2.50</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>8.51E-005</td>
<td>2.66</td>
<td>5.16E-005</td>
<td>2.52</td>
<td>9.20E-005</td>
<td>2.61</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.04E-005</td>
<td>3.03</td>
<td>6.77E-006</td>
<td>2.93</td>
<td>1.07E-005</td>
<td>3.10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.27E-004</td>
<td>-</td>
<td>7.31E-005</td>
<td>-</td>
<td>1.65E-004</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.87E-005</td>
<td>2.75</td>
<td>1.20E-005</td>
<td>2.61</td>
<td>2.10E-005</td>
<td>2.98</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.15E-006</td>
<td>2.17</td>
<td>2.69E-006</td>
<td>2.15</td>
<td>3.96E-006</td>
<td>2.40</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>9.25E-007</td>
<td>2.16</td>
<td>6.14E-007</td>
<td>2.13</td>
<td>9.04E-007</td>
<td>2.13</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>7.24E-006</td>
<td>-</td>
<td>4.69E-006</td>
<td>-</td>
<td>1.28E-005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.17E-007</td>
<td>5.06</td>
<td>1.77E-007</td>
<td>4.73</td>
<td>4.33E-007</td>
<td>4.88</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.51E-009</td>
<td>5.06</td>
<td>5.78E-009</td>
<td>4.94</td>
<td>1.21E-008</td>
<td>5.16</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.05E-010</td>
<td>4.99</td>
<td>1.82E-010</td>
<td>4.99</td>
<td>4.00E-010</td>
<td>4.92</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.46E-007</td>
<td>-</td>
<td>1.11E-007</td>
<td>-</td>
<td>3.32E-007</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>4.87E-009</td>
<td>4.91</td>
<td>4.24E-009</td>
<td>4.72</td>
<td>1.18E-008</td>
<td>4.81</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.58E-010</td>
<td>4.24</td>
<td>2.30E-010</td>
<td>4.20</td>
<td>5.56E-010</td>
<td>4.41</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.36E-011</td>
<td>4.25</td>
<td>1.21E-011</td>
<td>4.25</td>
<td>2.76E-011</td>
<td>4.33</td>
</tr>
</tbody>
</table>
Table 2: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\text{max}} = CFL \cdot h^2$ and final time $T=1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2.54E-000</td>
<td>-</td>
<td>1.12E-000</td>
<td>-</td>
<td>6.39E-001</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>9.25E-001</td>
<td>1.45</td>
<td>4.10E-001</td>
<td>1.45</td>
<td>2.33E-001</td>
<td>1.45</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.78E-001</td>
<td>1.73</td>
<td>1.23E-001</td>
<td>1.73</td>
<td>7.05E-002</td>
<td>1.73</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>9.51E-002</td>
<td>1.55</td>
<td>4.21E-002</td>
<td>1.55</td>
<td>2.40E-001</td>
<td>1.55</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7.71E-003</td>
<td>-</td>
<td>3.45E-003</td>
<td>-</td>
<td>3.05E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.28E-004</td>
<td>3.62</td>
<td>2.94E-004</td>
<td>3.55</td>
<td>4.14E-004</td>
<td>2.88</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.58E-005</td>
<td>3.26</td>
<td>3.38E-005</td>
<td>3.12</td>
<td>6.05E-005</td>
<td>2.77</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>8.80E-006</td>
<td>2.90</td>
<td>4.96E-006</td>
<td>2.77</td>
<td>9.55E-005</td>
<td>2.66</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.12E-004</td>
<td>-</td>
<td>6.27E-005</td>
<td>-</td>
<td>1.39E-004</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.59E-006</td>
<td>3.70</td>
<td>5.09E-006</td>
<td>3.62</td>
<td>1.15E-005</td>
<td>3.60</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>8.24E-007</td>
<td>3.38</td>
<td>5.12E-007</td>
<td>3.31</td>
<td>1.00E-006</td>
<td>3.52</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>8.90E-008</td>
<td>3.21</td>
<td>5.75E-008</td>
<td>3.15</td>
<td>9.58E-008</td>
<td>3.39</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>7.09E-006</td>
<td>-</td>
<td>4.21E-006</td>
<td>-</td>
<td>1.08E-005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.44E-007</td>
<td>4.86</td>
<td>1.62E-007</td>
<td>4.70</td>
<td>4.58E-007</td>
<td>4.56</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.27E-009</td>
<td>5.07</td>
<td>5.42E-009</td>
<td>4.90</td>
<td>1.43E-008</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.05E-010</td>
<td>5.15</td>
<td>1.76E-010</td>
<td>4.95</td>
<td>4.16E-010</td>
<td>5.10</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.42E-007</td>
<td>-</td>
<td>9.77E-008</td>
<td>-</td>
<td>2.64E-007</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.40E-009</td>
<td>5.89</td>
<td>1.92E-009</td>
<td>5.67</td>
<td>5.89E-009</td>
<td>5.49</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>5.42E-011</td>
<td>5.47</td>
<td>4.61E-011</td>
<td>5.38</td>
<td>1.37E-010</td>
<td>5.43</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.46E-012</td>
<td>5.22</td>
<td>1.28E-012</td>
<td>5.17</td>
<td>3.48E-012</td>
<td>5.30</td>
</tr>
</tbody>
</table>

The exact solution is

$$u(x,t) = \sin(x+t).$$  \hfill (4.2) 

We use the third order TVD Runge-Kutta time discretization with the time step $\Delta t = CFL \cdot h^3$. In this example, we choose $CFL = 0.001$. We use $\tau_{\text{max}} = CFL \cdot h$, $CFL \cdot h^2$, $CFL \cdot h^3$ to test the numerical schemes. The errors and numerical order of accuracy for $P^k$ elements with $1 \leq k \leq 5$ are listed in Tables 1-3. We observe that for $P^1$ polynomials, our scheme is not consistent when $\tau_{\text{max}} = CFL \cdot h$, and convergent but not optimally when $\tau_{\text{max}} = CFL \cdot h^2$, $\tau_{\text{max}} = CFL \cdot h^3$. For $k \geq 2$, the scheme is optimally convergent when $\tau_{\text{max}} = CFL \cdot h^3$, and $\tau_{\text{max}} = CFL \cdot h^2$ with even $k$.

For the non-dissipative case, the results are listed in Table 4. Similar to the case with $\tau_{\text{max}} = CFL \cdot h$, the CDG scheme without dissipative term is not consistent when $k=1$ and will also lose accuracy when $k$ is odd.
Table 3: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\max} = CFL \cdot h^3$ and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error order</th>
<th>$L^2$ error order</th>
<th>$L^\infty$ error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3.18E-000 -</td>
<td>1.41E-000 -</td>
<td>7.99E-001 -</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.23E-000 0.52</td>
<td>9.87E-001 0.52</td>
<td>5.58E-001 0.52</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.34E-000 0.73</td>
<td>5.96E-001 0.73</td>
<td>3.36E-001 0.73</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>7.42E-001 0.86</td>
<td>3.29E-001 0.86</td>
<td>1.86E-001 0.86</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1.13E-002 -</td>
<td>5.03E-003 -</td>
<td>3.60E-003 -</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.43E-003 2.99</td>
<td>6.33E-004 2.99</td>
<td>4.65E-004 2.95</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.78E-004 3.00</td>
<td>7.92E-005 3.00</td>
<td>5.97E-005 2.96</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.23E-005 3.00</td>
<td>9.90E-006 3.00</td>
<td>7.52E-005 2.99</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.07E-004 -</td>
<td>5.85E-005 -</td>
<td>1.21E-004 -</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.57E-006 4.03</td>
<td>3.63E-006 4.01</td>
<td>7.63E-006 3.99</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.11E-007 4.00</td>
<td>2.27E-007 4.00</td>
<td>4.79E-007 3.99</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.57E-008 4.00</td>
<td>1.42E-008 4.00</td>
<td>2.99E-008 4.00</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6.37E-006 -</td>
<td>3.56E-006 -</td>
<td>8.05E-006 -</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.05E-007 4.96</td>
<td>1.15E-007 4.95</td>
<td>2.75E-007 4.87</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.02E-010 5.00</td>
<td>1.13E-010 5.00</td>
<td>2.73E-010 5.00</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.50E-007 -</td>
<td>9.35E-008 -</td>
<td>2.06E-007 -</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.34E-009 6.00</td>
<td>1.48E-009 5.98</td>
<td>3.36E-009 5.94</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.66E-011 6.00</td>
<td>2.32E-011 6.00</td>
<td>5.30E-011 5.99</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>5.72E-013 6.00</td>
<td>3.63E-013 6.00</td>
<td>8.29E-013 6.00</td>
</tr>
</tbody>
</table>

We have also implemented non-uniform meshes, which are obtained by randomly perturbing each cell interface of a uniform mesh by up to 10%. The errors and numerical order of accuracy for $P^k$ elements with $1 \leq k \leq 5$ on non-uniform meshes are listed in Tables 5-8. The order of accuracy seems to be less clean for non-uniform meshes, however, similar conclusions can be obtained, which means that all cases perform well when $k$ is even and only the case with $\tau_{\max} = CFL \cdot h^3$ can achieve the optimal $(k+1)$-th order of accuracy for odd $k$ when $k > 1$.

**Example 4.2.** We solve the classical soliton solution of the nonlinear KdV equation

$$u_t - 3(u^2)_x + u_{xxx} = 0. \quad (4.3)$$

The exact solution is

$$u(x,t) = -2\text{sech}^2(x-4t). \quad (4.4)$$
Table 4: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.6), and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error order</th>
<th>$L^2$ error order</th>
<th>$L^\infty$ error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>3.83E-000</td>
<td>1.69E-000</td>
<td>1.02E-000</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.83E-000 0.00</td>
<td>1.70E-000 0.00</td>
<td>1.02E-000 0.00</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.83E-000 0.00</td>
<td>1.70E-000 0.00</td>
<td>1.03E-000 0.00</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>9.62E-003</td>
<td>5.06E-003</td>
<td>8.02E-003</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>9.22E-004 3.38</td>
<td>4.87E-004 3.37</td>
<td>6.78E-004 3.56</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.00E-004 3.20</td>
<td>6.08E-005 3.00</td>
<td>8.63E-005 2.97</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.02E-005 3.30</td>
<td>6.77E-006 3.17</td>
<td>1.02E-005 3.09</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>3.75E-004</td>
<td>2.37E-004</td>
<td>4.24E-004</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.86E-004 1.02</td>
<td>1.24E-004 0.94</td>
<td>1.88E-004 1.18</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.78E-006 4.78</td>
<td>4.51E-006 4.78</td>
<td>7.05E-006 4.73</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.47E-006 0.97</td>
<td>2.32E-006 0.96</td>
<td>3.47E-008 1.02</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>8.49E-006</td>
<td>6.64E-006</td>
<td>1.61E-005</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.15E-007 5.30</td>
<td>1.87E-007 5.15</td>
<td>4.14E-007 5.28</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.55E-009 5.04</td>
<td>5.85E-009 5.00</td>
<td>1.30E-008 4.99</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.05E-010 5.00</td>
<td>1.83E-010 5.00</td>
<td>4.07E-010 5.00</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.26E-006</td>
<td>1.11E-006</td>
<td>2.58E-006</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.60E-008 6.30</td>
<td>1.41E-008 6.30</td>
<td>2.97E-008 6.44</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.03E-009 3.95</td>
<td>9.27E-010 3.93</td>
<td>2.09E-009 3.83</td>
</tr>
</tbody>
</table>

We impose the boundary condition

$$u(10,t) = g_1(t), \quad u_x(12,t) = g_2(t), \quad u_{xx}(12,t) = g_3(t),$$

(4.5)

where $g_i(t)$ corresponds to the data from the exact solution. For this non-periodic case, the boundary condition is enforced by using the exact solution when available, and the numerical solution otherwise. From numerical experiments, the CDG schemes with $\tau_{\text{max}} = CFL \cdot h$ and non-dissipative case are unstable for all choice of $k$. If we take $\tau_{\text{max}} = CFL \cdot h^2$, the scheme is only stable for $k = 2$. Thus in Table 9, we only give the numerical error and order of accuracy with $\tau_{\text{max}} = CFL \cdot h^3$. Here we choose $CFL = 0.0001$ and $\Delta t = 0.8CFL \cdot h^3$. It is obvious that for $\tau_{\text{max}} = CFL \cdot h^3$ the optimal $(k+1)$-th order of accuracy is achieved for $k = 2, 3, 4$. This example demonstrates the non-periodic problems are more sensitive about numerical dissipations for CDG schemes. Actually similar results hold for CDG method for first order hyperbolic equations from our numerical experiment.
Table 5: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a non-uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\text{max}} = CFL \cdot \Delta t$ and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1.82E-000</td>
<td>-</td>
<td>8.06E-001</td>
<td>-</td>
<td>4.55E-001</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.83E-001</td>
<td>2.25</td>
<td>1.70E-001</td>
<td>2.25</td>
<td>1.00E-001</td>
<td>2.18</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.36E-001</td>
<td>0.70</td>
<td>1.04E-001</td>
<td>0.70</td>
<td>6.13E-002</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.92E-001</td>
<td>-0.73</td>
<td>1.74E-001</td>
<td>-0.73</td>
<td>1.01E-001</td>
<td>-0.72</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>6.12E-003</td>
<td>-</td>
<td>2.86E-003</td>
<td>-</td>
<td>3.76E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.09E-004</td>
<td>3.33</td>
<td>3.40E-004</td>
<td>3.07</td>
<td>6.84E-004</td>
<td>2.46</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.08E-004</td>
<td>2.49</td>
<td>6.36E-005</td>
<td>2.42</td>
<td>1.18E-004</td>
<td>2.54</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.44E-005</td>
<td>2.91</td>
<td>8.59E-006</td>
<td>2.89</td>
<td>1.54E-005</td>
<td>2.94</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.27E-004</td>
<td>-</td>
<td>7.60E-005</td>
<td>-</td>
<td>1.68E-004</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.01E-005</td>
<td>2.66</td>
<td>1.29E-005</td>
<td>2.56</td>
<td>2.55E-005</td>
<td>2.73</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.34E-006</td>
<td>3.90</td>
<td>1.30E-006</td>
<td>3.30</td>
<td>5.72E-006</td>
<td>2.12</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>4.66E-007</td>
<td>1.53</td>
<td>3.23E-007</td>
<td>2.01</td>
<td>8.82E-007</td>
<td>2.70</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6.47E-006</td>
<td>-</td>
<td>4.45E-006</td>
<td>-</td>
<td>1.31E-005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.02E-007</td>
<td>4.42</td>
<td>2.47E-007</td>
<td>4.17</td>
<td>6.03E-007</td>
<td>4.44</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>8.87E-009</td>
<td>5.09</td>
<td>7.50E-009</td>
<td>5.04</td>
<td>1.79E-008</td>
<td>5.07</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.20E-010</td>
<td>4.79</td>
<td>2.04E-010</td>
<td>5.20</td>
<td>4.09E-010</td>
<td>5.45</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.71E-007</td>
<td>-</td>
<td>1.33E-007</td>
<td>-</td>
<td>3.87E-007</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.58E-009</td>
<td>6.05</td>
<td>2.34E-009</td>
<td>5.83</td>
<td>1.04E-008</td>
<td>5.22</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.59E-011</td>
<td>6.17</td>
<td>2.99E-011</td>
<td>6.29</td>
<td>1.61E-010</td>
<td>6.02</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>6.81E-013</td>
<td>5.72</td>
<td>7.97E-013</td>
<td>5.23</td>
<td>7.03E-012</td>
<td>4.51</td>
</tr>
</tbody>
</table>

Example 4.3. We solve the KdV equation

$$u_t + uu_x + \varepsilon u_{xxx} = 0,$$  \hspace{1cm} (4.6)

with $\varepsilon = 1/24^2$. The computational domain is set to $[0,1]$. To check accuracy and convergence rates, two well-known solutions for (4.6) are used.

The first is a cnoidal-wave solution,

$$u(x,t) = a \operatorname{cn}^2(4K(x-\varepsilon t - x_0)),$$  \hspace{1cm} (4.7)

where $\operatorname{cn}(z) = \operatorname{cn}(z : m)$ is the Jacobi elliptic function with modulus $m \in (0,1)$ and the parameters have the values $a = 192 \varepsilon K(m)^2$ and $\varepsilon = 64 \varepsilon (2m - 1) K(m)^2$ whilst $x_0$ is an arbitrary, real translation. Here, the function $K = K(m)$ is the complete elliptic integral of
Table 6: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a non-uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\text{max}} = CFL \cdot h^2$ and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>2.53E-000</td>
<td>-</td>
<td>1.12E-000</td>
<td>-</td>
<td>6.35E-001</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.80E-001</td>
<td>1.53</td>
<td>3.89E-001</td>
<td>1.53</td>
<td>2.23E-001</td>
<td>1.51</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.75E-001</td>
<td>1.68</td>
<td>1.22E-001</td>
<td>1.68</td>
<td>7.02E-002</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>9.03E-002</td>
<td>1.61</td>
<td>4.00E-002</td>
<td>1.61</td>
<td>2.29E-001</td>
<td>1.61</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7.80E-003</td>
<td>-</td>
<td>3.55E-003</td>
<td>-</td>
<td>4.24E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.61E-004</td>
<td>3.56</td>
<td>3.04E-004</td>
<td>3.54</td>
<td>5.20E-004</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.58E-005</td>
<td>3.33</td>
<td>3.08E-005</td>
<td>3.30</td>
<td>5.41E-005</td>
<td>3.27</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>8.42E-006</td>
<td>2.97</td>
<td>4.68E-006</td>
<td>2.72</td>
<td>9.24E-005</td>
<td>2.55</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.12E-004</td>
<td>-</td>
<td>6.36E-005</td>
<td>-</td>
<td>1.42E-004</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.88E-006</td>
<td>3.67</td>
<td>5.19E-006</td>
<td>3.61</td>
<td>1.47E-005</td>
<td>3.28</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.99E-007</td>
<td>3.67</td>
<td>4.10E-007</td>
<td>3.66</td>
<td>9.84E-007</td>
<td>3.90</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>6.77E-008</td>
<td>3.37</td>
<td>4.35E-008</td>
<td>3.24</td>
<td>8.89E-008</td>
<td>3.47</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>5.88E-006</td>
<td>-</td>
<td>3.67E-006</td>
<td>-</td>
<td>1.09E-005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.47E-007</td>
<td>4.57</td>
<td>1.63E-007</td>
<td>4.49</td>
<td>4.66E-007</td>
<td>4.55</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.36E-009</td>
<td>5.07</td>
<td>5.57E-009</td>
<td>4.87</td>
<td>1.89E-008</td>
<td>4.63</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.73E-010</td>
<td>4.75</td>
<td>1.64E-010</td>
<td>5.08</td>
<td>3.24E-010</td>
<td>5.86</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.41E-007</td>
<td>-</td>
<td>9.83E-008</td>
<td>-</td>
<td>2.73E-007</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.66E-009</td>
<td>5.89</td>
<td>1.96E-009</td>
<td>5.65</td>
<td>5.86E-009</td>
<td>5.54</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.79E-011</td>
<td>5.80</td>
<td>4.17E-011</td>
<td>5.55</td>
<td>1.71E-010</td>
<td>5.10</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>7.88E-013</td>
<td>5.92</td>
<td>7.14E-013</td>
<td>5.87</td>
<td>4.03E-012</td>
<td>5.41</td>
</tr>
</tbody>
</table>

The first kind and the parameters are so organized that the solution $u$ has spatial period $1$.

The second is a classical solitary-wave solution

$$u(x,t) = A\sec^2(K(x-\sigma t-x_0)),$$

(4.8)

with $A = 1$, $v = A/3$, $K = \frac{1}{2}\sqrt{\frac{A}{3\epsilon}}$ and $x_0 = \frac{1}{2}$ so that the wave commences its evolution centered in the period domain. This traveling wave, too, is a stable solution of the KdV-equation (see [42] and [4] for the original proof of this fact). Of course, the later solution is not periodic in space, but owing to its exponential decay, it can be treated as periodic by simply restricting it to the computational domain $[0,1]$ and imposing periodic boundary conditions across $x = 0$ and $x = 1$. Tables 10-13 contain the numerical error and order of accuracy of cnoidal-wave problem of two schemes with $k = 2, 3, 4$. 
Table 7: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a non-uniform mesh of $N$ cells. Scheme (2.5), $\tau_{\text{max}} = CFL \cdot h^3$ and final time $T = 1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3.16E-000</td>
<td>-</td>
<td>1.40E-000</td>
<td>-</td>
<td>7.91E-001</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.12E-000</td>
<td>0.58</td>
<td>9.37E-001</td>
<td>0.58</td>
<td>5.29E-001</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.30E-000</td>
<td>0.70</td>
<td>5.77E-001</td>
<td>0.70</td>
<td>3.26E-001</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>7.07E-001</td>
<td>0.88</td>
<td>3.13E-001</td>
<td>0.88</td>
<td>1.77E-001</td>
<td>0.88</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1.01E-002</td>
<td>-</td>
<td>4.61E-003</td>
<td>-</td>
<td>5.30E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.25E-003</td>
<td>3.01</td>
<td>5.60E-004</td>
<td>3.04</td>
<td>7.04E-004</td>
<td>2.91</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.65E-004</td>
<td>2.93</td>
<td>7.38E-005</td>
<td>2.92</td>
<td>1.01E-004</td>
<td>2.80</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.15E-005</td>
<td>2.93</td>
<td>9.60E-006</td>
<td>2.94</td>
<td>8.33E-006</td>
<td>3.60</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.09E-004</td>
<td>-</td>
<td>5.94E-005</td>
<td>-</td>
<td>1.24E-004</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6.66E-006</td>
<td>4.04</td>
<td>3.61E-006</td>
<td>4.04</td>
<td>7.72E-006</td>
<td>4.01</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.67E-007</td>
<td>3.83</td>
<td>2.49E-007</td>
<td>3.86</td>
<td>4.82E-007</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.66E-008</td>
<td>4.14</td>
<td>1.43E-008</td>
<td>4.12</td>
<td>2.99E-008</td>
<td>4.01</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6.38E-006</td>
<td>-</td>
<td>3.53E-006</td>
<td>-</td>
<td>8.56E-006</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.52E-007</td>
<td>5.39</td>
<td>8.27E-008</td>
<td>5.42</td>
<td>2.28E-007</td>
<td>5.23</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.12E-010</td>
<td>4.84</td>
<td>1.14E-010</td>
<td>4.88</td>
<td>1.63E-010</td>
<td>5.46</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1.38E-007</td>
<td>-</td>
<td>9.88E-008</td>
<td>-</td>
<td>2.76E-007</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.62E-009</td>
<td>6.41</td>
<td>9.21E-010</td>
<td>6.74</td>
<td>2.88E-009</td>
<td>6.59</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.76E-011</td>
<td>5.43</td>
<td>2.46E-011</td>
<td>5.23</td>
<td>8.54E-011</td>
<td>5.07</td>
</tr>
</tbody>
</table>

For the cnoidal-wave test problem, Fig. 1 and Fig. 2 show the plots of the numerical solutions with scheme (2.5) with three choices of $\tau_{\text{max}}$ and non-dissipative case when $k = 2, 3$ at time $t = 10, 200$. There are obvious errors when $N$ is small for $k = 2$ if we take $\tau_{\text{max}} = CFL \cdot h^3$. But when $N$ increases the plot of approximation performs well for all choices of $\tau_{\text{max}}$. Also we have tried the nonuniform mesh which is obtained by randomly perturbing each cell interface of a uniform mesh by up to 10% with 80 cells. The comparison of the four numerical solutions are shown in Fig. 1, bottom right. And we can find that they all performed well. For $k = 3$ performance of $\tau_{\text{max}} = CFL \cdot h$, $CFL \cdot h^3$ and non-dissipative case is good, but if $t = 200$ the plot of approximation with $\tau_{\text{max}} = CFL \cdot h^2$ has a small phase error.

Fig. 3 and Fig. 4 show the numerical solutions of solitary-wave problem with scheme (2.5) with three choices of $\tau_{\text{max}}$ and non-dissipative case when $k = 2, 3$ at time $t = 25, 50$. 
Fig. 1: Numerical approximations of the cnoidal-wave problem with scheme (2.5), $\tau_{\text{max}} = \text{CFL} \cdot h$, $\text{CFL} \cdot h^2$, $\text{CFL} \cdot h^3$ and non-dissipative case; comparisons with the exact solution at time $t = 10$ with $\text{CFL} = 10$ and $k = 2$. Top left: 40 uniform cells; Top right: 80 uniform cells; Bottom left: 160 uniform cells; Bottom right: 80 nonuniform cells.

Fig. 2: Numerical approximations of the cnoidal-wave problem with scheme (2.5), $\tau_{\text{max}} = \text{CFL} \cdot h$, $\text{CFL} \cdot h^2$, $\text{CFL} \cdot h^3$ and non-dissipative case; comparisons with the exact solution with $\text{CFL} = 1$ and $k = 3$ and uniform meshes $N = 80$. Left: $t = 10$; Right: $t = 200$. 
Fig. 3: Numerical approximations of the solitary-wave problem with scheme (2.5). \( \tau_{\text{max}} = CFL \cdot h, CFL \cdot h^2, CFL \cdot h^3 \) and non-dissipative case; comparisons with the exact solution at time \( t = 25 \) with \( CFL = 10 \) and \( k = 2 \). Left: \( N = 40 \); Right: \( N = 80 \).

Fig. 4: Numerical approximations of the solitary-wave problem with scheme (2.5). \( \tau_{\text{max}} = CFL \cdot h, CFL \cdot h^2, CFL \cdot h^3 \) and non-dissipative case; comparisons with the exact solution with \( CFL = 1 \) and \( k = 3 \) and uniform meshes \( N = 80 \). Left: \( t = 25 \); Right: \( t = 50 \).

Fig. 5: Time history of the \( L^2 \)-norm of the numerical approximations for the cnoidal-wave problem with scheme (2.5). \( \tau_{\text{max}} = CFL \cdot h, CFL \cdot h^2, CFL \cdot h^3 \) and non-dissipative case with a uniform mesh with 80 cells. Left: \( k = 2 \) and \( CFL = 10 \); Right: \( k = 3 \) and \( CFL = 1 \).
Table 8: Errors and numerical orders of accuracy for Example 4.1 when using $P^k$ polynomials and Runge-Kutta third order time discretization on a non-uniform mesh of $N$ cells. Scheme (2.6), and final time $T=1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3.83E-000</td>
<td>-</td>
<td>1.74E-000</td>
<td>-</td>
<td>1.32E-000</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.86E-000</td>
<td>0.00</td>
<td>1.73E-000</td>
<td>0.01</td>
<td>1.19E-000</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.84E-000</td>
<td>0.01</td>
<td>1.71E-000</td>
<td>0.01</td>
<td>1.08E-000</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.84E-000</td>
<td>0.00</td>
<td>1.70E-000</td>
<td>0.00</td>
<td>9.64E-001</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>1.16E-002</td>
<td>-</td>
<td>6.17E-003</td>
<td>-</td>
<td>1.11E-002</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.41E-003</td>
<td>3.04</td>
<td>8.00E-004</td>
<td>2.95</td>
<td>1.70E-003</td>
<td>2.71</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.12E-004</td>
<td>3.65</td>
<td>7.20E-005</td>
<td>3.47</td>
<td>1.48E-004</td>
<td>3.52</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.27E-005</td>
<td>3.15</td>
<td>7.95E-006</td>
<td>3.18</td>
<td>1.53E-005</td>
<td>3.27</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>9.45E-004</td>
<td>-</td>
<td>6.42E-004</td>
<td>-</td>
<td>1.54E-003</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.97E-004</td>
<td>2.26</td>
<td>1.33E-004</td>
<td>2.27</td>
<td>2.34E-004</td>
<td>2.72</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.53E-006</td>
<td>7.01</td>
<td>1.44E-006</td>
<td>6.54</td>
<td>6.23E-006</td>
<td>5.23</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.89E-007</td>
<td>3.02</td>
<td>1.50E-007</td>
<td>3.26</td>
<td>7.07E-007</td>
<td>3.14</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>1.08E-005</td>
<td>-</td>
<td>7.84E-006</td>
<td>-</td>
<td>2.42E-005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.09E-007</td>
<td>5.13</td>
<td>2.63E-007</td>
<td>4.90</td>
<td>8.13E-007</td>
<td>4.90</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.09E-009</td>
<td>5.44</td>
<td>6.36E-009</td>
<td>5.37</td>
<td>1.52E-008</td>
<td>5.74</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.86E-010</td>
<td>4.63</td>
<td>2.06E-010</td>
<td>4.95</td>
<td>4.95E-010</td>
<td>4.94</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>6.90E-007</td>
<td>-</td>
<td>6.89E-007</td>
<td>-</td>
<td>1.91E-006</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.94E-008</td>
<td>4.55</td>
<td>2.43E-008</td>
<td>4.82</td>
<td>5.93E-008</td>
<td>5.01</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.49E-010</td>
<td>7.62</td>
<td>1.91E-010</td>
<td>6.99</td>
<td>1.08E-009</td>
<td>5.78</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>9.12E-013</td>
<td>7.35</td>
<td>1.28E-012</td>
<td>7.22</td>
<td>1.13E-011</td>
<td>6.58</td>
</tr>
</tbody>
</table>

For $k = 2$, when $N$ is small, $\tau_{\text{max}} = CFL \cdot h^2$, $CFL \cdot h^3$ show small phase errors and these errors will disappear as $N$ increases. With $k = 3$ and $CFL = 1$ all cases will give small phase errors when $t = 50$, which is probably caused by the boundary conditions of this problem.

The time evolutions of $L^2$-norm of numerical solutions for cnoidal-wave problem up to time $t = 10, 200$ with $k = 2, 3$ are shown in Fig. 5. The plot of $\tau_{\text{max}} = CFL \cdot h^3$ shows the biggest error when $k = 2$, when $k = 3$, $\tau_{\text{max}} = CFL \cdot h^2$ gives the biggest error mainly because of the reduced accuracy. The two graphs in Fig. 5 show that the scheme has better ability to conserve $L^2$-norm of solutions when $\tau_{\text{max}} = CFL \cdot h$. We also observe that the $L^2$-norm of solutions will increase a little as time goes on for the non-dissipative case when $k = 3$, which may be caused by the bad performance of non-dissipative case for $k$ is an odd number that contributes to poor resolutions at the boundary.
Table 9: Errors and numerical orders of accuracy for Example 4.2 when using $P_k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.5). $\tau_{\text{max}} = CFL \cdot h^3$ and final time $T = 0.1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error order</th>
<th>$L^2$ error order</th>
<th>$L^\infty$ error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20</td>
<td>8.47E-001</td>
<td>3.82E-001</td>
<td>3.40E-001</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>2.92E-001 1.54</td>
<td>1.45E-001 1.40</td>
<td>1.39E-001 1.30</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>6.24E-002 2.23</td>
<td>3.09E-002 2.23</td>
<td>3.05E-002 2.19</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>9.39E-003 2.73</td>
<td>4.32E-003 2.84</td>
<td>4.34E-003 2.81</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>1.23E-003 2.93</td>
<td>5.50E-004 2.97</td>
<td>5.51E-004 2.98</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>8.07E-002</td>
<td>4.02E-002</td>
<td>4.02E-002</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.41E-003 4.19</td>
<td>1.85E-003 4.44</td>
<td>2.15E-003 4.23</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.61E-004 4.78</td>
<td>6.66E-005 4.79</td>
<td>1.54E-004 3.81</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>5.82E-006 4.79</td>
<td>2.63E-006 4.67</td>
<td>8.74E-006 4.14</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>2.38E-007 4.61</td>
<td>1.28E-007 4.36</td>
<td>5.06E-007 4.11</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>1.19E-002</td>
<td>5.56E-003</td>
<td>6.72E-003</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.24E-004 6.58</td>
<td>7.37E-005 6.24</td>
<td>1.12E-004 5.90</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.40E-006 5.69</td>
<td>1.75E-006 5.40</td>
<td>3.24E-006 5.11</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>6.62E-008 5.18</td>
<td>5.11E-008 5.10</td>
<td>1.03E-007 4.97</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>2.22E-009 4.90</td>
<td>1.58E-009 5.01</td>
<td>3.21E-009 5.01</td>
</tr>
</tbody>
</table>

Table 10: Errors and numerical orders of accuracy for cnoidal-wave problem when using $P_k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.5). $\tau_{\text{max}} = CFL \cdot h^3$ and final time $T = 10$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error order</th>
<th>$L^2$ error order</th>
<th>$L^\infty$ error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>1.04E-000</td>
<td>1.17E-000</td>
<td>1.84E-000</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.77E-003 6.88</td>
<td>1.10E-002 6.74</td>
<td>3.32E-002 5.79</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>3.35E-003 1.39</td>
<td>4.12E-003 1.41</td>
<td>9.42E-003 1.82</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.81E-004 3.58</td>
<td>3.49E-004 3.56</td>
<td>9.10E-004 3.37</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>1.94E-005 3.85</td>
<td>2.48E-005 3.81</td>
<td>8.13E-005 3.48</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>6.36E-001</td>
<td>7.54E-001</td>
<td>1.24E-000</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.04E-002 5.93</td>
<td>1.27E-002 5.89</td>
<td>2.45E-002 5.66</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.70E-004 4.47</td>
<td>5.81E-004 4.45</td>
<td>1.45E-003 4.08</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>2.63E-005 4.16</td>
<td>3.38E-005 4.10</td>
<td>1.16E-004 3.64</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>1.72E-006 3.94</td>
<td>2.39E-006 3.82</td>
<td>1.12E-005 3.37</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>7.57E-003</td>
<td>9.24E-003</td>
<td>2.23E-002</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>4.34E-005 7.44</td>
<td>6.10E-005 7.24</td>
<td>3.80E-004 5.87</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>5.25E-007 6.37</td>
<td>1.42E-006 5.43</td>
<td>1.40E-005 4.76</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.45E-008 5.18</td>
<td>4.56E-007 4.96</td>
<td>4.51E-006 4.96</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>4.50E-010 5.01</td>
<td>1.44E-009 4.99</td>
<td>1.45E-008 4.96</td>
</tr>
</tbody>
</table>
Table 11: Errors and numerical orders of accuracy for cnoidal-wave problem when using \(P_k\) polynomials and Runge-Kutta third order time discretization on a uniform mesh of \(N\) cells. Scheme (2.5), \(\tau_{\text{max}} = CFL \cdot h^2\) and final time \(T = 10\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>(N)</th>
<th>(L^1) error order</th>
<th>(L^2) error order</th>
<th>(L^\infty) error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>5.74E-001</td>
<td>-</td>
<td>6.67E-001</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>7.29E-001</td>
<td>-0.35</td>
<td>8.57E-001</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.49E-002</td>
<td>3.28</td>
<td>9.06E-002</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>4.92E-003</td>
<td>3.93</td>
<td>5.95E-003</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>3.10E-004</td>
<td>3.99</td>
<td>3.76E-004</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>1.06E-000</td>
<td>-</td>
<td>1.20E-000</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>6.16E-002</td>
<td>4.10</td>
<td>7.46E-002</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.84E-004</td>
<td>4.08</td>
<td>2.23E-004</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>1.13E-005</td>
<td>4.02</td>
<td>1.38E-005</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>2.72E-002</td>
<td>-</td>
<td>3.29E-002</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.57E-004</td>
<td>7.44</td>
<td>1.93E-004</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.05E-006</td>
<td>7.22</td>
<td>1.72E-006</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.57E-008</td>
<td>6.07</td>
<td>4.57E-008</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>4.52E-010</td>
<td>5.12</td>
<td>1.43E-009</td>
</tr>
</tbody>
</table>

Table 12: Errors and numerical orders of accuracy for cnoidal-wave problem when using \(P_k\) polynomials and Runge-Kutta third order time discretization on a uniform mesh of \(N\) cells. Scheme (2.5), \(\tau_{\text{max}} = CFL \cdot h^3\) and final time \(T = 10\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>(N)</th>
<th>(L^1) error order</th>
<th>(L^2) error order</th>
<th>(L^\infty) error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>5.02E-001</td>
<td>-</td>
<td>5.74E-001</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>7.98E-001</td>
<td>-0.67</td>
<td>9.21E-001</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>5.04E-001</td>
<td>0.66</td>
<td>6.01E-001</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>7.26E-002</td>
<td>2.79</td>
<td>8.78E-002</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>9.23E-003</td>
<td>2.97</td>
<td>1.12E-002</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>8.43E-001</td>
<td>-</td>
<td>9.81E-001</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.48E-002</td>
<td>4.60</td>
<td>4.21E-002</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>1.04E-003</td>
<td>5.06</td>
<td>1.26E-003</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.23E-005</td>
<td>5.01</td>
<td>3.91E-005</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>1.01E-006</td>
<td>5.00</td>
<td>1.22E-006</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6.92E-002</td>
<td>-</td>
<td>8.37E-002</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>7.60E-004</td>
<td>6.51</td>
<td>9.21E-004</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>9.42E-006</td>
<td>6.33</td>
<td>1.15E-005</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>9.64E-008</td>
<td>6.61</td>
<td>1.19E-007</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>9.86E-010</td>
<td>6.61</td>
<td>1.47E-009</td>
</tr>
</tbody>
</table>
Table 13: Errors and numerical orders of accuracy for cnoidal-wave problem when using $P^k$ polynomials and Runge-Kutta third order time discretization on a uniform mesh of $N$ cells. Scheme (2.6), and final time $T = 10$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>$L^1$ error order</th>
<th>$L^2$ error order</th>
<th>$L^\infty$ error order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>1.01E-000</td>
<td>1.17E-000</td>
<td>1.85E-000</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.52E-002</td>
<td>3.56</td>
<td>1.03E-001</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>5.52E-003</td>
<td>3.95</td>
<td>6.73E-003</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.48E-004</td>
<td>3.99</td>
<td>4.29E-004</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>2.19E-005</td>
<td>3.99</td>
<td>2.78E-005</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>2.78E-001</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>8.69E-003</td>
<td>5.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>4.97E-004</td>
<td>4.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>4.05E-005</td>
<td>3.62</td>
<td></td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>4.15E-006</td>
<td>3.29</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>5.24E-003</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.79E-005</td>
<td>7.11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>5.14E-007</td>
<td>6.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>1.45E-008</td>
<td>5.15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>4.49E-010</td>
<td>5.01</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6: Time history of the $L^2$-norm of the numerical approximations for the solitary-wave problem with scheme (2.5), $\tau_{\max} = CFL \cdot h$, $CFL \cdot h^2$, $CFL \cdot h^3$ and non-dissipative case with a uniform mesh with 80 cells. Left: $k = 2$ and $CFL = 10$; Right: $k = 3$ and $CFL = 1$.

The time evolutions of the $L^2$-norm of numerical solutions for solitary-wave problem up to time $t = 25, 50$ are given in Fig. 6. The conclusions are similar to the cnoidal-wave problem.
5 Concluding remarks

A class of CDG method was proposed in this paper to solve generalized KdV equations based on the ultra-weak DG method in [14]. The scheme resorts to repeated integration by parts, and is defined on overlapping cells which avoids the introduction of numerical fluxes. We have performed $L^2$ stability analysis and \textit{a priori} error estimates for the dissipative and non-dissipative schemes. We established different convergence rate depending on the parameter choices and several projection operators are introduced in our proof. The numerical experiments have demonstrated that the results are sensitive about the choices of a dissipation parameter $\tau_{\text{max}}$. The results with $\tau_{\text{max}}=CFL\cdot h^3$ will have better convergence rate than other cases in most numerical tests and are more robust with boundary treatment. Long time simulation of solitary wave propagation seems to perform better with non-dissipative scheme similar as observed in [5]. Numerical results indicate higher convergence rate than what was established in the proof, which will be our future work. Another direction of future work is to simulate nonlinear dispersive wave equations in higher dimensions.

Appendix: Collection of proofs

In this appendix, we collect the proofs of some technical lemmas and propositions.

A.1 Proof of Lemma 3.1

Without loss of generality we will only consider $P^*_h$. Because $P^*_h$ is a local projection, so for existence and uniqueness, we only need to consider the projection defined on the reference interval $[-1,1]$. Note that the procedure to find the $P^*_hw \in P^k([-1,1])$ is to solve a linear system, so the existence and uniqueness are equivalent. Therefore, we only prove for the uniqueness.

We let $w_l(x) = P^*_h w(x)$ with $w(x) = 0$, and would like to prove that $w_l(x) = 0$. By the definition of $P^*_h$, we have
\[
\int_{-1}^{1} w_l \varphi_h dx = 0, \quad \forall \varphi_h \in P^{k-3}([-1,1]), \quad (A.1)
\]
\[
w_l(0) = (w_l)_x(0) = (w_l)_{xx}(0) = 0. \quad (A.2)
\]

Let $P_l(x)$ be the $l$-th order Legendre polynomials on $[-1,1]$. Then $w_l$ can be expressed in the form
\[
w_l(x) = \sum_{l=0}^{k} a_l P_l(x). \quad (A.3)
\]

(A.1) and the orthogonality of the Legendre polynomials yield that
\[
a_l = 0, \quad l = 0, \ldots, k-3. \quad (A.4)
\]
Hence, \( w_I(x) = \alpha_{k-2}P_{k-2}(x) + \alpha_{k-1}P_{k-1}(x) + \alpha_kP_k(x) \). It follows from (A.2) that
\[
\begin{align*}
\alpha_{k-2}P_{k-2}(0) + \alpha_{k-1}P_{k-1}(0) + \alpha_kP_k(0) &= 0, \\
\alpha_{k-2}P'_{k-2}(0) + \alpha_{k-1}P'_{k-1}(0) + \alpha_kP'_k(0) &= 0, \\
\alpha_{k-2}P''_{k-2}(0) + \alpha_{k-1}P''_{k-1}(0) + \alpha_kP''_k(0) &= 0.
\end{align*}
\] (A.5)

If \( k \) is an even number, we have
\[
\frac{(-1)^{k-2}}{4^{k-2}} \left( \frac{2k-4}{k-2} \right) a_{k-2} + \frac{(-1)^k}{4^k} \left( \frac{2k}{k} \right) a_k = 0,
\]
\[
a_{k-1} = 0,
\]
\[
\frac{(-1)^{k-2}(k-1)(k-2)}{4^{k-2}} \left( \frac{2k-4}{k-2} \right) a_{k-2} + \frac{(-1)^k(k+1)}{4^k} \left( \frac{2k}{k} \right) a_k = 0.
\] (A.6)

It is easy to verify that \( a_{k-2} = a_{k-1} = a_k = 0 \). Hence \( w_I(x) \equiv 0 \). Thus, we have finished the proof of uniqueness and existence. On the other hand, we note that when \( k \) is odd, the system (A.5) is not uniquely solvable.

Since \( P^*_h \) and \( Q^*_h \) are \( k \)-th degree polynomial preserving local projections, (3.15) follows from standard approximation theory [15].

### A.2 Proof of Lemma 3.2

**Proof.** We only consider \( P^*_h \), while the proof for \( Q^*_h \) follows similar lines. For \( \forall j \), we let \( \bar{\xi} = \frac{2(x-x_j)}{n} \) on \( I_j \), for a smooth function \( \omega(x) \) and a \( k \)-th order polynomial \( \varphi_h(x) \) on \( I_j \) define
\[
\bar{\omega}(\bar{\xi}) = \omega \left( \frac{h}{2} \bar{\xi} + x_j \right) = \omega(x),
\]
\[
\bar{\varphi}_h(\bar{\xi}) = \varphi_h \left( \frac{h}{2} \bar{\xi} + x_j \right) = \varphi_h(x).
\] (A.7)

Note that the procedure to find the \( P^*_h \bar{\omega} \in P^k([-1,1]) \) is to solve for a linear system, so the existence and uniqueness are equivalent. Thus, we only need to prove the uniqueness of the projection \( P^*_h \). We set \( \omega_I(\bar{\xi}) = P^*_h \bar{\omega}(\bar{\xi}) = P^*_h \omega(x) \) with \( \bar{\omega}(\bar{\xi}) = \omega(x) = 0 \), and would like to prove \( \omega_I(\bar{\xi}) = 0 \). Then by the definition of the projection \( P^*_h \), we have:
\[
\frac{h^2}{4} \bar{P}_h(\omega_I; \varphi_h) = \frac{h^3}{8k_{\text{max}}} \left( \int_{-1}^{0} \omega_I(\bar{\xi}+1)\varphi_h(\bar{\xi})d\bar{\xi} + \int_{0}^{1} \omega_I(\bar{\xi}-1)\varphi_h(\bar{\xi})d\bar{\xi} \right.
\]
\[
- \int_{-1}^{1} \omega_I(\bar{\xi})\varphi_h(\bar{\xi})d\bar{\xi} + \int_{-1}^{0} \omega_I(\bar{\xi}+1)(\varphi_h(\bar{\xi}))_{\bar{\xi}\bar{\xi}}d\bar{\xi}
\]
\[
+ \int_{0}^{1} \omega_I(\bar{\xi}-1)(\varphi_h(\bar{\xi}))_{\bar{\xi}\bar{\xi}}d\bar{\xi} - \omega_I(0)(\varphi_h(\bar{\xi})(1)
\]
\[
- (\varphi_h(\bar{\xi})(-1)) + (\omega_I(0)((\varphi_h(\bar{\xi})(1) - (\varphi_h(\bar{\xi})(-1))
\]
\[
- (\omega_I(\bar{\xi}))(0)(\varphi_h(1) - \varphi_h(-1)), \quad \forall \varphi_h(\bar{\xi}) \in P^k([-1,1]),
\] (A.8a)
Thus we rewrite the system resulting from (A.8) and combining with the standard approximation theory [15]. We skip the details, but note that (3.20) can be established because the choice of \( \tau_{\text{max}} \) does not break the correct scaling with respect to \( h \), therefore optimal convergence rate follows.
A.3 Proof of Proposition 3.1

Note that \( u_I = u \) and \( v_I = u \) when \( u(x) \in P^k([a,b]) \), by the uniqueness of the projections \( P_h^* \) and \( Q_h^* \). Therefore, we just need to prove \( u(x) = x^{l+1} \). First, we show a simple claim as following.

Claim A.1. With the same notations as those in Proposition 3.1, and \( u(x) = x^{l+1} \), then

\[
(x + \frac{h}{2})^{l+1} - v_I(x + \frac{h}{2}) = x^{l+1} - u_I(x)
\]

\[
= (x - \frac{h}{2})^{l+1} - v_I(x - \frac{h}{2}), \quad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \forall j.
\]

(A.13)

Proof. We just need to prove \( x^{l+1} - v_I(x) = (x - \frac{h}{2})^{l+1} - u_I(x - \frac{h}{2}), \forall x \in (x_j, x_{j+1}) \). We set \( \tilde{v}_I(x) = u_I(x - \frac{h}{2}) - (x - \frac{h}{2})^{l+1} + x^{l+1} \), then we just need to prove \( v_I(x) = \tilde{v}_I(x) \). By the uniqueness of the projection \( Q_h^* \), we just need to check the following equations:

\[
\int_{x_j}^{x_{j+1}} \tilde{v}_I(x) \, dx = \int_{x_j}^{x_{j+1}} u(x) \, dx,
\]

(A.14)

\[
\tilde{Q}_h(\tilde{v}_I; \psi_h)_{j+\frac{1}{2}} = \tilde{Q}_h(u; \psi_h)_{j+\frac{1}{2}}, \forall \psi_h \in P^{l+1}(I_{j+\frac{1}{2}}).
\]

The first equation can be verified as follows

\[
\int_{x_j}^{x_{j+1}} \tilde{v}_I(x) \, dx = \int_{x_j}^{x_{j+1}} u_I(x - \frac{h}{2}) - (x - \frac{h}{2})^{l+1} + x^{l+1} \, dx
\]

\[
= \int_{x_j}^{x_{j+1}} u_I(x) - (x_{j+\frac{1}{2}}) + \int_{x_j}^{x_{j+1}} x^{l+1} \, dx
\]

\[
= \int_{x_j}^{x_{j+1}} x^{l+1} \, dx = \int_{x_j}^{x_{j+1}} u(x) \, dx,
\]

where we have used the definition of the projection \( P_h^* \) in (3.17). The second equation can be verified as follows

\[
\tilde{Q}_h(\tilde{v}_I(x); \psi_h(x))_{j+\frac{1}{2}} = P_h\left( \tilde{u}_I(x) - u(x); \psi_h(x + \frac{h}{2}) \right) + \tilde{Q}_h(x^{l+1}; \psi_h(x))_{j+\frac{1}{2}}
\]

\[
= \tilde{Q}_h(x^{l+1}; \psi_h(x))_{j+\frac{1}{2}}, \forall \psi_h(x) \in P^{l+1}(I_{j+\frac{1}{2}}),
\]

where we have used the fact \( \psi_h(x + \frac{h}{2}) \in V_h^k \). Therefore the uniqueness of the projection \( Q_h^* \) implies that \( v_I(x) = \tilde{v}_I(x) \).

Now, we begin to prove Proposition 3.1. We will just prove one case \( \tilde{B}_I(u_I, v_I; \varphi_h) = \).
\[ \hat{B}_j(u, u_1; \varphi_h), \text{ as the other one follows the same lines. We use the claim to } \hat{B}_j(u_1, v_1; \varphi_h): \]

\[ \hat{B}_j(u_1, v_1; \varphi_h) = \frac{1}{\tau_{\max}} \left( \int_{x_{j-1/2}}^{x_j} v_1(x) \varphi_h dx + \int_{x_{j-1/2}}^{x_j} v_1(x) \varphi_h dx - \int_{x_{j-1/2}}^{x_j} u_1(x) \varphi_h dx \right) \]

\[ + \int_{x_{j-1/2}}^{x_j} v_1(x) (\varphi_h)_{xxx} dx + \int_{x_{j-1/2}}^{x_j} v_1(x) (\varphi_h)_{xxx} dx \]

\[ - v_1(x_{j+1/2}) (\varphi_h)_{xx}(x_{j+1/2}^-) + v_1(x_{j-1/2}) (\varphi_h)_{xx}(x_{j-1/2}^-) + (v_1)_{x}(x_{j+1/2}) (\varphi_h)_{x}(x_{j+1/2}^-) \]

\[ - (v_1)_{x}(x_{j-1/2}) (\varphi_h)_{x}(x_{j-1/2}^-) - (v_1)_{xx}(x_{j+1/2}) (\varphi_h)_{xx}(x_{j+1/2}^-) + (v_1)_{xx}(x_{j-1/2}) (\varphi_h)_{xx}(x_{j-1/2}^-) \]

\[ = \frac{1}{\tau_{\max}} \left( \int_{x_{j-1/2}}^{x_j} u_1(x) \left( \frac{h}{2} \right)^{k+1} + \frac{1}{2} \right) \varphi_h dx \]

\[ + \int_{x_{j-1/2}}^{x_j} \left( u_1(x - \frac{h}{2}) - (x - \frac{h}{2})^{k+1} + x^{k+1} \right) \varphi_h dx - \int_{x_{j-1/2}}^{x_j} u_1(x) \varphi_h dx \]

\[ + \int_{x_{j-1/2}}^{x_j} \left( u_1(x + \frac{h}{2}) - (x + \frac{h}{2})^{k+1} + x^{k+1} \right) (\varphi_h)_{xxx} dx \]

\[ + \int_{x_{j-1/2}}^{x_j} \left( u_1(x - \frac{h}{2}) - (x - \frac{h}{2})^{k+1} + x^{k+1} \right) (\varphi_h)_{xxx} dx \]

\[ - (u_1(x_j) - x_j^{k+1} + x_j^{k+1}) (\varphi_h)_{xx}(x_{j+1/2}) \]

\[ + (u_1(x_j) - x_j^{k+1} + x_j^{k+1}) (\varphi_h)_{xx}(x_{j-1/2}) \]

\[ + (u_1(x_j) - (k+1)x_j^k + (k+1)x_j^{k+1}) (\varphi_h)_{xx}(x_{j+1/2}) \]

\[ - (u_1(x_j) - (k+1)x_j^k + (k+1)x_j^{k+1}) (\varphi_h)_{xx}(x_{j-1/2}) \]

\[ - (u_1(x_j) - k(k+1)x_j^{k-1} + k(k+1)x_j^{k-1}) (\varphi_h)(x_{j+1/2}) \]

\[ + (u_1(x_j) - k(k+1)x_j^{k-1} + k(k+1)x_j^{k-1}) (\varphi_h)(x_{j-1/2}) \]

\[ = \hat{P}_h(u_1(x) - x^{k+1}; \varphi_h(x))_j + \hat{B}_j(x^{k+1}, x^{k+1}; \varphi_h) \]

\[ = \hat{B}_j(u(x), u(x); \varphi_h). \]

### A.4 Proof of Lemma 3.3

**Proof.** Similar to [51] and [45], to complete the proof of Lemma 3.3 we would like to use the following Taylor expansions:
where $f''_u$ and $f''_v$ are the mean values. These implies the following representation,
\[
\sum_j \int_{I_j} (f(u) - f(v_h))(\varphi_h)_x dx + \sum_j \sum_{\frac{j+1}{2}} (f(u) - f(v_h))[\varphi_h]_j \\
+ \sum_j \int_{I_j} (f(u) - f(v_h))(\varphi_h)_x dx + \sum_j \sum_{\frac{j+1}{2}} (f(u) - f(u_h))[\varphi_h]_j \\
= N_1 + N_2 + N_3,
\] (A.16)

where
\[
N_1 = \sum_j \int_{I_j} f'(u)\psi_h(\varphi_h)_x dx + \sum_j \int_{I_{j+\frac{1}{2}}} f'(u)\psi_h(\varphi_h)_x dx \\
+ \sum_j (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f'(u)\varphi_h[\psi_h])_j,
\]
\[
N_2 = -\left( \sum_j \int_{I_j} f'(u)\varphi_h(\psi_h)_x dx + \sum_j \int_{I_{j+\frac{1}{2}}} f'(u)\varphi_h(\varphi_h)_x dx \\
+ \sum_j (f'(u)\varphi_h[\psi_h])_{j+\frac{1}{2}} + \sum_j (f'(u)\varphi_h[\varphi_h])_j \right),
\]
\[
N_3 = -\frac{1}{2} \left( \sum_j \int_{I_j} f''_u(\varphi_h - \varphi_h)^2(\varphi_h)_x dx + \sum_j \int_{I_{j+\frac{1}{2}}} f''_u(\varphi_h - \varphi_h)^2(\varphi_h)_x dx \\
+ \sum_j (f''_u(\varphi_h - \varphi_h)^2[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f''_u(\varphi_h - \varphi_h)^2[\varphi_h])_j \right).
\]

- $N_1$ term. $N_1$ can be rewritten as
\[
N_1 = \sum_j \left( \int_{x_j}^{x_{j+\frac{1}{2}}} f'(u)\psi_h(\varphi_h)_x dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} f'(u)\psi_h(\varphi_h)_x dx \right) \\
+ \sum_j (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f'(u)\varphi_h[\psi_h])_j \\
= \sum_j \left( (f'(u)\psi_h\varphi_h^-)_{j+\frac{1}{2}} - (f'(u)\varphi_h\psi_h^+)_j + (f'(u)\varphi_h\psi_h^-)_{j+1} \\
- (f'(u)\psi_h\varphi_h^+)_j + (f'(u)\varphi_h[\varphi_h])_{j+\frac{1}{2}} + (f'(u)\varphi_h[\psi_h])_j \right) \\
- \sum j \int_{x_j}^{x_{j+1}} f'(u)\psi_h u_h dx.
\]
\[-\sum_{j}^{x_{j+1}} (f'(u))_{x} \psi_{h} \varphi_{h} dx \leq C \| \psi_{h} \| \| \varphi_{h} \| \leq C (\| \psi_{h} \|^{2} + \| \varphi_{h} \|^{2}). \] (A.17)

- **N\textsubscript{2} term.** By inverse properties and Young’s inequality, for \(N\textsubscript{2}\) we have
  \[N\textsubscript{2} \leq C (\| \psi' \| (\| \varphi_{h} \|_{x}) + \| \varphi' \| (\| \psi_{h} \|_{x}) + \| \psi' \|_{x} \| \varphi_{h} \|_{x} + \| \varphi' \|_{x} \| \psi_{h} \|_{x}) \leq C (h^{-1} \| \varphi' \| \| \varphi_{h} \| + h^{-1} \| \varphi' \| \| \psi_{h} \|) \leq C (\| \psi_{h} \|^{2} + \| \varphi_{h} \|^{2}) + Ch^{2k}. \] (A.18)

- **N\textsubscript{3} term.** \(N\textsubscript{3}\) is high order term in Taylor expansion, it’s easy to show that
  \[N\textsubscript{3} \leq C_{\ast} h^{-1} (\| e_{v} \|_{\infty} \| e_{u} \| \| \varphi_{h} \| + \| e_{u} \|_{\infty} \| e_{u} \| \| \varphi_{h} \|) \leq C_{\ast} h^{-1} \left( \| e_{v} \|_{\infty} \| \varphi_{h} \| + \| e_{u} \|_{\infty} \| \varphi_{h} \| + \| e_{u} \|_{\infty} \| \varphi_{h} \| + \| e_{u} \|_{\infty} \| \varphi_{h} \| \right) \leq C_{\ast} (h^{-1} \| e_{v} \|_{\infty} + h^{-1} \| e_{u} \|_{\infty}) (\| \varphi_{h} \|^{2} + \| \psi_{h} \|^{2}) + C_{\ast} (\| e_{v} \|_{\infty} + \| e_{u} \|_{\infty}) h^{2k+1}. \] (A.19)

Therefore, summing up the above estimates from (A.17)-(A.19), we complete the proof.

\[\square\]

References


