# An Efficient Positivity-Preserving Finite Volume Scheme for the Nonequilibrium Three-Temperature Radiation Diffusion Equations on Polygonal Meshes

Shuai Su<sup>1,\*</sup>, Huazhong Tang<sup>1</sup> and Jiming Wu<sup>2</sup>

 <sup>1</sup> Center for Applied Physics and Technology, HEDPS and LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China.
 <sup>2</sup> Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.

Received 15 May 2020; Accepted (in revised version) 7 December 2020

**Abstract.** This paper develops an efficient positivity-preserving finite volume scheme for the two-dimensional nonequilibrium three-temperature radiation diffusion equations on general polygonal meshes. The scheme is formed as a predictor-corrector algorithm. The corrector phase obtains the cell-centered solutions on the primary mesh, while the predictor phase determines the cell-vertex solutions on the dual mesh independently. Moreover, the flux on the primary edge is approximated with a fixed stencil and the nonnegative cell-vertex solutions are not reconstructed. Theoretically, our scheme does not require any nonlinear iteration for the linear problems, and can call the fast nonlinear solver (e.g. Newton method) for the nonlinear problems. The positivity, existence and uniqueness of the cell-centered solutions obtained on the corrector phase are analyzed, and the scheme on quasi-uniform meshes is proved to be  $L^2$ - and  $H^1$ -stable under some assumptions. Numerical experiments demonstrate the accuracy, efficiency and positivity of the scheme on various distorted meshes.

AMS subject classifications: 65M08, 65M22

**Key words**: Radiation diffusion equations, positivity-preserving, high efficiency, stability, finite volume method.

# 1 Introduction

The nonequilibrium three-temperature (3-T) radiation diffusion equations are a kind of strongly nonlinear partial differential equations, and are widely used to describe the energy evolution and exchanges among the electrons, ions and photons in a multi-material system. They simulate the radiation transport and arise in a wide range of applications

http://www.global-sci.com/cicp

<sup>\*</sup>Corresponding author. *Email addresses:* sushuai@math.pku.edu.cn (S. Su), hztang@math.pku.edu.cn (H. Tang), wu\_jiming@iapcm.ac.cn (J. Wu)

such as the astrophysics and the inertial confinement fusion (ICF), see e.g. [6,7,12,15], if the radiation field is not in thermodynamic equilibrium with the material and the material itself is not in equilibrium. Developing the high-order accurate and efficient numerical scheme for the nonequilibrium 3-T radiation diffusion equations is very challenging and has drawn many researchers' attention.

In recent years, some numerical methods have been developed for solving the 3-T radiation diffusion equations, such as the finite volume (FV) methods in [13, 21, 43], the finite difference methods in [2, 11] and the finite element method in [25]. In [21], a fully implicit finite volume scheme combined with parallel adaptive multigrid method was studied in the framework of UG. In [13], a symmetric finite volume element method (SFVE) was designed with a preconditioning technique, a mesh adaptation algorithm and a two-grid procedure. In [43], two substructuring nonoverlapping domain decomposition preconditioners were employed to solve the SFVE discretization of 2D 3-T radiation diffusion equations with strongly discontinuous coefficients. In [2,11], the Newton-Krylov method was used to solve the finite difference schemes for 2D 3-T radiation equations. In [25], the freezing coefficient method was adopted to linearize the nonlinear equations and then the resulting equations were solved by the Raviart-Thomas mixed finite element method. Unfortunately, those schemes only work on the rectangular or triangular meshes.

In many applications such as the radiation hydrodynamics solved by the Lagrangian method, the meshes become usually distorted and concave with or without hanging nodes due to the complex fluid flow. Developing an efficient numerical scheme for the 3-T radiation diffusion equations on arbitrary polygonal meshes is one of the significant requirements. Recently, some schemes have been proposed on general polygonal meshes. In [41], a Lions domain decomposition algorithm based on a cell functional minimization scheme was studied on non-matching multi-block grids for the nonlinear radiation diffusion equations. In [5], a set of numerical schemes were developed for the 3-T radiation diffusion equations in systems involving multi-materials. In [32], a monotone cell-centered scheme for diffusion equation was used for the 3-T radiation diffusion equations and the interpolation algorithms suggested in [9] was adopted to evaluate the cell-vertex unknowns.

In many situations, another significant requirement of the numerical schemes for the 3-T radiation diffusion equations is that the discrete solution should be nonnegative. Prior non-negative formulations for single diffusion problems were broadly studied over the years. A typical formulation is constrained optimization-based finite element method for transient diffusion equations, see standard linear finite element method [19], mixed finite element methods [24], the classical Galerkin formulation [22, 23]. It is not suitable for polygonal meshes. Le Potier [29] proposed a monotone cell-centered finite volume scheme for the steady-state diffusion equations on triangular meshes. Since then, this approach have been developed to obtain numerous positivity-preserving finite volume schemes on general grids, see [4, 10, 16, 17, 33, 44, 45] and the references therein. This paper is concerned with the finite volume schemes. There exist some positivity-preserving

finite volume works for 3-T radiation equations. The first scheme in [8] preserves the positivity of the temperature if and only if the maximal interior angle of the primary partition is not greater than  $\pi/2$ , while the second one in [8] preserves the solution positivity by two post-processing techniques, i.e., the global repair technique [37] and the cutoff method [20], which possibly destroy the accuracy and the energy conservation. Recently, a cell-centered nonlinear finite volume scheme satisfying the discrete maximum principle was proposed in [42] for the nonequilibrium 3-T equations. A cell-vertex positivity-preserving scheme was studied in [34] for the 2D 3-T radiation diffusion equations but it only preserved the conservation property on the dual mesh. A nonlinear positivity-preserving scheme was proposed in [28], which was an extension of the scheme in [10], and the truncation error was not analyzed.

In fact, the mechanism for all the aforementioned FV schemes that can preserve solution positivity is that the resulting coefficient matrix or its transpose is an *M*-matrix. Therefore, the nonlinear solver must be chosen in such a way that the structure of the M-matrix is not spoiled. At the present, the Picard method and its Anderson acceleration method [3] are widely used since they can maintain the *M*-matrix structure of the coefficient matrix. In [18], the Anderson acceleration was first applied to the Picard method in the framework of nonlinear monotone FV schemes for advection-diffusion problems. In [1], the Anderson acceleration of the Picard method was employed to solve the finite difference scheme of the 3-T equations and two strategies were used to improve its robustness. Unfortunately, in the case of strongly anisotropic media with the highly distorted mesh, the Picard method, even its Anderson acceleration, usually converges very slowly. As we know, the Newton method has a second-order convergence rate, and has been used to solve the nonlinear positivity-preserving FV schemes or FV schemes satisfying the discrete maximum principle, see e.g. [26,27,31,36] for two- and three-phase flows and so on. However, the positivity of the numerical solutions for the nonlinear system is uncertain, see [31] for a detailed discussion.

The aim of this paper is to improve the scheme proposed in [33] for the single diffusion equation and extend it to the nonequilibrium 3-T equations, with the positivitypreserving property. The newly resulting scheme is formed as a predictor-corrector algorithm. The corrector phase determines the cell-centered solutions on the primary mesh, while the predictor phase obtains the cell-vertex solutions on the dual mesh independently. Based on the original idea of the fixed-stencil flux discretizations in [10], the flux approximation is constructed in the corrector phase, which differs from that in [33]. Specifically, the flux approximation is based on the fixed decomposition of the co-normal and the temperature on the vertex is not required to be nonnegative. As a result, the truncation technique for the vertex values in [33] is avoided and the conservation on the dual mesh is maintained. In Comparison with the existing positivity-preserving or maximumpreserving schemes for the 3-T radiation equations, such as [28, 34, 42], our scheme does not need any nonlinear iteration for the linear problems and the general nonlinear iterative methods, such as the Newton method, can be used for the nonlinear problems. For the nonlinear diffusion coefficients with a flux-limiter, the Jacobian-free Newton-Krylov method can be selected to avoid computing Jacobian matrices.

The rest of this paper is organized as follows. In Section 2, a short description of the nonequilibrium 3-T radiation diffusion equations and the mesh notations are presented. The positivity-preserving scheme is constructed in Section 3, and in Section 4, the positivity, existence and uniqueness of the cell-centered unknowns are proved and the stability of the scheme is analyzed on quasi-uniform meshes. Finally, numerical examples are conducted in Section 5 to validate the effectiveness of our scheme and some concluding remarks are given in Section 6.

## 2 3-T radiation diffusion equations and mesh notations

Denote  $\Omega_T = \Omega \times (0,T]$ , where  $\Omega$  is an open bounded connected polygonal domain in  $\mathbb{R}^2$  and (0,T] is the time interval. Consider the nonequilibrium 3-T radiation diffusion equations [1,21,25,42] in  $\Omega_T$ 

$$\rho c_{ve} \frac{\partial T_e}{\partial t} - \nabla \cdot (\kappa_e \nabla T_e) = \rho \omega_{ei} (T_i - T_e) + \rho \omega_{er} (T_r - T_e) + Q_e, \qquad (2.1)$$

$$\rho c_{vi} \frac{\partial T_i}{\partial t} - \nabla \cdot (\kappa_i \nabla T_i) = \rho \omega_{ei} (T_e - T_i) + Q_i, \qquad (2.2)$$

$$\rho c_{vr} \frac{\partial T_r}{\partial t} - \nabla \cdot (\kappa_r \nabla T_r) = \rho \omega_{er} (T_e - T_r) + Q_r, \qquad (2.3)$$

where  $T_{\alpha}$ ,  $\alpha = e, i, r$ , are the electron, ion and photon temperatures, respectively,  $\rho$  denotes the material density,  $\omega_{ei}$  (resp.  $\omega_{er}$ ) is the energy exchange coefficient between electron and ion (resp. photon),  $\kappa_{\alpha} = \kappa_{\alpha}(T_{\alpha})$ ,  $\alpha = e, i, r$ , are scalar diffusion coefficients of the electron, ion and photon, respectively,  $c_{v\alpha}$ ,  $\alpha = e, i, r$ , are the thermal capacities of the electron, ion and photon, respectively, and  $Q_{\alpha}$ ,  $\alpha = e, i, r$ , are the source terms. All the aforementioned coefficients are assumed to be positive. The boundary and initial conditions are separately specified as follows:

$$\begin{cases} T_{\alpha}(\boldsymbol{x},t) = g_{\alpha}^{D}(\boldsymbol{x},t), & \text{on } \Gamma_{D}, \\ -\kappa_{\alpha} \frac{\partial T_{\alpha}(\boldsymbol{x},t)}{\partial n} = g_{\alpha}^{N}(\boldsymbol{x},t), & \text{on } \Gamma_{N}, \end{cases}$$
(2.4)

and

$$T_{\alpha}(\boldsymbol{x},0) = h_{\alpha}(\boldsymbol{x}), \quad \text{in } \Omega, \tag{2.5}$$

where  $\Gamma_D$  and  $\Gamma_N$  denote respectively Dirichlet and Neumann boundaries,  $\boldsymbol{n}$  is the unit outward normal vector to the boundary  $\partial \Omega$ , and  $g^D_{\alpha}(\boldsymbol{x},t)$ ,  $g^N_{\alpha}(\boldsymbol{x},t)$  and  $h_{\alpha}(\boldsymbol{x})$  are given functions,  $\alpha = e, i, r$ .

As shown in Fig. 1(a), the domain  $\Omega$  is partitioned into a number of non-overlapped polygonal cells that form the so-called *primary mesh*, depicted by solid line segments. Make sure that the line segments of the primary mesh are aligned with possible discontinuities. The hollow circles, solid circles, and hollow squares denote respectively the





(a) The primary mesh and its dual mesh.

(b) Notations related to a primary cell and a dual cell.

Figure 1: Meshes and the related notations.

cell centers, the vertices, and the edge midpoints of the primary mesh. Each cell in the primary mesh (i.e., the primary cell) is further partitioned into several quadrilateral subcells by connecting the cell center with the edge midpoints. All subcells sharing the same vertex in the primary mesh form a polygonal cell of the *dual mesh*. For example, the shadowed polygon in Fig. 1(a) stands for the dual cell associated with the vertex  $x_v$ . The dual mesh makes sense under the following geometric assumption:

(H1) All primary cells are star-shaped with respect to their cell centers, i.e., any ray emanating from the cell center intersects the cell boundary at exactly one point.

Denote  $\mathcal{M} = \{K\}$  the set of all primary cells, such that  $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K}$ . See Fig. 1(b), for a generic primary cell K in  $\mathcal{M}$  (the quadrilateral with solid line segments), its number of edges, cell center, measure, set of edges are denoted as  $n_K$ ,  $\mathbf{x}_K$ , |K| and  $\mathcal{E}_K$ , respectively. Define  $h_K = \max_{\mathbf{x}_A, \mathbf{x}_B \in \overline{K}} |\mathbf{x}_A \mathbf{x}_B|$  and denote the mesh size by  $h = \max_{K \in \mathcal{M}} h_K$ . For  $\sigma \in \mathcal{E}_K$ , its measure, unit outward normal vector, midpoint and two endpoints are denoted as  $|\sigma|$ ,  $n_{K,\sigma}$ ,  $\mathbf{x}_{\sigma}$ ,  $\mathbf{x}_{\nu}$  and  $\mathbf{x}_{\nu'}$ , respectively. For  $\mathbf{x}_{\nu} \in \overline{\Omega}$ , denote its associated dual cell by  $K_{\nu}^*$  and the unit outward normal vector to  $\partial K_{\nu}^*$  by  $\mathbf{n}_{\nu}^*$ .  $\sigma_K^*$  is the dual edge connecting  $\mathbf{x}_K$  and  $\mathbf{x}_{\sigma}$ .

Here we introduce two unknowns on the primary mesh and its dual counterpart, respectively. The *cell-vertex unknowns* are defined at the vertices of the primary mesh in  $\Omega$  or on  $\partial \Omega \setminus \Gamma_D$ . The *cell-centered unknowns* are defined at the cell centers of the primary mesh. Each cell-vertex or cell-centered unknown has an FV equation associated with it. In addition, we denote  $T_{\alpha,K}$  and  $T_{\alpha,\nu}$  as the approximation of  $T_{\alpha}$  at  $x_K$  and  $x_{\nu}$ , respectively.

Throughout this paper, the bold capital letter, such as **F**, **T**, denotes the vector or the matrix with one column, while the hollow capital letter, such as  $\mathbb{A}$ ,  $\mathbb{E}$ , represents the matrix with more than one column.

## 3 Positivity-preserving scheme

Our positivity-preserving scheme consists of the predictor and corrector phases. The cellvertex solutions are obtained in the predictor phase, while the cell-centered solutions are finally determined in the corrector phase. Hence, the FV equations for the cell-vertex and cell-centered unknowns are orderly constructed in those two phases. Hereafter, we use the tilde notation i.e.  $\tilde{T}$  for the temperatures obtained from the predictor phase to distinguish them from the corrector phase.

## 3.1 Predictor phase

Integrating (2.1)-(2.3) over the dual cell  $K_{\nu}^*$  with respect to  $x_{\nu}$  respectively, and using the Gauss divergence theorem, yields

$$\int_{K_{\nu}^{*}}\rho c_{\nu e}\frac{\partial T_{e}}{\partial t}d\mathbf{x} + \sum_{\sigma_{K}^{*}\in\partial K_{\nu}^{*}}\mathcal{F}_{K,\sigma^{*}}^{e} = \int_{K_{\nu}^{*}}\rho\omega_{ei}(T_{i}-T_{e})d\mathbf{x} + \int_{K_{\nu}^{*}}\rho\omega_{er}(T_{r}-T_{e})d\mathbf{x} + \int_{K_{\nu}^{*}}Q_{e}d\mathbf{x}, \quad (3.1)$$

$$\int_{K_{\nu}^{*}} \rho c_{\nu i} \frac{\partial T_{i}}{\partial t} d\mathbf{x} + \sum_{\sigma_{K}^{*} \in \partial K_{\nu}^{*}} \mathcal{F}_{K,\sigma^{*}}^{i} = \int_{K_{\nu}^{*}} \rho \omega_{ei} (T_{e} - T_{i}) d\mathbf{x} + \int_{K_{\nu}^{*}} Q_{i} d\mathbf{x},$$
(3.2)

$$\int_{K_{\nu}^{*}}\rho c_{\nu r} \frac{\partial T_{r}}{\partial t} d\mathbf{x} + \sum_{\sigma_{K}^{*} \in \partial K_{\nu}^{*}} \mathcal{F}_{K,\sigma^{*}}^{r} = \int_{K_{\nu}^{*}} \rho \omega_{er} (T_{e} - T_{r}) d\mathbf{x} + \int_{K_{\nu}^{*}} Q_{r} d\mathbf{x},$$
(3.3)

where  $\partial K_{\nu}^*$  is the boundary of  $K_{\nu}^*$ , and  $\mathcal{F}_{K,\sigma^*}^{\alpha}$  denotes the continuous flux on the dual edge  $\sigma_{K'}^*$  i.e.,

$$\mathcal{F}^{\alpha}_{K,\sigma^*} = -\int_{\sigma^*_K} \kappa_{\alpha} \nabla T_{\alpha} \cdot \boldsymbol{n}^*_{\nu} ds, \qquad (3.4)$$

where  $n_{\nu}^*$  is the unit outward normal vector to  $\partial K_{\nu}^*$ . In the following, we will discretize the continuous flux  $\mathcal{F}_{K,\sigma^*}^{\alpha}$  and construct the fully discrete FV equations for the cell-vertex unknowns.

#### 3.1.1 Flux approximations on the dual edges

Instead of building those flux approximations one after another, we gather all the ones inside *K* into a vector  $\mathbf{F}_{K}^{\alpha} = (F_{K,\sigma^*}^{\alpha}, \sigma \in \mathcal{E}_K)^{\mathsf{T}} \in \mathbb{R}^{n_K}$  where  $F_{K,\sigma^*}^{\alpha}$  denotes the flux approximation across the dual edge  $\sigma_K^{\alpha}$ , given by

$$F^{\alpha}_{K,\sigma^*} \approx -\int_{\sigma^*_K} \kappa_{\alpha} \nabla T_{\alpha} \cdot \boldsymbol{n}^*_{K,\sigma} ds,$$

where  $n_{K,\sigma}^*$  is a unit vector normal to  $\sigma_K^*$  and its direction is fixed (See Fig. 1(b)). All  $n_{K,\sigma}^*$  inside *K* are oriented clockwise so that  $n_{\nu}^* = -n_{K,\sigma}^*$  and  $n_{\nu'}^* = n_{K,\sigma}^*$ . From (3.4), we have

$$F_{K,\sigma^*}^{\alpha} \approx (\boldsymbol{n}_{K,\sigma}^* \cdot \boldsymbol{n}_{\nu}^*) \mathcal{F}_{K,\sigma^*}^{\alpha}.$$
(3.5)

We calculate  $F_K^{\alpha}$  by the following algebraic system

$$F_K^{\alpha} = \mathbb{A}_K^{\alpha} \delta \tilde{T}_K^{\alpha}, \tag{3.6}$$

where  $\mathbb{A}_{K}^{\alpha}$ , the cell matrix to be determined, is of size  $n_{K} \times n_{K}$  and  $\delta \tilde{T}_{K}^{\alpha} = (\tilde{T}_{\alpha,\nu'} - \tilde{T}_{\alpha,\nu}, \sigma = [x_{\nu};x_{\nu'}] \in \mathcal{E}_{K})^{\mathsf{T}} \in \mathbb{R}^{n_{K}}$  is a vector containing all the successive differences of the cell-vertex unknowns (ordered anticlockwise) along the boundary  $\partial K$ . Once the cell matrix  $\mathbb{A}_{K}^{\alpha}$  is specified, all the  $n_{K}$  flux discretizations inside K are obtained. In the following, we introduce a constructive algorithm for  $\mathbb{A}_{K}^{\alpha}$ .

According to the linearity-preserving criterion [39] (i.e., all derivations are exact whenever the solution is piecewise linear and the diffusion tensor is piecewise constant with respect to the primary mesh), we require that (3.6) is exact for the linear solution  $\tilde{T}_{\alpha}$  and a constant diffusion coefficient  $\kappa_{\alpha} = \kappa_{\alpha,K}$ , where  $\kappa_{\alpha,K}$  is the constant restriction of the diffusion coefficient  $\kappa_{\alpha}$  on *K*. Then we have

$$\kappa_{\alpha,K} \mathbb{F}_K = \mathbb{A}_K^{\alpha} \mathbb{X}_K, \tag{3.7}$$

where  $\mathbb{F}_K$  and  $\mathbb{X}_K$  are two  $n_K \times 2$  matrices, given by

$$\mathbb{F}_{K} = \left(-\left|\sigma_{K}^{*}\right| \boldsymbol{n}_{K,\sigma}^{*}, \sigma \in \mathcal{E}_{K}\right)^{\mathsf{T}}, \quad \mathbb{X}_{K} = \left(\boldsymbol{x}_{\nu'} - \boldsymbol{x}_{\nu}, \sigma \in \mathcal{E}_{K}\right)^{\mathsf{T}}, \tag{3.8}$$

here  $|\sigma_K^*|$  is the measure of  $\sigma_K^*$ . Let

$$\mathbb{A}_{K}^{\alpha} = \kappa_{\alpha,K} \mathbb{A}_{K},$$

where  $\mathbb{A}_K$  is not related to  $\tilde{T}_{\alpha,K}$  and  $\kappa_{\alpha,K}$ . As a result, we have

$$\mathbb{F}_K = \mathbb{A}_K \mathbb{X}_K. \tag{3.9}$$

Recalling the fact that (see Lemma 3.1 in [40])

$$\mathbb{F}_{K}^{\mathsf{T}} \mathbb{X}_{K} = |K| \mathbb{I}_{2}, \tag{3.10}$$

where  $\mathbb{I}_2$  is the 2×2 identity matrix, (3.10) can be rewritten as

$$\frac{1}{|K|} \mathbb{F}_K \mathbb{F}_K^\mathsf{T} \mathbb{X}_K = \mathbb{F}_K$$

Comparing it to (3.9) and adding a certain stabilized term [40] gives

$$\mathbb{A}_{K} = \frac{1}{|K|} \mathbb{F}_{K} \mathbb{F}_{K}^{\mathsf{T}} + \gamma_{K} \mathbb{C}_{K} \mathbb{C}_{K}^{\mathsf{T}}, \qquad (3.11)$$

where  $\gamma_K$  is a positive parameter and  $\mathbb{C}_K$  is an  $n_K \times n_K$  matrix, whose column vectors span the null space of  $\mathbb{X}_K^{\mathsf{T}}$  [40]. Generally speaking,  $\mathbb{C}_K$  is not unique and two candidates are listed below

$$\mathbb{C}_{K} = \mathbb{I}_{K} - \frac{1}{|K|} \mathbb{F}_{K} \mathbb{X}_{K'}^{\mathsf{T}}$$
(3.12)

or

$$\mathbf{C}_{K} = \mathbf{I}_{K} - \mathbf{X}_{K} (\mathbf{X}_{K}^{\mathsf{T}} \mathbf{X}_{K})^{-1} \mathbf{X}_{K}^{\mathsf{T}}, \tag{3.13}$$

where  $\mathbb{I}_K$  is the  $n_K \times n_K$  identity matrix. In [38], a formula for the inverse matrix of  $\mathbb{A}_K$  was given as follows

$$\mathbb{A}_{K}^{-1} = \frac{1}{|K|} \mathbb{X}_{K} \mathbb{X}_{K}^{\mathsf{T}} + \gamma_{K} \mathbb{C}_{K}^{\mathsf{T}} \mathbb{C}_{K}, \qquad (3.14)$$

where  $\mathbb{C}_K$  is defined by (3.12). In the numerical examples, we choose (3.14).  $\gamma_K$  is related to  $\underline{\varrho}_K$  in Lemma 4.1, see Theorem 5 in [38]. To balance the first and the stabilized terms in (3.14) and then improve the property of  $\mathbb{A}_K$ , a natural selection of  $\gamma_K$  is trace $((1/|K|) \mathbb{X}_K \mathbb{X}_K^T)$  [38].

**Lemma 3.1** (Theorem 3.2 in [40] or Theorem 2 in [38]). *The cell matrix*  $\mathbb{A}_K$ , *defined by* (3.11) *or* (3.14), *satisfies* (3.9) *and is symmetric and positive definite.* 

There are no finite volume equations associated with the vertices on the Dirichlet boundary. If  $x_{\nu} \in \Gamma_N$ , the dual cell of  $x_{\nu}$  has some edges located on the Neumann boundary and the Neumann boundary data are directly integrated into the corresponding finite volume equation.

**Remark 3.1.** It is emphasized that the nonlinear diffusion coefficients are defined at the centers of the primary cells. To calculate the nonlinear diffusion coefficients, the cell-centered temperatures are required. In order to decouple the cell-centered unknowns and the cell-vertex unknowns, here the cell-centered temperatures are interpolated by the cell-vertex unknowns. Under the assumption (H1), the second-order positivity-preserving interpolation algorithm proposed in [38] is a candidate. Here, for self-completeness of the present paper, we give its brief description. Let the vertices of cell *K* be denoted as  $x_i$  ( $1 \le i \le n_K$ ), which are ordered anticlockwisely. We have the following formula

$$\tilde{T}_{\alpha,K} = \frac{1}{\sum_{i=1}^{n_{K}} (\zeta_{i} + \eta_{i})} \sum_{i=1}^{n_{K}} (\zeta_{i} \tilde{T}_{\alpha,j(i)} + \eta_{i} \tilde{T}_{\alpha,j(i)+1}),$$
(3.15)

where  $\tilde{T}_{\alpha,l}(l=j(i),j(i)+1)$  denotes the approximation of  $T_{\alpha}$  at  $x_l$ 

$$\zeta_i = \frac{\mathcal{R}(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)^{\mathsf{T}} \mathcal{R}(\boldsymbol{x}_{j(i)+1} - \boldsymbol{x}_K)}{(\boldsymbol{x}_{j(i)} - \boldsymbol{x}_K)^{\mathsf{T}} \mathcal{R}(\boldsymbol{x}_{j(i)+1} - \boldsymbol{x}_K)}, \quad \eta_i = \frac{\mathcal{R}(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)^{\mathsf{T}} \mathcal{R}(\boldsymbol{x}_{j(i)} - \boldsymbol{x}_K)}{(\boldsymbol{x}_{j(i)+1} - \boldsymbol{x}_K)^{\mathsf{T}} \mathcal{R}(\boldsymbol{x}_{j(i)} - \boldsymbol{x}_K)}, \quad \mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and index j(i) is selected in such a way that the normal vector  $\mathcal{R}(\mathbf{x}_{i+1}-\mathbf{x}_i)$  is located between  $\mathbf{x}_{j(i)} - \mathbf{x}_K$  and  $\mathbf{x}_{j(i)+1} - \mathbf{x}_K$ . Seeing that  $\tilde{T}_{\alpha,l}$  may be negative, then  $\tilde{T}_{\alpha,K}$  may not be nonnegative by (3.15). To maintain the positivity of the nonlinear diffusion coefficients, we can use the cutoff method [20] to obtain the cell-centered diffusion coefficients by the following

$$\kappa_{\alpha,K} = \kappa_{\alpha}(\max\{\tilde{T}_{\alpha,K},\varepsilon\}), \qquad (3.16)$$

where  $\varepsilon$  is a positive number up to the machine precision.

#### 3.1.2 FV equations of the cell-vertex unknowns

Since the parameters  $\rho$ ,  $c_{v\alpha}$ ,  $\omega_{ei}$  and  $\omega_{er}$  are possibly discontinuous across the interfaces of different material areas, the approximate values of these parameters inside the dual cells need to be considered specially. Let  $\eta$  (= $\rho$ ,  $c_{v\alpha}$ ,  $\omega_{ei}$  or  $\omega_{er}$ ) be a possibly discontinuous parameter, then the restriction of  $\eta$  on the dual cell  $K_{\nu}^{*}$  is given by

$$\eta_{\nu}(t) := \frac{1}{|K_{\nu}^{*}|} \int_{K_{\nu}^{*}} \eta(\mathbf{x}, t) d\mathbf{x} \approx \frac{1}{|K_{\nu}^{*}|} \sum_{K \in \mathcal{M}_{\nu}} \eta(\mathbf{x}_{K}, t) |K_{\nu}^{*} \cap K|,$$
(3.17)

where  $M_{\nu}$  denotes the set of primary cells sharing the vertex  $x_{\nu}$ . If  $\eta$  is dependent of  $T_{\alpha}$ , (3.17) is updated by

$$\eta_{\nu}(t) \approx \frac{1}{|K_{\nu}^{*}|} \sum_{K \in \mathcal{M}_{\nu}} \eta(\mathbf{x}_{K}, \tilde{T}_{\alpha, K}, t) |K_{\nu}^{*} \cap K|, \qquad (3.18)$$

where  $\tilde{T}_{\alpha,K}$  is obtained by (3.15).

For the temporal discretization, we use the backward Euler method. Let [0,T] be split by  $0=t_0 < t_1 < \cdots < t_{N-1} < t_N = T$  with the time stepsize  $\tau$ . Let  $\mathcal{E}_{\nu}$  be the set of primary cell edges sharing the vertex  $\mathbf{x}_{\nu}$ . By the definition of the discrete flux  $\mathbf{F}_{K}^{\alpha} = (\mathbf{F}_{K,\sigma^*}^{\alpha}, \sigma \in \mathcal{E}_{K})^{\mathsf{T}}$  in (3.6), the FV equations with respect to the vertex  $\mathbf{x}_{\nu} \in \overline{\Omega} \setminus \Gamma_D$  are constructed as follows:

$$E_{e,v}^{n+1} \frac{\tilde{T}_{e,v}^{n+1} - \tilde{T}_{e,v}^{n}}{\tau} + \sum_{K \in \mathcal{M}_{v}} \sum_{\sigma \subset \mathcal{E}_{K} \cap \mathcal{E}_{v}} (\boldsymbol{n}_{K,\sigma}^{*} \cdot \boldsymbol{n}_{v}^{*}) F_{K,\sigma^{*}}^{e,n+1} \\ = W_{ei,v}^{n+1} (\tilde{T}_{i,v}^{n+1} - \tilde{T}_{e,v}^{n+1}) + W_{er,v}^{n+1} (\tilde{T}_{r,v}^{n+1} - \tilde{T}_{e,v}^{n+1}) + |K_{v}^{*}| Q_{e,v}^{n+1}, \quad (3.19)$$

$$E_{i,\nu}^{n+1} \frac{T_{i,\nu}^{n+1} - T_{i,\nu}^{n}}{\tau} + \sum_{K \in \mathcal{M}_{\nu} \sigma \subset \mathcal{E}_{K} \cap \mathcal{E}_{\nu}} \sum_{(n_{K,\sigma}^{*} \cdot n_{\nu}^{*})} F_{K,\sigma^{*}}^{i,n+1} = W_{ei,\nu}^{n+1} (\tilde{T}_{e,\nu}^{n+1} - \tilde{T}_{i,\nu}^{n+1}) + |K_{\nu}^{*}| Q_{i,\nu}^{n+1}, \quad (3.20)$$

$$E_{r,\nu}^{n+1} \frac{\tilde{T}_{r,\nu}^{n+1} - \tilde{T}_{r,\nu}^{n}}{\tau} + \sum_{K \in \mathcal{M}_{\nu}} \sum_{\sigma \subset \mathcal{E}_{K} \cap \mathcal{E}_{\nu}} (\boldsymbol{n}_{K,\sigma}^{*} \cdot \boldsymbol{n}_{\nu}^{*}) F_{K,\sigma^{*}}^{r,n+1} = W_{er,\nu}^{n+1} (\tilde{T}_{e,\nu}^{n+1} - \tilde{T}_{r,\nu}^{n+1}) + |K_{\nu}^{*}| Q_{r,\nu}^{n+1}, \quad (3.21)$$

where

$$Q_{\alpha,\nu}^{n+1} = \frac{1}{|K_{\nu}^{*}|} \int_{K_{\nu}^{*}} Q_{\alpha}(\mathbf{x}, t_{n+1}) d\mathbf{x},$$
  

$$E_{e,\nu}^{n+1} = \rho_{\nu}^{n+1} c_{ve,\nu}^{n+1} |K_{\nu}^{*}|, \quad E_{i,\nu}^{n+1} = \rho_{\nu}^{n+1} c_{vi,\nu}^{n+1} |K_{\nu}^{*}|, \quad E_{r,\nu}^{n+1} = \rho_{\nu}^{n+1} c_{vr,\nu}^{n+1} |K_{\nu}^{*}|.$$
  

$$W_{ei,\nu}^{n+1} = \rho_{\nu}^{n+1} \omega_{ei,\nu}^{n+1} |K_{\nu}^{*}|, \quad W_{er,\nu}^{n+1} = \rho_{\nu}^{n+1} \omega_{er,\nu}^{n+1} |K_{\nu}^{*}|.$$

Here  $\rho_{\nu}^{n+1}$ ,  $c_{v\alpha,\nu}^{n+1}$ ,  $\omega_{ei,\nu}^{n+1}$  and  $\omega_{er,\nu}^{n+1}$  are computed by (3.17) or (3.18) with  $t = t_{n+1}$ . Note that by (3.5), the factor  $(\mathbf{n}_{K,\sigma}^* \cdot \mathbf{n}_{\nu}^*)$  (=1 or -1) is added to preserve the true flux orientation.

#### 3.2 Corrector phase

Integrating (2.1)-(2.3) over the primary cell K, using the Gauss divergence theorem, we obtain

$$\int_{K} \rho c_{ve} \frac{\partial T_{e}}{\partial t} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_{K}} \mathcal{F}_{K,\sigma}^{e} = \int_{K} \rho \omega_{ei} (T_{i} - T_{e}) d\mathbf{x} + \int_{K} \rho \omega_{er} (T_{r} - T_{e}) d\mathbf{x} + \int_{K} Q_{e} d\mathbf{x}, \qquad (3.22)$$

$$\int_{K} \rho c_{vi} \frac{\partial T_{i}}{\partial t} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_{K}} \mathcal{F}_{K,\sigma}^{i} = \int_{K} \rho \omega_{ei} (T_{e} - T_{i}) d\mathbf{x} + \int_{K} Q_{i} d\mathbf{x}, \qquad (3.23)$$

$$\int_{K} \rho c_{vr} \frac{\partial T_{r}}{\partial t} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_{K}} \mathcal{F}_{K,\sigma}^{r} = \int_{K} \rho \omega_{er} (T_{e} - T_{r}) d\mathbf{x} + \int_{K} Q_{r} d\mathbf{x}, \qquad (3.24)$$

where  $\mathcal{F}_{K,\sigma}^{\alpha}$  denotes the continuous flux on the primary edge  $\sigma$ , i.e.,

$$\mathcal{F}^{\alpha}_{K,\sigma} = -\int_{\sigma} \kappa_{\alpha} \nabla T_{\alpha} \cdot \boldsymbol{n}_{K,\sigma} ds. \qquad (3.25)$$

Next we will approximate  $\mathcal{F}_{K,\sigma}^{\alpha}$  and construct the FV equations for the cell-centered unknowns. To preserve the positivity of the cell-centered unknowns, in the most existing work, such as [33], the discretization of the continuous primary flux is based on the convex decomposition of the co-normal  $\kappa_{\alpha} n_{K,\sigma}$ . Along this approach, the nonnegativity of the cell-vertex unknowns required. Hence, an additional procedure, such as the truncation technique, is adopted to maintain the nonnegativity of the cell-vertex unknowns, which leads to a possible violation of the local conservation on the dual mesh. To overcome this problem, motivated by [10], here we suggest a new approach to construct the FV equations for the cell-centered unknowns.

## 3.2.1 Flux approximations on the primary edges

As shown in Fig. 2, we recall that the primary cell *K* is a star-shaped cell with respect to its center  $x_K$ , and  $\sigma$  is an edge of *K* with the endpoints  $x_{\nu}$  and  $x_{\nu'}$ . Then for the normal vector  $n_{K,\sigma}$ , we have the following fixed decomposition

$$|\sigma|\mathbf{n}_{K,\sigma} = \beta_{K,\sigma} \mathbf{x}_K \mathbf{x}_\nu + \gamma_{K,\sigma} \mathbf{x}_K \mathbf{x}_{\nu'}, \qquad (3.26)$$



Figure 2: Notations of the normal decomposition.

S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

where  $x_K x_{\nu}$  (resp.  $x_K x_{\nu'}$ ) denotes the column vector pointing from  $x_K$  to  $x_{\nu}$  (resp.  $x_{\nu'}$ ) and

$$\beta_{K,\sigma} = \frac{|\sigma| \boldsymbol{n}_{K,\sigma} \cdot \mathcal{R} \boldsymbol{x}_K \boldsymbol{x}_{\nu'}}{\boldsymbol{x}_K \boldsymbol{x}_\nu \cdot \mathcal{R} \boldsymbol{x}_K \boldsymbol{x}_{\nu'}}, \quad \gamma_{K,\sigma} = \frac{|\sigma| \boldsymbol{n}_{K,\sigma} \cdot \mathcal{R} \boldsymbol{x}_K \boldsymbol{x}_\nu}{\boldsymbol{x}_K \boldsymbol{x}_{\nu'} \cdot \mathcal{R} \boldsymbol{x}_K \boldsymbol{x}_{\nu'}}.$$

**Lemma 3.2** (Theorem 3.1 in [10]). Assume that K is star-shaped with respect to its center  $x_K$ . Then, for the coefficients in (3.26), we have

$$\beta_{K,\sigma} + \gamma_{K,\sigma} > 0. \tag{3.27}$$

From (3.26) and the definition of the flux (3.25), we obtain the one-sided flux approximation

$$F_{K,\sigma}^{\alpha} = \kappa_{K}^{\alpha} (\beta_{K,\sigma} (T_{\alpha,K} - \tilde{T}_{\alpha,\nu}) + \gamma_{K,\sigma} (T_{\alpha,K} - \tilde{T}_{\alpha,\nu'})).$$
(3.28)

Similarly, if  $\sigma \in \mathcal{E}_L$  for the neighboring cell *L*, we can obtain another one-sided flux approximation

$$F_{L,\sigma}^{\alpha} = \kappa_{L}^{\alpha}(\beta_{L,\sigma}(T_{\alpha,L} - \tilde{T}_{\alpha,\nu}) + \gamma_{L,\sigma}(T_{\alpha,L} - \tilde{T}_{\alpha,\nu'})).$$
(3.29)

For an interior edge  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_L$ , we define a new and unique flux as a combination of the above two one-sided fluxes [29], i.e.

$$\widetilde{F}_{K,\sigma}^{\alpha} = \mu_{K,\sigma} F_{K,\sigma}^{\alpha} - \mu_{L,\sigma} F_{L,\sigma}^{\alpha}, \quad \widetilde{F}_{L,\sigma}^{\alpha} = \mu_{L,\sigma} F_{L,\sigma}^{\alpha} - \mu_{K,\sigma} F_{K,\sigma}^{\alpha}, \quad (3.30)$$

where  $\mu^{\alpha}_{K,\sigma}$  and  $\mu^{\alpha}_{L,\sigma}$  are two nonnegative parameters to be determined, satisfying

$$\mu_{K,\sigma}^{\alpha} + \mu_{L,\sigma}^{\alpha} = 1. \tag{3.31}$$

Obviously, we have the local conservation condition

$$\widetilde{F}^{\alpha}_{K,\sigma} + \widetilde{F}^{\alpha}_{L,\sigma} = 0. \tag{3.32}$$

Substituting (3.28) and (3.29) into the first equation of (3.30), gives

$$\tilde{F}^{\alpha}_{K,\sigma} = \mu^{\alpha}_{K,\sigma} \kappa^{\alpha}_{K} (\beta_{K,\sigma} + \gamma_{K,\sigma}) T_{\alpha,K} - \mu^{\alpha}_{L,\sigma} \kappa^{\alpha}_{L} (\beta_{L,\sigma} + \gamma_{L,\sigma}) T_{\alpha,L} + B^{\alpha}_{\sigma},$$
(3.33)

where

$$B^{\alpha}_{\sigma} = -\mu^{\alpha}_{K,\sigma} a^{\alpha}_{K,\sigma} + \mu^{\alpha}_{L,\sigma} a^{\alpha}_{L,\sigma} \tag{3.34}$$

and

$$a_{K,\sigma}^{\alpha} = \kappa_{K}^{\alpha}(\beta_{K,\sigma}\tilde{T}_{\alpha,\nu} + \gamma_{K,\sigma}\tilde{T}_{\alpha,\nu'}), \quad a_{L,\sigma}^{\alpha} = \kappa_{L}^{\alpha}(\beta_{L,\sigma}\tilde{T}_{\alpha,\nu} + \gamma_{L,\sigma}\tilde{T}_{\alpha,\nu'}).$$

To maintain the positivity-preserving property, a sufficient condition is that the basis of the entire formulation, i.e. the flux approximation (3.33) possesses two-point type with nonnegative coefficients. Based on (3.31) and (3.34), we choose

$$\mu_{K,\sigma}^{\alpha} = \begin{cases} 0.5, & \text{if } a_{K,\sigma}^{\alpha} = a_{L,\sigma}^{\alpha} = 0, \\ \frac{|a_{L,\sigma}^{\alpha}|}{|a_{K,\sigma}^{\alpha}| + |a_{L,\sigma}^{\alpha}|}, & \text{otherwise,} \end{cases}$$
(3.35)

and  $\mu_{L,\sigma}^{\alpha} = 1 - \mu_{K,\sigma}^{\alpha}$ . It is easy to see that  $B_{\sigma}^{\alpha} = 0$  if  $a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} \ge 0$  so that (3.33) possesses two-point type with nonnegative coefficients. Special treatment is needed for the case of  $a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} < 0$ . Define  $B_{\sigma}^{\alpha,\pm} = (|B_{\sigma}^{\alpha}| \pm B_{\sigma}^{\alpha})/2$  and rewrite  $B_{\sigma}^{\alpha}$  as

$$B_{\sigma}^{\alpha} = \frac{B_{\sigma}^{\alpha,+}}{|\tilde{T}_{\alpha,K}|} T_{\alpha,K} - \frac{B_{\sigma}^{\alpha,-}}{|\tilde{T}_{\alpha,L}|} T_{\alpha,L} + \widetilde{B}_{\sigma}^{\alpha},$$

where  $\tilde{T}_{\alpha,K}$  and  $\tilde{T}_{\alpha,L}$  are interpolated by (3.15), and

$$\widetilde{B}^{\alpha}_{\sigma} = \frac{B^{\alpha,+}_{\sigma}}{|\widetilde{T}_{\alpha,K}|} (|\widetilde{T}_{\alpha,K}| - T_{\alpha,K}) - \frac{B^{\alpha,-}_{\sigma}}{|\widetilde{T}_{\alpha,L}|} (|\widetilde{T}_{\alpha,L}| - T_{\alpha,L}).$$
(3.36)

We ignore  $\tilde{B}^{\alpha}_{\sigma}$  to reach a new definition of the unique edge flux. Finally, the flux approximations are given by

$$\widetilde{F}_{K,\sigma}^{\alpha} = A_{K,\sigma}^{\alpha} T_{\alpha,K} - A_{L,\sigma}^{\alpha} T_{\alpha,L}, \quad \widetilde{F}_{L,\sigma}^{\alpha} = A_{L,\sigma}^{\alpha} T_{\alpha,L} - A_{K,\sigma}^{\alpha} T_{\alpha,K}$$
(3.37)

where

$$A_{K,\sigma}^{\alpha} = \begin{cases} \mu_{K,\sigma}^{\alpha} \kappa_{K}^{\alpha} (\beta_{K,\sigma} + \gamma_{K,\sigma}), & a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} \ge 0, \\ \mu_{K,\sigma}^{\alpha} \kappa_{K}^{\alpha} (\beta_{K,\sigma} + \gamma_{K,\sigma}) + \frac{B_{\sigma}^{\alpha,+}}{|\hat{T}_{\alpha,K}|}, & a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} < 0, \end{cases} \\ A_{L,\sigma}^{\alpha} = \begin{cases} \mu_{L,\sigma}^{\alpha} \kappa_{L}^{\alpha} (\beta_{L,\sigma} + \gamma_{L,\sigma}), & a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} \ge 0, \\ \mu_{L,\sigma}^{\alpha} \kappa_{L}^{\alpha} (\beta_{L,\sigma} + \gamma_{L,\sigma}) + \frac{B_{\sigma}^{\alpha,-}}{|\hat{T}_{\alpha,L}|}, & a_{K,\sigma}^{\alpha} a_{L,\sigma}^{\alpha} < 0. \end{cases} \end{cases}$$

It is obvious that these two-point fluxes are nonlinear and the local conservation condition (3.32) is still maintained.

**Remark 3.2.** Here we evaluate the truncation term  $\tilde{B}^{\alpha}_{\sigma}$ . On the one hand,  $\tilde{T}_{\alpha,K} - T_{\alpha,K} = \mathcal{O}(h^2)$  for any  $K \in \mathcal{M}$  holds in the sense of truncation error. If  $T_{\alpha,K}$  is nonnegative (which will be proved in Theorem 4.1), we can have  $|\tilde{T}_{\alpha,K}| - T_{\alpha,K} \leq |\tilde{T}_{\alpha,K} - T_{\alpha,K}| = \mathcal{O}(h^2)$ . On the other hand, it is deduced from (3.35) and (3.34) that

$$|B^{\alpha}_{\sigma}| = \frac{-2a^{\alpha}_{K,\sigma}a^{\alpha}_{L,\sigma}}{|a^{\alpha}_{L,\sigma} - a^{\alpha}_{K,\sigma}|} \le \max_{\sigma \in \mathcal{E}} \{|a^{\alpha}_{K,\sigma}|, |a^{\alpha}_{L,\sigma}|\}$$

for the case of  $a_{K,\sigma}^{\alpha}a_{L,\sigma}^{\alpha} < 0$ . Here  $|a_{K,\sigma}^{\alpha}| \le \kappa_{K}^{\alpha}\max\{|\beta_{K,\alpha}|, |\gamma_{K,\alpha}|\}\max\{|\tilde{T}_{\alpha,\nu}|, |\tilde{T}_{\alpha,\nu'}|\}$ , where  $\beta_{K,\alpha}$  and  $\gamma_{K,\alpha}$  are independent of *h*. Hence, there exists a constant *C* which depends only on the mesh and  $\kappa_{\alpha}$ , such that  $|B_{\sigma}^{\alpha}| \le \max_{\sigma \in \mathcal{E}}\{|a_{K,\sigma}^{\alpha}|, |a_{L,\sigma}^{\alpha}|\} \le C$ . As a result, the term  $\widetilde{B}_{\sigma}^{\alpha} = \mathcal{O}(h^{2})$  can be obtained if  $\tilde{T}_{\alpha,K}\tilde{T}_{\alpha,L} \neq 0$ .

In practical computation, we introduce a small positive quantity  $\varepsilon$  such that  $\tilde{T}_{\alpha,K} = \varepsilon$  if  $\tilde{T}_{\alpha,K} = 0$ . In general,  $\varepsilon$  is chosen to be a small number, say,  $10^{-10}$  in double precision. Numerical results in Section 5 show that this procedure does not affect the accuracy.

#### S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

As for a boundary edge  $\sigma \in \mathcal{E}_K \cap \Gamma_D$ , we have  $B^{\alpha}_{\sigma} = -a^{\alpha}_{K,\sigma}$ . Then we simply set

$$\widetilde{F}_{K,\sigma}^{\alpha} = \kappa_K^{\alpha} (\beta_{K,\sigma} + \gamma_{K,\sigma}) T_{\alpha,K} + B_{\sigma}^{\alpha,+} - B_{\sigma}^{\alpha,-}, \qquad (3.38)$$

where  $B_{\sigma}^{\alpha,+}$  can be handled in a way analogous to that of an interior edge while  $B_{\sigma}^{\alpha,-}$  is moved to the right-hand side of the final FV equation. If  $\sigma \in \Gamma_N$ , it can be deduced from the boundary condition (2.4) that

$$\widetilde{F}^{\alpha}_{K,\sigma} = \int_{\sigma} g^{N}_{\alpha} ds.$$
(3.39)

From the above derivation, we find that the new flux approximation has a fixed stencil and the cell-vertex unknowns are not required to be nonnegative. These features will make our scheme more flexible and easy for implementation.

## 3.2.2 FV equations for the cell-centered unknowns

Recall that the parameters in (3.22)-(3.24) are continuous on each primary cell. For a parameter  $\eta$  (= $\rho$ ,  $c_{v\alpha}$ ,  $\omega_{ei}$  or  $\omega_{er}$ ), the restriction on the primary cell *K* is given by

$$\eta_K(t) = \frac{1}{|K|} \int_K \eta(\mathbf{x}, t) d\mathbf{x}.$$
(3.40)

If  $\eta$  is dependent of  $T_{\alpha}$ , (3.40) is updated by

$$\eta_K(t) = \frac{1}{|K|} \int_K \eta(\mathbf{x}, \tilde{T}_{\alpha, K}, t) d\mathbf{x}, \qquad (3.41)$$

where  $\tilde{T}_{\alpha,K}$  is computed by (3.15).

Based on the flux discretizations in (3.37)-(3.39), we obtain the following FV equations of the cell-centered unknown on the primary cell *K*:

$$E_{e,K}^{n+1} \frac{T_{e,K}^{n+1} - T_{e,K}^{n}}{\tau} + \sum_{\sigma \subset \mathcal{E}_{K}} \widetilde{F}_{K,\sigma}^{e,n+1} = W_{ei,K}^{n+1} (T_{i,K}^{n+1} - T_{e,K}^{n+1}) + W_{er,K}^{n+1} (T_{r,K}^{n+1} - T_{e,K}^{n+1}) + |K|Q_{e,K}^{n+1},$$
(3.42)

$$E_{i,K}^{n+1} \frac{T_{i,K}^{n+1} - T_{i,K}^{n}}{\tau} + \sum_{\sigma \in \mathcal{E}_{K}} \widetilde{F}_{K,\sigma}^{i,n+1} = W_{ei,K}^{n+1} (T_{e,K}^{n+1} - T_{i,K}^{n+1}) + |K|Q_{i,K}^{n+1},$$
(3.43)

$$E_{r,K}^{n+1} \frac{T_{r,K}^{n+1} - T_{r,K}^{n}}{\tau} + \sum_{\sigma \subset \mathcal{E}_{K}} \widetilde{F}_{K,\sigma}^{r,n+1} = W_{er,K}^{n+1} (T_{e,K}^{n+1} - T_{r,K}^{n+1}) + |K|Q_{r,K}^{n+1},$$
(3.44)

where

$$Q_{\alpha,K}^{n+1} = \frac{1}{|K|} \int_{K} Q_{\alpha}(\boldsymbol{x}, t_{n+1}) d\boldsymbol{x}$$

and

$$E_{e,K}^{n+1} = \rho_K^{n+1} c_{ve,K}^{n+1} |K|, \quad E_{i,K}^{n+1} = \rho_K^{n+1} c_{vi,K}^{n+1} |K|, \quad E_{r,K}^{n+1} = \rho_K^{n+1} c_{vr,K}^{n+1} |K|,$$
$$W_{ei,K}^{n+1} = \rho_K^{n+1} \omega_{ei,K}^{n+1} |K|, \quad W_{er,K}^{n+1} = \rho_K^{n+1} \omega_{er,K}^{n+1} |K|.$$

Here,  $\rho_{K}^{n+1}$ ,  $c_{v\alpha,K}^{n+1}$ ,  $\omega_{ei,K}^{n+1}$  and  $\omega_{er,K}^{n+1}$  are computed by (3.40) or (3.41) with  $t = t_{n+1}$ .

### 3.3 Implementation of the scheme

Denote the numbers of elements in set  $\mathcal{V}$  and  $\mathcal{M}$  by  $N_{\nu}$  and  $N_c$ , respectively, where  $\mathcal{V}$  is the set of vertices in  $\overline{\Omega}/\Gamma_D$ . Let

$$\boldsymbol{T}_{\nu}^{n+1} = \left(\tilde{T}_{e,\nu}^{n+1}, \tilde{T}_{i,\nu}^{n+1}, \tilde{T}_{r,\nu}^{n+1}, \boldsymbol{x}_{\nu} \in \mathcal{V}\right)^{\mathsf{T}}, \quad \boldsymbol{T}_{c}^{n+1} = \left(T_{e,K}^{n+1}, T_{i,K}^{n+1}, T_{r,K}^{n+1}, K \in \mathcal{M}\right)^{\mathsf{T}}$$

be the cell-vertex and cell-centered unknown vectors with dimensions  $3N_{\nu}$  and  $3N_{c}$ , respectively. Recalling once again that the diffusion coefficients are defined at the cell centers, we define a vector with respect to the diffusion coefficients by

$$\mathbf{K}_{c}^{n+1} = (\kappa_{e,K}^{n+1}, \kappa_{i,K}^{n+1}, \kappa_{r,K}^{n+1}, K \in \mathcal{M})^{\mathsf{T}},$$

where  $\kappa_{\alpha,K}^{n+1}$  is computed by (3.16). Using those notations, the FV equations in (3.19)-(3.21) and (3.42)-(3.44) can be rewritten in the following matrix forms

$$(\mathbb{E}_{\nu}^{n+1} + \tau \mathbb{A}_{\nu}^{n+1}(K_{c}^{n+1}))T_{\nu}^{n+1} = \mathbb{E}_{\nu}^{n+1}T_{\nu}^{n} + \tau F_{\nu}^{n+1},$$
(3.45)

and

$$(\mathbb{E}_{c}^{n+1} + \tau \mathbb{A}_{c}^{n+1}(K_{c}^{n+1}, T_{v}^{n+1}))T_{c}^{n+1} = \mathbb{E}_{c}^{n+1}T_{c}^{n} + \tau F_{c}^{n+1},$$
(3.46)

where  $\mathbb{E}_{\nu}^{n+1}$  and  $\mathbb{E}_{c}^{n+1}$  are the diagonal matrices, whose diagonal entry are composed of  $E_{\alpha,\nu}^{n+1}$  and  $E_{\alpha,K}^{n+1}$ , respectively.  $\mathbb{A}_{\nu}^{n+1}$  and  $\mathbb{A}_{c}^{n+1}$  (resp.  $F_{\nu}^{n+1}$  and  $F_{c}^{n+1}$ ) are the coefficient matrices, generated from the flux approximations and the exchange term (resp. the source terms and the boundary data). It is easy to deduce from Lemma 3.1 that the coefficient matrix of (3.45) is symmetric and positive definite, while the coefficient matrix of (3.46) is a strict *M*-matrix [10].

At each time step, the cell-vertex solutions are obtained by solving (3.45) at the predictor phase, while the cell-centered solutions are determined by solving (3.46) at the corrector phase.

The detailed procedures to implement the algorithm are summarized in Algorithm 1.

Remark 3.3. For the above FV scheme, we have the following comments.

If (2.1)-(2.3) is a linear problem, then K<sup>n+1</sup><sub>c</sub> is independent of T<sup>n+1</sup><sub>α</sub> and as a result, (3.45) is a linear system. Then the scheme does not any need nonlinear iteration and we only need to solve two linear systems of equations.

A	Algorithm 1: The predictor-corrector algorithm.							
1	Compute $T_{\nu}^{0}$ and $T_{c}^{0}$ by the initial condition.							
2	do $n = 0, 1, 2, \cdots$							
3	Evaluate $K_c^{n+1}$ by the formula (3.16). Solve the nonlinear system (3.45) to obtain $T_v^{n+1}$ and $K_c^{n+1}$ .	The predictor phase						
4	Substitute $T_{\nu}^{n+1}$ and $K_{c}^{n+1}$ into (3.46) and solve the resulting linear system to get $T_{c}^{n+1}$ .	The corrector phase						
5	end							

- Compared with the scheme in [33], the new scheme does not need the postpositivity-processing procedure for  $T_{\nu}^{n+1}$ , therefore the local conservation on the dual mesh is maintained.
- In the most existing cell-centered schemes, the cell-vertex unknowns are treated as auxiliary ones for the finite volume equations on the primary mesh. From this perspective, here the FV equations on the dual mesh can be considered as a certain predictor to obtain the cell-vertex unknowns.
- In practical calculation, the efficiency of the nonlinear solver is another important issue. In our scheme, certain nonlinear iterative method must be employed to solve the equations of the cell-vertex unknowns. Since the positivity-preserving mechanism of the cell-centered unknowns is independent of the nonlinear solver, we are allowed to choose an efficient one, such as the Newton method. As for the nonlinear diffusion coefficients with a flux limiter, the calculation of Jacobian matrix is very difficult. In this case, the Jacobian-free Newton-Krylov method [14] can be adopted. It should be noted that the Jacobian-free Newton-Krylov method can not provide arbitrary reduction due to the initial residual (usually, no more than  $6 \sim 7$  orders of magnitude). Thus the conservation property will be guaranteed only to this tolerance at each time step.

At last, we give a comparison of the existing positivity-preserving finite volume schemes for the 3-T radiation diffusion equations in Table 1, which shows the advantages of the newly resulting scheme.

# 4 Theoretical analysis

This section will analyze the positivity, existence and uniqueness of the cell-centered unknowns, and the  $L^2$ - and  $H^1$ -stability of the scheme on quasi-uniform meshes.

**Theorem 4.1.** Assume that  $Q_{\alpha} \ge 0$ ,  $g_{\alpha}^{D} \ge 0$  and  $g_{\alpha}^{N} \le 0$ . Then, for given  $T_{\nu}^{n+1}$  and  $K_{c}^{n+1}$ , (3.46) has a unique solution, and if  $T_{c}^{n} \ge 0$ , the solution is nonnegative.

Schomo	Unknowns	Interpolation	Local	Nonlinear iteration		
Scheme	Ulikilowiis	interpolation	conservation	Linear problems	Nonlinear problems	
[32]	Cell-centered	$\sqrt{a}$	Primary mesh	$\sqrt{c}$	$\sqrt{c}$	
$[8]^1$	Cell-vertex	×	Dual mesh	×	$\sqrt{c}$	
[42]	Cell-centered	$\sqrt{a}$	Primary mesh	$\sqrt{c}$	$\sqrt{c}$	
[34]	Cell-vertex	$\sqrt{b}$	Dual mesh	$\sqrt{c}$	$\sqrt{c}$	
[28]	Cell-centered	$\sqrt{b}$	Primary mesh	$\sqrt{c}$	$\sqrt{c}$	
The new scheme	Cell-vertex-centered	×	Primary and dual meshes	×	$\sqrt{d}$	

Table 1: Comparison of the existing positivity-preserving finite volume schemes for the 2D 3-T radiation diffusion problems.

[8]<sup>1</sup>: The first scheme in [8], which only works on the triangular mesh.

Cell-vertex-centered: The scheme has both cell-centered and cell-vertex unknowns.

 $\times$ : The interpolation or the nonlinear iteration does not required.

*a*: The interpolation algorithm for the auxiliary unknowns should be positive-preserving.

*b*: The interpolation algorithm does not need to be positive-preserving.

*c*: Only the iteration method that does not change the *M*-matrix structure can be used.

*d*: General nonlinear iteration method can be used.

*Proof.* For given  $T_{\nu}^{n+1}$  and  $K_{c}^{n+1}$ , (3.46) becomes a linear system which can be abbreviated as

$$(\mathbb{E}_c^{n+1}+\tau\mathbb{A}_c^{n+1})T_c^{n+1}=\mathbb{E}_c^{n+1}T_c^n+\tau F_c^{n+1},$$

where  $\mathbb{E}_{c}^{n+1}$ ,  $\mathbb{A}_{c}^{n+1}$  and  $F_{c}^{n+1}$  are independent of  $T_{c}^{n+1}$ . In view of the flux discretizations in (3.37)-(3.39),  $\mathbb{A}_{c}^{n+1}$  is an *M*-matrix [35]. Hence, the coefficient matrix  $\mathbb{E}_{c}^{n+1} + \tau \mathbb{A}_{c}^{n+1}$  is a nonsingular *M*-matrix so that (3.46) has a unique solution. Under the assumption of  $Q_{\alpha} \geq 0$ ,  $g_{\alpha}^{D} \geq 0$  and  $g_{\alpha}^{N} \leq 0$ , one can deduce that  $F_{c}^{n+1} \geq \mathbf{0}$  and then  $\mathbb{E}_{c}^{n+1}T_{c}^{n}+F_{c}^{n+1} \geq \mathbf{0}$ . Hence we obtain  $T_{c}^{n+1} \geq \mathbf{0}$ . This ends the proof.

**Remark 4.1.** If (2.1)-(2.3) is a linear problem, due to Lemma 3.1, we can find that the coefficient matrix of (3.45) is symmetric and positive definite, and then it has a unique solution. For the nonlinear case, (3.45) is a nonlinear system and the existence of the cell-vertex unknowns is still open.

Before giving the stability result on quasi-uniform meshes, we first introduce two discrete function spaces

$$\mathcal{V}_{h} = \{T_{\alpha,h}^{\upsilon}: T_{\alpha,h}^{\upsilon}(\boldsymbol{x}_{\nu}) = \tilde{T}_{\alpha,\nu}, \, \boldsymbol{x}_{\nu} \in \overline{\Omega}\}, \quad \mathcal{C}_{h} = \{T_{\alpha,h}^{c}: T_{\alpha,h}^{c}(\boldsymbol{x}_{K}) = T_{\alpha,K}, \, K \in \mathcal{M}\}.$$

The discrete  $L^2$  and  $H^1$  norms on  $\mathcal{V}_h$  and  $\mathcal{C}_h$  employed in the analysis are respectively defined by

$$\|T_{\alpha,h}^{n,v}\|_{0,\mathcal{M}} = \left(\sum_{x_{\nu}\in\overline{\Omega}} |K_{\nu}^{*}|(\tilde{T}_{\alpha,\nu}^{n})^{2}\right)^{1/2}, \quad \|T_{\alpha,h}^{n,c}\|_{0,\mathcal{M}} = \left(\sum_{K\in\mathcal{M}} |K|(T_{\alpha,K}^{n})^{2}\right)^{1/2},$$

S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

$$|T_{\alpha,h}^{n,v}|_{1,\mathcal{M}} = \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{K}}(\tilde{T}_{\alpha,\nu'}^{n} - \tilde{T}_{\alpha,\nu}^{n})^{2}\right)^{1/2} = \left(\sum_{K\in\mathcal{M}}\|\delta\tilde{T}_{K}^{\alpha,n}\|^{2}\right)^{1/2},$$

and

$$|T_{\alpha,h}^{n,c}|_{1,\mathcal{M}} = \left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} (\delta T_{\alpha,\sigma}^n)^2\right)^{1/2},$$

4 10

where  $\|\cdot\|$  denotes the Euclidean vector norm, and

$$\delta T_{\alpha,\sigma}^{n} = \begin{cases} T_{\alpha,K}^{n} - T_{\alpha,L'}^{n} & \sigma = \mathcal{E}_{K} \cap \mathcal{E}_{L}, \\ T_{\alpha,K'}^{n} & \sigma = \mathcal{E}_{K} \cap \partial \Omega. \end{cases}$$

Furthermore, we introduce the following assumptions:

(H2) Assume the subdivision  $\mathcal{M}$  is quasi-uniform, i.e., there exists a positive constant  $\underline{\alpha}$ , independent of the mesh size h, such that

$$|K| \geq \underline{\alpha}h^2, \quad \forall K \in \mathcal{M}.$$

(H3) There exists a positive constant  $\varsigma$ , independent of h, such that

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \subset \mathcal{E}_K} \widetilde{F}_{K,\sigma}^{\alpha,n+1} T_{\alpha,K}^{n+1} \geq \underline{\varsigma} |T_{\alpha,h}^{n+1,c}|_{1,\mathcal{M}}^2, \quad \forall T_{\alpha,h}^c \in \mathcal{C}_h.$$

**Remark 4.2.** To prove the stability of (3.19)-(3.21),  $\mathbb{A}_{K}^{\alpha}$  in (3.6) should satisfy (4.1). It should be noted that (**H2**) is required in the analysis of Lemma 4.1. Although the analysis is limited to the quasi-uniform condition, numerical results show that the scheme on meshes which do not satisfy (**H2**), can also get the desired convergence.

**Remark 4.3.** (H3) represents that the coercivity of (3.42)-(3.44) holds, which is required to prove the stability of (3.42)-(3.44). The proof of (H3) is very difficult for the cell-centered schemes so it is usually treated as an assumption, such as [28, 30]. In Section 5, it is verified numerically.

After the above assumptions, we give the following lemma.

**Lemma 4.1** (Theorem 5 in [38]). Assume that  $\kappa_{\alpha}(T_{\alpha}) \ge \underline{\kappa}$  and  $\mathbb{A}_{K}$  is defined by (3.14). Then, under the assumptions (H1)-(H2), there exists a positive constant  $\varrho_{\kappa'}$  independent of h, such that

$$v^{\mathsf{T}}\mathbb{A}_{K}^{\alpha}v \geq \underline{\varrho}_{K} \|v\|^{2}, \quad \forall v \in \mathbb{R}^{n_{K}}, \quad \forall K \in \mathcal{M}.$$

$$(4.1)$$

Finally, we give the following stability result.

**Theorem 4.2** (Stability). Assume that  $\kappa_{\alpha}(T_{\alpha}) \ge \underline{\kappa}$  and (2.1)-(2.3) is imposed with a homogenous Dirichlet boundary condition. Then, under assumptions (H1)-(H3), there exists a positive constant *C*, dependent only on the final time  $T = t_N$ , such that

$$\sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{n+1,v}\|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \min_{K\in\mathcal{M}} \underline{\varrho}_{K} \sum_{k=0}^{n} \sum_{\alpha=e,i,r} |T_{\alpha,h}^{k+1,v}|_{1,\mathcal{M}}^{2}$$

$$\leq C \left( \sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{0,v}\|_{0,\mathcal{M}}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_{0}^{2} \right), \qquad (4.2)$$

and

$$\sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{n+1,c}\|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \underline{c} \sum_{k=0}^{n} \sum_{\alpha=e,i,r} |T_{\alpha,h}^{k+1,c}|_{1,\mathcal{M}}^{2}$$

$$\leq C(\sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{0,c}\|_{0,\mathcal{M}}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_{0}^{2}), \qquad (4.3)$$

where  $\tau \leq 1/2$ ,  $\underline{c} = \min\{E_{\alpha,\nu}^{n+1}/|K_{\nu}^*|, E_{\alpha,K}^{n+1}/|K|, \forall K \in \mathcal{M}, x_{\nu} \in \overline{\Omega}\}$ ,  $n = 0, 1, \dots, N-1$ , and  $\|\cdot\|_0$  denotes the standard  $L^2$  norm and  $T_{\alpha,h}^{0,\nu}$  and  $T_{\alpha,h}^{0,\nu}$  are the discrete functions that coincide with the initial data  $T_{\alpha}(x,0)$  at the cell centers and vertices.

*Proof.* Notice that  $\tilde{T}_{\alpha,\nu}^{n+1} = 0$  if  $\mathbf{x}_{\nu} \in \partial \Omega$ . Then, multiplying both sides of (3.19) with  $\tilde{T}_{e,\nu}^{n+1}$  and summing over all the dual cells, gives

$$\sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} E_{e,\nu}^{n+1} \frac{\tilde{T}_{e,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n}}{\tau} \tilde{T}_{e,\nu}^{n+1} + \sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} \sum_{K\in\mathcal{M}_{\nu}\sigma\subset\mathcal{E}_{K}\cap\mathcal{E}_{\nu}} (\boldsymbol{n}_{K,\sigma}^{*}\cdot\boldsymbol{n}_{\nu}^{*}) F_{K,\sigma^{*}}^{e,n+1} \tilde{T}_{e,\nu}^{n+1}$$

$$= \sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} \left[ W_{ei,\nu}^{n+1}(\tilde{T}_{i,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n+1}) + W_{er,\nu}^{n+1}(\tilde{T}_{r,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n+1}) \right] \tilde{T}_{e,\nu}^{n+1} + \sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} |K_{\nu}^{*}| Q_{e,\nu}^{n+1} \tilde{T}_{e,\nu}^{n+1}.$$
(4.4)

For the first term in the left-hand side of (4.4), we have

$$\sum_{x_{\nu}\in\overline{\Omega}} E_{e,\nu}^{n+1} \frac{\overline{T}_{e,\nu}^{n+1} - \overline{T}_{e,\nu}^{n}}{\tau} \tilde{T}_{e,\nu}^{n+1} = \frac{1}{2\tau} \sum_{x_{\nu}\in\overline{\Omega}} E_{e,\nu}^{n+1} \left[ (\tilde{T}_{e,\nu}^{n+1})^{2} - (\tilde{T}_{e,\nu}^{n})^{2} + (\tilde{T}_{e,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n})^{2} \right] \\ \geq \frac{c}{2\tau} (\|T_{e,h}^{n+1,\nu}\|_{0,\mathcal{M}}^{2} - \|T_{e,h}^{n,\nu}\|_{0,\mathcal{M}}^{2}).$$
(4.5)

For the second term in the left-hand side of (4.4), shifting the summation to the primary cells yields

$$\sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}}\sum_{K\in\mathcal{M}_{\nu}}\sum_{\sigma\subset\mathcal{E}_{K}\cap\mathcal{E}_{\nu}}(\boldsymbol{n}_{K,\sigma}^{*}\cdot\boldsymbol{n}_{\nu}^{*})F_{K,\sigma^{*}}^{e,n+1}\tilde{T}_{e,\nu}^{n+1} = \sum_{K\in\mathcal{M}}\sum_{\sigma\subset\mathcal{E}_{K}}(\tilde{T}_{e,\nu}^{n+1}-\tilde{T}_{e,\nu'}^{n+1})F_{K,\sigma^{*}}^{e,n+1}$$
$$=\sum_{K\in\mathcal{M}}(\delta T_{K}^{e,n+1})^{\mathsf{T}}F_{K}^{e,n+1}$$
$$\geq \min_{K\in\mathcal{M}}\underline{\varrho}_{K}|T_{e,h}^{n+1,\nu}|_{1,\mathcal{M}}^{2}, \tag{4.6}$$

S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

where

$$\boldsymbol{n}_{K,\sigma}^{*} \cdot \boldsymbol{n}_{\nu}^{*} \tilde{T}_{e,\nu}^{n+1} + \boldsymbol{n}_{K,\sigma}^{*} \cdot \boldsymbol{n}_{\nu'}^{*} \tilde{T}_{e,\nu'}^{n+1} = \boldsymbol{n}_{K,\sigma}^{*} \cdot \boldsymbol{n}_{\nu'}^{*} (\tilde{T}_{e,\nu'}^{n+1} - \tilde{T}_{e,\nu}^{n+1}),$$

and  $\mathbf{n}_{K,\sigma}^* \cdot \mathbf{n}_{\nu'}^* = 1$  are used in the first equality, and (3.6) and (4.1) are used in the last inequality. As for the first term in right-hand side of (4.4),

$$\sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} \left[ W_{ei,\nu}^{n+1}(\tilde{T}_{i,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n+1}) + W_{er,\nu}^{n+1}(\tilde{T}_{r,\nu}^{n+1} - \tilde{T}_{e,\nu}^{n+1}) \right] \tilde{T}_{e,\nu}^{n+1}$$

$$\leq \frac{1}{2} \sum_{\boldsymbol{x}_{\nu}\in\overline{\Omega}} \left[ W_{ei,\nu}^{n+1}(\tilde{T}_{i,\nu}^{n+1})^{2} + W_{er,\nu}^{n+1}(\tilde{T}_{r,\nu}^{n+1})^{2} - (W_{ei,\nu}^{n+1} + W_{er,\nu}^{n+1})(\tilde{T}_{e,\nu}^{n+1})^{2} \right].$$
(4.7)

For the second term in right-hand side of (4.4), by Young inequality, we have

$$\frac{c}{x_{\nu}\in\overline{\Omega}} |K_{\nu}^{*}| Q_{e,\nu}^{n+1} \tilde{T}_{e,\nu}^{n+1} \leq \frac{c^{2}}{2} \sum_{x_{\nu}\in\overline{\Omega}} |K_{\nu}^{*}| (\tilde{T}_{e,\nu}^{n+1})^{2} + \frac{1}{2} \sum_{x_{\nu}\in\overline{\Omega}} |K_{\nu}^{*}| (Q_{e,\nu}^{n+1})^{2} \\
= \frac{c^{2}}{2} \|T_{e,h}^{n+1,\nu}\|_{0,\mathcal{M}}^{2} + \frac{1}{2} \|Q_{e}^{n+1}\|_{0}^{2}.$$
(4.8)

Substituting (4.5), (4.6), (4.7) and (4.8) into (4.4), yields

$$\frac{\underline{c}^{2}}{\tau} \left( \|T_{e,h}^{n+1,v}\|_{0,\mathcal{M}}^{2} - \|T_{e,h}^{n,v}\|_{0,\mathcal{M}}^{2} \right) + 2\underline{c} \min_{K \in \mathcal{M}} \underline{\varrho}_{K} |T_{e,h}^{n+1,v}|_{1,\mathcal{M}}^{2} \\
\leq \underline{c} \sum_{x_{v} \in \overline{\Omega}} \left[ W_{ei,v}^{n+1} (\tilde{T}_{i,v}^{n+1})^{2} + W_{er,v}^{n+1} (\tilde{T}_{r,v}^{n+1})^{2} - (W_{ei,v}^{n+1} + W_{er,v}^{n+1}) (\tilde{T}_{e,v}^{n+1})^{2} \right] \\
+ \underline{c}^{2} \|T_{e,h}^{n+1,v}\|_{0,\mathcal{M}}^{2} + \|Q_{e}^{n+1}\|_{0}^{2},$$
(4.9)

which implies

$$\underline{c}^{2} \| T_{e,h}^{n+1,v} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \min_{K \in \mathcal{M}} \underline{\varrho}_{K} \sum_{k=0}^{n} | T_{e,h}^{k+1,v} |_{1,\mathcal{M}}^{2} \\
\leq \underline{c}^{2} \| T_{e,h}^{0,v} \|_{0,\mathcal{M}}^{2} + \tau \underline{c} \sum_{k=0}^{n} \sum_{x_{v} \in \overline{\Omega}} \left[ W_{ei,v}^{k+1} (\tilde{T}_{i,v}^{k+1})^{2} + W_{er,v}^{k+1} (\tilde{T}_{r,v}^{k+1})^{2} - (W_{ei,v}^{k+1} + W_{er,v}^{k+1}) (\tilde{T}_{e,v}^{k+1})^{2} \right] \\
+ \tau \sum_{k=0}^{n} (\underline{c}^{2} \| T_{e,h}^{k+1,v} \|_{0,\mathcal{M}}^{2} + \| Q_{e}^{k+1} \|_{0}^{2}).$$
(4.10)

Similarly, we have

$$\underline{c}^{2} \| T_{i,h}^{n+1,v} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \min_{K\in\mathcal{M}} \underline{\varrho}_{K} \sum_{k=0}^{n} | T_{i,h}^{k+1,v} |_{1,\mathcal{M}}^{2} \\
\leq \underline{c}^{2} \| T_{i,h}^{0,v} \|_{0,\mathcal{M}}^{2} + \tau \underline{c} \sum_{k=0}^{n} \sum_{x_{v}\in\overline{\Omega}} W_{ei,v}^{k+1} \left[ (\tilde{T}_{e,v}^{k+1})^{2} - (\tilde{T}_{i,v}^{k+1})^{2} \right] \\
+ \tau \sum_{k=0}^{n} (\underline{c}^{2} \| T_{i,h}^{k+1,v} \|_{0,\mathcal{M}}^{2} + \| Q_{i}^{k+1} \|_{0}^{2}),$$
(4.11)

S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

$$\underline{c}^{2} \| T_{r,h}^{n+1,v} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \min_{K \in \mathcal{M}} \underline{\varrho}_{K} \sum_{k=0}^{n} | T_{r,h}^{k+1,v} |_{1,\mathcal{M}}^{2} \\
\leq \underline{c}^{2} \| T_{r,h}^{0} \|_{0,\mathcal{M}}^{2} + \tau \underline{c} \sum_{k=0}^{n} \sum_{x_{v} \in \overline{\Omega}} W_{er,v}^{k+1} \left[ (\tilde{T}_{e,v}^{k+1})^{2} - (\tilde{T}_{r,v}^{k+1})^{2} \right] \\
+ \tau \sum_{k=0}^{n} (\underline{c}^{2} \| T_{r,h}^{k+1,v} \|_{0,\mathcal{M}}^{2} + \| Q_{r}^{k+1} \|_{0}^{2}).$$
(4.12)

It deduces from (4.10), (4.11) and (4.12) that

$$\sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{n+1,v}\|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \min_{K\in\mathcal{M}} \underline{\varrho}_{K} \sum_{k=0}^{n} \sum_{\alpha=e,i,r} |T_{\alpha,h}^{n+1,v}|_{1,\mathcal{M}}^{2}$$

$$\leq \sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{0,v}\|_{0,\mathcal{M}}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_{0}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{k+1,v}\|_{0,\mathcal{M}}^{2}.$$
(4.13)

By the discrete Gronwall inequality, we have

$$\sum_{\alpha=e,i,r} \underline{c}^2 \|T_{\alpha,h}^{k+1,v}\|_{0,\mathcal{M}}^2 \le e^{2T} \left( \sum_{\alpha=e,i,r} \underline{c}^2 \|T_{\alpha,h}^{0,v}\|_{0,\mathcal{M}}^2 + \tau \sum_{k=0}^n \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_0^2 \right),$$
(4.14)

where  $\tau \leq \frac{1}{2}$ . Substituting (4.14) into the right-hand side of (4.13) gives (4.2). Multiplying both sides of (3.42) with  $T_{e,K}^{n+1}$  and summing over all the primary cells, yields

$$\sum_{K \in \mathcal{M}} E_{e,K}^{n+1} \frac{T_{e,K}^{n+1} - T_{e,K}^{n}}{\tau} T_{e,K}^{n+1} + \sum_{K \in \mathcal{M}} \sum_{\sigma \subset \mathcal{E}_{K}} \widetilde{F}_{K,\sigma}^{e,n+1} T_{e,K}^{n+1}$$

$$= \sum_{K \in \mathcal{M}} W_{ei,K}^{n+1} (T_{i,K}^{n+1} - T_{e,K}^{n+1}) T_{e,K}^{n+1}$$

$$+ \sum_{K \in \mathcal{M}} W_{er,K}^{n+1} (T_{r,K}^{n+1} - T_{e,K}^{n+1}) T_{e,K}^{n+1} + \sum_{K \in \mathcal{M}} |K| Q_{e,K}^{n+1} T_{e,K}^{n+1}.$$
(4.15)

For the first term in the left-hand side of (4.15), we have

$$\sum_{K \in \mathcal{M}} E_{e,K}^{n+1} \frac{T_{e,K}^{n+1} - T_{e,K}^{n}}{\tau} T_{e,K}^{n+1} \ge \frac{c}{2\tau} (\|T_{e,h}^{n+1,c}\|_{0,\mathcal{M}}^2 - \|T_{e,h}^{n,c}\|_{0,\mathcal{M}}^2).$$
(4.16)

For the second term in the left-hand side of (4.15), by (H3), we have

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \subset \mathcal{E}_K} \widetilde{F}_{K,\sigma}^{e,n+1} T_{e,K}^{n+1} \ge \underline{\varsigma} |T_{e,h}^{n+1,c}|_{1,\mathcal{M}}.$$
(4.17)

As for the right-hand side of (4.15),

$$\frac{c}{k \in \mathcal{M}} \sum_{K \in \mathcal{M}} W_{ei,K}^{n+1} (T_{i,K}^{n+1} - T_{e,K}^{n+1}) T_{e,K}^{n+1} + \sum_{K \in \mathcal{M}} W_{er,K}^{n+1} (T_{r,K}^{n+1} - T_{e,K}^{n+1}) T_{e,K}^{n+1} + \underline{c} \sum_{K \in \mathcal{M}} |K| Q_{e,K}^{n+1} T_{e,K}^{n+1} \\
\leq \frac{c}{2} \sum_{K \in \mathcal{M}} \left[ W_{ei,K}^{n+1} (T_{i,K}^{n+1})^2 + W_{er,K}^{n+1} (T_{r,K}^{n+1})^2 - (W_{ei,K}^{n+1} + W_{er,K}^{n+1}) (T_{e,K}^{n+1})^2 \right] \\
+ \frac{c^2}{2} \|T_{e,h}^{n+1,c}\|_{0,\mathcal{M}}^2 + \frac{1}{2} \|Q_e^{n+1}\|_0^2.$$
(4.18)

Substituting (4.16), (4.17) and (4.18) into (4.15) and summing it from k=0 to k=n, gives

$$\underline{c}^{2} \| T_{e,h}^{n+1,c} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \underline{c} \sum_{k=0}^{n} | T_{e,h}^{k+1,c} |_{1,\mathcal{M}}^{2}$$

$$\leq \underline{c}^{2} \| T_{e,h}^{0,c} \|_{0,\mathcal{M}}^{2} + \tau \underline{c} \sum_{k=0}^{n} \sum_{K \in \mathcal{M}} \left[ W_{ei,K}^{k+1} (T_{i,K}^{k+1})^{2} + W_{er,K}^{k+1} (T_{r,K}^{k+1})^{2} - (W_{ei,K}^{k+1} + W_{er,K}^{k+1}) (T_{e,K}^{k+1})^{2} \right]$$

$$+ \tau \sum_{k=0}^{n} \left[ \underline{c}^{2} \| T_{e,h}^{k+1,c} \|_{0,\mathcal{M}}^{2} + \| Q_{e}^{k+1} \|_{0}^{2} \right].$$

$$(4.19)$$

Similarly, we have

$$\begin{split} \underline{c}^{2} \| T_{i,h}^{n+1,c} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau_{\underline{c}} \sum_{k=0}^{n} \| T_{i,h}^{k+1,c} \|_{1,\mathcal{M}}^{2} \\ \leq \underline{c}^{2} \| T_{i,h}^{0,c} \|_{0,\mathcal{M}}^{2} + \tau_{\underline{c}} \sum_{k=0}^{n} \sum_{K \in \mathcal{M}} \left[ W_{ei,K}^{k+1} (T_{e,K}^{k+1})^{2} - W_{ei,K}^{k+1} (T_{i,K}^{k+1})^{2} \right] \\ + \tau \sum_{k=0}^{n} \left[ \underline{c}^{2} \| T_{i,h}^{k+1,c} \|_{0,\mathcal{M}}^{2} + \| Q_{i}^{k+1} \|_{0}^{2} \right], \end{split}$$
(4.20)  
$$\underline{c}^{2} \| T_{r,h}^{n+1,c} \|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau_{\underline{c}} \sum_{k=0}^{n} | T_{r,h}^{k+1,c} |_{1,\mathcal{M}}^{2} \\ \leq \underline{c}^{2} \| T_{r,h}^{0,c} \|_{0,\mathcal{M}}^{2} + \tau_{\underline{c}} \sum_{k=0}^{n} \sum_{K \in \mathcal{M}} \left[ W_{er,K}^{k+1} (T_{e,K}^{k+1})^{2} - W_{er,K}^{k+1} (T_{r,K}^{k+1})^{2} \right] \\ + \tau \sum_{k=0}^{n} \left[ \underline{c}^{2} \| T_{r,h}^{k+1,c} \|_{0,\mathcal{M}}^{2} + \| Q_{r}^{k+1} \|_{0}^{2} \right]. \tag{4.21}$$

It deduces from (4.19), (4.20) and (4.22) that

$$\sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{n+1,c}\|_{0,\mathcal{M}}^{2} + 2\underline{c}\tau \underline{c}\sum_{k=0}^{n} \sum_{\alpha=e,i,r} |T_{\alpha,h}^{k+1,c}|_{1,\mathcal{M}}^{2}$$

$$\leq \sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{0,c}\|_{0,\mathcal{M}}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_{0}^{2} + \tau \sum_{k=0}^{n} \sum_{\alpha=e,i,r} \underline{c}^{2} \|T_{\alpha,h}^{k+1,c}\|_{0,\mathcal{M}}^{2}.$$
(4.22)

By the discrete Gronwall inequality, we have

$$\sum_{\alpha=e,i,r} \underline{c}^2 \|T_{\alpha,h}^{n+1,c}\|_{0,\mathcal{M}}^2 \le e^{2T} \left( \sum_{\alpha=e,i,r} \underline{c}^2 \|T_{\alpha,h}^{0,c}\|_{0,\mathcal{M}}^2 + \tau \sum_{k=0}^n \sum_{\alpha=e,i,r} \|Q_{\alpha}^{k+1}\|_0^2 \right),$$
(4.23)

where  $\tau \leq \frac{1}{2}$ . Substituting (4.23) into the right-hand side of (4.22), gives (4.3).

# 5 Numerical results

Define the relative discrete error of the solutions and fluxes as follows:

$$\varepsilon_2^T = \left(\sum_{K \in \mathcal{M}} \sum_{\alpha = e, i, r} |K| (T_\alpha(\boldsymbol{x}_K) - T_{\alpha, K})^2\right)^{1/2} / \left(\sum_{K \in \mathcal{M}} \sum_{\alpha = e, i, r} |K| T_\alpha(\boldsymbol{x}_K)^2\right)^{1/2},$$

and

$$\varepsilon_{2}^{F} = \left(\sum_{\sigma \in \mathcal{E}} \sum_{\alpha = e, i, r} S_{\sigma} (\widetilde{F}_{K, \sigma}^{\alpha} - \mathcal{F}_{K, \sigma}^{\alpha})^{2}\right)^{1/2} / \left(\sum_{\sigma \in \mathcal{E}} \sum_{\alpha = e, i, r} S_{\sigma} (\mathcal{F}_{K, \sigma}^{\alpha})^{2}\right)^{1/2},$$

where  $S_{\sigma}$  is an area associated with the primary edge  $\sigma$  (for example, is equal to the area of the quadrilateral  $x_K x_\nu x_L x_{\nu'}$  as shown in Fig. 2 for interior edge or the area of the triangular  $x_K x_\nu x_{\nu'}$  for boundary edge), and  $\mathcal{F}_{K,\sigma}^{\alpha}$  is defined in (3.25). To investigate numerically the reliability of (**H3**), define

$$\underline{\varsigma}_{h} = \min_{\alpha=e,i,r} \frac{\sum_{K \in \mathcal{M}} \sum_{\sigma \subset \mathcal{E}_{K}} \overline{F}_{K,\sigma}^{\alpha,n+1} T_{\alpha,K}^{n+1}}{|T_{\alpha,h}^{n+1,c}|_{1,\mathcal{M}}^{2}}$$

If  $\underline{c}_{h}$  stays a positive constant, independent of *h*, then it is meaningful for us to assume (H3).

All the examples are performed in double precision, BCGSTAB is used to solve the linear systems with the stopping tolerance  $10^{-15}$ , while the possibly nonlinear iterative method, such as the Picard method or the Newton method, is carried out with the stopping tolerance  $10^{-6}$ . Throughout this section, the notations below will be used in the numerical examples.

- umin: the minimal value of the approximate solution;
- umax: the maximal value of the approximate solution;
- $\overline{iter}^{P}$ : the average value of the iteration number by the Picard method;
- $\overline{iter}^N$ : the average value of the iteration number by the Newton method.

## 5.1 Convergence

Since the 3-T radiation diffusion problem is relatively complex, it is extremely difficult to theoretically analyze the convergent accuracy of the scheme. Therefore, we can only discuss the approximation accuracy from the view of the numerical experiments. In this part, we investigate the convergence of the efficient positivity-preserving scheme (EPPS) by some radiation diffusion problems having non-zero source terms and the analytic solutions. To highlight the computational efficiency of the present scheme, we also test the cell-vertex positivity-preserving scheme (VPPS) in [34] for a comparison.

**Example 5.1** (Linear diffusion coefficients). Here we consider a linear radiation diffusion problem on  $\Omega = (0,1)^2$  with full Dirichlet boundary condition. Let the parameters in the 3-T radiation diffusion model (2.1)-(2.3) be specified by  $\rho = 1$ ,  $\omega_{ei} = \omega_{er} = 1$ ,  $c_{v\alpha} = 1$ ,  $\kappa_{\alpha} = 1$ . The associated analytical solutions are

$$T_{\alpha} = e^{t} \begin{cases} (x^{2}+1)(y^{2}+1), & \alpha = e, \\ (2x^{2}+1)(y^{2}+1), & \alpha = i, \\ (2x^{2}+1)(2y^{2}+1), & \alpha = r. \end{cases}$$

Here we remark that the source terms can be computed correspondingly. We choose the final time t = 0.5 and the time stepsize  $\tau = h^2$  where *h* is the mesh size. Four types of meshes are used in this example, see Fig. 3.

Numerical results of  $T_e$ ,  $T_i$  and  $T_r$  on the four meshes are graphically depicted in Fig. 4 as the log-log plots of the discrete errors versus the characteristic mesh size h. The scheme is approximately second-order accurate while the flux error achieves first-order accurate. The number of the nonlinear iterations of VPPS and the CPU cost of the two schemes are respectively listed in Table 2. As commented in Remark 3.3, the present scheme EPPS does not require nonlinear iteration for the linear problems and only need to solve two systems of linear equations at each time step. Hence, it consumes considerably less time than VPPS, especially on the large deformed meshes, such as the Kershaw mesh and the distorted mesh. Numerical results of  $\varsigma_h$  on all types of meshes are listed in Table 3, which

Mesh	Randon	Random mesh		Kershaw mesh		Distorted mesh		Polygonal mesh	
level	VPPS	EPPS	VPPS	EPPS	VPPS	EPPS	VPPS	EPPS	
1	9.25	-	22	-	53.38	-	6.80	-	
2	10	-	27.75	-	52.85	-	7	-	
3	10	-	34.71	-	49.98	-	7	-	
4	9.99	-	38.65	-	44.87	-	7	-	
5	9.25	-	25.98	-	39.97	-	8	-	
CPU(s)	13739.13	2069.40	5042.33	257.61	5758.55	143.98	40719.19	9875.25	

Table 2: Example 5.1: The number of the nonlinear iterations and the CPU time.



(c) Distorted mesh (6.29E-2, 3.10E-1)

(d) Polygonal mesh (1.52E-2, 2.30E-1)

Figure 3: Example 5.1: Four types of meshes. Each mesh is used with five successive mesh levels, and the range of associated characteristic mesh size  $h_i$  ( $i=1,\cdots,5$ ) is shown in the bracket as ( $h_1,h_5$ ) in the caption of each mesh.

Mesh level	Random mesh	Kershaw mesh	Distorted mesh	Polygonal mesh
1	0.54	0.26	0.56	0.24
2	0.51	0.23	0.23	0.25
3	0.50	0.21	0.19	0.24
4	0.50	0.21	0.18	0.24
5	0.50	0.21	0.18	0.24

Table 3: Example 5.1: The values of  $\underline{\varsigma}_h$ .

show that  $\underline{c}_h$  tends to a positive constant, independent of *h*, verifying the reliability of **(H3)**.

**Example 5.2** (Smooth nonlinear diffusion coefficients). Here we consider a radiation diffusion problem with nonlinear diffusion coefficients. The energy exchange coefficients and the diffusion coefficients are  $\omega_{ei} = \omega_{er} = T_e$  and  $\kappa_{\alpha} = 1 + T_{\alpha}^2$ . Other parameters and the



Figure 4: Example 5.1: Numerical errors of the temperatures at t = 0.5.

Mesh		Randor	n mesh			Kersha	w mesh	
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	6.49E-3	_	3.39E-2	_	1.92E-2	_	1.62E-1	-
2	1.75E-3	2.07	1.50E-2	1.28	5.85E-3	2.02	8.13E-2	1.17
3	4.31E-4	2.06	5.99E-3	1.35	1.50E-3	2.11	2.61E-2	1.77
4	1.19E-4	1.93	3.59E-3	0.77	3.83E-4	2.04	7.36E-2	1.89
5	2.91E-5	2.07	1.69E-3	1.10	9.65E-5	2.02	1.95E-3	1.95
Mesh	Distorted mesh				Polygonal mesh			
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	3.85E-3	_	3.64E-2	_	5.96E-3	_	6.49E-2	-
2	1.03E-3	1.93	1.14E-2	1.70	1.98E-3	1.67	1.98E-2	1.80
3	4.63E-4	1.97	5.35E-3	1.87	5.74E-4	1.81	6.98E-3	1.53
4	2.62E-4	1.99	3.08E-3	1.93	1.52E-4	1.93	2.60E-3	1.43
5	1.68E-4	2.00	1.99E-3	1.95	3.94E-5	1.95	9.45E-4	1.46

Table 4: Example 5.2: Numerical results of  $T_e$  at t = 0.5.

Table 5: Example 5.2: Numerical results of  $T_i$  at t = 0.5.

Mesh		Randor	n mesh			Kersha	w mesh	
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	1.15E-2	_	5.77E-2	_	2.71E-2	_	1.82E-1	-
2	3.10E-3	2.06	2.14E-2	1.56	8.64E-3	1.94	8.44E-2	1.31
3	7.55E-4	2.08	8.14E-3	1.42	2.20E-3	2.12	2.75E-2	1.74
4	2.05E-4	1.95	4.48E-3	0.89	5.58E-4	2.05	7.77E-3	1.89
5	4.98E-5	2.08	2.07E-3	1.13	1.40E-4	2.03	2.06E-3	1.95
Mesh	Distorted mesh				Polygonal mesh			
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	6.05E-3	_	3.87E-2	_	1.05E-2	_	8.39E-2	-
2	1.60E-3	1.95	1.17E-2	1.75	3.44E-3	1.70	2.52E-2	1.83
3	7.21E-4	1.98	5.46E-3	1.89	9.83E-4	1.83	8.72E-3	1.55
4	4.08E-4	1.99	3.14E-3	1.94	2.59E-4	1.93	3.23E-3	1.44
5	2.61E-4	2.00	2.03E-3	1.96	6.68E-5	1.95	1.17E-3	1.46

exact solutions are the same as that in the previous example. The source terms can be computed correspondingly.

Numerical results on the four types of meshes (Fig. 3) are listed in Tables 4-6. One can see that the scheme gets the second-order (resp. first-order) accuracy of the solution (resp. flux) error. The number of the nonlinear iterations by the present scheme EPPS

5

4.29E-4

1.99

2.80E-3

Mesh		Randor	n mesh			Kersha	w mesh	
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	1.43E-2	_	5.76E-2	_	4.23E-2	-	2.35E-1	-
2	4.10E-3	1.97	2.27E-2	1.47	1.36E-2	1.93	1.12E-1	1.27
3	1.03E-3	2.03	8.34E-3	1.47	3.52E-3	2.10	3.76E-2	1.69
4	2.79E-4	1.96	5.02E-3	0.76	8.96E-4	2.05	1.07E-2	1.88
5	6.67E-5	2.09	2.38E-3	1.09	2.26E-4	2.02	2.84E-3	1.95
Mesh		Distorte	ed mesh		Polygonal mesh			
level	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order	$\varepsilon_2^T$	order	$\varepsilon_2^F$	order
1	9.70E-3	_	5.19E-2	_	1.41E-2	-	1.05E-1	-
2	2.62E-3	1.92	1.60E-2	1.72	4.40E-3	1.76	2.99E-2	1.90
3	1.18E-3	1.97	7.51E-3	1.87	1.24E-4	1.85	1.03E-2	1.55
4	6.69E-4	1.99	4.32E-3	1.93	3.24E-4	1.94	3.89E-3	1.41

Table 6: Example 5.2: Numerical results of  $T_r$  at t = 0.5.

Table 7: Example 5.2: The number of the nonlinear iterations and the CPU time by EPPS with  $\tau = h^2$ .

1.95

8.36E-5

1.96

1.44E-3

1.44

Mesh	Mesh Random		Kershaw mesh		Distorted mesh		Polygonal mesh	
level	$\overline{iter}^P$	<i>iter</i> <sup>N</sup>	$\overline{iter}^P$	iter <sup>N</sup>	$\overline{iter}^P$	iter <sup>N</sup>	$\overline{iter}^P$	$\overline{iter}^N$
1	12.38	3.88	11.50	5.00	11.67	3.83	12.20	3.90
2	11.62	3.00	12.00	4.75	10.38	3.00	11.22	3.00
3	9.55	3.00	10.93	3.93	9.07	3.00	8.74	3.00
4	7.10	3.00	8.73	3.00	8.19	3.00	6.29	2.00
5	5.00	2.00	6.59	3.00	7.43	3.00	4.33	2.00
CPU(s)	7743.05	3827.45	1211.03	615.11	650.09	276.25	42803.19	20515.19

with the Picard method and the Newton method and CPU time are listed in Table 7, which implies that the Newton method can reduce the number of iterations and save more than half of the time consumed. Since the cell-vertex scheme VPPS with Picard method does not converge with  $\tau = h^2$  on the random mesh, here we choose the time stepsize  $\tau = 5 \times 10^{-4}$ . The number of the nonlinear iterations by EPPS with the Newton method and VPPS with the Picard method, and the CPU time are presented in Table 8. One can see that the computational efficiency of the proposed scheme EPPS is better than that of the scheme VPPS, especially in the large deformed grids. Moreover, the values of  $\varsigma_h$  are provided in Table 9 to support (H3).

Mesh	Iesh Random mesh		Kersha	w mesh	Distort	ed mesh	Polygonal mesh	
level	EPPS	VPPS	EPPS	VPPS	EPPS	VPPS	EPPS	VPPS
1	2	8.80	2	16.95	2	40.96	2	8.47
2	2	10.40	2	21.90	2	50.82	2	9.21
3	2	10.55	2	29.91	2	58.03	2	8.92
4	2	11.59	2	40.73	2	63.48	2	9.09
5	2	11.42	2	49.69	2	67.76	2	9.41
CPU(s)	3182.19	11408.84	2985.92	51148.30	2489.41	77682.91	11207.06	46408.41

Table 8: Example 5.2: The number of the nonlinear iterations and the CPU time by EPPS and VPPS with  $\tau\!=\!5\!\times\!10^{-4}.$ 

Table 9: Example 5.2: The values of  $\zeta_h$ .

Mesh level	Random mesh	Kershaw mesh	Distorted mesh	Polygonal mesh
1	3.42	1.46	1.18	1.63
2	3.42	1.47	1.19	1.63
3	3.43	1.48	1.20	1.63
4	3.43	1.48	1.21	1.64
5	3.43	1.48	1.21	1.64

**Example 5.3.** (Discontinuous diffusion coefficients) This example investigates a radiation diffusion system with discontinuous diffusion coefficients on  $\Omega = \Omega_1 \cup \Omega_2$  where  $\Omega_1 = (0,0.5) \times (0,1)$  and  $\Omega_2 = (0.5,1) \times (0,1)$ . Let the diffusion coefficients be

$$\kappa_e = \begin{cases} 5T_e, & \text{in } \Omega_1, \\ T_e, & \text{in } \Omega_2, \end{cases} \quad \kappa_i = \begin{cases} 5T_i, & \text{in } \Omega_1, \\ T_i, & \text{in } \Omega_2, \end{cases} \quad \kappa_r = \begin{cases} 5T_r, & \text{in } \Omega_1, \\ T_r, & \text{in } \Omega_2. \end{cases}$$

The exact solutions are

$$T_{e}(x,y,t) = \begin{cases} t(\sin \pi x + 1)\sin \pi y + 1, & \text{in } \Omega_{1}, \\ t(\sin 5\pi x + 1)\sin \pi y + 1, & \text{in } \Omega_{2}, \end{cases}$$
$$T_{i}(x,y,t) = \begin{cases} t(\sin \pi x + 1)(\sin 2\pi y + 1) + 1, & \text{in } \Omega_{1}, \\ t(\sin 5\pi x + 1)(\sin 2\pi y + 1) + 1, & \text{in } \Omega_{2}, \end{cases}$$
$$T_{r}(x,y,t) = \begin{cases} t(\sin \pi x + 1)(\sin 3\pi y + 1) + 1, & \text{in } \Omega_{1}, \\ t(\sin 5\pi x + 1)(\sin 3\pi y + 1) + 1, & \text{in } \Omega_{2}. \end{cases}$$

Other parameters are chosen as  $\rho = 1$ ,  $c_{v\alpha} = 1$ ,  $\omega_{ei} = \omega_{er} = 1$ .

Here we investigate the convergence on the triangular mesh and the locally refined mesh, see Fig. 5. The final time is t = 0.1 and the time stepsize is the same as that in the previous example.

Numerical errors of the example are graphically depicted in Fig. 6. It implies that our scheme EPPS achieves the desired order of convergence for the discontinuous problem.



(a) Triangular mesh (1.56E-2, 2.50E-1)

(b) Locally refined mesh (2.21E-2, 3.54E-1)

Figure 5: Example 5.3: Two types of meshes. Each mesh is used with five successive mesh levels, and the range of associated characteristic mesh size  $h_i$   $(i=1,\cdots,5)$  is shown in the bracket as  $(h_1,h_5)$  in the caption of each mesh.



Figure 6: Example 5.3: Numerical errors of the temperatures on the two types of meshes.

## 5.2 Applications

This subsection simulates two typical models of the 3-T radiation diffusion equations to verify the positivity-preserving property and high efficiency of the present scheme. Denote the first scheme proposed in [8] by FVEM-3T. In this subsection, the schemes VPPS and FVEM-3T are also implemented and the corresponding results are presented whenever there is a need for comparison.

**Example 5.4** (Positivity of the discrete solutions). This example simulates the following 3-T problem to verify the positivity-preserving property of our scheme.

The calculation domain consists of two parts:  $\Omega = \Omega_1 \cup \Omega_2$ , where the subdomains  $\Omega_1 = (0,250) \times (0,300)$  and  $\Omega_2 = (250,300) \times (0,300)$  are filled with two different materials *A* and *B*. The initial conditions are  $(h_{\alpha})_A = (h_{\alpha})_B = 3 \times 10^{-4}$ . The densities are  $\rho_A = 0.05$ ,  $\rho_B = 1.0$ . The diffusion coefficients are  $(\kappa_e)_A = (\kappa_e)_B = 10$ ,  $(\kappa_i)_A = (\kappa_i)_B = 10$ , and  $(\kappa_r)_A = 100$ ,  $(\kappa_e)_B = 10$ , respectively. The boundary conditions are specified as follows

$$-\kappa_e \frac{\partial T_e}{\partial \boldsymbol{n}}|_{\partial \Omega} = -\kappa_i \frac{\partial T_i}{\partial \boldsymbol{n}}|_{\partial \Omega} = 0, \quad T_r|_{y=0} = 100, \quad \frac{\partial T_r}{\partial \boldsymbol{n}}|_{\partial \Omega \setminus \{y=0\}} = 0.$$

Let  $\omega_{ei} = 10/\rho$ ,  $\omega_{er} = 100/\rho$ , and  $c_{v\alpha} = 1$ , and there is no source. The time stepsize  $\tau$  is chosen to be  $10^{-3}$ . The random triangular and quadrilateral meshes (see Fig. 7) are used in this example.

The contour plots of the photon temperature at time t=5 are shown in Fig. 8. The maximum and minimum temperatures of the cell-vertex unknowns on the predictor phase and cell-centered unknowns on the corrector phase are listed in Table 10-11. One can see that the cell-centered unknowns are positive while the cell-vertex unknowns are not, which is consistent with the positivity-preserving analysis of our scheme.



(a) Random triangular mesh

(b) Random quadrilateral mesh

Figure 7: Example 5.4: Two types of meshes.



Figure 8: Example 5.4: The contour plots of the photon temperature on the random triangular mesh (left) and quadrilateral mesh (right) at time t=5.

Table 10: Example 5.4: The maximum and minimum values of the numerical solutions on the random triangular mesh at t=5.

∐nknown .	Electron		Ior	ı	Photon	
CIRIOWII	umin	umax	umin	umax	umin	umax
cell-vertex	-1.853	99.974	-1.853	99.842	-1.853	92.877
cell-centered	2.984E-6	97.601	2.982E-6	97.580	2.984E-6	97.603

Table 11: Example 5.4: The maximum and minimum values of the numerical solutions on the random quadrilateral mesh at t=5.

Unknown	Electron		Ior	ı	Photon	
CHKHOWH	umin	umax	umin	umax	umin	umax
cell-vertex	-1.476	99.969	-1.476	99.818	-1.476	92.874
cell-centered	3.905E-7	95.961	3.905E-7	95.955	3.905E-7	95.962

Table 12: Example 5.4: The number of the nonlinear iterations and the CPU time.

Scheme	Tria	ngular	Quadrilateral		
Schenic	iter <sup>P</sup>	CPU(s)	iter <sup>P</sup>	CPU(s)	
EPPS	-	967.063	-	502.266	
VPPS	2.199	1071.844	2.227	956.656	

The average number of the nonlinear iterations for VPPS and the used time of the present scheme EPPS and VPPS are presented in Table 12, which shows that the present scheme is more efficient than the scheme VPPS. Since two systems need to be solved at each time step, the used time of EPPS does not take much less time than that of VPPS.

**Example 5.5** (Typical model of laser-driven implosion ICF). Here we simulate the typical model of the laser-driven implosion of the inertial confinement fusion in [21]. As shown in Fig. 9, the computational region  $\Omega$  with boundary  $\partial \Omega = \Gamma_1 \cup \Gamma_2$  is a half circle in the 2D Cartesian coordinate plane. The circle center is at the coordinate origin, and the diameter is overlapped with the *x*-axis. It is divided into three subregions  $\Omega_i$ , *i* = 1,2,3. The innermost, middle and outside subdomains cover with 0 < r < 90, 90 < r < 95 and 95 < r < 132 respectively, and are filled with deuterium gas (*DT*), glass (*SiO*<sub>2</sub>) and plastic foam (*CH*), respectively. The values of all parameters in (2.1)-(2.3) can be found in [34] and are not elaborated here. The source term  $Q_\alpha$  is zero. The boundary and initial conditions are

$$\begin{cases} -\kappa_e \frac{\partial T_e}{\partial n} = -\kappa_i \frac{\partial T_i}{\partial n} = 0, \quad T_r = 2, \quad \text{on } \Gamma_1, \\ -\kappa_\alpha \frac{\partial T_\alpha}{\partial n} = 0, \quad \text{on } \Gamma_2, \end{cases}$$

and

$$T_{\alpha}(\mathbf{x},0) = 3 \times 10^{-4}$$
,

respectively.

The time stepsize is taken as  $\tau = 5 \times 10^{-4}$ . Four types of meshes in Fig. 10 are used. They are the unstructured triangular mesh, quadrilateral mesh, polygonal mesh and non-matching mesh.

To test the energy conservation in the numerical simulations, we compute the relative energy conservative error [25] by

$$\left|\frac{\mathcal{E}_{enter}^{n} - (\mathcal{E}_{own}^{n} - \mathcal{E}_{own}^{0})}{\mathcal{E}_{enter}^{n} + \mathcal{E}_{own}^{0}}\right|,$$

where  $\mathcal{E}_{enter}^n$  denotes the total radiation energy importing from the Dirichlet boundary, and  $\mathcal{E}_{own}^n$  is the system energy at time  $t_n$  computed by

$$\mathcal{E}_{own}^{n} = \sum_{K \in \mathcal{M}} |K| \left( c_{ve,K}^{n} T_{e,K}^{n} + c_{vi,K}^{n} T_{i,K}^{n} + c_{vr,K}^{n} (T_{r,K}^{n})^{4} \right).$$



Figure 9: Example 5.5: Computational domain.



Figure 10: Example 5.5: Four types of meshes.

Note that the computation of  $\mathcal{E}_{enter}^n$  requires the discrete flux across the Dirichlet boundary. Here we first reconstruct the value of  $\nabla T_{\alpha}$ . By the Green formula and trapezoidal rule, we have

$$\int_{K} \nabla T_{\alpha} d\mathbf{x} = \sum_{\sigma \in \mathcal{E}_{K}} \int_{\sigma} T_{\alpha} \mathbf{n}_{K,\sigma} ds \approx \sum_{\sigma \in \mathcal{E}_{K}} \frac{1}{2} |\sigma| (T_{\alpha,\nu} + T_{\alpha,\nu'}) \mathbf{n}_{K,\sigma},$$

where  $T_{\alpha,\nu}$  and  $T_{\alpha,\nu'}$  are the approximations of  $T_{\alpha}$  at the endpoints of  $\sigma$ ,  $n_{K,\sigma}$  is the unit outward normal vector to the edge  $\sigma$ . Hence, we have

$$\nabla T_{\alpha,K} = \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{2|K|} (T_{\alpha,\nu} + T_{\alpha,\nu'}) \boldsymbol{n}_{K,\sigma}.$$

Assume that  $\sigma \in \mathcal{E}_K \cap \Gamma_D$ , then we results in the discrete flux across  $\sigma$  by

$$-\int_{\sigma}\kappa_{\alpha}\nabla T_{\alpha}\cdot\boldsymbol{n}_{K,\sigma}\approx-|\sigma|\kappa_{\alpha,K}\nabla T_{\alpha,K}\cdot\boldsymbol{n}_{K,\sigma}.$$

Numerical temperatures of the electron, ion and photon at  $\{x \ge 0, y = 0\}$ , t = 0.5 are shown in Fig. 11. The contour plots of the electron, ion and photon temperatures are depicted in Fig. 12 at the specified time, t = 0.5, 5, and 50 respectively on unstructured triangular mesh. The contour plots on other three meshes are similar so that we do not display to avoid redundancy. The electron, ion and photon temperatures are positive at each specified time, which verifies the positivity-preserving property of the present



Figure 11: Example 5.5: The temperatures of the electron (left), ion (middle) and photon (right) at  $\{x \ge 0, y=0\}$ , t=0.5.



Figure 12: Example 5.5: The temperature contours of the electron (top), ion (middle) and photon (bottom) on the unstructured triangular mesh at t = 0.5 (left), 5 (middle), 50 (right).

Scheme	t = 0.1	t = 1	t = 5	t = 10	t = 25	t = 50
VPPS [34]	10.91	5.52	2.11	1.30	0.76	0.24
FVEM-3T [8]	10.91	5.52	2.10	1.28	0.70	0.23
EPPS	10.88	5.52	2.11	1.30	0.76	0.24

Table 13: Example 5.5: Energy conservative errors (%) on the unstructured triangular mesh.

Table 14: Example 5.5: Energy conservative errors (%) on the quadrilateral mesh with four levels.

Mesh level	t = 0.1	t = 1	t = 5	t = 10	t = 20	t=30	t=40	t=50
1	65.17	23.21	11.21	8.31	6.97	6.05	5.44	5.01
2	16.73	9.25	5.06	3.93	3.27	2.91	2.66	2.49
3	7.26	4.31	2.44	1.92	1.60	1.44	1.33	1.25
4	3.35	2.08	1.20	0.94	0.79	0.72	0.66	0.62

scheme. Moreover, the photon temperature is higher than those of the electron and ion at each specified time and the temperature contours are all symmetric about the line x = 0.

The energy conservative errors of the schemes VPPS, FVEM-3T and EPPS at various specified times are listed in Table 13. Here we remark that the scheme FVEM-3T can only work on the unstructured triangular mesh. We find that those schemes possess the similar results. The energy conservative errors on the quadrilateral meshes are shown in Table 14. One can see that the energy conservative error becomes nearly half as the mesh is uniformly refined, and it is convergent of order one. This also confirms the correctness and robustness of our new scheme.

# 6 Conclusion

In this paper, an efficient and positivity-preserving finite volume scheme is proposed for the nonequilibrium radiation diffusion problems with three-temperature on general polygonal meshes with star-shaped cells. The scheme is formed as a predictor-corrector algorithm. The predictor phase determines the cell-vertex solutions on the dual mesh independently, while the corrector phase obtains the cell-centered solutions on the primary mesh. The  $L^2$ - and  $H^1$ -stability of the two phases are analyzed on quasi-uniform meshes, and the positivity, existence and uniqueness of the cell-centered unknowns on the corrector phase are proved. Since only two systems of linear equations are solved for the linear problems, and the general nonlinear iterative method can be used for the nonlinear problems, the computational efficiency of this scheme is higher than that of the existing nonlinear positivity-preserving finite volume schemes. In comparison with the scheme in [33], the primary flux approximation has a fixed stencil and the scheme does not need the additional step to keep the nonnegativity of the cell-vertex unknowns, which makes our scheme more flexible and easy for implementation. Numerical results show that the present scheme achieves second-order accuracy, generates the nonnegative discrete solutions at the cell centers of arbitrary star-shaped polygonal meshes. Moreover, the computational costs (the CPU time and the number of the nonlinear iterations) shows the high efficiency of the scheme. These results indicate that our positivity-preserving scheme is practical and attractive for solving the nonequilibrium 3-T radiation diffusion equations on the polygonal meshes. In the future, we will focus on the extension of the scheme to the 3-T radiation hydrodynamical equations.

# Acknowledgments

The first author was partially supported by China Postdoctoral Science Foundation (No. BX20190013). The second author was partially supported by the Special Project on Highperformance Computing under the National Key R&D Program (No. 2016YFB0200603), Science Challenge Project (No. TZ2016002), the National Natural Science Foundation of China (No. 11421101), and the Sino-German Research Group Project (No. 1465). The third author was partially supported by the National Natural Science Foundation of China (No. 11871009) and CAEP Foundation (No. CX2019028).

#### References

- [1] H. An, X. Jia, and H. F. Walker. Anderson acceleration and application to the three-temperature energy equations. *J. Comput. Phys.*, 347:1–19, 2017.
- [2] H. An, Z. Mo, X. Xu, and X. Liu. On choosing a nonlinear initial iterate for solving the 2-D 3-T heat conduction equations. *J. Comput. Phys.*, 228(9):3268–3287, 2009.
- [3] D. G. Anderson. Iterative procedures for nonlinear integral equations. J. Assoc. Comput. Mach., 12(4):547–560, 1965.
- [4] J. Camier and F. Hermeline. A monotone nonlinear finite volume method for approximating diffusion operators on general meshes. *Int. J. Numer. Methods Eng.*, 107:496–519, 2016.
- [5] William W. Dai and Anthony. J. Scannapieco. Interface- and discontinuity-aware numerical schemes for plasma 3-T radiation diffusion in two and three dimensions. *J. Comput. Phys.*, 300:643–664, 2015.
- [6] R. Paul Drake. *High-Energy-Density Physics: Fundamentals, Inertial Fusion, and Experimental Astrophysics, Second edition.* Springer Berlin Heidelberg, 2018.
- [7] S. Fu, S. Huang, and Y. Li. Numerical simulation of indirectly driven high convergence implosions. *Chinese J. Comput. Phys.*, 16:162–166, 1999.
- [8] Y. Gao, X. Zhao, and Y. Li. Finite volume element methods for two-dimensional threetemperature radiation diffusion equations. *Numer. Math. Theor. Meth. Appl.*, 9(3):470–496, 2016.
- [9] Z. Gao and J. Wu. A linearity-preserving cell-centered scheme for the heterogeneous and anisotropic diffusion equations on general meshes. *Int. J. Numer. Meth. Fluids*, 67(12):2157–2183, 2011.
- [10] Z. Gao and J. Wu. A second-order positivity-preserving finite volume scheme for diffusion equations on general meshes. *SIAM. J. Sci. Comput.*, 37:A420–A438, 2015.

- [11] T. Gu, Z. Dai, and X. Liu. Partial Newton-Krylov iterative methods for a system of energy equations with three-temperatures. *Numer. Heat Transf. Part B: Fundamentals*, 53(3):259–270, 2008.
- [12] J. A. Harte, W. E. Alley, D. S. Bailey, J. L. Eddleman, and G. B. Zimmerman. LASNEX-A 2-D physics code for modeling ICF. *In* 1996 ICF Annual Report, *Lawrence Livermore National Laboratory*, UCRL-LR-105821-96, pages 150–164, 1997.
- [13] J. Jiang, Y. Huang, S. Shu, and S. Zeng. Some new discretization and adaptation and multigrid methods for 2-D 3-T diffusion equations. *J. Comput. Phys.*, 224(1):168–181, 2007.
- [14] D. A. Knoll and D. E. Keyes. Jacobian-free Newton-Krylov methods: a survey of approaches and applications. J. Comput. Phys., 193:357–397, 2004.
- [15] S. H. Langer, H. A. Scott, M. M. Marinak, and O. L. Landen. Comparisons of line emission from 2- and 3-dimensional simulations of ICF capsules to experiments. J. Quant. Spectrosc. Radiat. Transfer, 81(1-4):275–286, 2003.
- [16] K. Lipnikov, M. Shashkov, D. Svyatskiy, and Yu. Vassilevski. Monotone finite volume schemes for diffusion equations on unstructured triangular and shape-regular polygonal meshes. J. Comput. Phys., 227(1):492–512, 2007.
- [17] K. Lipnikov, D. Svyatskiy, and Y. Vassilevski. Interpolation-free monotone finite volume method for diffusion equations on polygonal meshes. J. Comput. Phys., 228(3):703–716, 2009.
- [18] K. Lipnikov, D. Svyatskiy, and Y. Vassilevski. Anderson acceleration for nonlinear finite volume scheme for advection-diffusion problems. *SIAM J. Sci. Comput.*, 35(2):A1120–A1136, 2013.
- [19] R. Liska and M. Shashkov. Enforcing the discrete maximum principle for linear finite element solutions of second-order elliptic problems. *Commun. Comput. Phys.*, 3(4):852–877, 2008.
- [20] C. Lu, W. Huang, and E. S. V. Vleck. The cutoff method for the numerical computation of nonnegative solutions of parabolic PDEs with application to anisotropic diffusion and lubrication-type equations. *J. Comput. Phys.*, 242:24–36, 2013.
- [21] Z. Mo, L. Shen, and G. Wittum. Parallel adaptive multigrid algorithm for 2-D 3-T diffusion equations. *Int. J. Comput. Math.*, 81(3):361–374, 2004.
- [22] H. Nagarajan and K. B. Nakshatrala. Enforcing the non-negativity constraint and maximum principles for diffusion with decay on general computational grids. *Int. J. Numer. Meth. Fluids*, 67:820–847, 2011.
- [23] K. B. Nakshatrala, H. Nagarajan, and M Shabouei. A numerical methodology for enforcing maximum principles and the non-negative constraint for transient diffusion equations. *Commun. Comput. Phys.*, 19(1):53–93, 2016.
- [24] K. B. Nakshatrala and A. J. Valocchi. Non-negative mixed finite element formulations for a tensorial diffusion equation. J. Comput. Phys., 228(18):6726–6752, 2009.
- [25] C. Nie and H. Yu. A Raviart–Thomas mixed finite element scheme for the two-dimensional three-temperature heat conduction problems. *Int. J. Numer. Meth. Engng.*, 111:983–1000, 2017.
- [26] K. Nikitin, K. Novikov, and Y. Vassilevski. Nonlinear finite volume method with discrete maximum principle for the two-phase flow model. *Lobachevskii J. Math.*, 37(5):570–581, 2016.
- [27] K. A. Novikov. A maximum principle for multiphase flow models. Vychisl. Metody Programm., 18(2):138–145, 2017.
- [28] G. Peng, Z. Gao, W. Yan, and X. Feng. A positivity-preserving finite volume scheme for three-temperature radiation diffusion equations. *Appl. Numer. Math.*, 152:125–140, 2020.
- [29] C. Le Potier. Schéma volumes finis monotones pour des opérateurs de diffusion fortement anisotropes sur des maillages de triangles non structurs. C. R. Acad. Sci. Paris, Ser. I, 341:787–

S. Su, H. Tang and J. Wu / Commun. Comput. Phys., x (20xx), pp. 1-38

792, 2005.

- [30] M. Schneider, L. Agélas, G. Enchéry, and B. Flemisch. Convergence of nonlinear finite volume schemes for heterogeneous anisotropic diffusion on general meshes. *J. Comput. Phys.*, 351:80–107, 2017.
- [31] M. Schneider, B. Flemisch, and R. Helmig. Monotone nonlinear finite-volume method for nonisothermal two-phase two-component flow in porous media. *Int. J. Numer. Meth. Fluids*, 84:352–381, 2017.
- [32] C. D. Sijoy and S. Chaturvedi. TRHD: Three-temperature radiation-hydrodynamics code with an implicit non-equilibrium radiation transport using a cell-centered monotonic finite volume scheme on unstructured-grids. *Comput. Phys. Commun.*, 190:98–119, 2015.
- [33] S. Su, Q. Dong, and J. Wu. A decoupled and positivity-preserving discrete duality finite volume scheme for anisotropic diffusion problems on general polygonal meshes. *J. Comput. Phys.*, 372:773–798, 2018.
- [34] S. Su and J. Wu. A vertex-centered and positivity-preserving finite volume scheme for twodimensional three-temperature radiation diffusion equations on general polygonal meshes. *Numer. Math. Theor. Meth. Appl.*, 13:220–252, 2020.
- [35] R. S. Varga. *Matrix Iterative Analysis, Second revised and expanded edition*. Springer Berlin Heidelberg, 2000.
- [36] Y. Vassilevski, K. Terekhov, K. Nikitin, and I. Kapyrin. *Parallel Finite Volume Computation on General Meshes*. Springer International Publishing, 2020.
- [37] S. Wang, G. Yuan, Y. Li, and Z. Sheng. Discrete maximum principle based on repair technique for diamond type scheme of diffusion problems. *Int. J. Numer. Meth. Fluids*, 70(9):1188– 1205, 2012.
- [38] J. Wu. Vertex-centered linearity-preserving schemes for nonlinear parabolic problems on polygonal grids. *J. Sci. Comput.*, 71:499–524, 2017.
- [39] J. Wu and Z. Gao. A nine-point scheme with explicit weights for diffusion equations on distorted meshes. *Appl. Numer. Math.*, 61(7):844–867, 2011.
- [40] J. Wu, Z. Gao, and Z. Dai. A vertex-centered linearity-preserving discretization of diffusion problems on polygonal meshes. *Int. J. Numer. Meth. Fluids*, 81(3):131–150, 2016.
- [41] L. Yin, J. Wu, and Z. Dai. A Lions domain decomposition algorithm for radiation diffusion equations on non-matching grids. *Numer. Math. Theor. Meth. Appl.*, 8(4):530–548, 2015.
- [42] Y. Yu, X. Chen, and G. Yuan. A finite volume scheme preserving maximum principle for the system of radiation diffusion equations with three-temperture. *SIAM J. Sci. Comput.*, 41(1):B93–B113, 2019.
- [43] X. Yue, S. Shu, J. Wang, and Z. Zhou. Substructuring preconditioners with a simple coarse space for 2-D 3-T radiation diffusion equations. *Commun. Comput. Phys.*, 23(2):540–560, 2018.
- [44] X. Zhang, S. Su, and J. Wu. A vertex-centered and positivity-preserving scheme for anisotropic diffusion problems on arbitrary polygonal grids. *J. Comput. Phys.*, 344:419–436, 2017.
- [45] F. Zhao, X. Lai, G. Yuan, and Z. Sheng. A new interpolation for auxiliary unknowns of the monotone finite volume scheme for 3D diffusion equations. *Commun. Comput. Phys.*, 27:1201–1233, 2020.