Achieving Superconvergence by One-Dimensional Discontinuous Finite Elements: Weak Galerkin Method

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Abstract. A simple stabilizer free weak Galerkin (SFWG) finite element method for a one-dimensional second order elliptic problem is introduced. In this method, the weak function is formed by a discontinuous $k$-th order polynomial with additional unknowns defined on vertex points, whereas its weak derivative is approximated by a polynomial of degree $k + 1$. The superconvergence of order two for the SFWG finite element solution is established. It is shown that the elementwise lifted $P_{k+2}$ solution of the $P_k$ SFWG one converges at the optimal order. Numerical results confirm the theory.

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Key words: Finite element, weak Galerkin, stabilizer free.

1. Introduction

Weak Galerkin (WG) finite element methods, introduced and analyzed in [2,3], provide a general finite element technique for solving partial differential equations. The novelty of such methods consist in using weak functions and their weakly defined derivatives. Weak functions have the form $v = \{v_0, v_b\}$, with $v = v_0$ representing $v$ in the interior of each element and $v = v_b$ on the element boundary. The terms $v_0$ and $v_b$ are respectively approximated by polynomials $P_k(T)$ and $P_l(e)$, where $e$ refers to the edge or face of $T$. Weak derivative is specifically developed for weak function approximated by $P_l(T)$ polynomials. Each combination of the WG element ($P_k(T), P_l(e), [P_l(T)]^d$) leads to a weak Galerkin finite element method.

For special combinations of the WG elements ($P_k(T), P_l(e), [P_l(T)]^d$), $d \geq 2$, the corresponding WG method does not need a stabilizer. This leads to stabilizer free weak Galerkin (SFWG) methods. The first stabilizer free weak Galerkin method was introduced

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in [4] on polygonal and polyhedral meshes. It was shown that if the corresponding polygon has \(n\) sides, then one can eliminate the stabilizer term by using the WG elements \((P_k(T), P_e, [P_{k+n-1}(T)]^d)\). Al-Taweel and Wang [1] improved this result by reducing the polynomial degree for weak gradient on triangular meshes. Besides, the superconvergence is observed for special WG elements. In particular, order on the superconvergence of an SFWG polynomial degree for weak gradient on triangular meshes. Besides, the superconvergence is obtained for the WG elements \(P_k(T), P_{k+1}(e), [MRT_k(T)]^d\), where \(MRT_k(T)\) is a macro-Raviart-Thomas element on a polygon/polyhedron \(T\) [5]. Order two superconvergence is obtained for the WG elements \((P_k(T), P_{k+1}(e), [MP_{k+1}(T)]^d)\), where \([MP_{k+1}(T)]^d\) is a macro-BDM element on a polygon/polyhedron \(T\) [6].

We investigate the performance of the SFWG elements \((P_k(T), P_e, [P_l(T)]^d)\) in one dimension. In this case, the polynomial \(P_l(e)\) degenerates to a single value at end points of the interval. This work answers the question which WG elements \((P_k(T), P_0(x_i), P(I))\) maximize the order of convergence in one dimension, where \(x_i\) is the end point of the interval \(I\). More exactly, we show that in the one dimensional case, one can obtain two order higher convergence rate for the solution of the SFWG method with the WG element \((P_k(T), P_0(x_i), [P_{l+1}(T)]^d)\), i.e. the \(P_k\) SFWG solution converges with order \(k + 3\) in \(L^2\) norm and with order \(k + 2\) in \(H^1\) norm. Moreover, the \(P_k\) SFWG solution is lifted to a \(P_{k+2}\) solution elementwise, which converges with the optimal order. Numerical results confirms the theoretical findings.

2. SFWG Finite Element Schemes

Let \(\Omega = [a, b]\). We want to determine a function \(u\) such that

\[-u'' = f \quad \text{in} \ \Omega,\]
\[u = 0 \quad \text{on} \ \partial \Omega.\]  

(2.1) \hspace{1cm} (2.2)

Consider the splitting \(\Omega = \bigcup_{i=1}^N I_i, I_i = [x_{i-1}, x_i]\) of the interval \([a, b]\) and let \(\mathcal{R}_h := \{I_i | i = 1, \ldots, N\}\), where \(h = \max |I_i|\). For a given integer \(k \geq 1\), let \(V_h\) be the weak Galerkin finite element space associated with \(\mathcal{R}_h\), i.e.

\[V_h := \{v = \{v_0, v_b\} : v_0|_{I_i} \in P_k(I_i), v_b|_{x_i} \in \mathbb{R}, v_b|_{x_0} = v_b|_{x_N} = 0\}.\]  

(2.3)

For \(v \in V_h\), a weak derivative \(D_wv\) is a piecewise polynomial such that on each \(I_i\), \(D_wv \in P_{k+1}(I_i)\) satisfies the relations

\[(D_wv, q)_{I_i} = -(v_0, Dq)_{I_i} + (v_b, q)_{\partial I_i} \quad \text{for all} \quad q \in P_{k+1}(I_i),\]  

(2.4)

where \(Dv = dv/dx\) and \((v, w)_{\partial I_i} = v(x_i)w(x_i) - v(x_{i-1})w(x_{i-1})\).

The SFWG method for the problem (2.1)-(2.2) consists in finding of \(u_h \in V_h\) such that

\[(D_wu_h, D_wv) = (f, v_0) \quad \text{for all} \quad v = \{v_0, v_b\} \in V_h.\]  

(2.5)

In what follows, we adopt the following notations:

\[(v, w)_{\mathcal{R}_h} = \sum_{i=1}^N (v, w)_{I_i} = \sum_{i=1}^N \int_{I_i} vwdx,\]
Weak Galerkin Method

\[ \langle v, w \rangle_{\mathcal{E}^h} = \sum_{i=1}^{N} \langle v, w \rangle_{\mathcal{E}^h}, \]

and for any \( v \in \mathcal{V} \), we define the semi-norms

\[ \| v \|_h^2 = (D_w v, D_w v), \]  \( \text{(2.6)} \)

\[ \| v \|_{1,h}^2 = \sum_{I \in \mathcal{T}_h} \left( \| Dv_0 \|_{I}^2 + h_{I}^{-1} \| v_0 - v_b \|_{I}^2 \right). \] \( \text{(2.7)} \)

The following lemma shows that the above semi-norms are equivalent.

Lemma 2.1. There exist two positive constants \( C_1 \) and \( C_2 \) independent of \( h \), such that for any \( v \in \mathcal{V} \), we have

\[ C_1 \| v \|_{1,h} \leq \| v \| \leq C_2 \| v \|_{1,h}. \] \( \text{(2.8)} \)

Proof. We prove the upper bound first. Setting \( q := D_w v \) and using the norm equivalence in finite dimensional spaces and a scaling argument, we obtain

\[ \| v \|_h^2 = \sum_{I \in \mathcal{T}_h} \langle Dv_0, D_w v \rangle_I + \langle v_b - v_0, D_w v \rangle_{\mathcal{E}} \]

\[ \leq \sum_{I \in \mathcal{T}_h} \langle Dv_0, D_w v \rangle_I + \| v_b - v_0 \|_{I} \| D_w v \|_{\mathcal{E}} \]

\[ \leq \sum_{I \in \mathcal{T}_h} \left( \| Dv_0 \|_I + Ch^{-\frac{1}{2}} \| v_b - v_0 \|_{I} \right) \| D_w v \|_I \]

\[ \leq C \| v \|_{1,h} \| v \|, \]

and the upper bound is established. On the other hand, in order to derive the lower bound, we choose a special function \( q \), so that the above inequality can be reversed. Let \( q \in P_{k+1}(I_i) \) on each \( I_i \) be such that

\[ q(x_{i-1}) = h^{-1} \left( -v_b(x_{i-1}) + v_0(x_{i-1}) \right), \]

\[ q(x_i) = h^{-1} \left( v_b(x_{i-1}) - v_0(x_{i-1}) \right), \]

\[ (q, p_{k-1})_I = (Dv_0, p_{k-1})_I \quad \text{for all} \quad p_{k-1} \in P_{k-1}(I_i). \]

Using again the norm equivalence in finite dimensional spaces and a scaling argument yields

\[ \| q \|_0 \leq C \| v \|_{1,h}. \]

By the definition of weak derivative (2.4), with above \( q \), we get

\[ \| v \|_{1,h}^2 = (D_w v, q) \leq \| D_w v \|_0 \| q \|_0 \leq \| v \| \| v \|_{1,h}. \]

The proof is complete. \( \Box \)

It is easily seen that \( \| v \|_{1,h} \) defines a norm in \( \mathcal{V} \). Lemma 2.1 implies that the semi-norm \( \| \cdot \| \) is also a norm in \( \mathcal{V} \). Therefore the SFWG method is well posed.
3. Error Equation

Let \( \Pi_j \) be the elementwise defined \( L^2 \) projection onto \( P_j(I) \) for \( I \in \mathcal{T}_h \) and \( Q_h u = \{ \Pi_k u, u \} \in V_h \). We start this section with the following lemma.

**Lemma 3.1.** If \( \phi \in H^1(\Omega) \), then for any \( I \in \mathcal{T}_h \), we have

\[
D_w(Q_h \phi) = \Pi_{k+1}(D\phi) \tag{3.1}
\]

**Proof.** If \( q \in P_{k+1}(I) \), then (2.4) and integration by parts yield

\[
\left( D_w(Q_h \phi), q \right) = -\left( \Pi_k \phi, Dq \right)_I + \left( \phi, q \right)_{\partial I}
\]

\[
= -\left( \phi, Dq \right)_I + \left( \phi, q \right)_{\partial I}
\]

\[
= \left( D\phi, q \right)_I = \left( \Pi_{k+1}(D\phi), q \right)_I,
\]

which implies the Eq. (3.1). \( \square \)

Now we derive an equation for the error term \( e_h \in V_h \) defined by \( e_h := Q_h u - u_h \).

**Lemma 3.2.** For any \( v \in V_h \), the error \( e_h \) satisfies the equation

\[
(D_w e_h, D_w v) = \ell(u, v), \tag{3.2}
\]

where

\[
\ell(u, v) := (Du - \Pi_{k+1}Du, v_0 - v_b)_{\partial \mathcal{S}_h}.
\]

**Proof.** For \( v = \{v_0, v_b\} \in V_h \), testing (2.1) by \( v_0 \) and using the relation \( (Du, v_b)_{\partial \mathcal{S}_h} = 0 \), we obtain

\[
-(u'', v_0) = (Du, Dv_0)_{\mathcal{S}_h} - (Du, v_0 - v_b)_{\partial \mathcal{S}_h} = (f, v_0). \tag{3.3}
\]

Integration by parts and the Eqs. (2.4), (3.1) give

\[
(Du, Dv_0)_{\mathcal{S}_h} = (\Pi_{k+1}Du, Dv_0)_{\mathcal{S}_h}
\]

\[
= -(v_0, D(\Pi_{k+1}Du))_{\mathcal{S}_h} + (v_0, \Pi_{k+1}Du)_{\partial \mathcal{S}_h}
\]

\[
= (\Pi_{k+1}Du, D_w v)_{\mathcal{S}_h} + (v_0 - v_b, \Pi_{k+1}Du)_{\partial \mathcal{S}_h}
\]

\[
= (D_w(Q_h u), D_w v) + (v_0 - v_b, \Pi_{k+1}Du)_{\partial \mathcal{S}_h}. \tag{3.4}
\]

In addition, the Eqs. (3.3) and (3.4) yield

\[
-(u'', v_0) = (D_w Q_h u, D_w v) - \ell(u, v) = (f, v_0), \tag{3.5}
\]

so that

\[
(D_w Q_h u, D_w v) = (f, v) + \ell(u, v). \tag{3.6}
\]

Subtracting (2.5) from (3.6) gives the error equation (3.2). \( \square \)
4. Error Estimates in Energy Norm

If a function $\varphi \in H^1(I_i)$ with $I_i = [x_{i-1}, x_i]$, then it satisfies the trace inequality

$$|\varphi(x_j)|^2 \leq C \left( h_i^{-1} ||\varphi||_{I_i}^2 + h_i ||D\varphi||_{I_i}^2 \right), \quad j = i - 1, i. \tag{4.1}$$

Now we estimate the terms $\ell(u, v)$.

**Lemma 4.1.** If $w \in H^{k+3}(\Omega)$ and $v = \{v_0, v_b\} \in V_h$, then

$$|\ell(w, v)| \leq C h^{k+2} |w|_{k+3} ||v||. \tag{4.2}$$

**Proof.** Using the Cauchy-Schwarz inequality, the trace inequality (4.1), and (2.8), we write

$$|\ell(w, v)| = \left| \sum_{I \in \mathcal{T}_h} \langle Dw - \Pi_{k+1} Dw, v_0 - v_b \rangle_{\partial I} \right| \leq C \sum_{I \in \mathcal{T}_h} \|Dw - \Pi_{k+1} Dw\|_{\partial I} \|v_0 - v_b\|_{\partial I}$$

$$\leq C \left( \sum_{I \in \mathcal{T}_h} h_i \|Dw - \Pi_{k+1} Dw\|_{\partial I} \right)^{\frac{1}{2}} \left( \sum_{I \in \mathcal{T}_h} h_i^{-1} \|v_0 - v_b\|_{\partial I}^2 \right)^{\frac{1}{2}} \leq C h^{k+2} |w|_{k+3} ||v||. \quad \Box$$

**Theorem 4.1.** If $u_h \in V_h$ is the SFWG finite element solution of the Eq. (2.5), then there is a constant $C$ such that

$$||Q_h u - u_h|| \leq C h^{k+2} |u|_{k+3}. \tag{4.3}$$

**Proof.** Letting $v = e_h$ in (3.2) and using (4.2) implies

$$||e_h||^2 = (D_w e_h, D_w e_h) = |\ell(u, e_h)| \leq C h^{k+2} |u|_{k+3} ||e_h||,$$

and we are done. \quad \Box

5. Error Estimates in $L^2$ Norm

Recalling that $e_h = \{e_0, e_b\} = Q_h u - u_h$, we consider the respective dual problem — i.e. we want to find $\Phi \in H^1_0(\Omega)$ such that

$$-\Phi'' = e_0 \quad \text{in} \ \Omega. \tag{5.1}$$

Assume that the following $H^2$-regularity holds

$$||\Phi||_2 \leq C ||e_0||.$$

(5.2)
Theorem 5.1. Let \( u_h \in V_h \) be the SFWG finite element solution of (2.5). Assume that (5.2) holds true. Then there exists a constant \( C \) such that

\[
\|\Pi_k u - u_0\| \leq Ch^{k+3}|u|_{k+3}.
\] (5.3)

Proof. Testing (5.1) by \( e_0 \) gives

\[
\|e_0\|^2 = -(\Phi'', e_0).
\] (5.4)

Choosing \( u = \Phi \) and \( v = e_h \) in (3.5), we obtain

\[
-(\Phi'', e_0) = (D w Q_h \Phi, D w e_h) - \ell(\Phi, e_h).
\] (5.5)

It follows from (5.4) and (5.5) that

\[
\|e_0\|^2 = (D w Q_h \Phi, D w e_h) - \ell(\Phi, e_h),
\] (5.6)

and combining it with the Eq. (3.2) yields

\[
\|e_0\|^2 = \ell(u, Q_h \Phi) - \ell(\Phi, e_h).
\] (5.7)

Using the Cauchy-Schwarz inequality and the trace inequality (4.1), we write

\[
|\ell(u, Q_h \Phi)| = \left| \sum_{\ell \in T_{cal}} (Du - \Pi_{k+1} Du, \Pi_k \Phi - \Phi)_{\partial\ell} \right|
\]

\[
\leq C \sum_{\ell \in T_{cal}} \|Du - \Pi_{k+1} Du\|_{\partial\ell} \|\Pi_k \Phi - \Phi\|_{\partial\ell}
\]

\[
\leq C \left( \sum_{\ell \in T_{cal}} h_1 \|Du - \Pi_{k+1} Du\|^2_{\partial\ell} \right)^{\frac{1}{2}} \left( \sum_{\ell \in T_{cal}} h_1^{-1} \|\Pi_k \Phi - \Phi\|^2_{\partial\ell} \right)^{\frac{1}{2}}
\]

\[
\leq Ch^{k+3}|u|_{k+3}\|\Phi\|_2.
\] (5.8)

In order to estimate \( \ell(\Phi, e_h) \), we can invoke (4.2) thus obtaining

\[
|\ell(\Phi, e_h)| \leq Ch|\Phi|_2\|e_h\|.
\] (5.9)

It follows from (5.9) and (5.3) that

\[
|\ell(\Phi, e_h)| \leq Ch^{k+3}|u|_{k+3}\|\Phi\|_2.
\] (5.10)

Taking into account (5.7), (5.8) and (5.10) yields

\[
\|e_0\|^2 \leq Ch^{k+3}|u|_{k+3}\|\Phi\|_2,
\]

and recalling the regularity condition (5.2), we arrive at the optimal order error estimate (5.3). \( \square \)
6. A Locally Lifted $P_{k+2}$ Solution

Since the SFWG solution is two order superconvergent, we lift the solution to a $P_{k+2}$ solution which converges with the optimal order. Elementwise we compute a solution $\hat{u}_h \in \Pi_{I \in \mathbb{T}_h} P_{k+2}(I)$ by

$$
(D\hat{u}_h - D_w u_h, Dp) = 0 \quad \text{for all} \quad p \in \Pi_{I \in \mathbb{T}_h} P_{k+2}(I) \setminus P_0(I),
$$

(6.1)

$$
(\hat{u}_h - u_0, p) = 0 \quad \text{for all} \quad p \in \Pi_{I \in \mathbb{T}_h} P_0(I).
$$

(6.2)

The system of linear equations (6.1)-(6.2) is uniquely solvable. Indeed, this is a square system and if $u_h = 0$, the Eq. (6.1) implies $\|D\hat{u}_h\|_0^2 = 0$ and $\hat{u}_h$ is a constant on each $I$. By (6.2), this constant is zero, so that the system (6.1)-(6.2) has a unique solution.

**Theorem 6.1.** Let $u \in H^3(\Omega) \cap H^{k+3}$ and $u_h \in V_h$ be the solutions of the problem (2.1)-(2.2) and the SFWG equations (2.5), respectively. If $\hat{u}_h \in \Pi_{I \in \mathbb{T}_h} P_{k+2}(I)$ is the locally lifted solution of (6.1)-(6.2), then there exists a constant $C$ such that

$$
\|u - \hat{u}_h\| \leq Ch^{k+3}|u|_{k+3}.
$$

(6.3)

**Proof:** The problem (6.2) means that

$$
\Pi_0 \hat{u}_h = \Pi_0 u_h.
$$

Using the orthogonality of the projections $\Pi_0$ and $I - \Pi_0$, we write

$$
\|u - \hat{u}_h\|_0 \leq \|\Pi_0 (u - \hat{u}_h)\|_0 + \|(I - \Pi_0)(u - \hat{u}_h)\|_0.
$$

(6.4)

The first term in the right-hand side of (6.4) can be estimated by using (5.3), so that

$$
\|\Pi_0 (u - \hat{u}_h)\|_0 = \|\Pi_0 (\Pi_k u - u_h)\|_0 \leq \|\Pi_k u - u_h\|_0 \leq Ch^{k+3}|u|_{k+3}.
$$

Applying the triangle inequality to the second term in the right-hand side of (6.4), we write

$$
\|(I - \Pi_0)(u - \hat{u}_h)\|_0 \leq Ch\|D(u - \hat{u}_h)\|_0 \\
\leq Ch\|D(u - \Pi_{k+2} u)\|_0 + Ch\|D(\Pi_{k+2} u - \hat{u}_h)\|_0 \\
\leq Ch^{k+3}|u|_{k+3} + Ch\|D(\Pi_{k+2} u - \hat{u}_h)\|_0.
$$

It follows from (3.1), (6.1) and (4.3) that

$$
\|D(\Pi_{k+2} u - \hat{u}_h)\|_0^2 \\
= (D(\Pi_{k+2} u - u), D(\Pi_{k+2} u - \hat{u}_h)) + (D u - \Pi_{k+1} Du, D(\Pi_{k+2} u - \hat{u}_h)) \\
+ (D_w Q_h u - D_w u_h, D(\Pi_{k+2} u - \hat{u}_h)) \\
\leq (\|D(\Pi_{k+2} u - u)\|_0 + \|D u - \Pi_{k+1} Du\|_0 + \|Q_h u - u_h\|_0) \|D(\Pi_{k+2} u - \hat{u}_h)\|_0 \\
\leq Ch^{k+2}|u|_{k+3} \|D(\Pi_{k+2} u - \hat{u}_h)\|_0.
$$

Combining the three inequalities above, we arrive at the estimate (6.3). \qed
7. Numerical Experiments

We solve the 1D elliptic equation (2.1)-(2.2) on the domain \( \Omega = (0, 1) \). The function \( f \) is chosen so that the exact solution is

\[
u(x) = \sin(\pi x).
\]

(7.1)

Let the first level grid consist of one interval and any subsequent grid is the refinement of the previous one with each interval divided into two ones. We find the solution by using 5 SFWG finite elements. The results are shown in Table 1. In every case, we get two orders of superconvergence in both \( L^2 \) and \( H^1 \) norms.

After that, we lift the \( P_k \) weak Galerkin finite element solution to a \( P_{k+2} \) solution and show the corresponding convergence results in Table 2. Note that in all cases, the lifted solution converges two orders higher than the original \( P_k \) solution.

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Table 1: Error profiles and convergence rates for solution (7.1).

Table 2: Error profiles and convergence rates for solution (7.1).

References


