Sobolev Orthogonal Polynomials: Asymptotics and Symbolic Computation

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Abstract. The Sobolev polynomials, which are orthogonal with respect to an inner product involving derivatives, are considered. The theory about these nonstandard polynomials has been developed along the last 40 years. The local asymptotics of these polynomials can be described by the Mehler-Heine formulae, which connect the polynomials with the Bessel functions of the first kind. In recent years, the formulae have been computed for discrete Sobolev orthogonal polynomials in several particular cases. We improve various known results by unifying them. Besides, an algorithm to compute these formulae effectively is presented. The algorithm allows to construct a computer program based on *Mathematica*® language, where the corresponding Mehler-Heine formulae are automatically obtained. Applications and examples show the efficiency of the approach developed.

1. Introduction

Sobolev orthogonal polynomials were first considered in a paper concerning the simultaneous polynomial approximation of functions and their derivatives in 1947 [17]. In the sixties and seventies of the last century, the German school studied such polynomials, mainly from a theoretical point of view [4, 11, 26, 27]. At the beginning of the nineties, the topic attracted great interest after the seminal works of Iserles *et al.* [12, 13], where among other results the authors proposed a useful algorithm for computing the coefficients of the Fourier-Sobolev series. More details about the Sobolev orthogonality can be found mainly in the survey [21], but a very short and concise overview is given in the first two pages of [20]. More recently, several authors have found applications of this theory — e.g. in the study of second-order elliptic equations on the real line [25], in elliptic boundary

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value problems combining Jacobi-Sobolev polynomials and spectral methods [31], and in the Cauchy problem for ordinary differential equations [28,29].

In this paper we consider the Sobolev-type inner product or the discrete Sobolev inner product

$$(f,g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(c)g^{(j)}(c), \tag{1.1}$$

where μ is a finite positive Borel measure supported on an infinite subset of the real line, c is adequately located on the real axis, $j \in \mathbb{N} \cup \{0\}$, and $\{M_n\}_{n \geq 0}$ is a sequence of nonnegative real numbers satisfying a very general condition.

The orthonormal polynomials with respect to (1.1) are studied from a theoretical point of view in [19]. There the authors provided a scaled asymptotics around the point c where the perturbation of the standard inner product is introduced. This type of local asymptotics is known either as Mehler-Heine formula or Mehler-Heine asymptotics. As it is well known, it is quite useful to describe the asymptotic differences between the sequences of orthogonal polynomials with respect to (1.1) and the orthonormal polynomials with respect to μ. In fact, in that paper it was shown that the relative asymptotics or the Plancherel-Rotach type asymptotics do not provide exciting results for these polynomials. Now, our goal is more practical as we will describe below.

The paper [19] achieved two objectives. First, it presents a detailed description of the local asymptotic behavior of these Sobolev-type orthonormal polynomials or discrete Sobolev orthonormal polynomials (dSOP) in terms of a linear combination of special functions — viz. the Bessel functions of the first kind

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha}. \tag{1.2}$$

The results obtained are valid for measures with bounded and unbounded supports and generalize the works considering specifically chosen measures. Thus, paraphrasing [19], the paper provides "a final and global vision of the Mehler-Heine asymptotics" for these dSOP. Although the obtention of the Mehler-Heine formulae for the dSOP is constructive, many details are omitted. Thus a question arises: what else can we add to this topic?

Obtaining the limit functions in [19, Theorems 1 and 2] requires many calculations. Hence, the answer to the previous question is to provide an algorithm and the corresponding computer program, which would automatically determine the limit functions of these Mehler-Heine formulae — i.e. we used symbolic computation to obtain these limit functions involving Bessel functions. This program is implemented on a computer using the *Mathematica*® language 12.1.1 (also known as Wolfram language).[†] The program can be downloaded from the `https://w3.ual.es/GruposInv/Tapo/MHAS.nb`.

Numerical algorithms for computation are a necessary and relevant tool when we want to use orthogonal polynomials and special functions in applications. There are nice classic books and articles on this topic — cf. Refs. [1,9,14–16,23]. Besides, there are many others

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[†]The program in the previous versions of *Mathematica*®may not work properly — e.g. we found malfunctions in the 11-th version.

works where special cases are treated. As far as Sobolev orthogonality is concerned, the literature is not too wide. Thus, as was already mentioned, the seminal papers [12, 13] provide an algorithm to compute the coefficients of the Fourier-Sobolev expansion in the setting of coherence of measures. In [9, Section 2.5] we can find an algorithm to compute Sobolev orthogonal polynomials (SOP), and in [10] the zeros and critical points of SOP are studied from a numerical approach. Now we compute the local asymptotic for dSOP by using symbolic computation. As far as we know, it is the first work of this type in the context of SOP. In addition, the use of the program has also permitted to clarify some subtle theoretical aspects, see, for example, the comments in Subsection 4.2.1, Subsection 4.4 or Remark 4.1. Furthermore, using Hurwitz's theorem we obtain the asymptotic behavior of the zeros of these dSOP.

We structure the paper in the following way. In Section 2 we improve all theoretical results obtained in [19] that are necessary to construct the algorithm and the computer program. In Section 3 we explain how the algorithm works by including a process flow diagram. Finally, in Section 4 some applications and examples are shown.

2. Theoretical Results

We first summarize auxiliary results to make this section self-contained. Thus we improve various issues in [19] by providing more details needed for establishing concrete formulae.

Consider the discrete Sobolev inner product

$$(f, g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(c)g^{(j)}(c), \quad j \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

where μ is a finite positive Borel measure supported on an infinite subset of the real line, c a real number whose exact location will be determined later according to the measure μ, and $\{M_n\}_{n \geq 0}$ a sequence of nonnegative real numbers satisfying the general condition

$$\lim_{n \to \infty} M_n K_{n-1}^{(j,j)}(c, c) = L \in [0, +\infty], \tag{2.2}$$

where $K_{n-1}^{(\ell,k)}(x, y)$ are the partial derivatives of the n-th kernel for the sequence of orthonormal polynomials $\{p_n\}_{n \geq 0}$ with respect to μ, i.e.

$$K_n^{(\ell,k)}(x, y) = \frac{\partial^{\ell+k}}{\partial x^\ell \partial y^k} K_n(x, y) = \sum_{i=0}^n p_i^{(\ell)}(x)p_i^{(k)}(y), \quad \ell, k \in \mathbb{N} \cup \{0\}.$$

Since the inner product (2.1) is varying, there is a square tableau of orthonormal polynomials $\{q_k^{(M_n)}\}_{k \geq 0}$ for each n, but we deal with the diagonal of this tableau $\{q_n^{(M_n)}\}_{n \geq 0} =: \{q_n\}_{n \geq 0}$. Obtaining the Mehler-Heine formulae for these dSOP was the main task in [19]. To do this, it was necessary to deal with a third family of orthonormal polynomials. Thus, we denote by $\{p_n^{[2i]}\}_{n \geq 0}$ the sequence of orthonormal polynomials with respect to the measure μ_{2i} with $d\mu_{2i}(x) = (x - c)^{2i}d\mu(x)$, $i \in \mathbb{N} \cup \{0\}$. We will use the notation $f_n \simeq g_n$ to indicate that $\lim_{n \to \infty} f_n/g_n = 1$.

Taking into account the symmetry of measure μ, we distinguish two cases.

2.1. μ is a nonsymmetric measure

We denote by $I = \mathrm{supp}(\mu) \in \mathbb{R}$. Then, the point c is chosen either as $c = \inf(I)$ or $c = \sup(I)$. In [19] the results about Mehler-Heine asymptotics were given taking $c = \inf(I)$, although in Section 3 of that paper it was said "results can be also obtained with $c = \sup(I) \in \mathbb{R}$ making some changes". Now, to obtain a computer program that considers all the cases, we need to give the details that were not computed in [19].

We begin assuming that for the sequence $\{p_n^{[2i]}\}_{n \geq 0}$ we have the following Mehler-Heine asymptotics:

$$\lim_{n \to \infty} s_{n,c} \frac{a_{n,c}^{1/2}}{b_{n,c}^i} p_n^{[2i]} \left(c + s_c \frac{z^2}{b_{n,c}} \right) = z^{-(\nu_c + 2i)} J_{\nu_c + 2i}(2z), \quad i = 0, \ldots, j+1, \tag{2.3}$$

uniformly on compact subsets of $\mathbb{C}$, where $J_{\nu_c + 2i}$ are the Bessel functions of the first kind and order $\nu_c + 2i$ given by (1.2),

$$s_{n,c} = \begin{cases} (-1)^n, & \text{if} \quad c = \inf(I), \\ 1, & \text{if} \quad c = \sup(I), \end{cases} \qquad s_c = \begin{cases} 1, & \text{if} \quad c = \inf(I), \\ -1, & \text{if} \quad c = \sup(I), \end{cases} \tag{2.4}$$

and

$$a_{n,c}^{-1/2} \simeq A_c n^{a_c}, \quad b_{n,c} \simeq B_c n^{b_c}, \quad \nu_c > -1, \quad 2a_c + 1 = b_c(\nu_c + 1) \tag{2.5}$$

with $A_c, B_c, b_c > 0$.

Remark 2.1. Notice that conditions (2.3)-(2.5) are satisfied for a wide class of measures — e.g. Nevai-Blumenthal class [6, 22], the measures related to a Laguerre weight or to modified or generalized Jacobi weights [19, 24].

Remark 2.2. The differences between the cases $c = \inf(I)$ and $c = \sup(I)$ is demonstrated by an example in Section 4.

The proof of [24, Theorem 5], also used in [19], shows that if (2.3)-(2.5) are satisfied, then

$$\left(p_n^{[2i]} \right)^{(k)}(c) \simeq s_{n,c} \frac{(-s_c)^k A_c B_c^{k+i}}{\Gamma(k + \nu_c + 2i + 1)} n^{f_c(k+i)}, \quad 0 \leq k \leq n, \tag{2.6}$$

where $f_c(x) = b_c x + a_c$ is a strictly increasing function with $2f_c(0) + 1 = b_c(\nu_c + 1) > 0$. Therefore, the relation (2.6) can be written as

$$\left(p_n^{[2i]} \right)^{(k)}(c) \simeq C_{k,i} s_{n,c} n^{f_c(k+i)}, \quad 0 \leq k \leq n,$$

where

$$C_{k,i} = \frac{(-s_c)^k A_c B_c^{k+i}}{\Gamma(k + \nu_c + 2i + 1)}. \tag{2.7}$$

The constants $C_{k,i}$ also depend on c, so when $c = \sup(I)$ these constants are always positive and when $c = \inf(I)$ they alternate the sign according to the order of the derivatives k.

Theorem 2.1. *Assuming the conditions* (2.3)-(2.5)*, we have the following Mehler-Heine asymptotics for the dSOP q_n:*

$$\lim_{n\to\infty} s_{n,c} a_{n,c}^{1/2} q_n\left(c + s_c\frac{z^2}{b_{n,c}}\right) = \sum_{i=0}^{j+1}(-s_c)^i d_{i,c}(L)z^{-\nu_c}J_{\nu_c+2i}(2z) := \varphi_{c,\nu_c,j,L}(z), \qquad (2.8)$$

uniformly on compact subsets of $\mathbb{C}$, where J_{ν_c+2i} are the Bessel functions of the first kind given by (1.2)*. The coefficients $d_{i,c}(L)$ are known explicitly through the recursive process*

$$d_{i,c}(L) = (-1)^i\frac{\theta_{c,i,j,L} - \sum_{m=0}^{i-1}(-1)^m d_{m,c}(L)\binom{i}{m}m!\frac{C_{i-m,m}}{C_{i,0}}}{i!\frac{C_{0,i}}{C_{i,0}}}, \quad 0 \le i \le j+1, \qquad (2.9)$$

where the constants $C_{k,i}$ are given in (2.7)*, and*

$$\theta_{c,i,j,L} = \frac{L\left(f_c(i) - f_c(j)\right) + f_c(i) + f_c(j) + 1}{(1+L)(f_c(i) + f_c(j) + 1)} \qquad (2.10)$$

with L determined by (2.2) *denoting*

$$\theta_{c,i,j,+\infty} = \lim_{L\to+\infty}\frac{L(f_c(i) - f_c(j)) + f_c(i) + f_c(j) + 1}{(1+L)(f_c(i) + f_c(j) + 1)} = \frac{f_c(i) - f_c(j)}{f_c(i) + f_c(j) + 1}.$$

We note that the Mehler-Heine formulae (2.8) have been reported in [19, Theorem 1] but only for the case $c = \inf(I)$.

Remark 2.3. It is easily seen that if the asymptotic behavior of the sequences $a_{n,c}$ and $b_{n,c}$ is known, then $f_c(k)$ and $C_{k,i}$ can be computed straightforwardly. Furthermore, according to [19, Eq. (6)], one can determine L by using the relation

$$K_{n-1}^{(j,j)}(c,c) \simeq \frac{C_{j,0}^2}{2f_c(j)+1}n^{2f_c(j)+1}.$$

Consequently, we obtain

$$L = \lim_{n\to\infty} M_n\frac{C_{j,0}^2}{2f_c(j)+1}n^{2f_c(j)+1}. \qquad (2.11)$$

With all these ingredients we can automatize, through a symbolic computer program, the process for obtaining the limit function in (2.8).

2.2. μ is a symmetric measure

When μ is a symmetric positive Borel measure with respect to the origin, we take $c = 0$ in (2.1), i.e.

$$(f,g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(0)g^{(j)}(0). \qquad (2.12)$$

Therefore, the sequences of polynomials $\{p_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ are symmetric, and we have to consider the corresponding subsequences of polynomials of even and odd degree — i.e. $\{p_{2n}\}_{n \geq 0}$, $\{p_{2n+1}\}_{n \geq 0}$, $\{q_{2n}\}_{n \geq 0}$ and $\{q_{2n+1}\}_{n \geq 0}$. In this way, we can reproduce the method described for nonsymmetric measures with slight differences.

We begin by assuming that for all $i \geq 0$, the limits

$$\lim_{n \to \infty} (-1)^n \frac{a_n^{1/2}}{b_n^i} p_{2n}^{[2i]}\left(\frac{z}{b_n}\right) = z^{-(\nu+i)} J_{\nu+i}(2z), \tag{2.13}$$

$$\lim_{n \to \infty} (-1)^n \frac{a_n^{1/2}}{b_n^i} p_{2n+1}^{[2i]}\left(\frac{z}{b_n}\right) = z^{-(\nu+i)} J_{\nu+i+1}(2z) \tag{2.14}$$

hold uniformly on compact subsets of $\mathbb{C}$. Note that J_t is the Bessel function of the first kind and order t given by (1.2), and

$$a_n^{-1/2} \simeq A n^a, \quad b_n \simeq B n^b, \quad \nu > -1, \quad 2a + 1 = 2b(\nu + 1) \tag{2.15}$$

with $A, B, b > 0$.

The assumptions (2.13)-(2.15) are natural. For example, the generalized Freud polynomials satisfy these conditions [19, 24]. In fact, they only need to hold for $i = 0$, i.e. for the orthonormal polynomials with respect to μ, and the additional conditions

$$\lim_{n \to \infty} \frac{\gamma_{2n}}{\gamma_{2n+1}} \frac{b_n}{n} = \lim_{n \to \infty} \frac{\gamma_{2n+1}}{\gamma_{2n+2}} \frac{b_n}{n} = \frac{1}{2b},$$

where γ_n are the leading coefficients of p_n for $n \geq 0$, cf. [24, Lemma 3].

Similar to the nonsymmetric case, it is simple to deduce that there is a strictly increasing function $f(x) = bx + a$ with $2f(0) + 1 = b(\nu + 1) > 0$ such that for $0 \leq k \leq n$, one has

$$\left(p_{2n}^{[4i]}\right)^{(2k)}(0) \simeq C_{k,i}(-1)^n n^{f(2k+2i)} = C_{k,i}(-1)^n n^{g(k+i)}, \tag{2.16}$$

$$\left(p_{2n+1}^{[4i]}\right)^{(2k+1)}(0) \simeq \widetilde{C}_{k,i}(-1)^n n^{f(2k+2i+1)} = \widetilde{C}_{k,i}(-1)^n n^{g^*(k+i)}, \tag{2.17}$$

where g and g^* are strictly increasing functions defined as

$$g(k) := f(2k) \qquad \text{with} \quad 2g(0) + 1 > 0,$$
$$g^*(k) := f(2k+1) \quad \text{with} \quad 2g^*(0) + 1 > 0$$

and

$$C_{k,i} = \frac{(-1)^k (2k)! A B^{2(k+i)}}{k! \Gamma(k + \nu + 2i + 1)}, \quad \widetilde{C}_{k,i} = \frac{(-1)^k (2k+1)! A B^{2(k+i)+1}}{k! \Gamma(k + \nu + 2i + 2)}. \tag{2.18}$$

Therefore, [19, Theorem 2] concerning the Mehler-Heine asymptotics for the symmetric polynomials q_n can be now written as follows.

Theorem 2.2. *Let $\{q_n\}_{n \geq 0}$ be the sequence of orthonormal with respect to* (2.12). *Then, assuming* (2.13)-(2.15)*, we have uniformly on compact subsets of $\mathbb{C}$,*

- *when $j = 2r$,*

$$\lim_{n\to\infty} (-1)^n a_n^{1/2} q_{2n}\left(\frac{z}{b_n}\right) = \sum_{i=0}^{r+1} (-1)^i d_{i,1}(L) z^{-\nu} J_{\nu+2i}(2z) := \Phi_{\nu,r,L}(z), \qquad (2.19)$$

$$\lim_{n\to\infty} (-1)^n a_n^{1/2} q_{2n+1}\left(\frac{z}{b_n}\right) = z^{-\nu} J_{\nu+1}(2z);$$

- *when $j = 2r + 1$,*

$$\lim_{n\to\infty} (-1)^n a_n^{1/2} q_{2n}\left(\frac{z}{b_n}\right) = z^{-\nu} J_\nu(2z),$$

$$\lim_{n\to\infty} (-1)^n a_n^{1/2} q_{2n+1}\left(\frac{z}{b_n}\right) = \sum_{i=0}^{r+1} (-1)^i d_{i,2}(L) z^{-\nu} J_{\nu+2i+1}(2z) := \Phi_{\nu,r,L}^*(z). \quad (2.20)$$

The coefficients $d_{i,1}(L)$ and $d_{i,2}(L)$ are given in a recursive way — viz.

$$d_{i,1}(L) = (-1)^i \frac{\tau_{i,r,L} - (2i)! \sum_{m=0}^{i-1} d_{m,1}(L) \dfrac{(-1)^i C_{i-m,m}}{(2(i-m))!\, C_{i,0}}}{(2i)! \dfrac{C_{0,i}}{C_{i,0}}}, \qquad (2.21)$$

$$d_{i,2}(L) = (-1)^i \frac{\rho_{i,r,L} - (2i+1)! \sum_{m=0}^{i-1} d_{m,2}(L) \dfrac{(-1)^i \widetilde{C}_{i-m,m}}{(2(i-m)+1)!\, \widetilde{C}_{i,0}}}{(2i+1)! \dfrac{\widetilde{C}_{0,i}}{\widetilde{C}_{i,0}}}, \qquad (2.22)$$

where the constants $C_{k,i}$ and $\widetilde{C}_{k,i}$ are provided in (2.18), and

$$\tau_{i,r,L} = \frac{L(g(i) - g(r)) + g(i) + g(r) + 1}{(1+L)(g(i) + g(r) + 1)}, \qquad (2.23)$$

$$\varrho_{i,r,L} = \frac{L(g^*(i) - g^*(r)) + g^*(i) + g^*(r) + 1}{(1+L)(g^*(i) + g^*(r) + 1)} \qquad (2.24)$$

with

$$\tau_{i,r,+\infty} = \lim_{L\to+\infty} \tau_{i,r,L} = \frac{g(i) - g(r)}{g(i) + g(r) + 1},$$

$$\varrho_{i,r,+\infty} = \lim_{L\to+\infty} \varrho_{i,r,L} = \frac{g^*(i) - g^*(r)}{g^*(i) + g^*(r) + 1}.$$

J_t denotes again the Bessel function of the first kind and order t given by (1.2).

To compute the value of L we can proceed like in Remark 2.3. The symmetric case was illustrated in [19, Section 6] with an example where the measure μ corresponds to the classical Hermite weight on the real line. In the constructive process to obtain the Mehler-Heine asymptotics the generalized Hermite polynomials appeared and, just as we might expect, the sequences a_n and b_n are related to the Maskhar-Rakhmanov-Saff numbers cf. [8, p. 2].

The statements of the two theorems in this section have been given in detail to have all the necessary ingredients for the algorithm and the computer program that we provide in this paper.

3. Process Flow Diagram and Computer Program

In the previous section we have provided all the necessary results to obtain the Mehler-Heine asymptotics for the discrete Sobolev orthonormal polynomials considered in this work. Now, a natural question arises: which are the inputs that we need to run the computer program? The rule should be the lesser inputs, the better. A first look at these results tells us that it is mandatory to give the information of the symmetry (or nonsymmetry) of μ, the general term of the sequence $\{M_n\}_n$, and the values of the constants j and c (the last one is taken automatically as 0 when the measure is symmetric). These are basic inputs determined by the inner product (1.1). In addition, a second somewhat deeper look provides us with relevant information. Together with the previous inputs, if we know the general term of the sequences $a_{n,c}$ and $b_{n,c}$ (resp. a_n and b_n), we can compute explicitly all the functions and constants to obtain the Mehler-Heine formulae given in Theorem 2.1 (resp. Theorem 2.2). Therefore, the key inputs of the program are these two sequences.

In some situations both sequences are strongly connected: for example, for the generalized Freud orthonormal polynomials the sequence b_n is easily obtained from the sequence a_n [8]. But, in general, we choose to provide both sequences because they are obtained in a straightforward way through the Mehler-Heine formulae satisfied by the standard orthonormal polynomials with respect to μ (according to the assumptions in Theorems 2.1 and 2.2).

In some cases both sequences are well-known for the corresponding families of orthonormal polynomials — e.g. Laguerre [30], Jacobi [30], or generalized Freud [2–8]. For this reason, we preloaded these families in the program. Thus, we want to know the Mehler-Heine asymptotics for the discrete Sobolev orthonormal polynomials with respect to (1.1), where μ is one of the measures related to the families of standard polynomials just cited, then it is enough to indicate it without providing explicitly both sequences. In fact, we could also include the case corresponding to the so-called modified or generalized Jacobi orthonormal polynomials [7], i.e. when $d\mu = h(x)(1-x)^\alpha(1+x)^\beta dx$ with $\alpha, \beta > 0$, and where h is a real analytic function and strictly positive on $[-1, 1]$. However, the function h has no influence on the Mehler-Heine asymptotics [7, 24], so if we are interested in this type of asymptotics for the dSOP in this case, then it is enough to consider the Jacobi case with $\alpha, \beta > 0$.

In conclusion, to run the program, firstly, we choose either between the preloaded measures or the general case providing the key inputs — i.e. the sequences $a_{n,c}$ and $b_{n,c}$ or a_n and b_n. Thus,

- **General case** (Key inputs). After entering both sequences, the program asks about the symmetry (or nonsymmetry) of μ and for the basic inputs. For this case, we remark that the user must know that the sequences considered satisfy the assumptions (2.3)-(2.4) given in Theorem 2.1 (resp. the conditions (2.13)-(2.14) in Theorem 2.2). In other words, the user must know the corresponding Mehler-Heine formulae for those polynomials. Anyway, to avoid wrong results, the program checks automatically the condition (2.5) (resp. (2.15)).

- **Preloaded cases**. The program asks about some specific details of the preloaded measures and for the basic inputs.

Regarding the outputs, the program returns the corresponding limit function in the Mehler-Heine formula, together with three extras: the value of L, a list of the coefficients of $d_{i,c}(L)$, $d_{i,1}(L)$ or $d_{i,2}(L)$ as appropriate, and a plot of the limit function around the point c (just where this type of local asymptotics is more precise).

Notice that the program checks the admissibility of the given inputs. Thus, as we have commented before, it checks whether the sequences $a_{n,c}$ and $b_{n,c}$ (resp. a_n and b_n) satisfy the conditions (2.5) (resp. (2.15)), and whether j is a nonnegative integer. In the preloaded cases the program also checks that the involved parameters are admissible: for example, in the Laguerre case the parameter α must be greater than -1.

We summarize how the algorithm and the computer program work in the following process flow diagram — cf. Fig. 1.

4. Examples

We illustrate how the program works through examples where we only show some paths of the process flow diagram given in the previous section. Nevertheless, we are confident that it is sufficient to manage the program.

4.1. Example 1. Path: *General Case*

In this path we must provide the so-called key inputs. We tackle a case corresponding to a nonsymmetric measure belonging to the Nevai-Blumenthal class considered in [5]. When the *Mathematica*® notebook is evaluated, the first thing we see is the dialog notebook given in Fig. 2. In this example, we will show that it is enough to fill eight dialog notebooks where we will enter the basic and key inputs to get the desired limit function in the Mehler-Heine asymptotics. When the measure is symmetric we will only fill six dialog notebooks because $c = 0$. The number of dialog notebooks to fill does not include those asking for confirmation about the data entered — cf. Fig. 5.
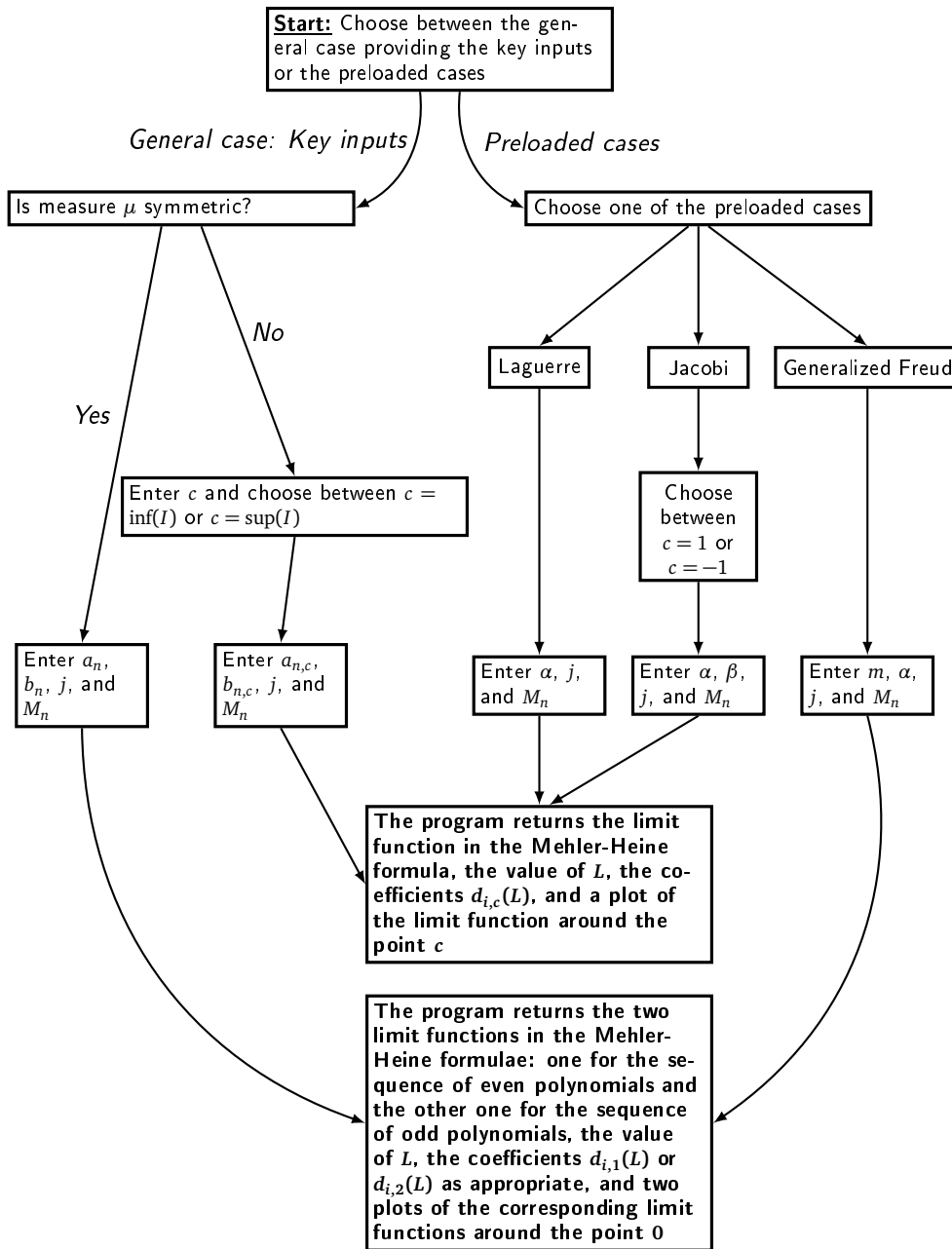
Figure 1: Flow diagram.

We click on *General case*. Next, the program asks about the symmetry of the measure — cf. Fig. 3. For this example, we choose *No*.

Next, the value and location (infimum or supremum) of point c are asked. We write 1 and click on *Supremum* in the corresponding dialog notebooks — cf. Figs. 4 and 6.
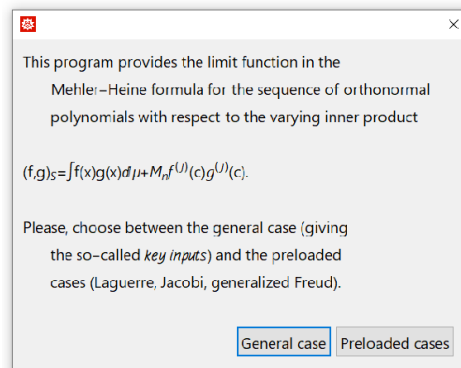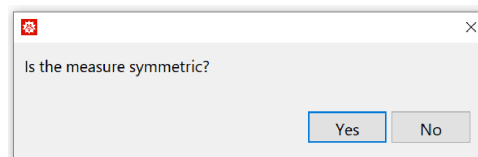
Figure 2: First dialog notebook.
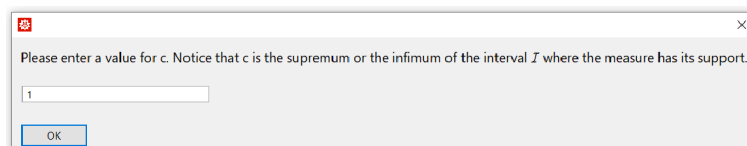

Figure 3: Second dialog notebook.

Figure 4: Third dialog notebook.
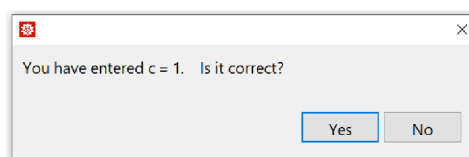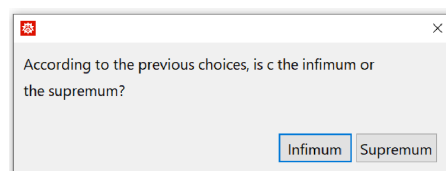

Figure 5: Dialog notebook asking for confirmation about the data.

Figure 6: Fourth dialog notebook.

Since the nonsymmetric measure considered belongs to the Nevai-Blumenthal class $M(0,1)$, the sequences (key inputs) are known [5,24]. Thus,

$$a_{n,1} = 4^{\nu_1} n^{-2\nu_1 - 1}, \quad b_{n,1} = \frac{n^2}{2}.$$

It is easy to check that the conditions (2.5) are satisfied by any $\nu_1 > -1$. For this example, we take $\nu_1 = 1$, so $a_{n,1} = 4/n^3$. In this way, we write these sequences in the dialog notebooks such as we see in Figs. 8 and 9. Obviously, and according to Theorem 2.1, the choice of ν_1 changes the limit function in the Mehler-Heine formula. Thus, one question arises: why does the program not ask about ν_1 (in general, for ν_c)? We think that it is better to give only two sequences and let the program check whether all the conditions required in (2.5) are satisfied. Thus, we trust the program — i.e. if we enter two sequences that do not satisfy (2.5), then the program gives a warning. For example, if we enter $a_{n,1} = n$ and $b_{n,1} = 3n^2$, then from (2.5) we would deduce that $\nu_1 = -1$, which is not an admissible value. Thus, the program returns the following warning:
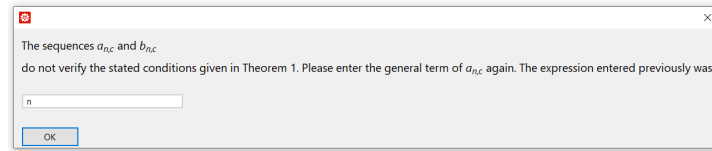

Figure 7: Dialog notebook when (2.5) are not satisfied.

Notice that the program does not continue with the evaluation of the notebook and asks us for a new expression for the sequence $a_{n,1}$ indicating the expression entered previously. Next, a new expression for $b_{n,1}$ will be requested.

Finally, the program asks for the remaining basic inputs such as the order of the derivatives, j, and the general term of the sequence M_n. In this example, we chose $j = 5$ and $M_n = n^2 - n + 3$. Notice that M_n must be a sequence of nonnegative real numbers. This relevant fact is warned in the corresponding dialog notebook, but it is not checked by the program. Thus, these data are entered as Figs. 10 and 11 show.

Eventually, in this easy way the program provides us with relevant information — viz. the case that we are tackling according to the value of L, the expression of the limit function in the Mehler-Heine asymptotics together with a list of the coefficients $(d_{i,c}(L))$ in that
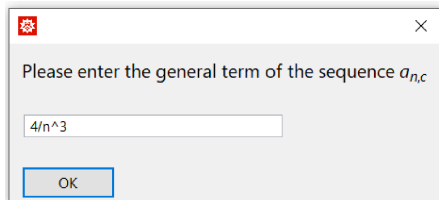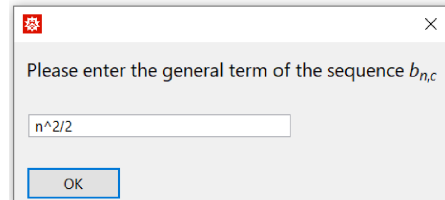

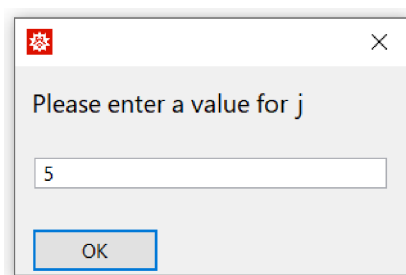Figure 8: Fifth dialog notebook.

Figure 9: Sixth dialog notebook.

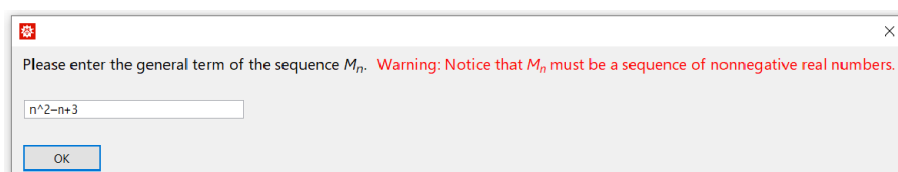Figure 10: Seventh dialog notebook.

Figure 11: Eighth dialog notebook.

expression, and a plot of the limit function around point c. In this example, the relevant outputs are collected in Figs. 12 and 13.
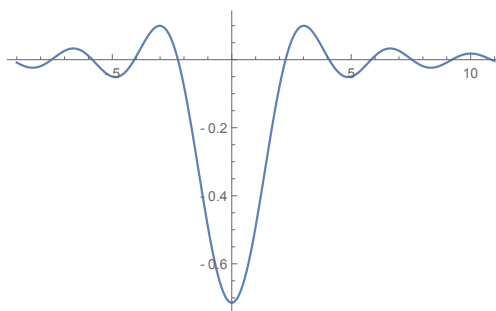

Figure 12: Graphic output of the program for this example.

We omitted all the dialog notebooks asking for confirmation about the data, except the one shown in Fig. 5 as an example. These notebooks are useful in practice because they help to avoid errors in input data.

4.2. Example 2. Path: *Preloaded cases:* Jacobi

For several standard orthonormal polynomials the sequences $a_{n,c}$ and $b_{n,c}$ (resp. a_n and b_n) are known. There is no need to provide them because they are already implemented in the program. Our second example corresponds to a classical bounded measure. More exactly, we obtain the Mehler-Heine asymptotics for the discrete Sobolev orthonormal polynomials for the measure μ determined by the Jacobi weight on $[-1, 1]$, i.e. $d\mu(x) = (1 - x)^{\alpha}(1 + x)^{\beta} dx$ with $\alpha, \beta > -1$.

Case L=

∞

The function $\varphi_{1,1,5,\infty}(x)$ is

$$-\frac{5\,\text{BesselJ}[1,2x]}{7x} - \frac{9\,\text{BesselJ}[3,2x]}{14x} - \frac{25\,\text{BesselJ}[5,2x]}{42x} - \frac{\text{BesselJ}[7,2x]}{3x} + \frac{9\,\text{BesselJ}[9,2x]}{77x} - \frac{\text{BesselJ}[11,2x]}{42x} + \frac{\text{BesselJ}[13,2x]}{462x}$$

Coefficient $d_{0,1}$

$-\dfrac{5}{7}$

Coefficient $d_{1,1}$

$-\dfrac{9}{14}$

Coefficient $d_{2,1}$

$\dfrac{25}{42}$

Coefficient $d_{3,1}$

$-\dfrac{1}{3}$

Coefficient $d_{4,1}$

$\dfrac{9}{77}$

Coefficient $d_{5,1}$

$-\dfrac{1}{42}$

Coefficient $d_{6,1}$

$\dfrac{1}{462}$

Plot of $\varphi_{1,1,5,\infty}(x)$ around of the point $c = 1$

Figure 13: Output of the program for this example.

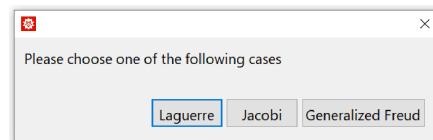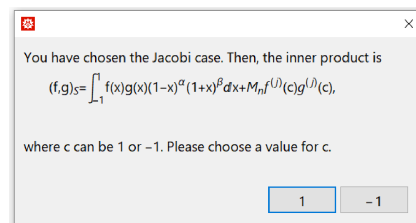Now in Fig. 2, we click on *Preloaded cases* and obtain the dialog notebook shown in Fig. 14.

Figure 14: Preloaded cases.

We choose *Jacobi*. In the next dialog notebook — cf. Fig. 15, we see the inner product considered and are asked to choose the value of c. Here, we set $c = -1$.

Figure 15: The nonstandard inner product and the choice of c.

In the next two steps we enter the values of the parameters α and β. For this example, we choose $\alpha = 2$ and $\beta = 1/5$, cf. Figs. 16 and 17.
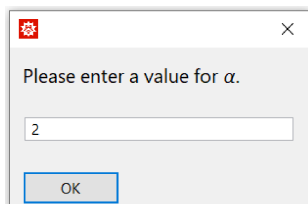

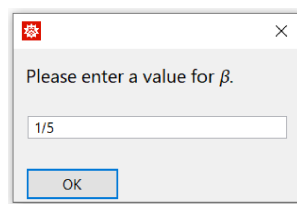Figure 16: The value of α.   Figure 17: The value of β.

The program checks whether the parameters required are admissible. It is well known that in this case the values α and β must be greater than -1. If we enter a non-admissible value, the program returns a warning — e.g. for $\beta = -5$ we obtain the answer shown in Fig. 18.
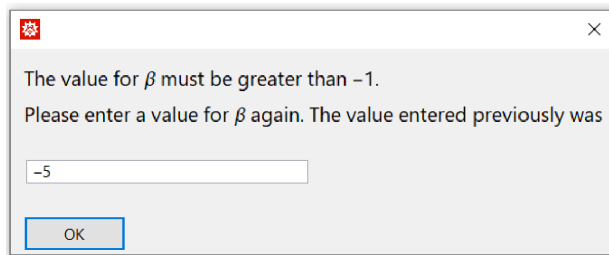
Figure 18: Dialog notebook when a non-admissible value of a parameter is entered.

Finally, similar to Example 4.1, the inputs j and M_n are required. We omit the corresponding dialog notebooks. Thus, introducing the values $j = 6$ and $M_n = n^{-132/5}$, the program returns (we omit the output corresponding to the coefficients $d_{i,-1}$ to save space):

Figure 19: Value of L and limit function in the Mehler-Heine formula.

Plot of $\varphi_{-1,\frac{1}{5},6,\frac{5}{2162\,688\,2^{1/5}\,\mathrm{Gamma}\left[\frac{36}{5}\right]^2}}$ (x) around of the point c = –1
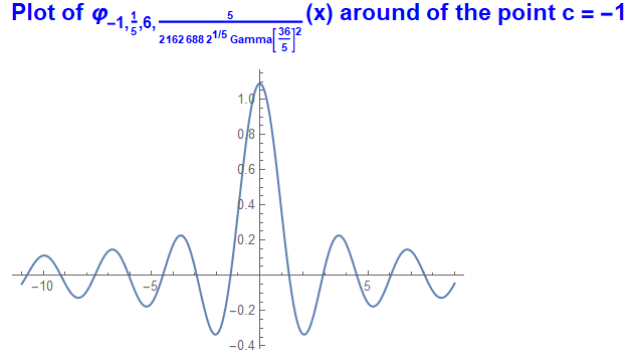
Figure 20: Graphic output of the program.

4.2.1. Differences between $c = -1$ and $c = 1$.

The choice of c is very relevant. When $c = -1$ the value of β plays a main role in the Mehler-Heine asymptotics. However, when $c = 1$, it is α which plays this role. Thus, the value of L changes according to the value of c. In the case $c = -1$ we have

$$f_{-1}(k) = 2k + \beta + \frac{1}{2}, \quad C_{k,0} = \frac{(-1)^k}{2^{\frac{2k+\alpha+\beta}{2}}\Gamma(k+\beta+1)}.$$

Now it follows from (2.11) that

$$L = \lim_{n\to\infty} n^{-132/5} \frac{C_{6,0}^2}{2f_{-1}(6)+1} n^{2f_{-1}(6)+1} = \frac{C_{6,0}^2}{2f_{-1}(6)+1} = \frac{5}{2162688\sqrt[5]{2}\,\Gamma^2\left(\frac{36}{5}\right)}.$$

On the other hand, if $c = 1$ then

$$f_1(k) = 2k + \alpha + \frac{1}{2}, \quad C_{k,0} = \frac{1}{2^{\frac{2k+\alpha+\beta}{2}}\Gamma(k+\alpha+1)}.$$

Consequently, we obtain

$$L = \lim_{n\to\infty} n^{-132/5} \frac{C_{6,0}^2}{2f_1(6)+1} n^{2f_1(6)+1}$$

$$= \lim_{n\to\infty} n^{-132/5} \frac{1}{799065243648000\sqrt[5]{2}} n^{30} = +\infty.$$

Repeating this example for $c = 1$ leads to the result shown in Fig. 21 visibly different from the one in Fig. 19 for $c = -1$. The outcome is consistent with expressions (2.8) related to Theorem 2.1.

We note that the Jacobi weights have been considered in [18] by using another standardization. However, if we orthonormalize the polynomials in [18], we obtain the same results as in Theorem 2.1.

Case L=

∞

The function $\varphi_{1,2,6,\infty}(x)$ is

$$-\frac{2\,\mathrm{BesselJ}[2,\,2\,x]}{3\,x^2} - \frac{2\,\mathrm{BesselJ}[4,\,2\,x]}{3\,x^2} + \frac{6\,\mathrm{BesselJ}[6,\,2\,x]}{11\,x^2} - \frac{10\,\mathrm{BesselJ}[8,\,2\,x]}{33\,x^2} + \frac{50\,\mathrm{BesselJ}[10,\,2\,x]}{429\,x^2} - \frac{30\,\mathrm{BesselJ}[12,\,2\,x]}{1001\,x^2} + \frac{2\,\mathrm{BesselJ}[14,\,2\,x]}{429\,x^2} - \frac{\mathrm{BesselJ}[16,\,2\,x]}{3003\,x^2}$$

Figure 21: Value of L and limit function in the Mehler-Heine formula when $c = 1$.

4.3. Example 3. Path: *Preloaded Cases*. Generalized Freud

To complete the examples, we consider a symmetric measure μ. In this way, we illustrate Theorem 2.2. We click on *Preloaded cases* and next on *Generalized Freud* in the dialog notebook shown in Fig. 14. Notice that $d\mu(x) = x^{2m}e^{-2|x|^{\alpha}}dx$ with $\alpha > 1$, $m \in \mathbb{N} \cup \{0\}$, and $c = 0$.

The program asks for the value of m, cf. Fig. 22. As an example we choose $m = 3$.
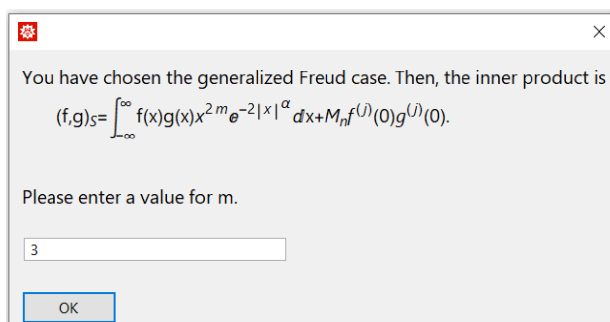
Figure 22: Generalized Freud case and the choice of m.

We omit the dialog notebooks where the program requires the values for α, j, and the general term M_n. Here, we choose $\alpha = 4$, $j = 6$, and $M_n = n$. Omitting again the output for the coefficients, the program returns, cf. Fig. 23 and 24.

Case L=

∞

Limit function in the Mehler–Heine formula for the subsequence of even Sobolev orthonormal polynomials: $\phi_{\frac{5}{2},3,\infty}(x)$

$$\frac{-22\,x\,\left(53\,865 - 35\,910\,x^2 + 7182\,x^4 - 684\,x^6 + 32\,x^8\right)\,\mathrm{Cos}[2\,x] + \left(592\,515 - 1\,185\,030\,x^2 + 395\,010\,x^4 - 52\,668\,x^6 + 3696\,x^8 - 64\,x^{10}\right)\,\mathrm{Sin}[2\,x]}{64\,\sqrt{\pi}\,x^{13}}$$

Limit function in the Mehler–Heine formula for the subsequence of odd Sobolev orthonormal polynomials

$$\frac{\mathrm{Cos}[2\,x] - \frac{15\,\mathrm{Cos}[2\,x]}{4\,x^2} + \frac{15\,\mathrm{Sin}[2\,x]}{8\,x^3} - \frac{3\,\mathrm{Sin}[2\,x]}{x}}{\sqrt{\pi}\,x^3}$$

Figure 23: Value of L and limit functions in the Mehler-Heine formula.

Remark 4.1. We make the following comments:

- According to Theorem 2.2, there are two Mehler-Heine formulae — viz. for even and odd polynomials. Since $j = 6$, we have $r = 3$. Therefore, in this example the odd case lacks interest.

Plot of $\phi_{\frac{5}{2},3,\infty}(x)$ **around the point 0 in the even case**

Plot of the limit function around the point 0 in the odd case
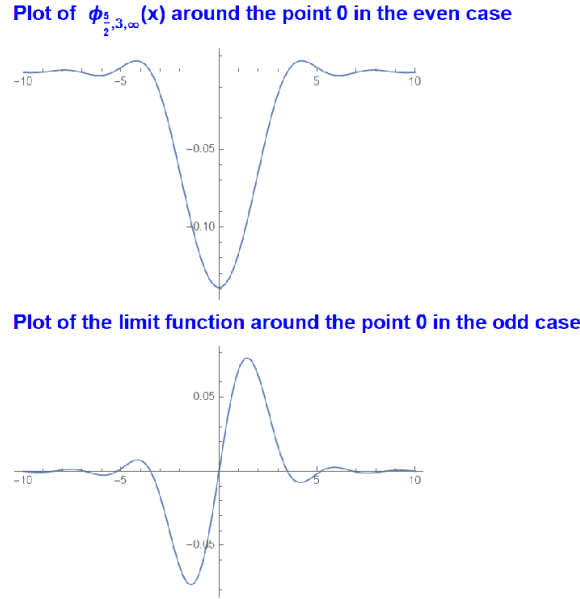
Figure 24: Graphic outputs of the program for this example.

- In [3] the authors prove that for standard generalized Freud orthogonal polynomials, the parameter α does not influence the limit function of the Mehler-Heine formula. As it is expected, this also occurs for the Sobolev case. However, the parameter m has an essential impact on the asymptotics because $\nu = m - 1/2$. In this way, the Mehler-Heine formulae do not change for these Sobolev polynomials if we choose any $\alpha > 1$. However, changing m in this example, we obtain different Mehler-Heine formulae for the even polynomials.

- Since $\nu = m - 1/2$, being m a nonnegative integer, the Bessel functions of the first kind can be expressed via trigonometric functions $\sin(x)$ and $\cos(x)$ as Fig. 23 shows. *Mathematica*® often uses this type of simplifications. In this example, it can be checked that

$$\frac{\cos(2x) - \frac{15\cos(2x)}{4x^2} + \frac{15\sin(2x)}{8x^3} - \frac{3\sin(2x)}{x}}{\sqrt{\pi}x^3} = x^{-5/2}J_{7/2}(2x),$$

which is consistent with Theorem 2.2.

4.4. Two other cases. Comments and warnings.

The simplest case within the Freud weights corresponds to classical Hermite weight. Thus, the authors of [19] consider the measure given by $d\mu(x) = e^{-x^2}dx$ to illustrate their results about the asymptotic behavior of the zeros of the orthonormal polynomials with respect to the discrete Sobolev inner product

$$(f,g)_S = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx + M_n f^{(j)}(0)g^{(j)}(0). \tag{4.1}$$

We know that the asymptotic behavior of the zeros is obtained from the corresponding Mehler-Heine formula by the Hurwitz's theorem. Since the above inner product is a particular case of (1.1), we could want to apply our program. Our first idea is to use the preloaded case *Generalized Freud*. However, choosing $m = 0$ and $\alpha = 2$ yields $d\mu = e^{-2x^2}dx$. This subtle difference between both measures makes it impossible to use the preloaded case directly. Nevertheless, there is a simple solution — viz. one can choose the *General case*, then symmetric measure, and introduce the key inputs as $a_n = b_n = \sqrt{n}$. Another option is to consider the relation

$$\tilde{q}_n(x) = 2^{1/4} q_n(\sqrt{2}x),$$

where q_n are the orthonormal polynomials with respect to the inner product (4.1) and $\tilde{q}_n$ are the orthonormal polynomials with respect to the Sobolev inner product

$$(f,g)_{S_1} = \int_{-\infty}^{+\infty} f(x)g(x)e^{-2x^2}dx + \widetilde{M}_n f^{(j)}(0)g^{(j)}(0), \tag{4.2}$$

where $\widetilde{M}_n = M_n/2^{j+1/2}$. Now one can use the preloaded case *Generalized Freud*, where we set $m = 0$, $\alpha = 2$, and when the general term of M_n is requested, we have to enter $\widetilde{M}_n$. As it is natural, both options lead to the same result. This occurs because the key inputs for $d\mu(x) = e^{-2x^2}dx$ are $a_n = \sqrt{n/2}$ and $b_n = \sqrt{2n}$. Therefore, the limits in Theorem 2.2 coincide for both families.

Another case of interest is related to the Gegenbauer measure

$$d\mu(x) = \left(1 - x^2\right)^{\alpha} dx, \quad \alpha > -1.$$

This can be viewed as a particular case of the Jacobi case when $\alpha = \beta$. Hence, we select the preloaded case *Jacobi* and choose either $c = -1$ or $c = 1$. On the other hand, since this measure is symmetric we can first select *General case*, $c = 0$, and using the corresponding sequences a_n and b_n gives the Mehler-Heine formula.

It is convenient to warn about some aspects of the program:

- If α, β (Laguerre and Jacobi cases) or m (generalized Freud case) are large enough, the limit functions in the corresponding Mehler-Heine formulae are very close to the zero function in the uniform norm. Then, the plot is not useful, and *Mathematica*® has problems with the plot construction and issues the warnings of the type "too small to represent as a normalized machine number; precision may be lost." This is not a malfunction because the limit function and the coefficients are correctly obtained, which are the main objectives of the program.

- To simplify the expression of the limit functions returned by the program we have used the command *FullSimplify* in the symmetric cases (general case and generalized Freud). In these situations, the simplification can take a substantial amount of time. We can gain time removing this command, although the expression obtained for the limit function can be enormous.

4.5. Zeros

Although it is not the objective of this paper, we guess that the location of the zeros of the orthonormal polynomials related to the inner product (1.1) is an interesting problem. Let us explain how to use the program to locate zeros of these dSOP. First, we need some theoretical results the proof of which is similar to [19, Section 4]. As usual we distinguish between a symmetric and nonsymmetric measure μ.

4.5.1. Nonsymmetric measure

Proposition 4.1. *The polynomial $q_n(x)$, $n \geq 1$, orthonormal with respect to (1.1), has n real simple zeros, at most one of which is located outside of I.*

In order to establish the asymptotic behavior of the zeros of q_n we have to consider two cases — viz. $c = \inf(I)$ and $c = \sup(I)$. The asymptotic result in the next statement is the consequence of applying the Hurwitz's theorem to the Mehler-Heine formula (2.8).

Proposition 4.2. *Let $j > 0$ and s_i refer to the i-th positive zero of the function $\varphi_{c,\nu_c,j,L}(z)$.*

(a) *Choose $c = \inf(I)$ and let $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ be the zeros of the polynomial $q_n(x)$ ordered in an increasing order. If n is large enough, then*

   (a1) *If*
$$0 \leq L < \frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)},$$
   then all zeros of $q_n(x)$ are located inside of I. Moreover,
$$\lim_{n \to \infty} b_{n,c}(y_{n,i} - c) = s_i^2, \quad i \geq 1.$$

   (a2) *If*
$$\frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)} \leq L \leq +\infty,$$
   we have
$$\lim_{n \to \infty} y_{n,1} = c, \quad \lim_{n \to \infty} b_{n,c}(y_{n,i} - c) = s_{i-1}^2, \quad i \geq 2.$$
   Moreover, if
$$L > \frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)},$$
   then $y_{n,1} < c$.

(b) *Choose $c = \sup(I)$ and let $y_{n,1} > y_{n,2} > \cdots > y_{n,n}$ be the zeros of the polynomial $q_n(x)$ ordered in a decreasing order. If n is large enough, then*

(b1) If

$$0 \leq L < \frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)},$$

then all zeros of $q_n(x)$ are located inside of I. Moreover,

$$\lim_{n \to \infty} b_{n,c}(c - y_{n,i}) = s_i^2, \quad i \geq 1.$$

(b2) If

$$\frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)} \leq L \leq +\infty,$$

we have

$$\lim_{n \to \infty} y_{n,1} = c, \quad \lim_{n \to \infty} b_{n,c}(c - y_{n,i}) = s_{i-1}^2, \quad i \geq 2.$$

Moreover, if

$$L > \frac{f_c(0) + f_c(j) + 1}{f_c(j) - f_c(0)},$$

then $y_{n,1} > c$.

Remark 4.2. If $j = 0$, then (1.1) is a standard inner product, so that all zeros of the corresponding orthonormal polynomials are located in I. Under the notation of Proposition 4.2, the following assertions hold:

(i) If $c = \inf(I)$ and $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ are the zeros of the polynomial $q_n(x)$ ordered in an increasing order, then

$$\lim_{n \to \infty} b_{n,c}(y_{n,i} - c) = s_i^2, \quad i \geq 1.$$

(ii) If $c = \sup(I)$ and $y_{n,1} > y_{n,2} > \cdots > y_{n,n}$ are the zeros of the polynomial $q_n(x)$ ordered in a decreasing order, then

$$\lim_{n \to \infty} b_{n,c}(c - y_{n,i}) = s_i^2, \quad i \geq 1.$$

4.5.2. Symmetric measure

In this case $c = 0$, $I = (-\rho, \rho)$ with $\rho > 0$, so we have the Sobolev-type inner product (2.12), i.e.

$$(f, g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(0)g^{(j)}(0).$$

It is clear that for even j, the polynomials q_{2n+1} coincide with the standard polynomials p_{2n+1}. Analogously, for odd j we have $q_{2n} = p_{2n}$ for all n. Thus, the Mehler-Heine formulae also coincide as Theorem 2.2 shows. Therefore, we only consider two other cases mentioned in (2.19)-(2.20). Applying the symmetrization process [2, 19], Theorem 2.2, Proposition 4.2, and the Hurwitz's theorem, we arrive at the following proposition.

Proposition 4.3. *Let $j > 0$, t_i refer to the i-th positive zero of the function $\Phi_{v,r,L}(z)$, and t_i^* to the i-th positive zero of the function $\Phi_{v,r,L}^*(z)$.*

(a) *For any even j, the polynomial q_{2n} has $2n-2$ real simple zeros and at most 2 purely imaginary zeros. Moreover, if n is large enough, then*

   (a1) *If*

$$0 \leq L < \frac{g(0) + g(j) + 1}{g(j) - g(0)},$$

   the polynomial q_{2n} has $2n$ real simple zeros. Denoting by $y_{2n,1} < y_{2n,2} < \cdots < y_{2n,n}$ the positive zeros of the polynomial q_{2n} ordered in an increasing order, we have

$$\lim_{n \to \infty} b_n y_{2n,i} = t_i, \quad i \geq 1.$$

   (a2) *If*

$$\frac{g(0) + g(j) + 1}{g(j) - g(0)} < L \leq +\infty,$$

   the polynomial q_{2n} has $2n-2$ real simple zeros and exactly 2 purely imaginary zeros. Denoting by $y_{2n,1}$ and $y_{2n,-1}$ the purely imaginary zeros and by $y_{2n,2} < y_{2n,3} < \cdots < y_{2n,n}$ the positive zeros of the polynomial q_{2n} ordered in an increasing order, we have

$$\lim_{n \to \infty} y_{2n,1} = \lim_{n \to \infty} y_{2n,-1} = 0, \quad \lim_{n \to \infty} b_n y_{2n,i} = t_{i-1}, \quad i \geq 2.$$

   If

$$L = \frac{g(0) + g(j) + 1}{g(j) - g(0)},$$

   then either q_{2n} has exactly 2 purely imaginary zeros, and we have the above asymptotic behavior, or all zeros of q_{2n} are real being $y_{2n,1} < y_{2n,2} < \cdots < y_{2n,n}$ their positive zeros with the asymptotic behavior

$$\lim_{n \to \infty} y_{2n,1} = 0, \quad \lim_{n \to \infty} b_n y_{2n,i} = t_{i-1}, \quad i \geq 2.$$

(b) *For any odd j, the polynomial q_{2n+1} has $2n-1$ real and simple zeros, one of which is located at the origin, and at most 2 purely imaginary zeros. If n is large enough, then*

   (b1) *If*

$$0 \leq L < \frac{g^*(0) + g^*(j) + 1}{g^*(j) - g^*(0)},$$

   the polynomial q_{2n+1} has $2n+1$ real simple zeros. Denoting by $y_{2n+1,1} < y_{2n+1,2} < \cdots < y_{2n+1,n}$ the positive zeros of the polynomial q_{2n+1} ordered in an increasing order, we have

$$\lim_{n \to \infty} b_n y_{2n+1,i} = t_i^*, \quad i \geq 1.$$

(b2) If

$$\frac{g^*(0) + g^*(j) + 1}{g^*(j) - g^*(0)} < L \le +\infty,$$

the polynomial q_{2n+1} has $2n - 1$ real simple zeros and exactly 2 purely imaginary zeros. Denoting by $y_{2n+1,1}$ and $y_{2n+1,-1}$ the purely imaginary zeros and by $y_{2n+1,2} < y_{2n+1,3} < \cdots < y_{2n+1,n}$ the positive zeros of the polynomial q_{2n+1} ordered in an increasing order, we have

$$\lim_{n\to\infty} y_{2n+1,1} = \lim_{n\to\infty} y_{2n+1,-1} = 0, \quad \lim_{n\to\infty} b_n y_{2n+1,i} = t_{i-1}^*, \quad i \ge 2.$$

If

$$L = \frac{g^*(0) + g^*(j) + 1}{g^*(j) - g^*(0)},$$

then either q_{2n+1} has exactly 2 purely imaginary zeros, and we have the above asymptotic behavior, or all zeros of q_{2n+1} are real being $y_{2n+1,1} < y_{2n+1,2} < \cdots < y_{2n+1,n}$ their positive zeros with the asymptotic behavior

$$\lim_{n\to\infty} y_{2n+1,1} = 0, \quad \lim_{n\to\infty} b_n y_{2n+1,i} = t_{i-1}^*, \quad i \ge 2.$$

Remark 4.3. The case $j = 0$ can be considered analogously to Remark 4.2.

Remark 4.4. Taking into account the Hurwitz's theorem, we note that the scaling sequence $b_{n,c}$ (or b_n) plays an important role in the description of asymptotic of the dSOP zeros.

4.5.3. Examples

In previous works we used numerical experiments to illustrate the asymptotic behavior of the zeros — cf. [19] and the references therein. In this paper we unify the notation (Theorems 2.1 and 2.2), so that we can proceed in a similar way. Now we want to show how the algorithm and the program can be used in locating the zeros of these dSOP. In order to illustrate Propositions 4.2 and 4.3, we present four examples but many more ones can be shown if needed.

Example 4.1 (Jacobi case)**.** We consider the inner product (1.1) with

$$d\mu(x) = (1 - x)^{-1/4}(1 + x)^{2/3} dx,$$

and $c = 1$, $j = 4$, $M_n = 10^7/(n + 1)^{35/2}$, i.e.

$$(f, g)_S = \int_{-1}^{1} f(x)g(x)(1 - x)^{-1/4}(1 + x)^{2/3} dx + \frac{10^7}{(n + 1)^{35/2}} f^{(4)}(1)g^{(4)}(1). \quad (4.3)$$

In this situation, $1 = \sup(I)$ and $b_{n,1} = n^2/2$ [24]. Therefore, we apply the program by choosing *Preloaded cases* → *Jacobi* and enter the data as above. The program returns (we omit the coefficients $d_{i,1}(L)$ and the plot of the limit function):

Case L=

$$\frac{15\,625}{7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2}$$

The function $\varphi_{1,-\frac{1}{4},4,\frac{15\,625}{7\,2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2}}(\mathbf{x})$ **is**

$$-\frac{3\,828\,125\,x^{1/4}\,\mathrm{BesselJ}\left[\frac{7}{4},\,2\,x\right]}{437\,\left(15\,625 + 7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2\right)} + \frac{43\,750\,000\,x^{1/4}\,\mathrm{BesselJ}\left[\frac{15}{4},\,2\,x\right]}{3933\,\left(15\,625 + 7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2\right)} - \frac{35\,000\,000\,x^{1/4}\,\mathrm{BesselJ}\left[\frac{23}{4},\,2\,x\right]}{5301\,\left(15\,625 + 7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2\right)} +$$

$$\frac{8\,000\,000\,x^{1/4}\,\mathrm{BesselJ}\left[\frac{31}{4},\,2\,x\right]}{3933\,\left(15\,625 + 7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2\right)} - \frac{32\,000\,000\,x^{1/4}\,\mathrm{BesselJ}\left[\frac{39}{4},\,2\,x\right]}{121\,923\,\left(15\,625 + 7 \times 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2\right)} + x^{1/4}\,\mathrm{BesselJ}\left[-\frac{1}{4},\,2\,x\right]\left(1 + \frac{1}{-\frac{19}{35} - \frac{19\cdot 2^{5/12}\,\mathrm{Gamma}\left[\frac{19}{4}\right]^2}{78\,125}}\right)$$

Figure 25: Output the program for the example corresponding to the Jacobi case.

It is clear that we are able to apply Proposition 4.2 (b). Since $f_1(k) = 2k + \alpha + 1 = 2k + 3/4$, we have

$$\frac{19}{16} = \frac{f_1(0) + f_1(4) + 1}{f_1(4) - f_1(0)} < L = \frac{15625}{7 \times 2^{5/12}\Gamma\left(\frac{19}{4}\right)^2} \approx 6.07853.$$

The term $(f_1(0) + f_1(4) + 1)/(f_1(4) - f_1(0))$ can be computed by the program by writing $(f[0]+f[j]+1)/(f[j]-f[0])$ in the *Command Window* of *Mathematica*® after running the program. Therefore, Proposition 4.2 (b2) provides the information about the asymptotic behavior of the zeros of the dSOP with respect to (4.3). Thus, for n large enough, we have $y_{n,1} > 1$ and

$$\lim_{n\to\infty} y_{n,1} = 1, \quad y_{n,i} \approx 1 - \frac{s_i^2}{b_{n,1}} = 1 - \frac{2s_i^2}{n^2}, \quad i \geq 2.$$

To obtain approximations of the zeros $y_{n,i}$, $i \geq 2$, one only has to determine the zeros s_i in some interval. For example, if we establish the first three zeros in the interval $[0,6]$, we can use the following *Mathematica*® command

$$\text{NSolve}[\text{MH}[j, x, \alpha] == 0 \,\&\&\, 0 < x < 6, x, \text{WorkingPrecision} \rightarrow 16],$$

thus obtaining

$$s_1 = 1.325920061278339,$$
$$s_2 = 3.244837348236775,$$
$$s_3 = 5.034929612268939.$$

As the result, for $n = 50$ we get the following approximations of the zeros $y_{n,i}$, $i = 2,3,4$:

$$y_{50,2} \approx 1 - \frac{2 \times 1.325920061278339^2}{50^2} = 0.998593548792880,$$
$$y_{50,3} \approx 1 - \frac{2 \times 3.244837348236775^2}{50^2} = 0.991576824466790,$$

$$y_{50,4} \approx 1 - \frac{2 \times 5.034929612268939^2}{50^2} = 0.979719587039598.$$

Mehler-Heine asymptotics is a type of local asymptotic, so we obtain better approximations for the zeros closer to point c. Since in this case $c = 1$, we get better approximations for the zeros closer to 1.

Example 4.2 (Laguerre case). In this case we choose $\alpha = 3$, $j = 5$, and $M_n = 1/(n+1)$, which gives the Sobolev-type inner product

$$(f,g)_S = \int_0^\infty f(x)g(x)x^3 e^{-x} dx + \frac{1}{n+1} f^{(5)}(0)g^{(5)}(0). \tag{4.4}$$

Noting that $c = 0 = \inf(I)$, we proceed similar to the Example 4.1. Thus, the program returns

Case L=

∞

The function $\varphi_{0,3,5,\infty}(x)$ is

$$-\frac{5\,\text{BesselJ}[3,\,2x]}{9\,x^3} - \frac{7\,\text{BesselJ}[5,\,2x]}{9\,x^3} + \frac{49\,\text{BesselJ}[7,\,2x]}{99\,x^3} - \frac{7\,\text{BesselJ}[9,\,2x]}{33\,x^3} + \frac{7\,\text{BesselJ}[11,\,2x]}{117\,x^3} - \frac{\text{BesselJ}[13,\,2x]}{99\,x^3} + \frac{\text{BesselJ}[15,\,2x]}{1287\,x^3}$$

Figure 26: Output the program for the example corresponding to the Laguerre case.

Now we can apply Proposition 4.2 (a2) with $L = +\infty$. Inasmuch as $b_{n,0} = n$ [24], the asymptotic behavior of the dSOP zeros of respect to (4.4) is

$$\lim_{n\to\infty} y_{n,1} = 0, \quad y_{n,i} \approx \frac{s_i^2}{b_{n,0}} = \frac{s_i^2}{n}, \quad i \geq 2$$

with $y_{n,1} < 0$ for n large enough. First, we search for the zeros of the limit function in the Mehler-Heine formula, say in the interval $[0,8]$. The command

$$\text{NSolve}[\text{MH}[j, x, \alpha] == 0 \,\&\&\, 0 < x < 8, x, \text{WorkingPrecision} \rightarrow 16]$$

returns

$$s_1 = 3.653924739560034,$$
$$s_2 = 5.533254604844355,$$
$$s_3 = 7.301999533325407,$$

and for $n = 150$ we have the following approximations of the zeros $y_{n,i}$, $i = 2, 3, 4$:

$$y_{150,2} \approx \frac{3.653924739560034^2}{150} = 0.089007773349126,$$

$$y_{150,3} \approx \frac{5.533254604844355^2}{150} = 0.204112710146875,$$

$$y_{150,4} \approx \frac{7.301999533325407^2}{150} = 0.355461314564563.$$

Example 4.3 (Hermite case). Hermite weight function is a particular case of the generalized Freud weight function $x^{2m}e^{-2|x|^\alpha}$ on the real line. We choose $m = 0$, $\alpha = 2$, $j = 8$ $(r = 4)$, $\tilde{M}_n = 1/(2560\sqrt{2}(n+2)^{17/2})$ and arrive at the following Sobolev inner product (4.2):

$$(f,g)_S = \int_{-\infty}^{+\infty} f(x)g(x)e^{-2x^2}dx + \frac{1}{2560\sqrt{2}(n+2)^{17/2}}f^{(8)}(0)g^{(8)}(0). \qquad (4.5)$$

Continuing similar to the previous examples, we get

Case L=

$$\frac{128\sqrt{2}}{85\pi}$$

Limit function in the Mehler–Heine formula for the subsequence of even Sobolev orthonormal polynomials: $\phi_{-\frac{1}{2},4,\frac{128\sqrt{2}}{85\pi}}(x)$

$$\frac{x\left(685440 - 456960 x^2 + 91392 x^4 - 8704 x^6 + \left(256 + 85\sqrt{2}\pi\right) x^8\right) \cos[2x] - 1088 \left(315 - 630 x^2 + 210 x^4 - 28 x^6 + 2 x^8\right) \sin[2x]}{\sqrt{\pi}\left(256 + 85\sqrt{2}\pi\right) x^9}$$

Figure 27: Output the program for the example corresponding to the Hermite case.

After that, we apply Proposition 4.3 (a) with $g(k) = k - 1/4$ and

$$\frac{128\sqrt{2}}{85\pi} = L < \frac{g(0)+g(4)+1}{g(4)-g(0)} = \frac{9}{8}.$$

Similar to the Jacobi case, the term $(g(0)+g(4)+1)/(g(4)-g(0))$ can be computed after the program start. Since $b_n = \sqrt{2n}$, by Proposition 4.3 (a1) for sufficiently large n, all dSOP zeros with respect to (4.5) are real, simple and

$$\lim_{n\to\infty}\sqrt{2n}y_{2n,i} = t_i, \quad i \geq 1,$$

cf. [24]. In order to find the first positive zeros of the limit function, we use the command

$$\text{NSolve}[\text{MH}[j/2, x, \nu] == 0 \,\&\&\, 0 < x < 6.5, x, \text{WorkingPrecision} \to 16],$$

thus obtaining

$$t_1 = 0.607643840651074,$$
$$t_2 = 2.017034485993581,$$
$$t_3 = 3.600436437073820,$$
$$t_4 = 5.222669794289976.$$

Choosing $n = 150$, we obtain

$$y_{300,1} \approx \frac{0.607643840651074}{\sqrt{300}} = 0.035082333497132,$$

$$y_{300,2} \approx \frac{2.017034485993581}{\sqrt{300}} = 0.116453540345315,$$

$$y_{300,3} \approx \frac{3.600436437073820}{\sqrt{300}} = 0.207871294614471,$$

$$y_{300,4} \approx \frac{5.222669794289976}{\sqrt{300}} = 0.301530981162185.$$

Unlike to Example 4.1, the Laguerre and Hermite weights are supported on unbounded sets. Therefore, in order to get good approximations in Examples 4.2 and 4.3, we take n bigger than in Example 4.1.

Example 4.4 (Nevai-Blumenthal case). If the measure μ is not explicitly given, Propositions 4.2 and 4.3 are even more interesting. For instance, in Subsection 4.1 we demonstrated the program action for the Sobolev inner product

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)d\mu + \left(n^2 - n + 3\right)f^{(5)}(1)g^{(5)}(1) \tag{4.6}$$

in (*General case*) with a measure from the Nevai-Blumenthal class $M(0,1)$. The outputs of the program are displayed in Figs. 12 and 13. Now, knowing that $L = +\infty$, we apply Proposition 4.2 (b2) thus obtaining

$$\lim_{n\to\infty} y_{n,1} = 1, \quad \lim_{n\to\infty} \frac{n^2}{2}(1 - y_{n,i}) = s_{i-1}^2, \quad i \geq 2,$$

where s_i is the i-th positive zero of the function $\varphi_{1,1,5,\infty}(x)$. Furthermore, $y_{n,1} > 1$ for n large enough. To illustrate this result, we compute the first three zeros by using the program

$$\text{NSolve}[\text{MH}[\text{j, x, }\nu] == 0 \ \&\& \ 0 < \text{x} < 6, \text{x, WorkingPrecision -> 16}]$$

and obtain

$$s_1 = 2.253003516051088,$$
$$s_2 = 4.077745620336094,$$
$$s_3 = 5.835148392017446.$$

Above, ν is the notation in the program for the term ν_c. In this example $\nu_c = \nu_1 = 1$. Taking $n = 50$, we arrive at the following approximations of the first dSOP zeros with respect to (4.6):

$$y_{50,2} \approx 1 - \frac{2 \times 2.253003516051088^2}{50^2} = 0.995939180125329,$$
$$y_{50,3} \approx 1 - \frac{2 \times 4.077745620336094^2}{50^2} = 0.986697592524664,$$
$$y_{50,4} \approx 1 - \frac{2 \times 5.835148392017446^2}{50^2} = 0.972760834594509.$$

In conclusion, we hope the program will be useful to obtain Mehler-Heine asymptotics for dSOP in a simple and symbolic way and to help locate the zeros of these nonstandard polynomials. As we have commented previously, the program is free and it can be downloaded from the website `https://w3.ual.es/GruposInv/Tapo/MHAS.nb`.

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