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A MIXED VIRTUAL ELEMENT METHOD FOR THE BOUSSINESQ PROBLEM ON POLYGONAL MESHES*

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Abstract

In this work we introduce and analyze a mixed virtual element method (mixed-VEM) for the two-dimensional stationary Boussinesq problem. The continuous formulation is based on the introduction of a pseudostress tensor depending nonlinearly on the velocity, which allows to obtain an equivalent model in which the main unknowns are given by the aforementioned pseudostress tensor, the velocity and the temperature, whereas the pressure is computed via a postprocessing formula. In addition, an augmented approach together with a fixed point strategy is used to analyze the well-posedness of the resulting continuous formulation. Regarding the discrete problem, we follow the approach employed in a previous work dealing with the Navier-Stokes equations, and couple it with a VEM for the convection-diffusion equation modelling the temperature. More precisely, we use a mixed-VEM for the scheme associated with the fluid equations in such a way that the pseudostress and the velocity are approximated on virtual element subspaces of $\mathbb{H}(\mathrm{div})$ and $\mathbf{H}^1$, respectively, whereas a VEM is proposed to approximate the temperature on a virtual element subspace of H^1. In this way, we make use of the L^2-orthogonal projectors onto suitable polynomial spaces, which allows the explicit integration of the terms that appear in the bilinear and trilinear forms involved in the scheme for the fluid equations. On the other hand, in order to manipulate the bilinear form associated to the heat equations, we define a suitable projector onto a space of polynomials to deal with the fact that the diffusion tensor, which represents the thermal conductivity, is variable. Next, the corresponding solvability analysis is performed using again appropriate fixed-point arguments. Further, Strang-type estimates are applied to derive the *a priori* error estimates for the components of the virtual element solution as well as for the fully computable projections of them and the postprocessed pressure. The corresponding rates of convergence are also established. Finally, several numerical examples illustrating the performance of the mixed-VEM scheme and confirming these theoretical rates are presented.

Mathematics subject classification: 65N30, 65N12, 65N15, 65N99, 76M25, 76S05.
Key words: Boussinesq problem, Pseudostress-based formulation, Augmented formulation, Mixed virtual element method, High-order approximations.

1. Introduction

In [36] we developed a mixed-VEM for a pseudostress-velocity formulation of the two-dimensional Navier-Stokes equations. There, we employed a dual-mixed approach based on the in-

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troduction of a nonlinear pseudostress linking the usual linear one for the Stokes equations and the convective term. In this way, the resulting continuous scheme is augmented with Galerkin type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, all them multiplied by suitable stabilization parameters, so that the Banach fixed-point and Lax-Milgram theorems are applied to establish the well-posedness of the continuous scheme (cf. [24]). Regarding the discrete problem we proposed there the simultaneous use of virtual element subspaces for $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$ in order to approximate the velocity and the pseudostress, respectively. Then, the discrete bilinear and trilinear forms involved, their main properties, and the associated mixed virtual scheme were defined, and the corresponding solvability was performed by applying similar techniques to those for the continuous formulation. Other contributions dealing with VEM for nonlinear models include [13, 14, 21, 25, 37, 45]. In particular, a virtual element method employing a low-order approximation of the displacement field is introduced in [13] for nonlinear elastic and inelastic materials. Additionally, a mixed-VEM for quasi-Newtonian Stokes flows is proposed in [21], whereas its extension to a nonlinear Brinkman model of porous media flow is developed in [37]. In turn, a virtual element method dealing with quasilinear elliptic problems is studied in [25]. Finally, an H^1-conforming VEM for the Navier-Stokes equations was introduced in [14], and a nonconforming one was provided in [45].

On the other hand, concerning general numerical methods for approximating the solution of the Boussinesq system, and among the large amount of contributions in the literature, we highlight here [2, 15, 19, 27, 41, 42], which include the primal velocity-pressure-temperature formulations introduced in [2,15,19]. In particular, it is established in [15] that the use of any pair of stable Stokes elements for the fluid unknowns and Lagrange elements for the temperature leads to a convergent scheme. In turn, piecewise linear conforming velocities and temperatures, and piecewise constant pressures are proposed in [2], in such a way that the pressure stabilization is carried out by penalizing inter-element jumps. Furthermore, the development of new mixed finite element methods for the Boussinesq model has constituted a very active research topic in recent years [3–5, 29–31]. In particular, an augmented mixed-primal formulation is introduced and analyzed in [29], where the sought quantities are the pseudostress, the velocity, the temperature, and the normal heat flux through the boundary. Under sufficiently small data, it is proved there that when Raviart-Thomas, Lagrange, and discontinuous piecewise finite elements are used to approximate the above unknowns, then the resulting Galerkin method is well-posed and optimally-convergent. Similarly, two formulations for this model, based on a dual-mixed formulation for the momentum equation, and either a primal or a mixed-primal one for the energy equation, are proposed in [30]. In this case, the velocity, the trace-free gradient, and the normal heat flux are approximated by discontinuous piecewise polynomials, whereas Raviart-Thomas and Lagrange elements are employed for the stress and the temperature, which guarantees the stability and the optimal convergence of the finite element methods. In turn, the Boussinesq problem with temperature-dependent parameters was studied in [3] for the two-dimensional case. There, the authors propose an augmented mixed-primal finite element method that approximates the pseudostress tensor with Raviart-Thomas elements of order $k + 1$, the velocity and the temperature with Lagrange elements of order k, and the vorticity tensor and normal heat flux on the boundary with discontinuous piecewise polynomials of degree $\leq k$, thus obtaining optimal a priori error estimates as well. Later on, the approach from [3] is suitably modified in [4] to derive an augmented mixed-primal finite element method for the n-dimensional case, $n \in \{2, 3\}$, in which the incorporation of the strain rate tensor as an auxiliary unknown plays a key role in the analysis. Discontinuous piecewise polynomial functions of degree $\leq k$, together with Raviart-Thomas and Lagrange elements of order k and $k+1$, respectively, are utilized in [4] to approximate the strain rate, the vorticity, the normal heat flux, the pseudostress, the velocity, and the temperature of the fluid.

Now, concerning the advantages of using both dual-mixed formulations and VEM techniques, we first stress that the former, and particularly those employing the pseudostress and the velocity as the main unknowns, allow to compute further variables of physical meaning, and hence of wide interest in applications, by applying simple postprocessing formulae. Moreover, the above is achieved without any loss of accuracy since the same rates of convergence of the original variables are obtained for the postprocessed ones. In the case of the standard FEM, discrete versions of these additional unknowns can only be computed by performing numerical differentiation procedures, which are known to deteriorate the exactness of the resulting approximations. In turn, the capability of VEM of being able to utilize general polygonal/polyhedral meshes to design convergent Galerkin schemes constitutes one of its main strength. This is specially valid for coupled problems arising from the interaction between diverse linear and nonlinear models, such as Navier-Stokes, convection-diffusion, bioconvective flows, and porous media flows (for instance Darcy and Darcy-Forchheimer) (see e.g. [26, 32, 35, 38]). Indeed, the fact that a virtual element method allows hanging nodes and nonconvex elements, makes much easier the eventual interaction between meshes coming from different domains or from different parts of a given region. In particular, the simulation of fluid-structure interaction problems in the case of an immersed structure that interacts with an incompressible fluid is a very interesting example of it. The reason of the latter is the overlapping that arises between the meshes of the structure and the fluid, which generates polygonal elements that require to be adequately manipulated. This situation occurs in some engineering applications as well as in biomedical applications, such as the simulation of heart valves [6, 44].

According to the above discussion and in order to continue extending the applicability of VEM to nonlinear models in fluids mechanics, we now generalize the approach from [36] to the case of the Boussinesq problem, which, bearing in mind the aforementioned advantages of combining dual-mixed approaches with VEM, can be seen as a particular coupling between the Navier-Stokes and convection-diffusion equations. More precisely, we consider the equations and the variational formulation from [29], and then adapt the approach from [36] to propose, up to our knowledge by the first time, a mixed-VEM for Boussinesq. In fact, the pseudostress and the velocity of the fluid are approximated by virtual element subspaces of $\mathbb{H}(\mathrm{div})$ and $\mathbf{H}^1$, respectively, whereas a virtual element subspace of H^1 is employed to approximate the temperature. Needless to say, and similarly as remarked in [36], we also stress here that the simultaneous use of virtual element subspaces of $\mathbb{H}(\mathrm{div})$ and $\mathbf{H}^1$ constitutes another motivation for solving now the Boussinesq model via a mixed virtual element method.

Thus, similarly as in the aforementioned references, fixed-point arguments are utilized to develop the corresponding solvability analysis, whereas Strang-type estimates are applied to derive the corresponding a priori error estimates for the components of the virtual element solution as well as for their fully calculable projections and the postprocessed pressure. Finally, we point out that our analysis is performed in 2D just for sake of simplicity in the exposition of the ideas and the computational implementation of the method. We recall that the construction of three-dimensionals nodal virtual elements is studied in [1] for the lowest order case, whereas the high-order one can be found in [11]. In addition, as it was remarked in [9], the definition and properties of the three-dimensional $\mathrm{H}(\mathrm{div})$-conforming spaces follow very naturally the approach of their two-dimensional counterpart. Thereby, all the ideas presented here can be extended to 3D up to some minor changes in the definition of the virtual spaces and their degrees of freedom. Perhaps the main challenge has to do with the implementation computational. Further discussions on this matter should form part of future works.

1.1. Outline

The rest of this work is organized as follows. At the end of the present section we provide some useful notations. In Section 2 we describe our nonlinear model, recall from [29] the derivation of the augmented formulation to be employed, as well as the corresponding well-posedness result. Then, in Section 3 we introduce the virtual element subspaces approximating the temperature, the velocity and the pseudostress in H^1, $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$, respectively, state their approximation properties, and define the L^2-projectors and remaining ingredients that are needed for the discrete analysis. In turn, computable discrete versions of the bilinear and trilinear forms involved, and of the corresponding functional on the right-hand side of the formulation, are locally and then globally defined in Section 4. Next, in Section 5 we define the associated mixed virtual element scheme, and perform its solvability analysis by using suitable fixed-point arguments. Moreover, we apply Strang-type estimates to derive the *a priori* error estimates for both the virtual element solution and the fully computable projections of its components. The corresponding rates of convergence are then readily established by using the approximation properties of the subspaces introduced in Sections 3 and 4. Finally, several numerical examples illustrating the performance of the mixed-VEM scheme are reported in Section 6.

1.2. Notations

For any vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$, we set the gradient, divergence and tensor product operators as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \qquad \mathrm{div}\,(\mathbf{v}) := \sum_{j=1}^{2}\frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,2}\,,$$

respectively. In addition, denoting by $\mathbb{I}$ the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathbf{t}} := (\tau_{ji})\,, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2}\tau_{ii}, \quad \boldsymbol{\tau}^{\mathbf{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\,, \quad \text{and} \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{2}\tau_{ij}\zeta_{ij},$$

which corresponds, respectively, to the transpose, the trace, and the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. Next, given a bounded domain $\mathcal{O} \subseteq \mathrm{R}^2$ with boundary $\partial\mathcal{O}$, we let $\mathbf{n}$ be the outward unit normal vector on $\partial\mathcal{O}$. Also, given $r \geq 0$ and $1 < p \leq \infty$, we let $\mathrm{W}^{r,p}(\mathcal{O})$ be the standard Sobolev space with norm $\|\cdot\|_{r,p,\mathcal{O}}$ and seminorm $|\cdot|_{r,p,\mathcal{O}}$. In particular, for $r = 0$ we let $\mathrm{L}^p(\mathcal{O}) := \mathrm{W}^{0,p}(\mathcal{O})$ be the usual Lebesgue space, and for $p = 2$ we let $\mathrm{H}^s(\mathcal{O}) := \mathrm{W}^{r,2}(\mathcal{O})$ be the classical Hilbertian Sobolev space with norm $\|\cdot\|_{s,\mathcal{O}}$ and seminorm $|\cdot|_{s,\mathcal{O}}$. Furthermore, given a generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts, respectively, whose norms and seminorms are denoted exactly as those of M. On the other hand, letting $\mathbf{div}$ (resp. $\mathbf{rot}$) be the usual divergence operator div (resp. rotational operator rot) acting along the rows of a given tensor, we recall that the space

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\mathcal{O}): \quad \mathbf{div}\,(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})\Big\},$$
$$\mathbb{H}(\mathbf{rot};\mathcal{O}) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\mathcal{O}): \quad \mathbf{rot}\,(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})\Big\},$$

equipped with the usual norms

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\mathcal{O}} := \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{div}\,(\boldsymbol{\tau})\|^2_{0,\mathcal{O}} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O}),$$
$$\|\boldsymbol{\tau}\|^2_{\mathbf{rot};\mathcal{O}} := \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{rot}\,(\boldsymbol{\tau})\|^2_{0,\mathcal{O}} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{rot};\mathcal{O}),$$

are Hilbert spaces. Also, we define

$$\mathbb{H}_0(\mathbf{div};\mathcal{O}) := \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O}): \int_{\mathcal{O}} \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\},$$

and recall (see [18, 34]) that there holds the decomposition

$$\mathbb{H}(\mathbf{div};\mathcal{O}) = \mathbb{H}_0(\mathbf{div};\mathcal{O}) \oplus \mathrm{R}\,\mathbb{I}. \tag{1.1}$$

More precisely, for each $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\mathcal{O})$ there exist unique $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\mathcal{O})$ and $c := \frac{1}{2|\mathcal{O}|}\int_{\mathcal{O}}\mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}$, where $|\mathcal{O}|$ denotes the measure of $\mathcal{O}$, such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathbb{I}$. Finally, in what follows we employ $\mathbf{0}$ to denote a generic null vector, null tensor or null operator, and use C to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The Model Problem and Its Continuous Formulation

Let Ω be a bounded polygonal domain in R^2 with boundary Γ. We consider the stationary Boussinesq problem, that is, given an external force per unit mass $\mathbf{g}\in\mathbf{L}^\infty(\Omega)$ and the boundary data $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$, we are interested in finding the velocity $\mathbf{u}$, the pressure p and the temperature φ of a fluid occupying the region Ω, such that

$$-\mu\,\Delta\,\mathbf{u} + (\nabla\,\mathbf{u})\,\mathbf{u} + \nabla\,p - \mathbf{g}\,\varphi = 0 \quad \text{in} \quad \Omega, \tag{2.1a}$$
$$\mathrm{div}\,(\mathbf{u}) = 0 \quad \text{in} \quad \Omega, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \tag{2.1b}$$
$$-\,\mathrm{div}\,(\mathbb{K}\,\nabla\,\varphi) + \mathbf{u}\cdot\nabla\,\varphi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \varphi = 0 \quad \text{on} \quad \Gamma, \tag{2.1c}$$

where $\mu > 0$ is the fluid viscosity and $\mathbb{K}\in\mathbb{L}^\infty(\Omega)$ is a uniformly positive definite tensor describing the thermal conductivity. Note that from the incompressibility condition (cf. first equation in (2.1b)) the data $\mathbf{u}_D$ must satisfy the compatibility condition $\int_\Gamma \mathbf{u}_D\cdot\mathbf{n} = 0$. In addition, the uniqueness of a pressure solution of (2.1), is ensured in the space $\mathrm{L}^2_0(\Omega) := \{q\in \mathrm{L}^2(\Omega): \int_\Omega q = 0\}$.

Now, proceeding as in [29, Section II], we introduce the pseudostress tensor

$$\boldsymbol{\sigma} := \mu\,\nabla\,\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega, \tag{2.2}$$

and use the incompressibility condition to eliminate the pressure, so that then our model problem (2.1) can be rewritten equivalently as

$$\boldsymbol{\sigma}^{\mathbf{d}} + (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} = \mu\nabla\mathbf{u} \quad \text{in} \quad \Omega, \tag{2.3a}$$

$$-\operatorname{\mathbf{div}}(\boldsymbol{\sigma}) - \mathbf{g}\,\varphi = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \tag{2.3b}$$

$$-\operatorname{div}(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi = 0 \quad \text{in} \quad \Omega, \tag{2.3c}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0\,, \tag{2.3d}$$

where the pressure p can be approximated by the postprocessing formula

$$p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega\,. \tag{2.4}$$

Notice that we can easily compute other variables of physical interest as well, such as the shear stress tensor, the velocity gradient, and the vorticity, which are given respectively, by

$$\widetilde{\boldsymbol{\sigma}} := \boldsymbol{\sigma}^{\mathbf{d}} + (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} + \boldsymbol{\sigma}^{\mathbf{t}} + (\mathbf{u}\otimes\mathbf{u})\,,$$

$$\nabla\mathbf{u} := \frac{1}{\mu}\left(\boldsymbol{\sigma}^{\mathbf{d}} + (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}}\right)\,, \quad \boldsymbol{\omega} := \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathbf{t}})\,.$$

Next, following [29, Section III], and motivated by the decomposition (1.1), we test the first, second and fourth equation of (2.3), with $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{v} \in \mathbf{H}^1(\Omega)$, and $\psi \in \mathrm{H}^1_0(\Omega)$, respectively. Then, we integrate by parts, use the boundary conditions, and enrich the resulting variational formulation with the incorporation of the following redundant terms

$$\kappa_1 \int_\Omega \left\{\mu\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} - \boldsymbol{\sigma}^{\mathbf{d}}\right\} : \nabla\mathbf{v} = 0 \qquad \forall\,\mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

$$\kappa_2 \int_\Omega \operatorname{\mathbf{div}}(\boldsymbol{\sigma})\cdot\operatorname{\mathbf{div}}(\boldsymbol{\tau}) + \kappa_2 \int_\Omega \mathbf{g}\,\varphi\cdot\operatorname{\mathbf{div}}(\boldsymbol{\tau}) = 0 \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,,$$

$$\kappa_3 \int_\Gamma \mathbf{u}\cdot\mathbf{v} = \kappa_3 \int_\Gamma \mathbf{u}_D\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in \mathbf{H}^1(\Omega)\,,$$

with κ_1, κ_2 and κ_3 positives parameters to be specified later. In this way, we arrive at the following augmented formulation: Find $(\vec{\boldsymbol{\sigma}}, \varphi) := ((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H}\times\mathrm{H}$ such that

$$\begin{aligned}
&\mathbf{A}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) + \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) = \mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}) + \mathbf{F}_D(\vec{\boldsymbol{\tau}}) \quad && \forall\,\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H} := \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega),\\
&\mathbf{a}(\varphi,\psi) = \mathrm{F}(\mathbf{u},\varphi;\psi) && \forall\,\psi \in \mathrm{H} := \mathrm{H}^1_0(\Omega)\,,
\end{aligned} \tag{2.5}$$

where the forms $\mathbf{A}, \mathbf{B}$ and $\mathbf{a}$ are defined, respectively as

$$\begin{aligned}
\mathbf{A}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) := &\int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} + \kappa_2 \int_\Omega \operatorname{\mathbf{div}}(\boldsymbol{\sigma})\cdot\operatorname{\mathbf{div}}(\boldsymbol{\tau}) + \kappa_1\,\mu \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{u}\\
&- \mu\int_\Omega \mathbf{v}\cdot\operatorname{\mathbf{div}}(\boldsymbol{\sigma}) + \mu\int_\Omega \mathbf{u}\cdot\operatorname{\mathbf{div}}(\boldsymbol{\tau}) - \kappa_1 \int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \nabla\mathbf{v} + \kappa_3\int_\Gamma \mathbf{v}\cdot\mathbf{u}\,,
\end{aligned} \tag{2.6}$$

$$\mathbf{B}(\mathbf{z};\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) := \int_\Omega (\mathbf{u}\otimes\mathbf{z})^{\mathbf{d}} : \left\{\boldsymbol{\tau} - \kappa_1\nabla\mathbf{v}\right\}, \tag{2.7}$$

$$\mathbf{a}(\varphi,\psi) := \int_\Omega \mathbb{K}\nabla\varphi\cdot\nabla\psi \tag{2.8}$$

for all $\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{H}$, for all $\mathbf{z} \in \mathbf{H}^1(\Omega)$, and for all $\varphi,\psi \in \mathrm{H}$. In turn, $\mathbf{F}(\varphi)$ (with a given $\varphi \in \mathrm{H}^1_0(\Omega)$), and $\mathrm{F}(\mathbf{u},\varphi)$ (with a given $(\mathbf{u},\varphi) \in \mathbf{H}^1(\Omega)\times\mathrm{H}^1_0(\Omega)$), are the linear functionals defined by

$$\mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}) := \int_\Omega \mathbf{g}\,\varphi\cdot\left\{\mu\,\mathbf{v} - \kappa_2\operatorname{\mathbf{div}}(\boldsymbol{\tau})\right\}, \tag{2.9}$$

and

$$\mathrm{F}(\mathbf{u},\varphi;\psi) := -\int_\Omega (\mathbf{u}\cdot\nabla\varphi)\,\psi\,, \tag{2.10}$$

respectively, whereas $\mathbf{F}_D$ is given by

$$\mathbf{F}_D(\vec{\boldsymbol{\tau}}) := \kappa_3 \int_\Gamma \mathbf{u}_D\cdot\mathbf{v} + \mu\,\langle \boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle\,. \tag{2.11}$$

We recall here that the choice of $\mathbf{H}^1(\Omega)$ and $\mathrm{H}_0^1(\Omega)$ as tests functions spaces for the velocity $\mathbf{u}$ and the temperature φ, is motivated by the convective terms at the first and fourth equation in (2.3), which require $\mathbf{u}$ and φ to be in spaces smaller than $\mathbf{L}^2(\Omega)$ and $\mathrm{L}^2(\Omega)$, respectively. In fact, this is possible thanks to the Cauchy-Schwarz and Hölder inequalities, and the compact (and hence continuous) injections (see [24, 29] for more details)

$$\mathbf{i}_c\,:\,\mathbf{H}^1(\Omega)\to\mathbf{L}^4(\Omega) \quad \text{and} \quad \mathrm{i}_c\,:\,\mathrm{H}^1(\Omega)\to\mathrm{L}^4(\Omega)\,. \tag{2.12}$$

In this way, according to (2.7) and (2.12), we have that

$$|\mathbf{B}(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \le C_{\mathbf{B}}\|\mathbf{z}\|_{1,\Omega}\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \qquad \forall\,\mathbf{z}\in\mathbf{H}^1(\Omega), \qquad \forall\,\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}\in\mathbf{H}\,, \tag{2.13}$$

with $C_{\mathbf{B}} := \|\boldsymbol{i}_c\|^2(1+\kappa_1^2)^{1/2}$.

In addition, the analysis of the continuous formulation (2.5) is analogous to [29, Section III], and therefore up to minor changes caused by the homogeneous Dirichlet condition for φ, its well-posedness is developed through a fixed-point strategy based on decoupling the fluid and heat equations, and then combining the classical Banach Theorem and the Lax-Milgram Theorem. In particular, it was proved there (cf. [29, Lemma 3.3]) that for $\kappa_1\in(0,2\mu)$ and $\kappa_2,\kappa_3\in(0,\infty)$, there exists $\alpha_{\mathbf{A}}>0$ (cf. [29, eq. 3.30]), depending on $\kappa_1,\kappa_2,\kappa_3,\mu$ and the constants $c_1(\Omega)$ and $c_2(\Omega)$ (cf. Lemma 4.3 below), such that

$$\mathbf{A}(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) \ge \alpha_{\mathbf{A}}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}^2 \quad \forall\,\vec{\boldsymbol{\tau}}\in\mathbf{H}\,, \tag{2.14}$$

which together with (2.13), yielded the $\mathbf{H}$-ellipticity of the bilinear form $\mathbf{A}+\mathbf{B}(\mathbf{z};\cdot,\cdot)$ for sufficiently small $\mathbf{z}$, that is, for each $\mathbf{z}\in\mathbf{H}^1(\Omega)$ such that $\|\mathbf{z}\|_{1,\Omega}\le\dfrac{\alpha_{\mathbf{A}}}{2C_{\mathbf{B}}}$, there holds (cf. [29, eq. 3.32])

$$\mathbf{A}(\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) + \mathbf{B}(\mathbf{z};\vec{\boldsymbol{\tau}},\vec{\boldsymbol{\tau}}) \ge \frac{\alpha_{\mathbf{A}}}{2}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}^2 \quad \forall\,\vec{\boldsymbol{\tau}}\in\mathbf{H}\,.$$

In turn, the boundedness of the bilinear form $\mathbf{A}$ (cf. (2.6)) is obtained with a constant $C_{\mathbf{A}}>0$, depending on $\kappa_1,\kappa_2,\kappa_3,\mu$ and $\|\gamma_0\|$, where $\gamma_0:\mathbf{H}^1(\Omega)\to\mathbf{H}^{1/2}(\Gamma)$ is the usual trace operator, that is, there holds

$$|\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \le C_{\mathbf{A}}\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \quad \forall\,\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}\in\mathbf{H}\,. \tag{2.15}$$

Furthermore, given $\phi\in\mathrm{H}_0^1(\Omega)$, it follows from the Cauchy-Schwarz inequality and the trace theorems in $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{H}^1(\Omega)$, that

$$\|\mathbf{F}(\phi;\cdot)\| \le C_{\mathbf{F}}\|\mathbf{g}\|_{\infty,\Omega}\|\phi\|_{0,\Omega}\,, \tag{2.16a}$$

$$\|\mathbf{F}_D\| \le \kappa_3\|\gamma_0\|\|\mathbf{u}_D\|_{0,\Gamma} + \mu\|\mathbf{u}_D\|_{1/2,\Gamma}\,, \tag{2.16b}$$

with $C_{\mathbf{F}} := (\mu^2+\kappa_2^2)^{1/2}$. In this way, denoting $M_{\mathbf{F}} := \max\Big\{C_{\mathbf{F}},\kappa_3\|\gamma_0\|\Big\}$, we get

$$\|\mathbf{F}(\phi;\cdot)+\mathbf{F}_D\| \le M_{\mathbf{F}}\Big\{\|\mathbf{g}\|_{\infty,\Omega}\|\phi\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{2.17}$$

On the other hand, it is clear from (2.8), the properties of the tensor $\mathbb{K}$, and the Poincaré inequality, that $\mathbf{a}$ is a bounded and H-elliptic bilinear form with constants $\|\mathbb{K}\|_{\infty,\Omega}$ and $\alpha_{\mathbf{a}}$, respectively. In addition, it follows from (2.10) and (2.12), that for a given $(\mathbf{z},\phi) \in \mathbf{H}^1(\Omega) \times \mathrm{H}^1_0(\Omega)$, there holds

$$\|\mathrm{F}(\mathbf{z},\phi)\| \le M_{\mathrm{F}} \|\mathbf{z}\|_{1,\Omega} |\phi|_{1,\Omega}\,,$$

with $M_{\mathrm{F}} := \|\boldsymbol{i}_c\| \|i_c\|$. Finally, by using the aforementioned arguments we can conclude the following result.

Theorem 2.1. *Let $\kappa_1 \in (0, 2\,\mu)$ and $\kappa_2\,,\,\kappa_3 \in (0,\infty)$. Given $\rho \in (0, \alpha_{\mathbf{A}}/(2C_{\mathbf{B}}))$, let W_ρ be the closed ball in $\mathbf{H}^1(\Omega) \times \mathrm{H}^1_0(\Omega)$ defined by $W_\rho := \Big\{ (\mathbf{z},\phi) \in \mathbf{H}^1(\Omega) \times \mathrm{H}^1_0(\Omega) : \quad \|(\mathbf{z},\phi)\| \le \rho \Big\}$. In addition, assume that the data satisfy the assumptions*

$$c_{\mathbf{T}} \Big\{ \|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \le \rho\,,$$
$$C_{\mathbf{T}} \Big\{ \|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} < 1\,,$$

where $c_{\mathbf{T}} := c_{\mathbf{T}}(\rho, M_{\mathbf{F}}, \alpha_{\mathbf{A}}, M_{\mathrm{F}}, \alpha_{\mathbf{a}})$ and $C_{\mathbf{T}} := C_{\mathbf{T}}(\rho, C_{\mathbf{F}}, C_{\mathbf{B}}, \alpha_{\mathbf{a}}, \alpha_{\mathbf{A}}, M_{\mathbf{F}})$ are positive constants. Then, problem (2.5) has a unique solution $((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H} \times \mathrm{H}$ with $(\mathbf{u},\varphi) \in W_\rho$. Moreover, there hold

$$\|(\boldsymbol{\sigma},\mathbf{u})\|_{\mathbf{H}} \le \frac{2M_{\mathbf{F}}}{\alpha_{\mathbf{A}}} \Big\{ \rho\,\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}\,, \tag{2.18}$$

$$\|\varphi\|_{1,\Omega} \le \frac{2M_{\mathrm{F}}\rho}{\alpha_{\mathbf{a}}} \|\mathbf{u}\|_{1,\Omega}\,. \tag{2.19}$$

Proof. We omit details and refer to [29, Theorem 3.9]. □

3. The Virtual Element Subspaces

In this section we introduce suitable virtual element subspaces for $\mathrm{H}^1_0(\Omega)$, $\mathbf{H}^1(\Omega)$, and $\mathbb{H}_0(\mathbf{div};\Omega)$, together to their respective approximation properties. To this end, we will assume the basic assumptions on meshes that are standard in this context (cf. [7, 17]), that is, given $\{\mathcal{T}_h\}_{h>0}$ a family of decompositions of Ω in polygonal elements K, and given a particular $K \in \mathcal{T}_h$, we denote its barycenter, diameter, and number of edges by $\mathbf{x}_K$, h_K, and d_K, respectively, and define, as usual, $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$.

Now, given an integer $\ell \ge 0$ and $\mathcal{O} \subseteq \mathrm{R}^2$, we let $\mathrm{P}_\ell(\mathcal{O})$ be the space of polynomials on $\mathcal{O}$ of degree up to ℓ, and according to the notations introduced in Section 1.2, we set $\mathbf{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^2$ and $\mathbb{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^{2\times 2}$. Also, in what follows we use the multi-index notation, that is, given $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \mathrm{R}^2$ and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)^{\mathbf{t}}$, with non-negative integers α_1, α_2, we let $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2$. Furthermore, given $K \in \mathcal{T}_h$ and an edge $e \in \partial K$ with barycentric x_e and diameter h_e, we introduce the following sets of $(\ell{+}1)$ normalized monomials on e

$$\mathcal{B}_\ell(e) := \left\{ \left(\frac{x - x_e}{h_e} \right)^j \right\}_{0 \le j \le \ell}\,,$$

and $\frac{1}{2}(\ell+1)(\ell+2)$ normalized monomials on K

$$\mathcal{B}_\ell(K) \;:=\; \left\{ \left(\frac{\mathbf{x}-\mathbf{x}_K}{h_K}\right)^{\boldsymbol{\alpha}} \right\}_{0\le|\boldsymbol{\alpha}|\le\ell},$$

which constitute basis of $\mathrm{P}_\ell(e)$ and $\mathrm{P}_\ell(K)$, respectively. In addition, denoting $\widetilde{\mathcal{B}}_0(K) := \mathcal{B}_1(K)$, we define for each integer $\ell \ge 1$,

$$\widetilde{\mathcal{B}}_\ell(K) \;:=\; \mathcal{B}_{\ell+1}(K) \setminus \mathcal{B}_{\ell-1}(K)\,,$$

which is a basis of the subspace of polynomials on K of degree exactly $\ell+1$ or ℓ. In turn, the corresponding vector and tensor versions of the foregoing sets of monomials are given by

$$\begin{aligned}
\boldsymbol{\mathcal{B}}_\ell(e) &:= \Big\{(q,0)^{\mathbf{t}}: \quad q\in\mathcal{B}_\ell(e)\Big\} \cup \Big\{(0,q)^{\mathbf{t}}: \quad q\in\mathcal{B}_\ell(e)\Big\}\,,\\
\boldsymbol{\mathcal{B}}_\ell(K) &:= \Big\{(\mathbf{q},0)^{\mathbf{t}}: \quad \mathbf{q}\in\boldsymbol{\mathcal{B}}_\ell(K)\Big\} \cup \Big\{(0,\mathbf{q})^{\mathbf{t}}: \quad \mathbf{q}\in\boldsymbol{\mathcal{B}}_\ell(K)\Big\}\,,\\
\widetilde{\boldsymbol{\mathcal{B}}}_\ell(K) &:= \Big\{(\mathbf{q},0)^{\mathbf{t}}: \quad \mathbf{q}\in\widetilde{\boldsymbol{\mathcal{B}}}_\ell(K)\Big\} \cup \Big\{(0,\mathbf{q})^{\mathbf{t}}: \quad \mathbf{q}\in\widetilde{\boldsymbol{\mathcal{B}}}_\ell(K)\Big\}\,.
\end{aligned}$$

On the other hand, for each integer $\ell \ge 0$, we let $\mathcal{G}_\ell(K)$ be a basis of $\big(\nabla\mathrm{P}_{\ell+1}(K)\big)^\perp \cap \mathbf{P}_\ell(K)$, which is the $\mathbf{L}^2(K)$-orthogonal of $\nabla\mathrm{P}_{\ell+1}(K)$ in $\mathbf{P}_\ell(K)$, and denote its vectorial counterparts as follow:

$$\boldsymbol{\mathcal{G}}_\ell(K) \;:=\; \left\{\begin{pmatrix}\mathbf{q}\\ \mathbf{0}\end{pmatrix} : \mathbf{q}\in\mathcal{G}_\ell(K)\right\} \cup \left\{\begin{pmatrix}\mathbf{0}\\ \mathbf{q}\end{pmatrix} : \mathbf{q}\in\mathcal{G}_\ell(K)\right\}\,.$$

We remark that, alternatively, one could also consider another choices, not necessarily orthogonal, that have been proposed recently, such as $\mathbf{P}_k(K) = \nabla\mathrm{P}_{k+1} \oplus \mathbf{x}^\perp\mathrm{P}_{k-1}(K)$, where, given $\mathbf{x} := (x_1,x_2) \in \mathrm{R}^2$, $\mathbf{x}^\perp$ denotes the rotated vector $(-x_2,x_1)$. Actually, it is not difficult to see that it suffices to choose any space $\mathcal{G}(K)$ such that $\mathbf{P}_\ell(K) = \nabla\mathrm{P}_{\ell+1} \oplus \mathcal{G}(K)$.
Finally, we let

$$\mathrm{H}^1(\mathcal{T}_h) \;:=\; \Big\{\psi\in\mathrm{L}^2(\Omega): \quad \psi|_K\in\mathrm{H}^1(K) \quad \forall\, K\in\mathcal{T}_h\Big\}\,,$$

and consider the H^1-broken seminorm

$$|\psi|_{1,h} \;:=\; \left\{\sum_{K\in\mathcal{T}_h} \|\nabla\psi\|_{0,K}^2\right\}^{1/2} \qquad \forall\,\psi\in\mathrm{H}^1(\mathcal{T}_h)\,.$$

3.1. The virtual subspaces of $\mathrm{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$

Given $K\in\mathcal{T}_h$ and an integer $k\ge 0$, we first let $\mathcal{R}_k^K : \mathrm{H}^1(K) \to \mathrm{P}_{k+1}(K)$ be the projection operator defined for each $\psi\in\mathrm{H}^1(K)$ as the unique polynomial $\mathcal{R}_k^K(\psi) \in \mathrm{P}_{k+1}(K)$ satisfying (cf. [7,10])

$$\int_K \nabla\mathcal{R}_k^K(\psi)\cdot\nabla q = \int_K \nabla\psi\cdot\nabla q \qquad \forall\, q\in\mathrm{P}_{k+1}(K)\,, \tag{3.1a}$$

$$\int_{\partial K} \mathcal{R}_k^K(\psi) = \int_{\partial K} \psi\,. \tag{3.1b}$$

It is readily seen from (3.1a) that

$$|\mathcal{R}_k^K(\psi)|_{1,K} \;\le\; |\psi|_{1,K} \qquad \forall\,\psi\in\mathrm{H}^1(K)\,.$$

In turn, we let $\boldsymbol{\mathcal{R}}_k^K : \mathbf{H}^1(K) \to \mathbf{P}_{k+1}(K)$ be the vectorial version of $\mathcal{R}_k^K$. Then, we recall from [10, Lemma 5.1] that for integers $m \in [2, k+2]$ and $\ell \in [1, m]$, there hold the approximation properties

$$\|\psi - \mathcal{R}_k^K(\psi)\|_{\ell,K} \,\le\, C\, h_K^{m-\ell}\, |\psi|_{m,K} \qquad \forall\, \psi \in \mathrm{H}^m(K)\,, \quad \forall\, K \in \mathcal{T}_h\,, \tag{3.2}$$

$$\|\mathbf{v} - \boldsymbol{\mathcal{R}}_k^K(\mathbf{v})\|_{\ell,K} \,\le\, C\, h_K^{m-\ell}\, |\mathbf{v}|_{m,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^m(K)\,, \quad \forall\, K \in \mathcal{T}_h\,, \tag{3.3}$$

where (3.3) is certainly a straightforward consequence of (3.2). Furthermore, we now consider the finite-dimensional subspace of $\mathrm{C}(\partial K)$ given by

$$\mathrm{B}_k(\partial K) \,:=\, \Big\{ \psi \in \mathrm{C}(\partial K) : \quad \psi|_e \in \mathrm{P}_{k+1}(e)\,, \quad \forall \text{ edge } e \subseteq \partial K \Big\}\,, \tag{3.4}$$

define the following local virtual element space (see, e.g. [1])

$$\begin{aligned} \mathcal{Q}_k^K \,:=\, \Big\{ \psi \in \mathrm{H}^1(K) : \quad \psi|_{\partial K} \in \mathrm{B}_k(\partial K)\,, \quad \Delta\, \psi \in \mathrm{P}_{k+1}(K)\,, \\ \text{and} \quad \int_K \big\{ \mathcal{R}_k^K(\psi) - \psi \big\}\, q \,=\, 0 \qquad \forall\, q \in \widetilde{\mathcal{B}}_k(K) \Big\}\,, \end{aligned} \tag{3.5}$$

and recall from [1] the following degrees of freedom for a given $\psi \in \mathcal{Q}_k^K$

$$\begin{array}{ll} i) & \text{the value of } \psi \text{ at the } i\text{th vertex of } K\,, \quad \forall\, i \text{ vertex of } K\,, \\ ii) & \text{the values of } \psi \text{ at } k \text{ uniformly spaced points on } e\,,\ \forall\, e \in \partial K\,,\ \text{for } k \ge 1\,, \\ iii) & \text{the moments } \displaystyle\int_K \psi\, q\,,\ \forall\, q \in \mathcal{B}_{k-1}(K)\,,\ \text{for } k \ge 1\,. \end{array} \tag{3.6}$$

It is well-known that for each $\psi \in \mathcal{Q}_k^K$ the projection $\mathcal{R}_k^K(\psi) \in \mathrm{P}_{k+1}(K)$ is fully computable using only the degrees of freedom (3.6) (cf. [1,7]). In addition, for each $K \in \mathcal{T}_h$ and $\psi \in \mathrm{H}^1(K)$, we denote its $\mathcal{Q}_k^K$-interpolant by ψ_I. Also, we let V_k^K be the vectorial version of $\mathcal{Q}_k^K$, whose degrees of freedom, for a given $\mathbf{v} \in V_k^K$, are defined componentwise as in (3.6), and denote by $\mathbf{v}_I$ the V_k^K-interpolant of $\mathbf{v} \in \mathbf{H}^1(K)$. Next, we recall from [1, Proposition 4] (see also [16] or [40]) the approximation properties of both interpolants. We just stress in advance that (3.8) below follows directly from (3.7).

Lemma 3.1. *Let k, ℓ and m be integers such that $\ell \in [0,1]$ and $m \in [2, k+2]$. Then, there exists a constant $C > 0$, independent of K, such that for each $K \in \mathcal{T}_h$, there holds*

$$\|\psi - \psi_I\|_{\ell,K} \,\le\, C\, h_K^{m-\ell}\, |\psi|_{m,K} \qquad \forall\, \psi \in \mathrm{H}^m(K)\,, \tag{3.7}$$

$$\|\mathbf{v} - \mathbf{v}_I\|_{\ell,K} \,\le\, C\, h_K^{m-\ell}\, |\mathbf{v}|_{m,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^m(K)\,. \tag{3.8}$$

3.2. The virtual subspaces of $\mathbb{H}_0(\mathbf{div};\Omega)$

For each $K \in \mathcal{T}_h$ and $k \ge 0$, we introduce the local virtual space H_k^K as follows (see, e.g. [8])

$$\begin{aligned} H_k^K \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};K) \cap \mathbb{H}(\mathbf{rot};K) : \quad \boldsymbol{\tau}\mathbf{n}|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \in \partial K, \\ \mathbf{div}\,(\boldsymbol{\tau}) \in \mathbf{P}_k(K)\,, \quad \text{and} \quad \mathbf{rot}\,(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(K) \Big\}\,, \end{aligned} \tag{3.9}$$

whose local degrees of freedom, for a given $\boldsymbol{\tau} \in H_k^K$, are given by

$$\int_e \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{q} \qquad \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_k(e)\,, \quad \forall \text{ edge } e \in \partial K\,, \tag{3.10a}$$

$$\int_K \boldsymbol{\tau} : \nabla\mathbf{q} \qquad \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_k(K) \setminus \{(1,0)^{\mathbf{t}}, (0,1)^{\mathbf{t}}\}\,, \tag{3.10b}$$

$$\int_K \boldsymbol{\tau} : \boldsymbol{\rho} \qquad \forall\, \boldsymbol{\rho} \in \boldsymbol{\mathcal{G}}_k(K)\,. \tag{3.10c}$$

Now, for each $K \in \mathcal{T}_h$ and $\boldsymbol{\tau} \in \mathbb{H}^1(K)$, we denote its H_k^K-interpolant by $\boldsymbol{\tau}_I$, which has the following approximation properties: for each integer $r \in [1, k+1]$ there exists $C > 0$, independent of K, such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_I\|_{0,K} \;\le\; C\, h_K^r\, |\boldsymbol{\tau}|_{r,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K)\,. \tag{3.11}$$

In addition, for each integer $r \in [0, k+1]$ there exists $C > 0$, independent of K, such that

$$\|\mathbf{div}\,(\boldsymbol{\tau}) - \mathbf{div}\,(\boldsymbol{\tau}_I)\|_{0,K} \le C\, h_K^r\, |\mathbf{div}\,(\boldsymbol{\tau})|_{r,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^1(K) \text{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^r(K)\,. \tag{3.12}$$

Then, the foregoing estimate together with (3.11) yields the following result.

Lemma 3.2. *For each integer $r \in [1, k+1]$ there exists $C > 0$, independent of K, such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_I\|_{\mathbf{div};K} \le C\, h_K^r \Big\{ |\boldsymbol{\tau}|_{r,K} + |\mathbf{div}\,(\boldsymbol{\tau})|_{r,K} \Big\} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K) \textit{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^r(K)\,.$$

Proof. It follows straightforwardly from (3.11) and (3.12). □

3.3. The global virtual subspaces

We now set the global virtual element subspaces of $\mathbb{H}_0(\mathbf{div};\Omega)$, $\mathrm{H}_0^1(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively, that is

$$H_k^h \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}\big|_K \in H_k^K \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \tag{3.13}$$

$$\mathcal{Q}_k^h \;:=\; \Big\{ \psi \in \mathrm{H}_0^1(\Omega) : \quad \psi\big|_K \in \mathcal{Q}_k^K \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \tag{3.14}$$

$$V_k^h \;:=\; \Big\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \quad \mathbf{v}\big|_K \in V_k^K \quad \forall\, K \in \mathcal{T}_h \Big\}\,. \tag{3.15}$$

Then, from Lemmas 3.1 and 3.2, the approximation properties of (3.13)–(3.15) are given, respectively by

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each integer $r \in [1, k+1]$ there holds

$$\mathrm{dist}(\boldsymbol{\sigma}, H_k^h) := \inf_{\boldsymbol{\zeta}_h \in H_k^h} \|\boldsymbol{\sigma} - \boldsymbol{\zeta}_h\|_{\mathrm{div};\Omega} \le\; C\, h^r \Big\{ \sum_{K\in\mathcal{T}_h} \Big(|\boldsymbol{\sigma}|_{r,K}^2 + |\mathbf{div}\,(\boldsymbol{\sigma})|_{r,K}^2 \Big) \Big\}^{1/2}$$

for all $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$ such that $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$ and $\mathbf{div}\,(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$, for all $K \in \mathcal{T}_h$.

($\mathbf{AP}_h^{\varphi}$) there exists $C > 0$, independent of h, such that for each integer $m \in [2, k+2]$ there holds

$$\operatorname{dist}(\varphi, \mathcal{Q}_k^h) := \inf_{\phi_h \in \mathcal{Q}_k^h} \|\varphi - \phi_h\|_{1,\Omega} \;\leq\; C h^{m-1} \left\{ \sum_{K \in \mathcal{T}_h} |\varphi|_{m,K}^2 \right\}^{1/2}$$

for all $\varphi \in \mathrm{H}_0^1(\Omega)$ such that $\varphi\big|_K \in \mathrm{H}^m(K) \quad \forall\, K \in \mathcal{T}_h$.

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C > 0$, independent of h, such that for each integer $s \in [2, k+2]$ there holds

$$\operatorname{dist}(\mathbf{u}, V_k^h) := \inf_{\mathbf{w}_h \in V_k^h} \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} \;\leq\; C h^{s-1} \left\{ \sum_{K \in \mathcal{T}_h} |\mathbf{u}|_{s,K}^2 \right\}^{1/2}$$

for all $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u}\big|_K \in \mathbf{H}^s(K) \quad \forall\, K \in \mathcal{T}_h$.

3.4. L^2-orthogonal projections

For each $k \geq 0$, we let $\mathcal{P}_k^K : \mathrm{L}^2(K) \to \mathrm{P}_k(K)$ be the $\mathrm{L}^2(K)$-orthogonal projector, which, given $\psi \in \mathrm{L}^2(K)$, is characterized by

$$\mathcal{P}_k^K(\psi) \in \mathrm{P}_k(K) \quad \text{and} \quad \int_K \mathcal{P}_k^K(\psi)\, q = \int_K \psi\, q \quad \forall\, q \in \mathrm{P}_k(K)\,.$$

In addition, it is well-known that, given integers k, s, and ℓ such that $k \geq 0$, $s \in [1, k+1]$, and $\ell \in [0, s]$, there holds the following approximation property

$$\|\psi - \mathcal{P}_k^K(\psi)\|_{\ell,K} \;\leq\; C\, h_K^{s-\ell}\, |\psi|_{s,K} \qquad \forall\, \psi \in \mathrm{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{3.16}$$

Further, letting $\boldsymbol{\mathcal{P}}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$ and $\mathbb{P}_k^K : \mathbb{L}^2(K) \to \mathbb{P}_k(K)$ be the vectorial and tensorial versions of the orthogonal projector $\mathcal{P}_k^K$, respectively, as consequence of (3.16) we have that, given integers k, s, and ℓ such that $k \geq 0$, $s \in [1, k+1]$, and $\ell \in [0, s]$, there hold

$$\|\mathbf{v} - \boldsymbol{\mathcal{P}}_k^K(\mathbf{v})\|_{\ell,K} \;\leq\; C\, h_K^{s-\ell}\, |\mathbf{v}|_{s,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,, \tag{3.17}$$

and

$$\|\boldsymbol{\tau} - \mathbb{P}_k^K(\boldsymbol{\tau})\|_{\ell,K} \;\leq\; C\, h_K^{s-\ell}\, |\boldsymbol{\tau}|_{s,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{3.18}$$

The following lemma establishes the approximation properties of the projector $\boldsymbol{\mathcal{P}}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$ with respect to more general Sobolev norms

Lemma 3.3. *Let $K \in \mathcal{T}_h$ and k, s, m, and p be integers such that $k \geq 0$, $s \in [0, k+1]$, $\ell \in [s, k+1]$, and $p \in [2, +\infty)$. Then, there exists a constant $C > 0$, independent of K, such that*

$$|\mathbf{v} - \boldsymbol{\mathcal{P}}_k^K(\mathbf{v})|_{\ell,p,K} \;\leq\; C\, h_K^{s-\ell}\, |\mathbf{v}|_{s,p,K} \qquad \forall\; \mathbf{v} \in \mathbf{W}^{s,p}(K)\,. \tag{3.19}$$

Proof. See [36, Lemma 3.7]. □

As a consequence of the previous lemma, we have the following result.

Lemma 3.4. *Let $K \in \mathcal{T}_h$ and k, s, and p be integers such that $k \geq 0$, $s \in [0, k+1]$, and $p \in [2, +\infty)$. Then, there exists a constant $C_{\mathrm{k}} \geq 1$, independent of K, such that*

$$|\boldsymbol{\mathcal{P}}_k^K(\mathbf{v})|_{s,p,K} \;\leq\; C_{\mathrm{k}}\, |\mathbf{v}|_{s,p,K} \qquad \forall\; \mathbf{v} \in \mathbf{W}^{s,p}(K)\,. \tag{3.20}$$

Proof. See [36, Lemma 3.8]. □

In addition, we now recall, as it was remarked in [1] (respectively [8]) that the degrees of freedom introduced in (3.6) (respectively (3.10)) do allow the explicit calculation of $\mathcal{P}_{k+1}^K(\psi)$ (respectively $\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})$) for each $\psi \in \mathcal{Q}_k^K$ (respectively for each $\boldsymbol{\tau} \in H_k^K$). Further, as consequence of the above it is clear that the vectorial version of the degrees of freedom (3.6) ensures the computability of the $\boldsymbol{\mathcal{P}}_{k+1}^K(\mathbf{v})$ for each $\mathbf{v} \in V_k^K$. Furthermore, also it is possible to compute $\boldsymbol{\mathcal{P}}_k^K(\nabla\psi)$ and $\boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v})$ for each $\psi \in \mathcal{Q}_k^K$ and $\mathbf{v} \in V_k^K$, respectively. More details can be found in [1, 8, 36, 37]).

4. The Discrete Forms

We proceed as in [36, Section 4]. Indeed, we introduce a global virtual element subspace of $\mathbf{H} := \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$. More precisely, given $k \geq 0$, we set $\mathbf{H}_k^h := H_k^h \times V_k^h$, where H_k^h and V_k^h have been defined in (3.13) and (3.15), respectively. Further, defining $\mathbf{H}_k^K := H_k^K \times V_k^K$, it is clear that

$$\mathbf{H}_k^h := \left\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H} : \quad \vec{\boldsymbol{\tau}}\big|_K \in \mathbf{H}_k^K \quad \forall\, K \in \mathcal{T}_h \right\}.$$

Now, we observe that for each $K \in \mathcal{T}_h$ the local version $\mathbf{A}^K : \mathbf{H}_k^K \times \mathbf{H}_k^K \to \mathrm{R}$ of the bilinear form $\mathbf{A}$ (cf. (2.6)), which is defined for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^K$ by

$$\begin{aligned} \mathbf{A}^K(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \int_K \boldsymbol{\zeta}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} + \kappa_2 \int_K \mathbf{div}\,(\boldsymbol{\zeta}) \cdot \mathbf{div}\,(\boldsymbol{\tau}) + \kappa_1\mu \int_K \nabla\mathbf{w} : \nabla\mathbf{v} - \mu \int_K \mathbf{v} \cdot \mathbf{div}\,(\boldsymbol{\zeta}) \\ + \mu \int_K \mathbf{w} \cdot \mathbf{div}\,(\boldsymbol{\tau}) - \kappa_1 \int_K \boldsymbol{\zeta}^{\mathbf{d}} : \nabla\mathbf{v} + \kappa_3 \int_{\partial K \cap \Gamma} \mathbf{w} \cdot \mathbf{v} \end{aligned}$$

is not computable since the tensors $\boldsymbol{\zeta}^{\mathbf{d}}$, $\boldsymbol{\tau}^{\mathbf{d}}$, $\nabla\mathbf{w}$ and $\nabla\mathbf{v}$ are not known on each $K \in \mathcal{T}_h$. This is the reason why in what follows we define a discrete computable versions of $\mathbf{A}^K$ in terms of some suitable projection operators. Then, proceeding as in [36], by using the analysis from Section 3.4, we can introduce a local discrete bilinear form $\mathbf{A}_h^K : \mathbf{H}_k^K \times \mathbf{H}_k^K \to \mathrm{R}$, as

$$\begin{aligned} \mathbf{A}_h^K(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \mathbf{A}_h^{K,\mathbf{d}}(\boldsymbol{\zeta}, \boldsymbol{\tau}) + \kappa_2 \int_K \mathbf{div}\,(\boldsymbol{\zeta}) \cdot \mathbf{div}\,(\boldsymbol{\tau}) + \mathbf{A}_h^{K,\nabla}(\mathbf{w}, \mathbf{v}) - \mu \int_K \mathbf{v} \cdot \mathbf{div}\,(\boldsymbol{\zeta}) \\ + \mu \int_K \mathbf{w} \cdot \mathbf{div}\,(\boldsymbol{\tau}) - \kappa_1 \int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) + \kappa_3 \int_{\partial K \cap \Gamma} \mathbf{w} \cdot \mathbf{v} \end{aligned}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^K$, where $\mathbf{A}_h^{K,\mathbf{d}} : H_k^K \times H_k^K \to \mathrm{R}$ and $\mathbf{A}_h^{K,\nabla} : V_k^K \times V_k^K \to \mathrm{R}$ are the bilinear forms given by

$$\mathbf{A}_h^{K,\mathbf{d}}(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}))^{\mathbf{d}} + \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})) \quad \forall\, \boldsymbol{\zeta}, \boldsymbol{\tau} \in H_k^K, \tag{4.1}$$

and

$$\mathbf{A}_h^{K,\nabla}(\mathbf{w}, \mathbf{v}) := \kappa_1\mu \int_K \nabla\boldsymbol{\mathcal{R}}_k^K(\mathbf{w}) : \nabla\boldsymbol{\mathcal{R}}_k^K(\mathbf{v}) + \mathcal{S}^{K,\nabla}(\mathbf{w} - \boldsymbol{\mathcal{R}}_k^K(\mathbf{w}), \mathbf{v} - \boldsymbol{\mathcal{R}}_k^K(\mathbf{v})) \quad \forall\, \mathbf{w}, \mathbf{v} \in V_k^K, \tag{4.2}$$

respectively, with $\mathcal{S}^{K,\mathbf{d}} : H_k^K \times H_k^K \to \mathrm{R}$ and $\mathcal{S}^{K,\nabla} : V_k^K \times V_k^K \to \mathrm{R}$ being symmetric and positive bilinear forms verifying (see [7, Section 4.6] or [8, Section 3.3])

$$\widehat{c}_0 \|\boldsymbol{\zeta}\|_{0,K}^2 \le \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \le \widehat{c}_1 \|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\zeta} \in H_k^K,$$

and

$$\widetilde{c}_0\, |\mathbf{w}|_{1,K}^2 \le \mathcal{S}^{K,\nabla}(\mathbf{w},\mathbf{w}) \le \widetilde{c}_1\, |\mathbf{w}|_{1,K}^2 \qquad \forall\, \mathbf{w} \in V_k^K\,,$$

where $\widehat{c}_0, \widehat{c}_1, \widetilde{c}_0, \widetilde{c}_1 > 0$ are constants depending only on $C_{\mathcal{T}}$. In particular, we can take $\mathcal{S}^{K,\mathbf{d}}$ (respectively $\mathcal{S}^{K,\nabla}$) as the bilinear form whose associated matrix with respect to the canonical basis of H_k^K (respectively V_k^K) determined by the opportunely scaled degrees of freedom (3.10) (respectively (3.6)), is the identity matrix.

In addition, the bilinear form $\mathcal{S}^{K,\nabla}$, which stabilizes the term $\kappa_1\mu \int_K \nabla\boldsymbol{\mathcal{R}}_k^K(\mathbf{w}) : \nabla\boldsymbol{\mathcal{R}}_k^K(\mathbf{v})$, does not need to be multiplied by $\kappa_1\mu$, since the constant that provides the ellipticity of $\mathbf{A}_h$ (cf. Lemma 4.4 below), involve the parameters κ_2 and κ_3, and the unknowns constants $c_1(\Omega)$ and $c_2(\Omega)$ (cf. Lemma 4.3). More information about this fact can be found in [36, Section 4.1] or [37, Section 3.4].

Now, the following two lemmas establish the properties of the bilinear forms $\mathbf{A}^{K,\mathbf{d}}$ (cf. (4.1)) and $\mathbf{A}^{K,\nabla}$ (cf. (4.2)), respectively.

Lemma 4.1. *For each $K \in \mathcal{T}_h$, there holds*

$$\mathbf{A}_h^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau}) \;=\; \mathbf{A}^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau}) \qquad \forall\, \mathbf{p} \in \mathbb{P}_k(K)\,, \quad \forall\, \boldsymbol{\tau} \in H_k^K\,.$$

In addition, there exist constants $\alpha_1,\ \alpha_2 > 0$, independent of h and K, such that

$$\begin{aligned} |\mathbf{A}_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \;&\le\; \alpha_2\, \|\boldsymbol{\zeta}\|_{0,K} \|\boldsymbol{\tau}\|_{0,K} && \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_k^K\,,\\ \alpha_1\, \|\boldsymbol{\zeta}^{\mathbf{d}}\|_{0,K}^2 \;\le\; \mathbf{A}_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \;&\le\; \alpha_2\, \|\boldsymbol{\zeta}\|_{0,K}^2 && \forall\, \boldsymbol{\zeta} \in H_k^K\,. \end{aligned}$$

Proof. See [36, Lemma 4.2]. □

Lemma 4.2. *For each $K \in \mathcal{T}_h$ there holds*

$$\mathbf{A}_h^{K,\nabla}(\mathbf{q},\mathbf{v}) \;=\; \mathbf{A}^{K,\nabla}(\mathbf{q},\mathbf{v}) \quad \forall\, \mathbf{q} \in \mathbf{P}_k(K)\,, \quad \forall\, \mathbf{v} \in V_k^K\,,$$

and there exist positive constants β_1, β_2, independent of h and K, such that

$$\begin{aligned} |\mathbf{A}_h^{K,\nabla}(\mathbf{w},\mathbf{v})| \;&\le\; \beta_2\, |\mathbf{w}|_{1,K} |\mathbf{v}|_{1,K},\\ \beta_1\, |\mathbf{w}|_{1,K}^2 \;\le\; \mathbf{A}_h^{K,\nabla}(\mathbf{w},\mathbf{w}) \;&\le\; \beta_2\, |\mathbf{w}|_{1,K}^2 \end{aligned}$$

for all $\mathbf{w},\mathbf{v} \in V_k^K$.

Proof. See [36, Lemma 4.4]. □

Hence, we define the global discrete bilinear form $\mathbf{A}_h : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$ as

$$\mathbf{A}_h(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;:=\; \sum_{K\in\mathcal{T}_h} \mathbf{A}_h^K(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \qquad \forall\, \vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h\,.$$

In turn, in what follows, for each $k \geq 0$ we denote by $\mathcal{P}_k^h$, $\boldsymbol{\mathcal{P}}_k^h$, and $\mathbb{P}_k^h$, the global counterparts of the projections $\mathcal{P}_k^K$, $\boldsymbol{\mathcal{P}}_k^K$, and $\mathbb{P}_k^K$, respectively, which were introduced in Section 3.4. In other words, for each $K \in \mathcal{T}_h$ we let

$$\mathcal{P}_k^h(\psi)|_K := \mathcal{P}_k^K(\psi|_K), \quad \boldsymbol{\mathcal{P}}_k^h(\mathbf{v})|_K := \boldsymbol{\mathcal{P}}_k^K(\mathbf{v}|_K), \quad \text{and} \quad \mathbb{P}_k^h(\boldsymbol{\tau})|_K := \mathbb{P}_k^K(\boldsymbol{\tau}|_K),$$

for all $\psi \in \mathrm{L}^2(\Omega)$, $\mathbf{v} \in \mathbf{L}^2(\Omega)$, and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$. Next, we observe that using the properties of the projector $\mathbb{P}_k^h$ and the Lemmas 4.1 and 4.2, we can deduce the boundedness of the bilinear form $\mathbf{A}_h$, that is, there exists a positive constant $\widetilde{C}_{\mathbf{A}}$, depending only on $\kappa_1, \kappa_2, \kappa_3, \mu, \alpha_2, \beta_2$ and $\|\gamma_0\|$, such that

$$|\mathbf{A}_h(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})| \leq \widetilde{C}_{\mathbf{A}} \|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}} \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h. \tag{4.3}$$

Now, in order to prove the $\mathbf{H}_k^h$-ellipticity of the bilinear form $\mathbf{A}_h$, we require the following results.

Lemma 4.3. *There exist constants $c_1(\Omega)$, $c_2(\Omega) > 0$, independent of h, such that*

$$\begin{aligned} c_1(\Omega)\, \|\boldsymbol{\tau}\|_{0,\Omega}^2 &\leq \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega), \\ c_2(\Omega)\, \|\mathbf{v}\|_{1,\Omega}^2 &\leq |\mathbf{v}|_{1,\Omega}^2 + \|\mathbf{v}\|_{0,\Gamma}^2 && \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega). \end{aligned}$$

Proof. See [18, Proposition 3.1, Chapter IV] and [33, Lemma 3.3], respectively. □

Lemma 4.4. *Assume that $\kappa_2, \kappa_3 > 0$ and $0 < \kappa_1 < 2\min\{\alpha_1, \beta_1\}$, where α_1 and β_1 are the positive constants from Lemmas 4.1 and 4.2, respectively. Then, there holds*

$$\mathbf{A}_h(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \geq \widetilde{\alpha}_{\mathbf{A}} \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}^2 \qquad \forall\, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h, \tag{4.4}$$

with $\widetilde{\alpha}_{\mathbf{A}} := \min\Big\{\alpha_1 - \frac{\kappa_1}{2}, \frac{\kappa_2}{2}, \beta_1 - \frac{\kappa_1}{2}, \kappa_3\Big\} \min\Big\{1, c_1(\Omega), c_2(\Omega)\Big\}$.

Proof. See [36, Lemma 4.11]. □

Regarding an optimal choice of the parameters κ_1, κ_2, and κ_3, we follow the approach from [29] (see also [22] and [23]) and adopt the criterion of maximizing some of the constants defining $\widetilde{\alpha}_{\mathbf{A}}$. In this way, κ_1 is taken as the midpoint of its range, that is $\kappa_1 = \min\{\alpha_1, \beta_1\}$, and then both κ_3 and $\frac{\kappa_2}{2}$ are chosen equal to $\frac{1}{2}\min\{\alpha_1, \beta_1\}$. If the constants α_1 and β_1 are not known explicitly, then we proceed as in the continuous case (see [29]) and replace $\min\{\alpha_1, \beta_1\}$ above by μ, thus yielding heuristic choices for these stabilization parameters.

We now introduce a computable discrete version of the form $\mathbf{B}$ defined in (2.7). Indeed, for each $\mathbf{z} \in V_k^h$ we let $\mathbf{B}_h(\mathbf{z};\cdot,\cdot) : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$ be the bilinear form defined by

$$\mathbf{B}_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) := \int_\Omega \big(\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}) \otimes \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{z})\big)^{\mathbf{d}} : \big\{\mathbb{P}_k^h(\boldsymbol{\tau}) - \kappa_1 \mathbb{P}_k^h(\nabla \mathbf{v})\big\}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$. In addition, the boundedness of the form $\mathbf{B}_h$ is established by (cf. [36, Lemma 4.13])

$$|\mathbf{B}_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})| \leq \widetilde{C}_{\mathbf{B}} \|\mathbf{z}\|_{1,\Omega} \|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}} \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \tag{4.5}$$

for all $\mathbf{z} \in V_k^h$ and $\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h$, with $\widetilde{C}_{\mathbf{B}} := \|\boldsymbol{i}_c\|^2 C_{\mathbf{k}}^2 (1+\kappa_1^2)^{1/2}$. Finally, for a given $\phi \in \mathcal{Q}_k^h$, we introduce the computable discrete version $\mathbf{F}_h(\phi;\cdot) : \mathbf{H}_k^h \to \mathrm{R}$ of the functional $\mathbf{F}(\phi;\cdot)$ (cf. (2.9)) given by

$$\mathbf{F}_h(\phi;\vec{\boldsymbol{\tau}}) := \int_\Omega \mathbf{g}\,\mathcal{P}_{k+1}^h(\phi)\cdot\Big\{\mu\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{v}) - \kappa_2\,\mathbf{div}\,(\boldsymbol{\tau})\Big\} \qquad \forall\,\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H}_k^h\,. \tag{4.6}$$

We remark here that the functional $\mathbf{F}_D : \mathbf{H}_k^h \to \mathrm{R}$ (cf. (2.11)) is fully computable using the degrees of freedom (3.6) and (3.10). On the other hand, since the local version $\mathbf{a}^K : \mathcal{Q}_k^K \times \mathcal{Q}_k^H \to \mathrm{R}$ of the bilinear form $\mathbf{a}$ (cf. (2.8)), which is defined for all $\varphi,\psi \in \mathcal{Q}_k^K$ by

$$\mathbf{a}^K(\varphi,\psi) := \int_K \mathbb{K}\nabla\varphi\cdot\nabla\psi\,, \tag{4.7}$$

is not computable, in what follows we aim to define a computable version $\mathbf{a}_h : \mathcal{Q}_k^h \times \mathcal{Q}_k^h \to \mathrm{R}$ of the bilinear form $\mathbf{a}$ (cf. (2.8)). To this end, motivated by the fact that the tensor $\mathbb{K}$ (cf. Section 2) is not constant, we follow the approach from [12, Section 3.4]. Indeed, for each $q \in \mathrm{P}_{k+1}(K)$ and $\psi \in \mathcal{Q}_k^K$, and bearing in mind the orthogonal projector $\boldsymbol{\mathcal{P}}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$, a simple integration by parts yields

$$\int_K \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q)\cdot\nabla\psi = -\int_K \mathrm{div}(\boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q))\psi + \int_{\partial K}(\boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q))\cdot\mathbf{n}\,\psi\,. \tag{4.8}$$

Then, using the fact that $\mathrm{div}(\boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q)) \in \mathrm{P}_{k-1}(K)$ and $(\boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q))\cdot\mathbf{n} \in \mathrm{P}_k(e)$ for each edge $e \in \partial K$, together with the knowledge of the degrees of freedom (3.6), we deduce that the expression (4.8) is fully computable. Therefore, we can introduce the projection operator $\Pi_k^K : \mathcal{Q}_k^K \to \mathrm{P}_{k+1}(K)$ defined for each $\psi \in \mathcal{Q}_k^K$ as the unique polynomial $\Pi_k^K(\psi) \in \mathrm{P}_{k+1}(K)$ satisfying (cf. [12, eq. 3.22])

$$\int_K \mathbb{K}\nabla\Pi_k^K(\psi)\cdot\nabla q = \int_K \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla q)\cdot\nabla\psi \qquad \forall\,q \in \mathrm{P}_{k+1}(K)\,, \tag{4.9a}$$

$$\overline{\Pi_k^K(\psi)} = \overline{\psi}\,. \tag{4.9b}$$

with $\overline{\psi} := \dfrac{1}{d_K}\displaystyle\sum_{\mathbf{x}\in\mathcal{V}(K)}\psi(\mathbf{x})$, where d_K and $\mathcal{V}(K)$ denote the number of edges and the set of vertices of K, respectively. Notice that it is clear from (4.8) and (4.9) that $\Pi_k^K(\psi)$ is well-defined for each $\psi \in \mathcal{Q}_k^K$, and that Π_k^K is indeed a projection operator. Also, it easy to see from the first equation of (4.9) and the properties of the tensor $\mathbb{K}$ that there exists $C_{\mathbb{K}} > 0$, depending only on $\mathbb{K}$, such that

$$|\Pi_k^K(\psi)|_{1,K} \;\le\; C_{\mathbb{K}}|\psi|_{1,K} \qquad \forall\;\psi \in \mathrm{H}^1(K)\,. \tag{4.10}$$

In addition, the approximation properties of Π_k^K are established in [12, Section 4], that is, given integers k, m and ℓ such that $k \ge 0$, $m \in [2,k+2]$ and $\ell \in [1,m]$, there holds

$$\|\psi - \Pi_k^K(\psi)\|_{\ell,K} \;\le\; C\,h_K^{m-\ell}\,|\psi|_{m,K} \qquad \forall\,\psi \in \mathrm{H}^m(K)\,, \quad \forall\,K \in \mathcal{T}_h\,. \tag{4.11}$$

Now, we can introduce a local discrete bilinear form $\mathbf{a}_h^K : \mathcal{Q}_k^K \times \mathcal{Q}_k^K \to \mathrm{R}$, which is defined by

$$\mathbf{a}_h^K(\varphi,\psi) := \mathbf{a}^K(\Pi_k^K(\varphi),\Pi_k^K(\psi)) + \mathcal{S}^{K,\Pi}(\varphi - \Pi_k^K(\varphi), \psi - \Pi_k^K(\psi)) \tag{4.12}$$

for all $\varphi, \psi \in \mathcal{Q}_k^K$, where $\mathcal{S}^{K,\Pi} : \mathcal{Q}_k^K \times \mathcal{Q}_k^K \to \mathrm{R}$ is a positive and symmetric bilinear form verifying

$$\bar{c}_0 \, |\psi|_{1,K}^2 \;\leq\; \mathcal{S}^{K,\Pi}(\psi,\psi) \;\leq\; \bar{c}_1 \, |\psi|_{1,K}^2 \qquad \forall\, \psi \in \mathcal{Q}_k^K \,, \tag{4.13}$$

with $\bar{c}_0$ $\bar{c}_1$ positives constant depending only on $C_{\mathcal{T}}$.

At this point we remark that, alternatively to the present definition of the local discrete bilinear form $\mathbf{a}_h^K$ (cf. (4.12)), which has been motivated only by the previous introduction of the operator Π_k^K in [12, Section 3.4], we could simply proceed as in [10, eq. (4.14)] and define the first component of $\mathbf{a}_h^K$ by replacing $\nabla\varphi$ and $\nabla\psi$ in (4.7) by $\boldsymbol{\mathcal{P}}_k^K(\nabla\varphi)$ and $\boldsymbol{\mathcal{P}}_k^K(\nabla\psi)$, respectively. In turn, the stabilizing bilinear form $\mathcal{S}^{K,\Pi}$ would employ $\mathcal{R}_k^K$ instead of Π_k^K. Needless to say, the estimate (5.32) below will certainly need to be modified accordingly.

The following lemma establishes the properties of the bilinear form (4.12). (cf. [12]).

Lemma 4.5. *There holds*

$$\mathbf{a}_h^K(p,\psi) = \int_K \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla p)\cdot\nabla\psi \qquad \forall\, p \in \mathrm{P}_{k+1}(K)\,, \quad \forall\, \psi \in \mathcal{Q}_k^K\,, \quad \forall\, K \in \mathcal{T}_h\,,$$

and there exist constants $\alpha_*, \alpha^* > 0$, such that

$$\alpha_* \mathbf{a}^K(\psi,\psi) \leq \mathbf{a}_h^K(\psi,\psi) \leq \alpha^* \mathbf{a}^K(\psi,\psi) \qquad \forall\, \psi \in \mathcal{Q}_k^K\,, \quad \forall K \in \mathcal{T}_h\,.$$

Proof. See [12, Section 3.4].

In this way, we define the global discrete bilinear form $\mathbf{a}_h : \mathcal{Q}_k^h \times \mathcal{Q}_k^h \to \mathrm{R}$ as

$$\mathbf{a}_h(\varphi,\psi) \;:=\; \sum_{K\in\mathcal{T}_h} \mathbf{a}_h^K(\varphi,\psi) \qquad \forall\, \varphi,\psi \in \mathcal{Q}_k^h\,.$$

In turn, it is clear from (2.10) that, given $(\mathbf{z},\phi) \in V_k^h \times \mathcal{Q}_k^h$, the functional $\mathrm{F}(\mathbf{z},\phi;\cdot) : \mathcal{Q}_k^h \to \mathrm{R}$ (cf. (2.10)) is not computable. Therefore, we introduce a computable discrete version $\mathrm{F}_h(\mathbf{z},\phi;\cdot)$, which is given by

$$\mathrm{F}_h(\mathbf{z},\phi;\psi) := -\int_\Omega (\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{z})\cdot\boldsymbol{\mathcal{P}}_k^h(\nabla\phi))\mathcal{P}_{k+1}^h(\psi) \tag{4.14}$$

for all $\psi \in \mathcal{Q}_k^h$.

5. The Virtual Element Scheme and Its Stability Analysis

We now use the discrete forms analyzed in the previous section to introduce our mixed virtual element scheme associated with (2.5), which reads: Find $(\vec{\boldsymbol{\sigma}}_h,\varphi_h) := ((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_k^h \times \mathcal{Q}_k^h$ such that

$$\begin{aligned} &\mathbf{A}_h(\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) + \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) = \mathbf{F}_h(\varphi_h;\vec{\boldsymbol{\tau}}_h) + \mathbf{F}_D(\vec{\boldsymbol{\tau}}_h) \qquad && \forall\, \vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbf{H}_k^h\,,\\ &\mathbf{a}_h(\varphi_h,\psi_h) = \mathrm{F}_h(\mathbf{u}_h,\varphi_h;\psi_h) && \forall\, \psi_h \in \mathcal{Q}_k^h\,. \end{aligned} \tag{5.1}$$

For the stability analysis of the Galerkin scheme (5.1), we follow the approach from [29, Section III.B] and employ a fixed-point strategy. Indeed, we define the discrete operators $\mathbf{S}_h : V_k^h \times \mathcal{Q}_k^h \to \mathbf{H}_k^h$ and $\widetilde{\mathbf{S}}_h : V_k^h \times \mathcal{Q}_k^h \to \mathcal{Q}_k^h$, respectively, as

$$\mathbf{S}_h(\mathbf{z}_h,\phi_h) \;:=\; (\mathbf{S}_{1,h}(\mathbf{z}_h,\phi_h), \mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h)) = \vec{\boldsymbol{\sigma}}_h,$$

and

$$\widetilde{\mathbf{S}}_h(\mathbf{z}_h,\phi_h) := \varphi_h$$

for all $(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h$, where $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{H}_k^h$ and $\varphi_h \in \mathcal{Q}_k^h$ are the unique solutions of the discrete problems:

$$\mathbf{A}_h(\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) \,+\, \mathbf{B}_h(\mathbf{z}_h;\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) = \mathbf{F}_h(\phi_h;\vec{\boldsymbol{\tau}}_h) + \mathbf{F}_D(\vec{\boldsymbol{\tau}}_h) \qquad \forall\, \vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h\,, \tag{5.2}$$

and

$$\mathbf{a}_h(\varphi_h,\psi_h) = \mathrm{F}_h(\mathbf{z}_h,\phi_h;\psi_h) \quad \forall\, \psi_h \in \mathcal{Q}_k^h, \tag{5.3}$$

respectively. Next, we introduce the operator $\mathbf{T}_h : V_k^h \times \mathcal{Q}_k^h \to V_k^h \times \mathcal{Q}_k^h$ as

$$\mathbf{T}_h(\mathbf{z}_h,\phi_h) := (\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h), \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h)) \quad \forall\, (\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h\,, \tag{5.4}$$

and realize that (5.1) can be rewritten as the fixed-point problem: Find $(\mathbf{u}_h,\varphi_h) \in V_k^h \times \mathcal{Q}_k^h$ such that

$$\mathbf{T}_h(\mathbf{u}_h,\varphi_h) = (\mathbf{u}_h,\varphi_h)\,. \tag{5.5}$$

The following two lemmas establish the well-posedness of (5.2) and (5.3), and hence the well-definedness of the operator $\mathbf{T}_h$.

Lemma 5.1. *Suppose that the parameters κ_1,κ_2 and κ_3, satisfy the conditions required by Lemma 4.4 and let $\rho \in \Big(0, \widetilde{\alpha}_{\mathbf{A}}/(2\widetilde{C}_{\mathbf{B}})\Big)$. Then, the problem (5.2) has a unique solution $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{H}_k^h$ for each $(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h$ such that $\|\mathbf{z}_h\|_{1,\Omega} \le \rho$. Further, there exists a constant $c_{\mathbf{S}} > 0$, independent of $\mathbf{z}_h,\phi_h$, and h, such that*

$$\|\mathbf{S}_h(\mathbf{z}_h,\phi_h)\|_{\mathbf{H}} = \|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \le\; c_{\mathbf{S}} \Big\{ \|\mathbf{g}\|_{\infty,\Omega}\|\phi_h\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}\,. \tag{5.6}$$

Proof. We proceed as in [29, Lemma 3.3] (see also [36, Lemma 5.1]). In fact, given $\rho \in \Big(0, \widetilde{\alpha}_{\mathbf{A}}/(2\widetilde{C}_{\mathbf{B}})\Big)$ and $(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h$ such that $\|\mathbf{z}_h\|_{1,\Omega} \le \rho$, we can deduce, using (4.4) and (4.5), that the ellipticity of the bilinear form $\mathbf{A}_h + \mathbf{B}_h(\mathbf{z}_h;\cdot,\cdot)$ is ensured with the constant $\widetilde{\alpha}_{\mathbf{A}}/2$. In addition, we have that

$$\|\mathbf{F}_h(\phi_h;\cdot)\| \le C_{\mathbf{F}}\|\mathbf{g}\|_{\infty,\Omega}\|\phi_h\|_{0,\Omega} \qquad \forall \varphi_h \in \mathcal{Q}_k^h\,, \tag{5.7}$$

with $C_{\mathbf{F}}$ the bound in (2.16). Then, there holds

$$\|\mathbf{F}_h(\phi_h;\cdot) + \mathbf{F}_D\| \le M_{\mathbf{F}}\Big\{\|\mathbf{g}\|_{\infty,\Omega}\|\phi_h\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,,$$

where $M_{\mathbf{F}}$ is the constant in (2.17). Then, a direct application of the Lax-Milgram theorem implies the existence of a unique solution $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{H}_k^h$ of (5.2), which satisfies (5.6) with $c_{\mathbf{S}} := 2M_{\mathbf{F}}/\widetilde{\alpha}_{\mathbf{A}}$. □

Lemma 5.2. *For each $(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h$, there exists a unique $\varphi_h \in \mathcal{Q}_k^h$ solution of (5.3), and there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h,\phi_h)\|_{1,\Omega} = \|\varphi_h\|_{1,\Omega} \le\; c_{\widetilde{\mathbf{S}}}\, \|\mathbf{z}_h\|_{1,\Omega}|\phi_h|_{1,\Omega}\,, \tag{5.8}$$

with $c_{\widetilde{\mathbf{S}}}$ independent of $\mathbf{z}_h,\phi_h$ and h.

Proof. From Lemma 4.5 we deduce the boundedness and ellipticity of the bilinear form $\mathbf{a}_h$ with constants $\alpha^*\|\mathbf{a}\| = \alpha^*\|\mathbb{K}\|_{\infty,\Omega}$ and $\alpha_*\alpha_{\mathbf{a}}$, respectively. Further, for each $(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h$, we find from (4.14), (2.12) and Lemma 3.4, that

$$\|\mathrm{F}_h(\mathbf{z}_h,\phi_h;\cdot)\| \le \widetilde{M}_{\mathrm{F}}\|\mathbf{z}_h\|_{1,\Omega}|\phi_h|_{1,\Omega}\,, \tag{5.9}$$

with $\widetilde{M}_{\mathrm{F}} := \|\boldsymbol{i}_c\|\|i_c\|C_{\mathbf{k}}^2$ (cf. (2.12) and (3.20)). In this way, the Lax-Milgram theorem guarantees the existence of a unique solution $\varphi_h \in \mathcal{Q}_k^h$ of (5.3), and a positive constant $c_{\widetilde{\mathbf{S}}} := \widetilde{M}_{\mathrm{F}}/(\alpha_*\alpha_{\mathbf{a}})$ such that (5.8) holds. □

Having proved the well-definedness of $\mathbf{T}_h$, we now aim to establish the existence of a unique fixed point for this operator. We begin with the following result.

Lemma 5.3. *Suppose that the parameters κ_1,κ_2 and κ_3, satisfy the conditions required by Lemma 4.4 and let $\rho \in \Big(0, \widetilde{\alpha}_{\mathbf{A}}/(2\widetilde{C}_{\mathbf{B}})\Big)$. Also, let W_ρ^h be the closed ball in $V_k^h \times \mathcal{Q}_k^h$ defined by*

$$W_\rho^h := \Big\{(\mathbf{z}_h,\phi_h) \in V_k^h \times \mathcal{Q}_k^h : \quad \|(\mathbf{z}_h,\phi_h)\| \le \rho\Big\}\,, \tag{5.10}$$

and assume that the data satisfy

$$\widetilde{c}_{\mathbf{T}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} \le \rho\,, \tag{5.11}$$

where $\widetilde{c}_{\mathbf{T}} := \max\Big\{(1 + c_{\widetilde{\mathbf{S}}}\rho)\max\Big\{1,\rho\Big\}c_{\mathbf{S}}, c_{\widetilde{\mathbf{S}}}\Big\}$. Then, there holds $\mathbf{T}_h(W_\rho^h) \subseteq W_\rho^h$.

Proof. It follows by similar arguments to those used in the proof of [29, Lemma 3.5]. □

Lemma 5.4. *Suppose that the parameters κ_1,κ_2 and κ_3, satisfy the conditions required by Lemma 4.4. In addition, let $\rho \in \Big(0, \widetilde{\alpha}_{\mathbf{A}}/(2\widetilde{C}_{\mathbf{B}})\Big)$ and W_ρ^h as in Lemma 5.3 (cf. (5.10)). Then, there exists a positive constant $\widetilde{C}_{\mathbf{T}}$, such that*

$$\begin{aligned}&\|\mathbf{T}_h(\mathbf{z}_h,\phi_h) - \mathbf{T}_h(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|\\ \le&\widetilde{C}_{\mathbf{T}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\|(\mathbf{z}_h,\phi_h) - (\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|\end{aligned} \tag{5.12}$$

for all $(\mathbf{z}_h,\phi_h), (\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h) \in W_\rho^h$.

Proof. We proceed as in [29, Lemma 3.8]. In fact, from the definition of $\mathbf{T}_h$ (cf. (5.4)) we first observe that

$$\begin{aligned}&\|\mathbf{T}_h(\mathbf{z}_h,\phi_h) - \mathbf{T}_h(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|\\ \le&\|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|_{1,\Omega}\|\widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h) - \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h),\widetilde{\phi}_h)\|_{1,\Omega}\,.\end{aligned} \tag{5.13}$$

The two expressions on the right-hand side of (5.13) are bounded in what follows. Indeed, letting $(\boldsymbol{\sigma}_h,\mathbf{u}_h) := \mathbf{S}_h(\mathbf{z}_h,\phi_h)$ and $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h) := \mathbf{S}_h(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)$ be the corresponding solutions of problem (5.2), and reasoning similarly as in [36, Lemma 5.2], we deduce that

$$\begin{aligned}&\|\mathbf{S}_h(\mathbf{z}_h,\phi_h) - \mathbf{S}_h(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|_{\mathbf{H}}\\ \le&\frac{2}{\widetilde{\alpha}_{\mathbf{A}}}\Big\{\widetilde{C}_{\mathbf{B}}\|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h)\|_{1,\Omega}\|\mathbf{z}_h - \widetilde{\mathbf{z}}_h\|_{1,\Omega} + C_{\mathbf{F}}\|\mathbf{g}\|_{\infty,\Omega}\|\phi_h - \widetilde{\phi}_h\|_{0,\Omega}\Big\}.\end{aligned}$$

Then, from the foregoing inequality and Lemma 5.1, we get

$$\|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|_{1,\Omega} \tag{5.14}$$

$$\leq \frac{2}{\widetilde{\alpha}_{\mathbf{A}}}\Big\{c_{\mathbf{S}}\widetilde{C}_{\mathbf{B}}\left(\rho\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right)\|\mathbf{z}_h - \widetilde{\mathbf{z}}_h\|_{1,\Omega} + C_{\mathbf{F}}\|\mathbf{g}\|_{\infty,\Omega}\|\phi_h - \widetilde{\phi}_h\|_{0,\Omega}\Big\}$$

$$\leq C_{\mathbf{S}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\|(\mathbf{z}_h,\phi_h) - (\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|\,,$$

with $C_{\mathbf{S}} := \dfrac{2(1+\rho)}{\widetilde{\alpha}_{\mathbf{A}}}\max\Big\{c_{\mathbf{S}}\widetilde{C}_{\mathbf{B}}, C_{\mathbf{F}}\Big\}$.

On the other hand, since $\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h), \mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h) \in V_k^h$ we let $\varphi_h := \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h)$ and $\widetilde{\varphi}_h := \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h),\widetilde{\phi}_h)$ be the corresponding solutions of problem (5.3). Then, using the ellipticity of the bilinear form $\mathbf{a}$, Lemma 4.5, and adding and subtracting suitable terms, we get

$$\begin{aligned}
&\|\widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h) - \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h),\widetilde{\phi}_h)\|_{1,\Omega}^2\\
=&\|\varphi_h - \widetilde{\varphi}_h\|_{1,\Omega}^2 \leq (\alpha_*\alpha_{\mathbf{a}})^{-1}\mathbf{a}_h(\varphi_h - \widetilde{\varphi}_h,\varphi_h - \widetilde{\varphi}_h)\\
\leq&(\alpha_*\alpha_{\mathbf{a}})^{-1}|\mathrm{F}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h - \widetilde{\phi}_h;\varphi_h - \widetilde{\varphi}_h) + \mathrm{F}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h),\widetilde{\phi}_h;\varphi_h - \widetilde{\varphi}_h)|\,.
\end{aligned}$$

Then, from the foregoing inequality, the boundedness of F_h (cf. (5.9)), the estimates (5.6) and (5.14), and the fact that $\phi_h, \widetilde{\phi}_h \in W_\rho^h$, we obtain

$$\begin{aligned}
&\|\widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h),\phi_h) - \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h),\widetilde{\phi}_h)\|_{1,\Omega}\\
\leq&(\alpha_*\alpha_{\mathbf{a}})^{-1}\widetilde{M}_{\mathrm{F}}\Big\{\|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h)\|_{1,\Omega}|\phi_h - \widetilde{\phi}_h|_{1,\Omega} + \rho\|\mathbf{S}_{2,h}(\mathbf{z}_h,\phi_h) - \mathbf{S}_{2,h}(\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|_{1,\Omega}\Big\}\\
\leq&C_{\widetilde{\mathbf{S}}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\|(\mathbf{z}_h,\phi_h) - (\widetilde{\mathbf{z}}_h,\widetilde{\phi}_h)\|\,,
\end{aligned} \tag{5.15}$$

where $C_{\widetilde{\mathbf{S}}} := (\alpha_*\alpha_{\mathbf{a}})^{-1}\widetilde{M}_{\mathrm{F}}(1+\rho)\max\Big\{c_{\mathbf{S}}, \rho C_{\mathbf{S}}\Big\}$. Therefore, from (5.13)-(5.15) we conclude (5.12) with $\widetilde{C}_{\mathbf{T}} := \max\{C_{\mathbf{S}}, C_{\widetilde{\mathbf{S}}}\}$. □

We are ready to prove that our discrete scheme (5.1) (equivalently, the fixed-point operator equation (5.5)) is well-posed. More precisely, we have the following result.

Theorem 5.1. *Suppose that the parameters κ_1, κ_2 and κ_3, satisfy the conditions required by Lemma 4.4, and let $\rho \in \Big(0, \widetilde{\alpha}_{\mathbf{A}}/(2\widetilde{C}_{\mathbf{B}})\Big)$. Also, let W_ρ^h as in Lemma 5.3 (cf. (5.10)), and assume that the data satisfy the assumptions (5.11) and*

$$\widetilde{C}_{\mathbf{T}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} < 1\,,$$

with $\widetilde{C}_{\mathbf{T}}$ given by Lemma 5.4. Then, the mixed virtual element scheme (5.1) has a unique solution $((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_k^h \times \mathcal{Q}_k^h$, with $(\mathbf{u}_h,\varphi_h) \in W_\rho^h$, and there hold

$$\|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}} \leq \frac{2M_{\mathbf{F}}}{\widetilde{\alpha}_{\mathbf{A}}}\Big\{\rho\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,, \tag{5.16}$$

$$\|\varphi_h\|_{1,\Omega} \leq \frac{\widetilde{M}_{\mathrm{F}}}{\alpha_*\alpha_{\mathbf{a}}}\,\rho\|\mathbf{u}_h\|_{1,\Omega}\,. \tag{5.17}$$

Proof. The desired results follow from Lemmas 5.3 and 5.4, the Banach fixed-point theorem, and the estimates (5.6) and (5.8). □

5.1. A priori error estimates

We now aim to derive the *a priori* estimates for the error

$$\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\| := \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} + \|\varphi - \varphi_h\|_{1,\Omega}\,, \tag{5.18}$$

where $(\vec{\boldsymbol{\sigma}},\varphi) := ((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H}\times \mathrm{H}$ and $(\vec{\boldsymbol{\sigma}}_h,\varphi_h) := ((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_k^h \times \mathcal{Q}_k^h$ are the unique solutions of the continuous and discrete schemes (2.5) and (5.1), respectively. In this regard, and as suggested by Theorems 2.1 and 5.1, we first define

$$\rho_0 := \min\left\{\frac{\alpha_{\mathbf{A}}}{2C_{\mathbf{B}}},\frac{\widetilde{\alpha}_{\mathbf{A}}}{2\widetilde{C}_{\mathbf{B}}}\right\}, \tag{5.19}$$

and observe that, under the assumptions that $\kappa_2,\kappa_3>0$, and $0<\kappa_1<2\min\{\mu,\alpha_1,\beta_1\}$, the existence of $(\vec{\boldsymbol{\sigma}},\varphi)$ and $(\vec{\boldsymbol{\sigma}}_h,\varphi_h)$ is guaranteed within the respective balls centered at the origin and with radius $\rho \in (0,\rho_0)$.

Next, recalling that the local projectors $\boldsymbol{\mathcal{R}}_k^K : V_k^K \to \mathbf{P}_{k+1}(K)$ and $\Pi_k^K : \mathcal{Q}_k^K \to \mathrm{P}_{k+1}(K)$ are introduced in Sections 3.1 and 4, respectively, we now denote by $\boldsymbol{\mathcal{R}}_k^h$ and Π_k^h its global counterparts, respectively, that is, given $\mathbf{v}\in V_k^h$ and $\psi\in\mathcal{Q}_k^h$, we let

$$\boldsymbol{\mathcal{R}}_k^h(\mathbf{v})|_K := \boldsymbol{\mathcal{R}}_k^K(\mathbf{v}|_K) \quad \text{and} \quad \Pi_k^h(\psi)|_K := \Pi_k^K(\psi|_K) \quad \forall\, K\in\mathcal{T}_h\,.$$

We begin our analysis with some preliminary lemmas.

Lemma 5.5. *There exist positive constants $L_{\mathbf{A}}$, $C_{\mathtt{p}}$, and $C_{\mathtt{q}}$, independent of h, such that*

$$\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbf{H}_k^h\\ \vec{\boldsymbol{\tau}}_h\neq\mathbf{0}}} \frac{|(\mathbf{A}-\mathbf{A}_h)(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \leq L_{\mathbf{A}}\Big\{\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + |\mathbf{u}-\boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h}\Big\}, \tag{5.20}$$

and

$$\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbf{H}_k^h\\ \vec{\boldsymbol{\tau}}_h\neq\mathbf{0}}} \frac{|(\mathbf{B}-\mathbf{B}_h)(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \leq C_{\mathtt{p}}\left\{\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + |\mathbf{u}-\boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h} + \|\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\right\} \tag{5.21}$$

for all $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathbf{H}_k^h$, and

$$\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbf{H}_k^h\\ \vec{\boldsymbol{\tau}}_h\neq\mathbf{0}}} \frac{|(\mathbf{F}-\mathbf{F}_h)(\varphi;\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \leq C_{\mathtt{q}}\Big\{\|\mathbf{div}\,(\boldsymbol{\sigma})-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{div}\,(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\varphi-\mathcal{P}_{k+1}^h(\varphi)\|_{0,\Omega}\Big\}. \tag{5.22}$$

Proof. Firstly, using [36, Lemma 4.8], and by adding and subtracting suitable terms (see also [36, eq. (5.21)]), we get (5.20) with $L_{\mathbf{A}} := 3\max\Big\{\alpha_2+\kappa_1,\beta_2\Big\}$, where α_2 and β_2 are the constants from Lemmas 4.1 and 4.2, respectively. In turn, in order to prove (5.21), we proceed as in [36, Lemma 4.12] by adding and subtracting suitable terms, which yields

$$\begin{aligned}(\mathbf{B}-\mathbf{B}_h)(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) &= \int_\Omega \Big\{(\mathbf{w}_h\otimes\mathbf{u})^{\mathbf{d}} - \boldsymbol{\mathcal{P}}_k^h((\mathbf{w}_h\otimes\mathbf{u})^{\mathbf{d}})\Big\} : \Big\{\boldsymbol{\tau}_h-\kappa_1\nabla\mathbf{v}_h\Big\} \\ &\quad + \int_\Omega \Big(\mathbf{w}_h\otimes\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h)\otimes\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\Big)^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}_h-\kappa_1\nabla\mathbf{v}_h)\,.\end{aligned} \tag{5.23}$$

The two expressions on the right-hand side of (5.23) are bounded in what follows. In fact, adding and subtracting $\mathbf{u}$, it follows that

$$\begin{aligned}&(\mathbf{w}_h \otimes \mathbf{u}) - \mathcal{P}_k^h(\mathbf{w}_h \otimes \mathbf{u})\\ =&(\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u} - \mathcal{P}_k^h((\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u}) + (\mathbf{u} \otimes \mathbf{u}) - \mathcal{P}_k^h(\mathbf{u} \otimes \mathbf{u})\,.\end{aligned}$$

Then, using the foregoing expression, and the first equation of (2.3), we arrive at

$$\int_\Omega \Big\{(\mathbf{w}_h \otimes \mathbf{u})^{\mathbf{d}} - \mathcal{P}_k^h((\mathbf{w}_h \otimes \mathbf{u})^{\mathbf{d}})\Big\} : \Big\{\boldsymbol{\tau}_h - \kappa_1 \nabla \mathbf{v}_h\Big\} \tag{5.24}$$

$$= \int_\Omega \Big\{(\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u} - \mathcal{P}_k^h((\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u})\Big\}^{\mathbf{d}} : \Big\{\boldsymbol{\tau}_h - \kappa_1 \nabla \mathbf{v}_h\Big\}$$

$$+ \mu \int_\Omega \Big\{\nabla \mathbf{u} - \mathcal{P}_k^h(\nabla \mathbf{u})\Big\} : \Big\{\boldsymbol{\tau}_h - \kappa_1 \nabla \mathbf{v}_h\Big\} - \int_\Omega \Big\{\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\Big\}^{\mathbf{d}} : \Big\{\boldsymbol{\tau}_h - \kappa_1 \nabla \mathbf{v}_h\Big\}\,.$$

In this way, replacing (5.24) into (5.23), using the Cauchy-Schwarz and Hölder inequalities, employing the compact injection (2.12) and the fact that $\nabla \boldsymbol{\mathcal{R}}_k^h(\mathbf{u})\big|_K \in \mathbb{P}_k(K)$ for all $K \in \mathcal{T}_h$, and then bounding $\|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega}$ and $\|\mathbf{u}\|_{1,\Omega}$ by $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}$ and ρ_0, respectively, we deduce

$$\begin{aligned}&|(\mathbf{B} - \mathbf{B}_h)(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)|\\ \le&\widehat{C}_{\mathrm{p}}\Big\{\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + |\mathbf{u} - \boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h} + \|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\\ &\qquad + \|(\mathbf{w}_h \otimes \mathbf{u}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h) \otimes \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,\Omega}\Big\}\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}\,,\end{aligned} \tag{5.25}$$

with $\widehat{C}_p := (1 + \kappa_1^2)^{1/2} \max\Big\{1, 2\rho_0\|\boldsymbol{i}_c\|^2, \mu\Big\}$. On the other hand, adding and subtracting $\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})$, employing the Cauchy-Schwarz and Hölder inequalities, Lemma 3.4, and the compact injection (2.12), we find that

$$\begin{aligned}&\|(\mathbf{w}_h \otimes \mathbf{u}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h) \otimes \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,\Omega}\\ \le&\|\boldsymbol{i}_c\|C_{\mathrm{k}}\Big\{\|\mathbf{w}_h\|_{1,\Omega}\|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega} + \|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}_h - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h)\|_{0,4,\Omega}\Big\}\,.\end{aligned} \tag{5.26}$$

Furthermore, using similar arguments, and bounding $\|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega}$ and $\|\mathbf{u}\|_{1,\Omega}$ by $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}$ and ρ_0, respectively, we get

$$\begin{aligned}&\|\mathbf{w}_h\|_{1,\Omega}\|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\\ &\quad\le \|\boldsymbol{i}_c\|(1 + C_{\mathrm{k}})\|\mathbf{u}\|_{1,\Omega}\|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u}\|_{1,\Omega}\|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\\ &\quad\le \rho_0 \max\Big\{1, \|\boldsymbol{i}_c\|(1 + C_{\mathrm{k}})\Big\}\Big\{\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\,,\end{aligned} \tag{5.27}$$

$$\begin{aligned}&\|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}_h - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h)\|_{0,4,\Omega}\\ &\quad\le \rho_0\Big\{\|\boldsymbol{i}_c\|(1 + C_{\mathrm{k}})\|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\\ &\quad\le \rho_0 \max\Big\{1, \|\boldsymbol{i}_c\|(1 + C_{\mathrm{k}})\Big\}\Big\{\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\,.\end{aligned} \tag{5.28}$$

Therefore, replacing (5.27) and (5.28) back into (5.26), we get

$$\begin{aligned}&\|(\mathbf{w}_h \otimes \mathbf{u}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{w}_h) \otimes \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,\Omega}\\ \leq &\overline{C}_{\mathtt{p}}\Big\{\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\,,\end{aligned} \tag{5.29}$$

with $\overline{C}_{\mathtt{p}} := \|\boldsymbol{i}_c\| C_{\mathtt{k}}\rho_0 \max\Big\{1, \|\boldsymbol{i}_c\|(1 + C_{\mathtt{k}})\Big\}$. Finally, replacing (5.29) into (5.25), and taking the supremum on $\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h$, we deduce (5.21) with $C_{\mathtt{p}} := \widehat{C}_{\mathtt{p}}(1 + \overline{C}_{\mathtt{p}})$. Next, in order to deal with (5.22), we observe from (2.9) and (4.6) that

$$\begin{aligned}&(\mathbf{F} - \mathbf{F}_h)(\varphi; \vec{\boldsymbol{\tau}}_h)\\ =&\int_\Omega \mathbf{g}\,\varphi \cdot \mu\mathbf{v}_h - \int_\Omega \mathbf{g}\,\mathcal{P}_{k+1}^h(\varphi) \cdot \mu\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{v}_h) - \kappa_2 \int_\Omega \mathbf{g}\,\Big\{\varphi - \mathcal{P}_{k+1}^h(\varphi)\Big\} \cdot \mathbf{div}\,(\boldsymbol{\tau}_h)\,.\end{aligned} \tag{5.30}$$

Next, adding and subtracting $\mu \displaystyle\int_\Omega \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{g}\varphi) \cdot \mathbf{v}_h$, and using the second equation in (2.3), we deduce that

$$\begin{aligned}&\int_\Omega \mathbf{g}\,\varphi \cdot \mu\mathbf{v}_h - \int_\Omega \mathbf{g}\,\mathcal{P}_{k+1}^h(\varphi) \cdot \mu\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{v}_h)\\ =&\mu \int_\Omega \Big\{\mathbf{g}\,\varphi - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{g}\,\varphi)\Big\} \cdot \mathbf{v}_h + \mu \int_\Omega \mathbf{g}\,\Big\{\varphi - \mathcal{P}_{k+1}^h(\varphi)\Big\} \cdot \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{v}_h)\\ =&-\mu \int_\Omega \Big\{\mathbf{div}\,(\boldsymbol{\sigma}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{div}\,(\boldsymbol{\sigma}))\Big\} \cdot \mathbf{v}_h + \mu \int_\Omega \mathbf{g}\,\Big\{\varphi - \mathcal{P}_{k+1}^h(\varphi)\Big\} \cdot \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{v}_h)\,.\end{aligned} \tag{5.31}$$

Finally, replacing (5.31) into (5.30), and applying the Cauchy-Schwarz inequality, we get (5.22) with $C_{\mathtt{q}} := (4\mu^2 + \kappa_2^2)^{1/2} \max\Big\{1, \|\mathbf{g}\|_{\infty,\Omega}\Big\}$. □

Lemma 5.6. *There exist positive constants $L_{\mathbf{a}}$ and $\widetilde{C}_{\mathtt{q}}$, independent of h, such that*

$$\begin{aligned}&\sup_{\substack{\psi_h \in \mathcal{Q}_k^h\\ \psi_h \neq 0}} \frac{|(\mathbf{a} - \mathbf{a}_h)(\phi_h, \psi_h)|}{\|\psi_h\|_{1,\Omega}}\\ \leq &L_{\mathbf{a}}\Big\{\|\varphi - \phi_h\|_{1,\Omega} \,+\, \|\mathbb{K}\nabla\varphi - \boldsymbol{\mathcal{P}}_k^h(\mathbb{K}\nabla\varphi)\|_{0,\Omega} \,+\, |\varphi - \Pi_k^h(\varphi)|_{1,h}\Big\}\end{aligned} \tag{5.32}$$

for all $\phi_h \in \mathcal{Q}_k^h$, and

$$\begin{aligned}&\sup_{\substack{\psi_h \in \mathcal{Q}_k^h\\ \psi_h \neq 0}} \frac{|(\mathrm{F} - \mathrm{F}_h)(\mathbf{u}, \varphi; \psi_h)|}{\|\psi_h\|_{1,\Omega}}\\ \leq &\widetilde{C}_{\mathtt{q}}\Big\{h\|\mathrm{div}(\mathbb{K}\nabla\varphi) - \mathcal{P}_{k+1}^h(\mathrm{div}(\mathbb{K}\nabla\varphi))\|_{0,\Omega} + |\varphi - \Pi_k^h(\varphi)|_{1,h} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\,.\end{aligned} \tag{5.33}$$

Proof. Given $K \in \mathcal{T}_h$, using the symmetry of the bilinear form $\mathbf{a}^K$ (cf.(4.7)), and the first equation in (4.9) with $q := \Pi_k^K(\phi_h) \in \mathrm{P}_{k+1}(K)$, the local bilinear form $\mathbf{a}_h^K$ (cf. (4.12)) can be rewritten as

$$\mathbf{a}_h^K(\phi_h, \psi_h) = \int_K \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla\Pi_k^K(\phi_h)) \cdot \nabla\psi_h + \mathcal{S}^{K,\Pi}(\phi_h - \Pi_k^K(\phi_h), \psi_h - \Pi_k^K(\psi_h)) \tag{5.34}$$

for all $\phi_h, \psi_h \in \mathcal{Q}_k^K$. Then, from (2.8) and (5.34), and adding and subtracting $\int_K \mathbb{K}\nabla\Pi_k^K(\phi_h)\cdot\nabla\psi_h$, we get

$$
\begin{aligned}
&(\mathbf{a}^K - \mathbf{a}_h^K)(\phi_h,\psi_h)\\
&= \int_K \mathbb{K}\nabla(\phi_h - \Pi_k^K(\phi_h))\cdot\nabla\psi_h - \mathcal{S}^{K,\Pi}(\phi_h - \Pi_k^K(\phi_h), \psi_h - \Pi_k^K(\psi_h))\\
&\qquad + \int_K \Big\{\mathbb{K}\nabla\Pi_k^K(\phi_h) - \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla\Pi_k^K(\phi_h))\Big\}\cdot\nabla\psi_h,
\end{aligned}
$$

whence, applying the Cauchy-Schwarz inequality, using the symmetry of the bilinear form $\mathcal{S}^{K,\Pi}$, the upper bound in (4.13), and the estimate (4.10), we obtain

$$
\begin{aligned}
&|(\mathbf{a}-\mathbf{a}_h)(\phi_h,\psi_h)|\\
&\le \Big\{\|\mathbb{K}\nabla\phi_h - \mathbb{K}\nabla\Pi_k^K(\phi_h)\|_{0,K} + \|\mathbb{K}\nabla\Pi_k^K(\phi_h) - \boldsymbol{\mathcal{P}}_k^K(\mathbb{K}\nabla\Pi_k^K(\phi_h))\|_{0,K}\\
&\qquad + (1+C_{\mathbb{K}})\,\bar{c}_1\,|\phi_h - \Pi_k^K(\phi_h)|_{1,K}\Big\}|\psi_h|_{1,K}\,.
\end{aligned}
$$

Further, adding and subtracting suitable terms, summing over all $K \in \mathcal{T}_h$, and then taking supremum over $\psi_h \in \mathcal{Q}_k^h$, we deduce the estimate (5.32) with $L_{\mathbf{a}}$ depending only on $\mathbb{K}$ and $\bar{c}_1$. On the other hand, from (2.10) and (4.14), adding and subtracting $\int_\Omega \mathcal{P}_{k+1}^h(\mathbf{u}\cdot\nabla\varphi)\psi_h$, and using the fourth equation in (2.3), we find that

$$
\begin{aligned}
&(\mathrm{F}-\mathrm{F}_h)(\mathbf{u},\varphi;\psi_h)\\
&= -\int_\Omega(\mathbf{u}\cdot\nabla\varphi)\psi_h + \int_\Omega(\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\cdot\boldsymbol{\mathcal{P}}_k^h(\nabla\varphi))\mathcal{P}_{k+1}^h(\psi_h)\\
&= -\int_\Omega\Big\{(\mathbf{u}\cdot\nabla\varphi) - \mathcal{P}_{k+1}^h(\mathbf{u}\cdot\nabla\varphi)\Big\}\psi_h - \int_\Omega\Big\{(\mathbf{u}\cdot\nabla\varphi) - (\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\cdot\boldsymbol{\mathcal{P}}_k^h(\nabla\varphi))\Big\}\mathcal{P}_{k+1}^h(\psi_h)\\
&= -\int_\Omega\Big\{\mathrm{div}(\mathbb{K}\nabla\varphi) - \mathcal{P}_{k+1}^h(\mathrm{div}(\mathbb{K}\nabla\varphi))\Big\}(\psi_h - \mathcal{P}_{k+1}^h(\psi_h)) - \int_\Omega\Big\{\Big(\mathbf{u}\\
&\qquad - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\Big)\cdot\nabla\varphi\Big\}\mathcal{P}_{k+1}^h(\psi_h) - \int_\Omega\Big\{\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\cdot\Big(\nabla\varphi - \boldsymbol{\mathcal{P}}_k^h(\nabla\varphi)\Big)\Big\}\mathcal{P}_{k+1}^h(\psi_h)\,,
\end{aligned}
$$

whence, applying the Cauchy-Schwarz and Hölder inequalities, the approximation properties (3.16), Lemma 3.4, the fact that $\nabla\Pi_k^h(\varphi)\big|_K \in \mathbf{P}_k(K)$ for all $K \in \mathcal{T}_h$, and finally bounding $|\varphi|_{1,\Omega}$ and $\|\mathbf{u}\|_{1,\Omega}$ by ρ_0, we get (5.33) with $\widetilde{C}_{\mathtt{q}} := \max\Big\{\widehat{C}, C_{\mathtt{k}}\|i_c\|\rho_0, C_{\mathtt{k}}^2\|\boldsymbol{i}_c\|\|i_c\|\rho_0\Big\}$, where $\widehat{C}$ is the constant obtained when (3.16) is applied with $\psi_h \in \mathrm{H}^1(\Omega)$. □

Next, since we are interested in obtaining an upper bound for the error $\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|$ (cf. (5.18)), we first rearrange (2.5) and (5.1) as the following pairs of continuous and discrete formulations

$$\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) + \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) = \mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}) + \mathbf{F}_D(\vec{\boldsymbol{\tau}}) \qquad \forall\,\vec{\boldsymbol{\tau}} \in \mathbf{H}\,, \tag{5.35a}$$

$$\mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) + \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) = \mathbf{F}_h(\varphi_h;\vec{\boldsymbol{\tau}}_h) + \mathbf{F}_D(\vec{\boldsymbol{\tau}}_h) \qquad \forall\,\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h\,, \tag{5.35b}$$

and

$$\mathbf{a}(\varphi,\psi) = \mathrm{F}(\mathbf{u},\varphi;\psi) \qquad \forall\, \psi \in \mathrm{H}\,, \tag{5.36a}$$

$$\mathbf{a}_h(\varphi_h,\psi_h) = \mathrm{F}_h(\mathbf{u}_h,\varphi_h;\psi_h) \qquad \forall\, \psi_h \in \mathcal{Q}_k^h\,. \tag{5.36b}$$

Then, we have the following lemma establishing a preliminary estimate for $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$.

Lemma 5.7. *There exist positive constants $C_{\mathtt{d}}$ and $C_{\mathtt{r}}$, independent of h, such that*

$$\begin{aligned}
&\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \\
&\le C_{\mathtt{d}} \Big\{ \|\mathbf{div}\,(\boldsymbol{\sigma}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{div}\,(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\varphi - \mathcal{P}_{k+1}^h(\varphi)\|_{0,\Omega} + \|\boldsymbol{\sigma} - \mathbb{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \\
&\qquad + |\mathbf{u} - \boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h} + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega} + \mathrm{dist}(\vec{\boldsymbol{\sigma}}, \mathbf{H}_k^h) \Big\} \\
&\qquad + C_{\mathtt{r}} \Big\{ \|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} + \|\mathbf{g}\|_{\infty,\Omega} \Big\} \|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|\,.
\end{aligned} \tag{5.37}$$

Proof. Employing the bounds provided by (2.13)-(2.15) and (4.3)-(4.5), the fact that $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{u}_h\|_{1,\Omega}$ are bounded by ρ_0 (cf.(5.19)), and recalling that $C_{\mathtt{k}} \ge 1$ (cf. Lemma 3.4), we deduce that $\mathbf{A} + \mathbf{B}(\mathbf{u};\cdot,\cdot)$ and $\mathbf{A}_h + \mathbf{B}_h(\mathbf{u}_h;\cdot,\cdot)$ are bounded and elliptic with the common constants L_{B} and L_{E}, respectively, both independent of h, which are given by

$$L_{\mathrm{B}} \;:=\; \max\big\{C_{\mathbf{A}}, \widetilde{C}_{\mathbf{A}}\big\} + \widetilde{C}_{\mathbf{B}}\rho_0 \quad \text{and} \quad L_{\mathrm{E}} \;:=\; \frac{1}{2}\min\{\alpha_{\mathbf{A}}, \widetilde{\alpha}_{\mathbf{A}}\}\,.$$

In turn, $\mathbf{F}(\varphi;\cdot) + \mathbf{F}_D$ and $\mathbf{F}_h(\varphi_h;\cdot) + \mathbf{F}_D$ are bounded linear functionals in $\mathbf{H}$ and $\mathbf{H}_k^h$, respectively. Then, a straightforward application of the first Strang lemma for linear problems (see [28, Theorem 4.1.1] or [43, Theorem 11.1]) to the context (5.35) gives

$$\begin{aligned}
&\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \\
&\le C_{\mathtt{st}} \left\{ \sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}_h) - \mathbf{F}_h(\varphi_h;\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} + \inf_{\vec{\boldsymbol{\zeta}}_h \in \mathbf{H}_k^h} \left(\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \right.\right. \\
&\qquad \left.\left. + \sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|(\mathbf{A} - \mathbf{A}_h)(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) + \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \right) \right\},
\end{aligned} \tag{5.38}$$

where $C_{\mathtt{st}} \;:=\; L_{\mathrm{E}}^{-1} \max\big\{1, L_{\mathrm{E}} + L_{\mathrm{B}}\big\}$. Next, adding and subtracting $\mathbf{F}_h(\varphi;\vec{\boldsymbol{\tau}}_h)$, we find that

$$\mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}_h) - \mathbf{F}_h(\varphi_h;\vec{\boldsymbol{\tau}}_h) = (\mathbf{F} - \mathbf{F}_h)(\varphi;\vec{\boldsymbol{\tau}}_h) + \mathbf{F}_h(\varphi - \varphi_h;\vec{\boldsymbol{\tau}}_h)\,. \tag{5.39}$$

Then, from (5.39), the estimate (5.22) in Lemma 5.5, and the boundedness of $\mathbf{F}_h$ (cf. (5.7)), we deduce that

$$\begin{aligned}
&\sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{F}(\varphi;\vec{\boldsymbol{\tau}}_h) - \mathbf{F}_h(\varphi_h;\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \\
&\le C_{\mathtt{q}} \Big\{ \|\mathbf{div}\,(\boldsymbol{\sigma}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{div}\,(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\varphi - \mathcal{P}_{k+1}^h(\varphi)\|_{0,\Omega} \Big\} + C_{\mathbf{F}}\|\mathbf{g}\|_{\infty,\Omega}\|\varphi - \varphi_h\|_{1,\Omega}\,.
\end{aligned} \tag{5.40}$$

Also, adding and subtracting suitable terms, we find that

$$\begin{aligned}&|\mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)-\mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|\\ \leq\,&|(\mathbf{B}-\mathbf{B}_h)(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|+|\mathbf{B}_h(\mathbf{u}-\mathbf{u}_h;\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)-\mathbf{B}_h(\mathbf{u}-\mathbf{u}_h;\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|\,.\end{aligned}\tag{5.41}$$

Furthermore, by using the boundedness of $\mathbf{B}_h$ (cf. (4.5)), and bounding $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{u}_h\|_{1,\Omega}$ by ρ_0, we get

$$\begin{aligned}&|\mathbf{B}_h(\mathbf{u}-\mathbf{u}_h;\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)-\mathbf{B}_h(\mathbf{u}-\mathbf{u}_h;\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|\\ &\leq \widetilde{C}_{\mathbf{B}}\left(\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}+\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}\right)\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}\\ &\leq \widetilde{C}_{\mathbf{B}}\left(\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}+2\rho_0\,\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}\right)\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}\,.\end{aligned}$$

Then, from (5.41), the foregoing expression, and the estimate (5.21) from Lemma 5.5, we get

$$\begin{aligned}&\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbf{H}_k^h\\ \vec{\boldsymbol{\tau}}_h\neq\mathbf{0}}}\frac{|\mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)-\mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}}\\ &\leq\widetilde{C}_{\mathtt{p}}\Big\{\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}\,+\,\|\boldsymbol{\sigma}-\mathbb{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}+|\mathbf{u}-\boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h}\\ &\qquad+\,\|\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}+\widetilde{C}_{\mathbf{B}}\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}\,,\end{aligned}\tag{5.42}$$

with $\widetilde{C}_{\mathtt{p}}:=C_{\mathtt{p}}+2\rho_0\widetilde{C}_{\mathbf{B}}$. In this way, replacing (5.40), (5.20) and (5.42) into (5.38), we deduce the estimate (5.37) with

$$C_{\mathtt{d}}:=C_{\mathbf{st}}\max\Big\{C_{\mathtt{q}},L_{\mathbf{A}},\widetilde{C}_{\mathtt{p}}\Big\}\quad\text{and}\quad C_{\mathtt{r}}:=C_{\mathbf{st}}\max\Big\{\widetilde{C}_{\mathbf{B}},C_{\mathbf{F}}\Big\}\,.\tag{5.43}$$

This completes the proof of the lemma. □

Next, as for the error $\|\varphi-\varphi_h\|_{1,\Omega}$ arising from (5.36), we have the following result.

Lemma 5.8. *There exist positive constants $\widetilde{C}_{\mathtt{d}}$ and $\widetilde{C}_{\mathtt{r}}$, independent of h, such that*

$$\begin{aligned}&\|\varphi-\varphi_h\|_{1,\Omega}\\ &\leq\widetilde{C}_{\mathtt{d}}\Big\{\|\mathbb{K}\nabla\varphi-\boldsymbol{\mathcal{P}}_k^h(\mathbb{K}\nabla\varphi)\|_{0,\Omega}+h\|\mathrm{div}\,(\mathbb{K}\nabla\varphi)-\mathcal{P}_{k+1}^h(\mathrm{div}(\mathbb{K}\nabla\varphi))\|_{0,\Omega}\\ &\qquad+|\varphi-\Pi_k^h(\varphi)|_{1,h}+\|\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}+\,\mathrm{dist}(\varphi,\mathcal{Q}_k^h)\Big\}\\ &\qquad+\widetilde{C}_{\mathtt{r}}\Big\{\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}+\|\varphi\|_{1,\Omega}\Big\}\|(\vec{\boldsymbol{\sigma}},\varphi)\,-\,(\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|\,.\end{aligned}\tag{5.44}$$

Proof. We first observe that the boundedness and ellipticity of the bilinear form $\mathbf{a}$ and Lemma 4.5 guarantee that the family $\{\mathbf{a}\}\cup\{\mathbf{a}_h\}_{h>0}$ is uniformly bounded and uniformly elliptic with constants, independent of h, given by

$$L_{\mathrm{B}}\;:=\;\max\Big\{1,\alpha^*\Big\}\|\mathbf{a}\|\quad\text{and}\quad L_{\mathrm{E}}\;:=\;\min\{1,\alpha_*\}\alpha_{\mathbf{a}}$$

respectively. Hence, proceeding as in Lemma 5.7, and applying again the first Strang lemma to the context given by (5.36), we find that

$$\|\varphi-\varphi_h\|_{1,\Omega} \le \widetilde{C}_{\mathtt{st}} \left\{ \sup_{\substack{\psi_h\in \mathcal{Q}_k^h\\ \psi_h\neq 0}} \frac{|\mathrm{F}(\mathbf{u},\varphi;\psi_h)-\mathrm{F}_h(\mathbf{u}_h,\varphi_h;\psi_h)|}{\|\psi_h\|_{1,\Omega}} + \inf_{\phi_h\in\mathcal{Q}_k^h}\left(\|\varphi-\phi_h\|_{1,\Omega} + \sup_{\substack{\psi_h\in \mathcal{Q}_k^h\\ \psi_h\neq 0}} \frac{|(\mathbf{a}-\mathbf{a}_h)(\phi_h,\psi_h)|}{\|\psi_h\|_{1,\Omega}}\right)\right\}. \tag{5.45}$$

Next, adding and subtracting $\mathrm{F}_h(\mathbf{u}-\mathbf{u}_h,\varphi,\psi_h)$ we find that

$$\begin{aligned}&|\mathrm{F}(\mathbf{u},\varphi;\psi_h)-\mathrm{F}_h(\mathbf{u}_h,\varphi_h;\psi_h)|\\ =&|(\mathrm{F}-\mathrm{F}_h)(\mathbf{u},\varphi;\psi_h)+\mathrm{F}_h(\mathbf{u}_h,\varphi-\varphi_h;\psi_h)+\mathrm{F}_h(\mathbf{u}-\mathbf{u}_h,\varphi;\psi_h)|\,.\end{aligned} \tag{5.46}$$

For the second and third term on the right-hand side of (5.46) we apply the bound of F_h (cf. (5.9)) to obtain

$$\begin{aligned}&|\mathrm{F}_h(\mathbf{u}_h,\varphi-\varphi_h;\psi_h)+\mathrm{F}_h(\mathbf{u}-\mathbf{u}_h,\varphi;\psi_h)|\\ \le&\widetilde{M}_{\mathrm{F}}\Big\{\|\mathbf{u}_h\|_{1,\Omega}|\varphi-\varphi_h|_{1,\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}|\varphi|_{1,\Omega}\Big\}\|\psi_h\|_{1,\Omega}\,.\end{aligned}$$

In addition, thanks to (5.33) from Lemma 5.6, and the foregoing inequality, we get

$$\begin{aligned}&\sup_{\substack{\psi_h\in \mathcal{Q}_k^h\\ \psi_h\neq 0}} \frac{|\mathrm{F}(\mathbf{u},\varphi;\psi_h)-\mathrm{F}_h(\mathbf{u}_h,\varphi_h;\psi_h)|}{\|\psi_h\|_{1,\Omega}}\\ \le&\widetilde{C}_{\mathtt{q}}\Big\{h\|\mathrm{div}(\mathbb{K}\nabla\varphi)-\mathcal{P}_{k+1}^h(\mathrm{div}(\mathbb{K}\nabla\varphi))\|_{0,\Omega}+|\,\varphi-\Pi_k^h(\varphi)\,|_{1,h}+\|\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\\ &+\widetilde{C}_{\mathrm{F}}\Big\{\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}\|\varphi-\varphi_h\|_{1,\Omega}+\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}\|\varphi\|_{1,\Omega}\Big\}\,.\end{aligned} \tag{5.47}$$

Then, replacing (5.47) into (5.45), and using the estimate (5.32) from Lemma 5.6, we conclude the proof with

$$\widetilde{C}_{\mathtt{d}}:=\widetilde{C}_{\mathtt{st}}\max\Big\{\widetilde{C}_{\mathtt{q}},L_{\mathbf{a}}\Big\}\quad\text{and}\quad\widetilde{C}_{\mathtt{r}}:=\widetilde{C}_{\mathtt{st}}\widetilde{M}_{\mathrm{F}}\,. \tag{5.48}$$

This completes the proof of the lemma. □

We are now in a position to derive an estimation for the global error $\|(\vec{\boldsymbol{\sigma}},\varphi)-(\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|$, which was defined in (5.18). Indeed, bearing in mind the terms in Lemmas 5.7 and 5.8 that are multiplying $\|(\vec{\boldsymbol{\sigma}},\varphi)-(\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|$, using the bounds for $\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}$, $\|\varphi\|_{1,\Omega}$, and $\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$, given by (2.18), (2.19), and (5.16), respectively, the fact that $\|\mathbf{u}\|_{1,\Omega}\le\rho_0$ (cf. (5.19)), and performing some algebraic manipulations, we find that

$$\begin{aligned}&C_{\mathtt{r}}\Big\{\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}+\|\mathbf{g}\|_{\infty,\Omega}\Big\}+\widetilde{C}_{\mathtt{r}}\Big\{\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}+\|\varphi\|_{1,\Omega}\Big\}\\ \le&\ \mathbf{C}_{\mathtt{r}}\Big\{\|\mathbf{g}\|_{\infty,\Omega}+\|\mathbf{u}_D\|_{0,\Gamma}+\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,,\end{aligned} \tag{5.49}$$

where

$$\mathbf{C}_{\mathbf{r}} := \mathbf{C}_{1,\mathbf{r}}\mathbf{C}_{2,\mathbf{r}}\mathbf{C}_{3,\mathbf{r}}\,, \quad \mathbf{C}_{1,\mathbf{r}} := \max\Big\{C_{\mathtt{r}}, \widetilde{C}_{\mathtt{r}}\Big\}\,, \tag{5.50a}$$

$$\mathbf{C}_{2,\mathbf{r}} := \max\left\{1, 2M_{\mathbf{F}}\left(\frac{1}{\alpha_{\mathbf{A}}} + \frac{1}{\widetilde{\alpha}_{\mathbf{A}}}\right), \frac{2M_{\mathrm{F}}}{\alpha_{\mathbf{a}}}\right\}, \tag{5.50b}$$

$$\mathbf{C}_{3,\mathbf{r}} := (1+\rho)^2 \max\left\{1, \rho\frac{2M_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\right\}. \tag{5.50c}$$

In this way, since the constant $\mathbf{C}_{\mathbf{r}}$ depends linearly on the data $\mathbf{g}$ and $\mathbf{u}_D$, we conclude from the foregoing analysis, the following result.

Theorem 5.2. *Let $\mathbf{C}_{\mathbf{r}}$ be the constant from* (5.50)*, and assume that the data $\mathbf{g}, \mathbf{u}_D$ are such that*

$$\mathbf{C}_{\mathbf{r}}\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} \le \frac{1}{2}\,. \tag{5.51}$$

Then, there exists a positive constant C, depending on $C_{\mathtt{d}}$ (cf. (5.43)*) and $\widetilde{C}_{\mathtt{d}}$ (cf.* (5.48)*), such that*

$$\begin{aligned}
&\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\| \\
&\le C\,\Big\{\|\mathbf{div}\,(\boldsymbol{\sigma}) - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{div}\,(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\boldsymbol{\sigma} - \mathbb{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + |\mathbf{u} - \boldsymbol{\mathcal{R}}_k^h(\mathbf{u})|_{1,h} \\
&\quad + \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega} + \|\mathbb{K}\nabla\varphi - \boldsymbol{\mathcal{P}}_k^h(\mathbb{K}\nabla\varphi)\|_{0,\Omega} + h\|\mathrm{div}(\mathbb{K}\nabla\varphi) - \mathcal{P}_{k+1}^h(\mathrm{div}(\mathbb{K}\nabla\varphi))\|_{0,\Omega} \\
&\quad + \|\varphi - \mathcal{P}_{k+1}^h(\varphi)\|_{0,\Omega} + |\varphi - \Pi_k^h(\varphi)|_{1,h} + \mathrm{dist}(\vec{\boldsymbol{\sigma}}, \mathbf{H}_k^h) + \mathrm{dist}(\varphi, \mathcal{Q}_k^h)\Big\}\,.
\end{aligned} \tag{5.52}$$

Proof. It suffices to add the estimates (5.37) and (5.44) from Lemmas 5.7 and 5.8, respectively, and to use the estimate (5.49) together with the assumption given by (5.51). □

Having established Theorem 5.2, we now provide the rates of convergence for $\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\|$ (cf. (5.18)).

Theorem 5.3. *Let $(\vec{\boldsymbol{\sigma}},\varphi) := ((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H}\times\mathrm{H}$ and $(\vec{\boldsymbol{\sigma}}_h,\varphi_h) := ((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_k^h \times \mathcal{Q}_k^h$ be the unique solutions of the continuous and discrete schemes* (2.5) *and* (5.1)*, respectively. Assume that for integers $r \in [1, k+1]$, $s \in [2, k+2]$, and $m \in [2, k+2]$, there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$, $\mathbf{div}\,(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$, $\mathbf{u}|_K \in \mathbf{H}^s(K)$, $\varphi\big|_K \in \mathrm{H}^m(K)$, and $\mathbb{K}\big|_K \in \mathbb{W}^{m-1,\infty}(K)$, for each $K \in \mathcal{T}_h$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}
&\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\| \\
&\le C\,h^{\min\{r,s-1,m-1\}}\left\{\sum_{K\in\mathcal{T}_h}\Big(|\boldsymbol{\sigma}|_{r,K}^2 + |\mathbf{div}\,(\boldsymbol{\sigma})|_{r,K}^2 + |\mathbf{u}|_{s,K}^2 + |\varphi|_{m,K}^2\Big)\right\}^{1/2} \\
&\quad + C\,h^{s-1}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{s-1,4,K}^4\right\}^{1/4}.
\end{aligned} \tag{5.53}$$

Proof. It follows from (5.18) and (5.52), and the approximation properties (3.3), (3.16)-(3.19), (4.11), $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\varphi})$, and $(\mathbf{AP}_h^{\mathbf{u}})$. □

5.2. Computable approximations of $\boldsymbol{\sigma}$, u, φ and p

We first introduce the fully computable approximations of $\boldsymbol{\sigma}_h$, $\mathbf{u}_h$ and φ_h given by

$$\widehat{\boldsymbol{\sigma}}_h \;:=\; \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma}_h), \quad \widehat{\mathbf{u}}_h \;:=\; \boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u}_h), \quad \text{and} \quad \widehat{\varphi}_h \;:=\; \mathcal{P}_{k+1}^h(\varphi_h)\,, \tag{5.54}$$

and establish the corresponding *a priori* error estimates for the errors

$$\begin{aligned}
&\|((\boldsymbol{\sigma},\mathbf{u}),\varphi) \,-\, ((\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h),\widehat{\varphi}_h)\|_{0,\Omega} := \|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{u}-\widehat{\mathbf{u}}_h\|_{0,\Omega} + \|\varphi-\widehat{\varphi}_h\|_{0,\Omega}\,,\\
&|(\mathbf{u},\varphi) \,-\, (\widehat{\mathbf{u}}_h,\widehat{\varphi}_h)|_{1,h} := |\mathbf{u}-\widehat{\mathbf{u}}_h|_{1,h} + |\varphi-\widehat{\varphi}_h|_{1,h}\,.
\end{aligned}$$

As shown below in Theorem 5.6, they yield exactly the same rate of convergence given by Theorem 5.3. Then, we begin the analysis with the following result.

Theorem 5.4. *Let* $(\vec{\boldsymbol{\sigma}},\varphi) := ((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H}\times \mathrm{H}$ *and* $(\vec{\boldsymbol{\sigma}}_h,\varphi_h) := ((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_h \times \mathcal{Q}_k^h$ *be the unique solutions of the continuous and discrete schemes* (2.5) *and* (5.1)*, respectively. In addition, let* $\widehat{\boldsymbol{\sigma}}_h$*,* $\widehat{\mathbf{u}}_h$*, and* $\widehat{\varphi}_h$ *be the discrete approximations introduced in* (5.54)*. Then there exists a positive constant* $C>0$*, independent of* h*, such that*

$$\begin{aligned}
&\|((\boldsymbol{\sigma},\mathbf{u}),\varphi) - ((\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h),\widehat{\varphi}_h)\|_{0,\Omega} \;+\; |(\mathbf{u},\varphi) - (\widehat{\mathbf{u}}_h,\widehat{\varphi}_h)|_{1,h}\\
&\le C\Bigg\{ \|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\| \;+\; \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\\
&\qquad +\;\Big\{\sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^K(\mathbf{u})\|_{1,K}^2\Big\}^{1/2} + \Big\{\sum_{K\in\mathcal{T}_h} \|\varphi - \mathcal{P}_{k+1}^K(\varphi)\|_{1,K}^2\Big\}^{1/2}\Bigg\}.
\end{aligned}$$

Proof. It follows by using similar arguments from [36, Theorem 5.4]. □

Next, proceeding as in [36, Section 5.3], and according to (2.4) and the decomposition of $\boldsymbol{\sigma}$ provided by (1.1), we suggest the following computable approximation of the pressure:

$$\widehat{p}_h \;:=\; -\frac{1}{2}\,\mathrm{tr}\Big(\widehat{\boldsymbol{\sigma}}_h \,+\, \widehat{c}_h\mathbb{I} \,+\, \widehat{\mathbf{u}}_h\otimes\widehat{\mathbf{u}}_h\Big) \quad \text{in} \quad \Omega\,, \tag{5.55a}$$

with

$$\widehat{c}_h \;:=\; -\frac{1}{2|\Omega|}\,\|\widehat{\mathbf{u}}_h\|_{0,\Omega}^2\,. \tag{5.55b}$$

The following lemma establishes the corresponding *a priori* error estimate.

Theorem 5.5. *There exists a positive constant* $C>0$*, independent of* h*, such that*

$$\|p-\widehat{p}_h\|_{0,\Omega} \;\le\; C\Big\{\|(\vec{\boldsymbol{\sigma}},\varphi) - (\vec{\boldsymbol{\sigma}}_h,\varphi_h)\| \;+\; \|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \;+\; \|\mathbf{u}-\boldsymbol{\mathcal{P}}_{k+1}^h(\mathbf{u})\|_{0,4,\Omega}\Big\}\,.$$

Proof. See [36, Theorem 5.5]. □

We end this section by providing the theoretical rates of convergence for $\widehat{\boldsymbol{\sigma}}_h$, $\widehat{\mathbf{u}}_h$, $\widehat{\varphi}_h$ and $\widehat{p}_h$.

Theorem 5.6. *Let* $(\vec{\boldsymbol{\sigma}},\varphi) := ((\boldsymbol{\sigma},\mathbf{u}),\varphi) \in \mathbf{H}\times \mathrm{H}$ *and* $(\vec{\boldsymbol{\sigma}}_h,\varphi_h) := ((\boldsymbol{\sigma}_h,\mathbf{u}_h),\varphi_h) \in \mathbf{H}_h \times \mathcal{Q}_k^h$ *be the unique solutions of the continuous and discrete schemes* (2.5) *and* (5.1)*, respectively. In addition, let* $((\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h),\widehat{\varphi}_h)$*, and* $\widehat{p}_h$ *be the discrete approximations introduced in* (5.54) *and (5.55), respectively. Assume that for integers* $r\in[1,k+1]$*,* $s\in[2,k+2]$*, and* $m\in[2,k+2]$*, there hold* $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$*,* $\mathbf{div}\,(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$*,* $\mathbf{u}|_K\in\mathbf{H}^s(K)$*,* $\varphi\big|_K \in \mathrm{H}^m(K)$*, and* $\mathbb{K}\big|_K \in$

$\mathbb{W}^{m-1,\infty}(K)$, *for each* $K \in \mathcal{T}_h$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}\|((\boldsymbol{\sigma},\mathbf{u}),\varphi) - ((\widehat{\boldsymbol{\sigma}}_h,\widehat{\mathbf{u}}_h),\widehat{\varphi}_h)\|_{0,\Omega} \,+\, |(\mathbf{u},\varphi) - (\widehat{\mathbf{u}}_h,\widehat{\varphi}_h)|_{1,h} \,+\, \|p-\widehat{p}_h\|_{0,\Omega} \\ \le C\,h^{\min\{r,s-1,m-1\}}\Bigg\{\sum_{K\in\mathcal{T}_h}\Big(|\boldsymbol{\sigma}|_{r,K}^2 \,+\, |\mathbf{div}(\boldsymbol{\sigma})|_{r,K}^2 \,+\, |\mathbf{u}|_{s,K}^2 \,+\, |\varphi|_{m,K}^2\Big)\Bigg\}^{1/2} \\ +\, C\,h^{s-1}\Bigg\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{s-1,4,K}^4\Bigg\}^{1/4}. \end{aligned} \tag{5.56}$$

Proof. It follows from Theorems 5.3 to 5.5, and the approximation properties provided along the paper. In particular, applying (3.16) and (3.17), we readily find that

$$\Bigg\{\sum_{K\in\mathcal{T}_h}\|\varphi - \mathcal{P}_{k+1}^K(\varphi)\|_{1,K}^2\Bigg\}^{1/2} \,\le\, C\,h^{m-1}\Bigg\{\sum_{K\in\mathcal{T}_h}|\varphi|_{m,K}^2\Bigg\}^{1/2},$$

$$\Bigg\{\sum_{K\in\mathcal{T}_h}\|\mathbf{u} - \boldsymbol{\mathcal{P}}_{k+1}^K(\mathbf{u})\|_{1,K}^2\Bigg\}^{1/2} \,\le\, C\,h^{s-1}\Bigg\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{s,K}^2\Bigg\}^{1/2},$$

respectively. □

5.3. A convergent approximation of $\boldsymbol{\sigma}$ in the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm

In what follows we proceed as in [20, Section 5.3] and propose a second approximation $\widetilde{\boldsymbol{\sigma}}_h$ of the pseudostress $\boldsymbol{\sigma}$, which yields the same rate of convergence from Theorems 5.3 and 5.6 in the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm. For this purpose, for each $K\in\mathcal{T}_h$ we let $(\cdot,\cdot)_{\mathbf{div};K}$ be the usual $\mathbb{H}(\mathbf{div};K)$-inner product with induced norm $\|\cdot\|_{\mathbf{div};K}$. Then, we let $\widetilde{\boldsymbol{\sigma}}_h \in \mathbb{L}^2(\Omega)$ be the tensor defined locally as $\widetilde{\boldsymbol{\sigma}}_h|_K := \widetilde{\boldsymbol{\sigma}}_{h,K}$, where $\widetilde{\boldsymbol{\sigma}}_{h,K}\in\mathbb{P}_{k+1}(K)$ is the unique solution of the problem

$$(\widetilde{\boldsymbol{\sigma}}_{h,K},\boldsymbol{\tau}_h)_{\mathbf{div};K} \,=\, \int_K \widehat{\boldsymbol{\sigma}}_h : \boldsymbol{\tau}_h \,+\, \int_K \mathbf{div}(\boldsymbol{\sigma}_h)\cdot\mathbf{div}(\boldsymbol{\tau}_h) \qquad \forall\,\boldsymbol{\tau}_h\in\mathbb{P}_{k+1}(K). \tag{5.57}$$

Note here that the right-hand side of (5.57), and hence $\widetilde{\boldsymbol{\sigma}}_{h,K}$, is fully computable since both $\widehat{\boldsymbol{\sigma}}_h$ and $\mathbf{div}(\boldsymbol{\tau}_h)$ are. In addition, it is important to remark that $\widetilde{\boldsymbol{\sigma}}_{h,K}$ can be calculated for each $K\in\mathcal{T}_h$, independently. Then, the rate of convergence for the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm of $\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}_h$ is established as follows.

Lemma 5.9. *Assume that the hypotheses of Theorem* 5.3 *are satisfied. Then, there exists a positive constant* C, *independent of* h, *such that*

$$\Bigg\{\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}_{h,K}\|_{\mathbf{div};K}^2\Bigg\}^{1/2} \,=\, \mathcal{O}(h^{\min\{r,s-1,m-1\}}). \tag{5.58}$$

Proof. See [36, Theorem 5.7]. □

6. Numerical Examples

In this section we present some numerical tests illustrating the performance of the augmented mixed virtual element scheme (5.1), which was introduced and analyzed in Section 5.

In addition, and similarly as in [37], the zero integral mean condition for tensors in the space (3.13) is imposed via a real Lagrange multiplier. Concerning the decomposition of Ω employed in our computations, we use two families of uniformly generated meshes: distorted squares and uniform triangular (see Fig. 6.1).

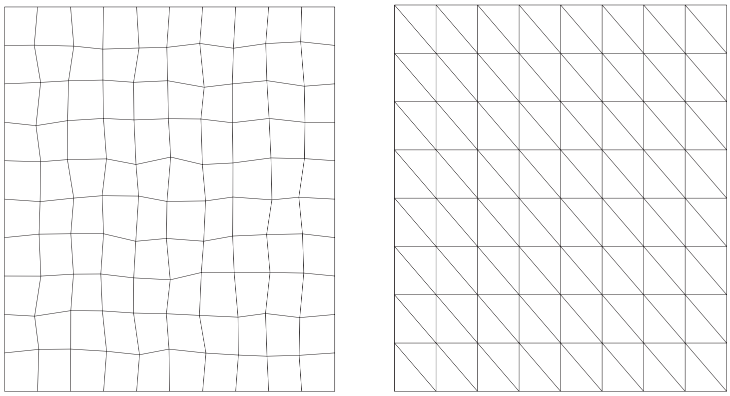

Fig. 6.1. Sample meshes: distorted squares (left), triangular (right).

We begin by introducing some notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (5.1). Further, the individual errors are defined by

$$\mathsf{e}_0(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}, \quad \mathsf{e}_0(\mathbf{u}) := \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega}, \quad \mathsf{e}_1(\mathbf{u}) := |\mathbf{u} - \widehat{\mathbf{u}}_h|_{1,\Omega},$$
$$\mathsf{e}_0(\varphi) := \|\varphi - \widehat{\varphi}\|_{0,\Omega}, \qquad \mathsf{e}_1(\varphi) := |\varphi - \widehat{\varphi}_h|_{1,\Omega},$$
$$\mathsf{e}(p) := \|p - \widehat{p}_h\|_{0,\Omega}, \quad \text{and} \quad \mathsf{e}(\widetilde{\boldsymbol{\sigma}}) := \left\{ \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\|^2_{\mathbf{div};K} \right\}^{1/2},$$

where $\widehat{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\sigma}}_h, \widehat{p}_h, \widehat{\mathbf{u}}_h$ and $\widehat{\varphi}_h$ are computed as in Section 5.2, whereas the associated experimental rates of convergence are given by

$$\mathtt{r}(\cdot) := \frac{\log(\mathsf{e}(\cdot)/\mathsf{e}'(\cdot))}{\log(h/h')},$$

where e and e' denote the corresponding errors for two consecutive meshes with sizes h and h', respectively. In turn, the nonlinear algebraic systems obtained are solved by the Newton method with a tolerance of 10^{-6} and taking as initial iteration the solution $(\mathbf{u}_h, \varphi_h) = (\mathbf{0}, 0)$. The numerical results presented below were obtained using a MATLAB code.

In Example 1, we consider the square $\Omega := (0,1)^2$, set $\mu = 1$, $\mathbf{g} = (0,1)^{\mathbf{t}}$, the thermal conductivity $\mathbb{K} = \mathbb{I}$ in Ω, and adequately manufacture the data so that the exact solution is given by the smooth functions

$$\varphi(\mathbf{x}) = x_1^2 + x_2^4, \quad p(\mathbf{x}) = e^{x_2}\left(x_1 - \frac{1}{2}\right)^3,$$

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} 2x_1^2 x_2(2x_2 - 1)(x_2 - 1)(x_1 - 1)^2 \\ -2x_1 x_2^2 (x_2 - 1)^2 (2x_1 - 1)(x_1 - 1) \end{pmatrix},$$

for all $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$. For this first example, we use meshes composed of distorted squares (see Fig. 6.1).

In Example 2, we consider the square $\Omega := (-1/2, 3/2) \times (0, 2)$, set $\mu = 1$, $\mathbf{g} = (0, 1)^{\mathbf{t}}$, the thermal conductivity $\mathbb{K}(\mathbf{x}) = e^{(x_1+x_2)}\mathbb{I} \quad \forall\, \mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$. Then, the terms on the right-hand sides are adjusted so that the exact solution is given by the functions

$$\varphi(\mathbf{x}) = x_1^2(x_2^2 + 1)\,, \quad \mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} 1 - e^{\lambda x_1}\cos(2\pi x_2) \\ \\ \dfrac{\lambda}{2\pi} e^{\lambda x_1}\sin(2\pi x_2) \end{pmatrix}, \quad p(\mathbf{x}) \;=\; -\frac{1}{2} e^{2\lambda x_1} + p_0\,,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$ where $\lambda := -8\pi^2/(1 + \sqrt{1 + 16\pi^2})$ and p_0 is such that $\int_\Omega p = 0$. Notice that $(\mathbf{u}, p)$ is the well known analytical solution for the Navier-Stokes problem obtained by Kovasznay in [39]. Here we make use of triangular meshes (see Fig. 6.1).

Table 6.1: Example 1: Convergence history using distorted squares with $k = 0$.

h	N	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}_0(\mathbf{u})$	$\mathtt{r}_0(\mathbf{u})$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_0(\varphi)$	$\mathtt{r}_0(\varphi)$
0.2001	532	4.0438e-02	−−	2.0899e-03	−−	2.1666e-02	−−	8.7405e-03	−−
0.1008	1956	2.0448e-02	0.9934	5.9382e-04	1.8331	1.0365e-02	1.0742	2.1844e-03	2.0201
0.0504	7492	9.9385e-03	1.0427	1.5886e-04	1.9055	4.8940e-03	1.0845	5.4847e-04	1.9972
0.0252	29316	4.7872e-03	1.0539	4.0979e-05	1.9549	2.3559e-03	1.0548	1.3770e-04	1.9940
0.0126	115972	2.3363e-03	1.0353	1.0364e-05	1.9839	1.1560e-03	1.0275	3.4472e-05	1.9985

$\mathtt{e}_1(\varphi)$	$\mathtt{r}_1(\varphi)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}_1(\widetilde{\boldsymbol{\sigma}})$	iter
2.0770e-01	−−	2.3446e-02	−−	1.7541e-01	−−	3
1.0475e-01	0.9973	1.2254e-02	0.9453	8.8354e-02	0.9991	3
5.2347e-02	1.0025	6.0438e-03	1.0215	4.4173e-02	1.0019	3
2.6240e-02	0.9964	2.9143e-03	1.0523	2.2076e-02	1.0007	3
1.3132e-02	0.9990	1.4191e-03	1.0384	1.1031e-02	1.0012	3

Table 6.2: Example 2: Convergence history using triangles with $k = 0$.

h	N	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}_0(\mathbf{u})$	$\mathtt{r}_0(\mathbf{u})$	$\mathtt{e}_1(\mathbf{u})$	$\mathtt{r}_1(\mathbf{u})$	$\mathtt{e}_0(\varphi)$	$\mathtt{r}_0(\varphi)$
0.4000	504	8.4858e+01	−−	4.0905e+00	−−	4.8767e+01	−−	9.2742e-02	−−
0.2000	1904	5.8586e+01	0.5345	1.3543e+00	1.5947	3.0955e+01	0.6557	2.2886e-02	2.0187
0.1000	7404	3.2513e+01	0.8495	3.4294e-01	1.9816	1.6366e+01	0.9195	5.6053e-03	2.0296
0.5000	29204	1.6533e+01	0.9757	9.0330e-02	1.9247	8.3920e+00	0.9636	1.3887e-03	2.0130
0.0250	116004	8.2805e+00	0.9975	2.3206e-02	1.9607	4.2200e+00	0.9918	3.4822e-04	1.9957

$\mathtt{e}_1(\varphi)$	$\mathtt{r}_1(\varphi)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}_1(\widetilde{\boldsymbol{\sigma}})$	iter
1.2798e+00	−−	4.5138e+01	−−	8.4914e+01	−−	5
6.2869e-01	1.0255	2.6777e+01	0.7533	5.8594e+01	0.5352	6
3.1236e-01	1.0092	1.4876e+01	0.8481	3.2514e+01	0.8497	8
1.5595e-01	1.0021	7.4528e+00	0.9971	1.6533e+01	0.9757	8
7.7953e-02	1.0005	3.7126e+00	1.0054	8.2805e+00	0.9975	8

In Tables 6.1 and 6.2 we summarize the convergence history of our virtual scheme applied to Example 1 and 2. We notice there that the rate of convergence $\mathcal{O}(h)$ predicted by Theorem

5.6 is achieved by all the unknowns, with exception of $\|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,\Omega}$ and $\|\varphi - \widehat{\varphi}_h\|_{0,\Omega}$, which have rate of convergence $\mathcal{O}(h^2)$. In addition, we can observe that our postprocessed stress $\widetilde{\boldsymbol{\sigma}}_h$ provided the correct order in the broken $\mathbb{H}(\mathbf{div})$-norm. Moreover, we can observe the robustness of our method with respect to the mesh shape. Finally, in order to graphically illustrate the accurateness of our discrete scheme, in Figs. 6.2 and 6.3 we display some components of the approximate solutions for the examples. They all correspond to those obtained with the last mesh of each kind (distorted squares and triangles, respectively).

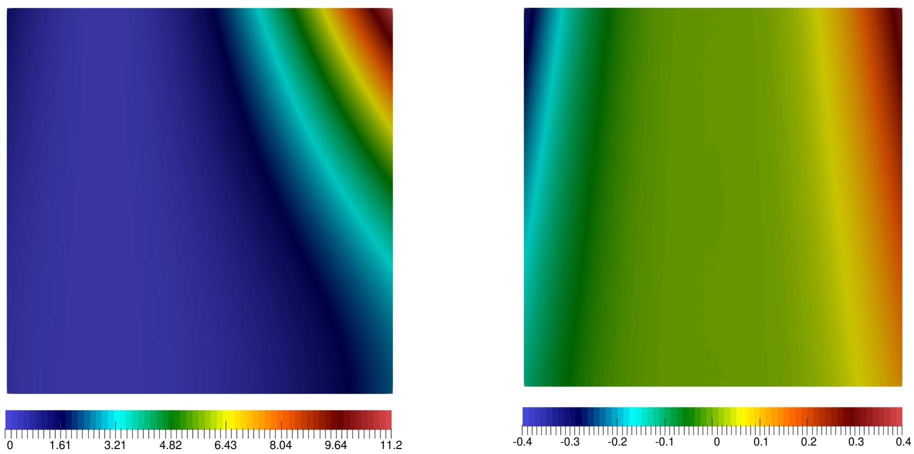

Fig. 6.2. Example 1: $\widehat{p}_h$ (left) and $|\widehat{\mathbf{u}}_h|$ (right) with $N = 115972$.

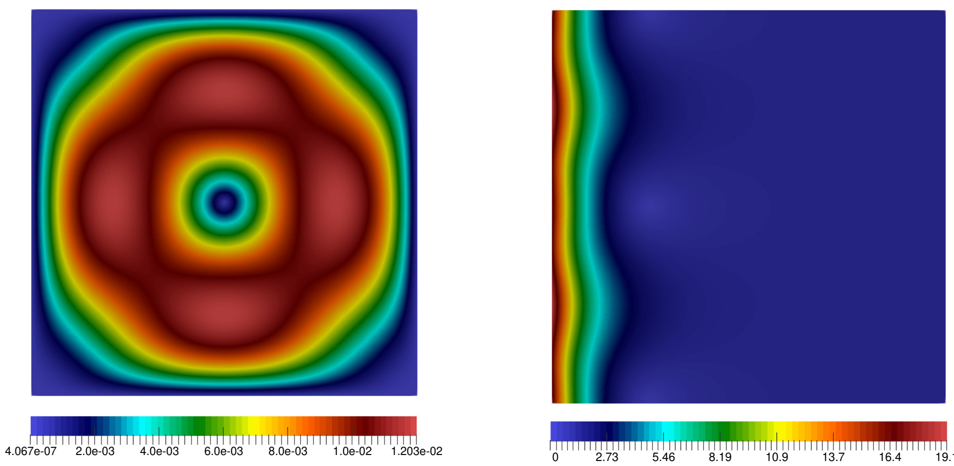

Fig. 6.3. Example 2: $\widehat{\varphi}_h$ (left) and $|\widehat{\mathbf{u}}_h|$ (right) with $N = 116004$.

We end the paper by stressing that numerical results for other more realistic scenarios and corresponding comparisons with alternative methods available in the literature will be addressed in a separate work.

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