A FAST COMPACT DIFFERENCE METHOD FOR
TWO-DIMENSIONAL NONLINEAR SPACE-FRACTIONAL
COMPLEX GINZBURG-LANDAU EQUATIONS*

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Abstract
This paper focuses on a fast and high-order finite difference method for two-dimensional
space-fractional complex Ginzburg-Landau equations. We firstly establish a three-level
finite difference scheme for the time variable followed by the linearized technique of the
nonlinear term. Then the fourth-order compact finite difference method is employed to
discretize the spatial variables. Hence the accuracy of the discretization is $O(\tau^2 + h_1^4 + h_2^4)$ in $L^2$-norm, where $\tau$ is the temporal step-size, both $h_1$ and $h_2$ denote spatial mesh
sizes in $x$- and $y$- directions, respectively. The rigorous theoretical analysis, including the
uniqueness, the almost unconditional stability, and the convergence, is studied via the
energy argument. Practically, the discretized system holds the block Toeplitz structure.
Therefore, the coefficient Toeplitz-like matrix only requires $O(M_1M_2)$ memory storage,
and the matrix-vector multiplication can be carried out in $O(M_1M_2(\log M_1 + \log M_2))$
computational complexity by the fast Fourier transformation, where $M_1$ and $M_2$ denote the numbers of the spatial grids in two different directions. In order to solve the resulting
Toeplitz-like system quickly, an efficient preconditioner with the Krylov subspace method
is proposed to speed up the iteration rate. Numerical results are given to demonstrate the
well performance of the proposed method.

Key words: Space-fractional Ginzburg-Landau equation, Compact scheme, Boundedness,
Convergence, Preconditioner, FFT.

1. Introduction

In this paper, we develop a fast high-order compact finite difference scheme for two-dimensional
space-fractional complex Ginzburg-Landau equations [32] in the truncated domain as follows

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\[ \partial_t u - (\nu + i\eta)(\partial_x^\alpha + \partial_y^\beta)u + (\kappa + i\zeta)|u|^2u - \gamma u = 0, \quad (x, y) \in \Omega, \ 0 < t \leq T, \quad (1.1) \]
\[ u(x, y, t) = 0, \quad (x, y) \in \mathbb{R}^2 \setminus \Omega, \ 0 < t \leq T, \quad (1.2) \]
\[ u(x, y, 0) = \varphi(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (1.3) \]

where \( 1 < \alpha, \beta \leq 2, \nu > 0, \kappa > 0, \eta, \zeta, \gamma \) are given real constants, \( \Omega = (x_l, x_r) \times (y_d, y_u) \) is the rectangular region with the boundary \( \partial \Omega \), \( i = \sqrt{-1} \) is the imaginary unit, \( u(x, y, t) \) is the complex-valued function, and \( \varphi(x, y) \) is a given smooth function with compact support vanishing in \( \mathbb{R}^2 \setminus \Omega \). Furthermore, \( \partial_x^\alpha \) in (1.1) denotes the Riesz fractional derivative operator and is defined as [8]
\[ \partial_x^\alpha u(x, y, t) = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} |x - \xi|^{1-\alpha} u(\xi, y, t) d\xi, \]
or equivalently,
\[ \partial_x^\alpha u(x, y, t) = -\frac{1}{2\cos(\alpha\pi/2)} \left[ -\infty \partial_x^\alpha u(x, y, t) + x \partial_x^\alpha \partial_\infty u(x, y, t) \right], \]

where \( -\infty \partial_x^\alpha u(x, y, t) \) denotes the left Riemann-Liouville fractional derivative [25]
\[ -\infty \partial_x^\alpha u(x, y, t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{u(\xi, y, t)}{(x - \xi)^{\alpha-1}} d\xi; \]
and \( x \partial_\infty^\alpha u(x, y, t) \) denotes the right Riemann-Liouville fractional derivative
\[ x \partial_\infty^\alpha u(x, y, t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{x}^{+\infty} \frac{u(\xi, y, t)}{(\xi - x)^{\alpha-1}} d\xi; \]

Analogously,
\[ \partial_y^\beta u(x, y, t) = -\frac{1}{2\cos(\beta\pi/2)\Gamma(2-\beta)} \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} |y - \xi|^{1-\beta} u(\xi, x, t) d\xi \]
is defined.

Problems (1.1)–(1.3) were firstly proposed by Tarasov and Zaslavsky in recent years [30, 31] due to the fast development of fractional quantum mechanics [13, 14], which are mainly related to the quantum phenomena in fractal environments. Compared with the conventional complex Ginzburg-Landau equations arising from the fractality of the Brownian trajectories [1], fractional complex Ginzburg-Landau equations originate from the variational Euler-Lagrange equation for fractal media over the Lévy paths [7, 35]. In the past few years, the theoretical properties of the fractional complex Ginzburg-Landau equations have been extensively investigated; for example, the well-posedness, dynamics and inviscid limit behavior of solution [9, 10, 26], the asymptotic dynamics for two-dimensional case and the random attractor for the multiplicative noise [19–21], the asymptotic analysis in the bounded domains [23], and exact and soliton solutions [2, 18, 28].

There are a lot of works on the numerical solutions of the fractional complex Ginzburg-Landau equations in the literature, specially for the one-dimensional cases; see [11, 15, 16, 33, 34, 38]. Nevertheless, only a few investigations for the multi-dimensional cases with higher-order discretizations. Mohebbi [24] proposed a numerical algorithm based on Fourier spectral.
method. However, the analysis of stability and convergence is missing. Wang and Huang developed a split-step quasi-compact finite difference method to solve the space-fractional complex Ginzburg-Landau equation in one and two dimensions, where the alternating direction implicit scheme is constructed for two-dimensional problems [32]. Yet only the linear part (in the split-step scheme) can confirm the unconditioned stability. Lu et al. utilized Fourier Galerkin spectral method to discretize the three-dimensional fractional complex Ginzburg-Landau equation, in which only the periodic initial boundary value condition is studied [22].

Although researchers have developed a significant amount of relevant works for the higher-dimensional cases, they usually focus on the low-order discretized schemes, only with the conditional stability, and by the direct solvers. For all above aspects, there is still a need for constructing numerical schemes with high-order schemes and unconditionally stable fast solvers for high-dimensional problems. The fundamental theoretical and computational challenges for high-dimensional models lie in the theoretical analysis and performance including the derivation, accuracy, stability, convergence, and fast implementation of the numerical scheme.

The main contributions of the current paper are to derive a three-level linearized compact finite difference scheme with a fast iteration method to solve the two-dimensional nonlinear fractional complex Ginzburg-Landau equations. In our proposed method, the fourth-order compact operator for the spatial discretization, which was firstly proposed by Zhao, Sun, and Hao in [42], is involved. In their method, the alternate direction implicit (ADI) technique is exploited for the aim to reduce the complexity for solving the discretized system. However, the ADI scheme cannot guarantee the unconditional stability of their method; see [11] for more explains. In order to develop the unconditionally stable or almost unconditionally stable algorithm, we abandon the ADI strategy. Instead of it, we construct a superfast iteration method to implement our scheme that is one of the motivation in this paper. The contribution of the current paper is listed specifically below:

- The proposed scheme is a linearized implicit method, which can naturally avoid the iteration of the nonlinear systems resulted from temporal discretizations.

- The proposed method can be proved that it is almost unconditionally stable and convergent with fourth-order convergence in space and second-order convergence in time.

- The superfast preconditioning strategy is proposed to accelerate the iteration rate, where the block Toeplitz matrix with Toeplitz blocks arising from the scheme can be efficiently approximated by a block circulant matrix with circulant blocks.

- The fast Fourier transformation (FFT) in two dimension is utilized and embedded in the calculation with only $O(M_1M_2(\log M_1 + \log M_2))$ computational complexity and $O(M_1M_2)$ memory storage via the Krylov subspace method, where $M_1$ and $M_2$ denote the numbers of the spatial grids in two different directions.

The main novelty of our paper aims at that we have proved the scheme is almost unconditionally stable and convergent with convergence orders four in both dimensions. Moreover, we have established a preconditioning strategy, in which the block Toeplitz matrix with Toeplitz blocks arising from the scheme can be efficiently approximated by a block circulant matrix with circulant blocks with only $O(M_1M_2(\log M_1 + \log M_2))$ computational complexity by two-dimensional FFT. Numerical results of convergence orders and CPU time verify the effectiveness of the proposed scheme.
The rest of the paper is organized as follows. In Section 2, a second-order central difference approximation and fourth-order compact operator for the space Riesz derivative with order $\alpha$ ($1 < \alpha \leq 2$) is introduced. Some basic lemmas, inner product and norm for the following numerical analysis are also prepared. Section 3 is the main body of the paper, where detailed derivation of the three-level linearized compact scheme is carried out and theoretical results are proved, including uniqueness, almost unconditional stability and convergence. In Section 4, a preconditioned strategy is illustrated and an efficient preconditioner is proposed. In Section 5, numerical results are provided about the accuracy, efficiency and almost unconditional stability. We end the paper in Section 6 with some conclusion remarks on future work.

2. Preliminary

For ease of the finite difference method, we first introduce some useful notations and lemmas for the theoretical and numerical analysis.

2.1. Notations

For the notations in temporal direction, suppose $N$ is a given positive integer and let $t_k = k\tau$, $0 \leq k \leq N$, where $\tau = T/N$. Denote $\Omega_t = \{t_k | 0 \leq k \leq N\}$ and give a grid function $v = \{v^k | 0 \leq k \leq N\}$ on $\Omega_t$. Denote $\Delta_tv^k = \frac{1}{2}(v^{k+1} - v^{k-1})$, $v^k = \frac{1}{2}(v^{k-1} + v^{k+1})$, $v^{k+\frac{1}{2}} = \frac{1}{2}(v^{k+1} + v^k)$.

For the notations in spatial direction, suppose $M_1$, $M_2$ are two positive integers and let $h_1 = (x_i - x_{i-1})/M_1$, $h_2 = (y_j - y_{j-1})/M_2$, and denote $x_i = x_0 + ih_1$, $y_j = y_0 + jh_2$, $0 \leq i \leq M_1$, $0 \leq j \leq M_2$. Denote $\Omega_h = \{(x_i, y_j) | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, $\Omega = \Omega_h \bigcap \Omega$, $\partial \Omega_h = \Omega_h \bigcap \partial \Omega$, $\omega = \{(i, j) | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, $\omega = \{(i, j) | (x_i, y_j) \in \Omega_h\}$, $\partial \omega = \partial \omega \setminus \omega$. Define $\mathcal{V}_h = \{v | v = \{v_{ij}\}, 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, $\mathcal{V}_h = \{v \in \mathcal{V}_h, v_{ij} = 0 \text{ if } (i, j) \in \partial \omega\}$.

For any grid functions $u \in \mathcal{V}_h$ and $v \in \mathcal{V}_h$, define the inner product and $L_2$-norm as follows

$$
(u, v) = h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{ij}v_{ij}, \quad \|u\| = \sqrt{(u, u)}.
$$

The key idea for constructing high-order accurate scheme is based on the following compact operators

$$
A_{ij}^{\alpha}v_{ij} = \begin{cases}
\frac{\alpha}{24}v_{i-1,j} + \left(1 - \frac{\alpha}{12}\right)v_{ij} + \frac{\alpha}{24}v_{i+1,j}, & 1 \leq i \leq M_1 - 1, 0 \leq j \leq M_2, \\
0, & i = 0 \text{ or } M_1, 0 \leq j \leq M_2,
\end{cases}
$$

$$
A_{ij}^{\beta}v_{ij} = \begin{cases}
\frac{\beta}{24}v_{i,j-1} + \left(1 - \frac{\beta}{12}\right)v_{ij} + \frac{\beta}{24}v_{i,j+1}, & 1 \leq j \leq M_2 - 1, 0 \leq i \leq M_1, \\
0, & j = 0 \text{ or } M_2, 0 \leq i \leq M_1.
\end{cases}
$$

Moreover, we denote

$$
A_{ij}^{\alpha, \beta}v_{ij} = A_{x,x}^{\alpha}A_{y,y}^{\beta}v_{ij}, \quad A_{ij}^{\alpha}v_{ij} = (A_{y,y}^{\alpha} + A_{x,x}^{\beta})v_{ij}, \quad (x_i, y_j) \in \Omega_h.
$$

Then we will have the following lemmas, which are the key tools for the theoretical analysis of the high-order numerical scheme.
Lemma 2.1 (Lemma 3.11 in [42]). For any grid function \( v \in \tilde{V}_h \), there exists a fractional symmetric positive quotient operator \( \Lambda_h^\frac{1}{2} \) such that
\[-(\Lambda_h v, v) = (\Lambda_h^\frac{1}{2} v, \Lambda_h^\frac{1}{2} v).\]

Lemma 2.2 (Lemmas 3.3 and 3.4 in [42]). For any grid function \( v \in \tilde{V}_h \), it holds that
\[\|A^{\alpha,\beta} v\| \leq \|v\|,\]
and
\[\frac{1}{3}\|v\|^2 \leq (A^{\alpha,\beta} v, v) \leq \|v\|^2.\]

2.2. Approximation of the spatial Riesz fractional derivative

Introduce
\[C^{n+\alpha}(\mathbb{R}) = \left\{ f \right| \left. \int_{-\infty}^{\infty} (1 + |\omega|)^{n+\alpha} |\hat{f}(\omega)| d\omega < \infty, \ f \in L^1(\mathbb{R}) \right\}, \]
where \( \hat{f}(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t) f(t) dt \) denotes the Fourier transformation of \( f(t) \).

Riesz fractional derivative in two dimension is discreted by the second-order centered difference method, which is appeared in C. Çelik, and M. Duman’s widely cited paper [3], see also [39, 41] for example.

Lemma 2.3 ([3]). Suppose the function \( f \in C^{2+\alpha}(\mathbb{R}) \) and the fractional central difference operator is defined as follows
\[\delta_x^{\alpha} f(x) = -h^{-\alpha} \sum_{k=-\infty}^{+\infty} g^{(\alpha)}_k f(x - kh).\]
Then we have
\[\delta_x^{\alpha} f(x) = \partial_x^{\alpha} f(x) + O(h^2).\]
Here, \( g^{(\alpha)}_k \) is defined as
\[g^{(\alpha)}_k = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\alpha/2 - k + 1) \Gamma(\alpha/2 + k + 1)}, \quad k \in \mathbb{Z}. \quad (2.1)\]

The coefficients for \( g^{(\alpha)}_k \) satisfy the following lemma.

Lemma 2.4 (Property 2.1 in [3]). Let \( 1 < \alpha < 2 \) and \( g^{(\alpha)}_k \) be defined in (2.1). Then it holds
\[(1) \ g^{(\alpha)}_0 \geq 0, \quad g^{(\alpha)}_k = g^{(\alpha)}_{-k} \leq 0 \ for \ all \ |k| \geq 1; \]
\[(2) \ \sum_{k=-\infty}^{+\infty} g^{(\alpha)}_k = 0. \]

Moreover, for the high-order numerical scheme, we have the following lemma.
Lemma 2.5 (Theorem 2.4 in [42]). Suppose the function \( f \in C^{4+\alpha}(\mathbb{R}) \), then it holds
\[
\delta_x^\alpha f(x) = A_2^\alpha \partial_x^\alpha f(x) + O(h^4).
\]

The elementary lemmas for the stability and convergence are listed as follows.

Lemma 2.6 (Lemma 4.2 in [34]). For any grid function \( v \in \hat{V}_h \), there exist fractional symmetric positive quotient operators \( \delta_x^{\alpha/2} \) and \( \delta_y^{\beta/2} \), such that
\[
(-\delta_x^\alpha v, v) = (\delta_x^{\alpha/2} v, \delta_x^{\alpha/2} v), \quad (-\delta_y^\beta v, v) = (\delta_y^{\beta/2} v, \delta_y^{\beta/2} v).
\]

Lemma 2.7 (Lemma 2.1 in [17]). Let \( g \in C^3[t_{k-1}, t_{k+1}] \), it holds that
\[
\frac{1}{2} \left[ g'(t_{k+1}) + g'(t_{k-1}) \right] = \frac{1}{2\tau} \left[ g(t_{k+1}) - g(t_{k-1}) \right] + \frac{\tau^2}{4} \int_0^1 \left[ g''(t_k + s\tau) + g''(t_k - s\tau) \right] (1 - s^2)ds.
\]

Lemma 2.8 (Gronwall inequality, [29]). Let \( \{G^k\}_{k=0}^\infty \) be a nonnegative sequence and satisfies
\[
G^{k+1} \leq (1 + c\tau)G^k + \tau d, \quad k = 0, 1, 2, \ldots.
\]

Then we have
\[
G^k \leq \exp (ck\tau) \left( G^0 + \frac{d}{c} \right), \quad k = 1, 2, \ldots,
\]
where \( c \) and \( d \) are nonnegative constants.

3. The Derivation of the Compact Scheme and Theoretical Analysis

Based on the defined notations and introduced lemmas above, we now consider the derivation of the numerical scheme. In current paper, we suppose the truncated problems (1.1)–(1.3) are compact supported in \( \mathbb{R}^2 \setminus \Omega \). There are also some researchers designing suitable artificial boundary methods or Neumann boundary condition to solve related problems on \( \mathbb{R}^2 \), see [6,36,37]. Define \( U^k = u(x, y, t_k) \), we suppose the exact solution is smooth enough that there exists a positive constant \( c_0 \) such that
\[
\|U^k\|_\infty \leq c_0, \quad 0 \leq k \leq N,
\]
where \( c_0 = \sup_{(x,y) \in \Omega, t \in [0,T]} |u(x,y,t)| \).

3.1. The derivation of the compact scheme

Firstly, define the grid function
\[
U^k_{ij} = u(x_i, y_j, t_k), \quad (i, j) \in \omega, \quad 0 \leq k \leq N.
\]

Then we consider (1.1) at the point \( (x_i, y_j, t_{1/2}) \), we have
\[
\begin{align*}
\partial_t u(x_i, y_j, t_{1/2}) - (\nu + i\eta)(\partial_x^\alpha + \partial_y^\beta)u(x_i, y_j, t_{1/2}) + (\kappa + i\zeta)(|u|^2u)(x_i, y_j, t_{1/2}) & - \gamma u(x_i, y_j, t_{1/2}) = 0, \quad (i, j) \in \omega,
\end{align*}
\]
\](3.2)
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Operating $A_h^{\alpha,\beta}$ on both sides of (3.2), we have

$$A_h^{\alpha,\beta} \partial_t u(x_i, y_j, t_{\frac{k}{2}}) - (\nu + \eta) A_h^{\alpha,\beta} (\partial_x^\alpha + \partial_y^\beta) u(x_i, y_j, t_{\frac{k}{2}}) + (\kappa + i \varsigma) A_h^{\alpha,\beta} (|u|^2 u)(x_i, y_j, t_{\frac{k}{2}}) - \gamma A_h^{\alpha,\beta} u(x_i, y_j, t_{\frac{k}{2}}) = 0, \quad (i, j) \in \omega. \quad (3.3)$$

Combining Lemma 2.5 and Taylor expansion, we have

$$A_h^{\alpha,\beta} \partial_t u(x_i, y_j, t_{\frac{k}{2}}) = A_h^{\alpha,\beta} \delta_t U_{ij}^\frac{k}{2} + \mathcal{O}(\tau^2 + h_{\frac{k}{2}}^4), \quad (i, j) \in \omega, \quad (3.4)$$

$$A_h^{\alpha,\beta} \delta_t \partial_x^\alpha U_{ij}^\frac{k}{2} = A_h^{\alpha,\beta} \delta_y^\beta U_{ij}^\frac{k}{2} + \mathcal{O}(\tau^2 + h_{\frac{k}{2}}^4), \quad (i, j) \in \omega, \quad (3.5)$$

$$\partial_t u(x_i, y_j, t_{\frac{k}{2}}) = \delta_t U_{ij}^\frac{k}{2} + \mathcal{O}(\tau^2), \quad (3.6)$$

and

$$u(x_i, y_j, t_{\frac{k}{2}}) = u(x_i, y_j, 0) + \frac{\tau}{2} u_t(x_i, y_j, \xi_{\frac{k}{2}}^\frac{1}{2}), \quad (3.7)$$

where $\xi_{\frac{k}{2}}^\frac{1}{2} \in [0, t_k]$. Using the smoothness of the exact solution in (3.1), and combining (3.7) with (3.8), we have

$$|u|^2 u(x_i, y_j, t_{\frac{k}{2}}) = |U_{ij}^0|^2 U_{ij}^\frac{k}{2} + \mathcal{O}(\tau). \quad (3.9)$$

Substituting (3.4)–(3.6), (3.9) into (3.3), we have

$$A_h^{\alpha,\beta} \delta_t U_{ij}^\frac{k}{2} - (\nu + \eta) \left( A_h^{\alpha,\beta} \delta_x^\alpha U_{ij}^\frac{k}{2} + A_h^{\alpha,\beta} \delta_y^\beta U_{ij}^\frac{k}{2} \right) + (\kappa + i \varsigma) A_h^{\alpha,\beta} (|U_{ij}^0|^2 U_{ij}^\frac{k}{2}) - \gamma A_h^{\alpha,\beta} U_{ij}^\frac{k}{2} = r_{ij}^k, \quad (i, j) \in \omega. \quad (3.10)$$

There exists a positive constant $c_1$ such that

$$|r_{ij}^k| \leq c_1 (\tau + h_{\frac{k}{2}}^4 + h_{\frac{k}{2}}^2), \quad (i, j) \in \omega. \quad (3.11)$$

Similarly, we consider (1.1) again at $(x_i, y_j, t_k)$ and then operate $A_h^{\alpha,\beta}$ on both sides. Using the similar argument, we arrive at

$$A_h^{\alpha,\beta} \Delta_t U_{ij}^k - (\nu + \eta) \left( A_h^{\alpha,\beta} \delta_x^\alpha U_{ij}^k + A_h^{\alpha,\beta} \delta_y^\beta U_{ij}^k \right) + (\kappa + i \varsigma) A_h^{\alpha,\beta} (|U_{ij}^k|^2 U_{ij}^k) - \gamma A_h^{\alpha,\beta} U_{ij}^k = r_{ij}^{k+1}, \quad (i, j) \in \omega, \quad 1 \leq k \leq N - 1, \quad (3.12)$$

where

$$|r_{ij}^{k+1}| \leq c_2 (\tau^2 + h_{\frac{k}{2}}^4 + h_{\frac{k}{2}}^2), \quad (i, j) \in \omega, \quad 1 \leq k \leq N - 1, \quad (3.13)$$

with $c_2$ being a positive constant.

Using the initial and boundary conditions (1.2) and (1.3), we have

$$U_{ij}^0 = \varphi(x_i, y_j), \quad (i, j) \in \bar{\omega}, \quad (3.13)$$

$$U_{ij}^k = 0, \quad (i, j) \in \partial \omega, \quad 0 \leq k \leq N. \quad (3.14)$$
Noticing (3.13)-(3.14), omitting the small term \( r_{ij}^k \) in (3.10) and \( r_{ij}^{k+1} \) in (3.12), replacing the exact solution \( U_{ij}^k \) with the numerical solution \( u_{ij}^k \), a three-level linearized compact finite difference scheme reads

\[
\begin{align*}
A_h^{\alpha,\beta} u_{ij}^k - (\nu + i\eta) \left( A_h^{\alpha,\beta} u_{ij}^k + A_h^{\alpha,\beta} u_{ij}^k \right) + (\kappa + i\zeta) A_h^{\alpha,\beta} (|u_{ij}^k|^2 u_{ij}^k) \\
- \gamma A_h^{\alpha,\beta} u_{ij}^k = 0, \quad (i, j) \in \omega. \tag{3.15}
\end{align*}
\]

\[
\begin{align*}
A_h^{\alpha,\beta} A_t u_{ij}^k - (\nu + i\eta) \left( A_h^{\alpha,\beta} u_{ij}^k + A_h^{\alpha,\beta} u_{ij}^k \right) + (\kappa + i\zeta) A_h^{\alpha,\beta} (|u_{ij}^k|^2 u_{ij}^k) \\
- \gamma A_h^{\alpha,\beta} u_{ij}^k = 0, \quad (i, j) \in \omega, \quad 1 \leq k \leq N - 1, \tag{3.16}
\end{align*}
\]

\[
\begin{align*}
u_{ij}^0 = \varphi(x_i, y_j), \quad (i, j) \in \partial\omega, \quad u_{ij}^1 = 0, \quad (i, j) \in \partial\omega, \quad 0 \leq k \leq N. \tag{3.18}
\end{align*}
\]

### 3.2. The unique solvability of the compact scheme

**Theorem 3.1.** Denote \( c_r = \frac{1}{\sqrt{\kappa^2 + \zeta^2 \tau^2}} \). When \( \tau \leq \frac{1}{6c_r} \) for \( \gamma < 0 \) or \( \tau \leq \frac{1}{3\gamma + 6c_r} \) for \( \gamma > 0 \), the linearized finite difference scheme (3.15)-(3.18) is uniquely solvable.

**Proof.** From (3.17), the conclusion is valid for \( k = 0 \). When \( k = 1 \), the corresponding homogeneous linear system of equations is

\[
\begin{align*}
\frac{1}{2} A_h^{\alpha,\beta} u_{ij}^1 - \frac{1}{2} (\nu + i\eta) A_h u_{ij}^1 + \frac{1}{2} (\kappa + i\zeta) A_h^{\alpha,\beta} (|u_{ij}^0|^2 u_{ij}^1) - \gamma A_h^{\alpha,\beta} u_{ij}^1 = 0, \quad (i, j) \in \omega. \tag{3.19}
\end{align*}
\]

\[
\begin{align*}
u_{ij}^1 = 0, \quad (i, j) \in \partial\omega. \tag{3.20}
\end{align*}
\]

Taking the inner product of (3.19) with \( u^1 \) and taking the real part, we have

\[
\begin{align*}
\frac{1}{2} \text{Re} \left( A_h^{\alpha,\beta} u^1, u^1 \right) - \frac{1}{2} \text{Re} \left( (\nu + i\eta) (A_h u^1, u^1) \right) + \frac{1}{2} \text{Re} \left( (\kappa + i\zeta) (A_h^{\alpha,\beta} (|u^0|^2 u^1), u^1) \right) \\
- \frac{\gamma}{2} \text{Re} (A_h^{\alpha,\beta} u^1, u^1) = 0.
\end{align*}
\]

In view of Lemma 2.2 and Lemma 2.1, we obtain

\[
\frac{1}{3\tau} \|u^1\|^2 \leq \frac{1}{2} \text{Re} \left( (\kappa + i\zeta) (A_h^{\alpha,\beta} (|u^0|^2 u^1), u^1) \right) + \frac{\gamma}{2} \text{Re} (A_h^{\alpha,\beta} u^1, u^1)
\]

\[
\leq \frac{1}{2} \sqrt{\kappa^2 + \zeta^2 \tau^2} \|u^1\|^2 + \frac{\gamma}{2} \|u^1\|^2.
\]

Rearranging the above inequality, we have

\[
\left( \frac{1}{3\tau} - \frac{\gamma}{2} - c_r \right) \|u^1\|^2 \leq 0.
\]

When \( \tau \leq \frac{1}{6c_r} \) for \( \gamma \leq 0 \), we have \( \|u^1\| = 0 \). When \( \tau \leq \frac{2}{3\gamma + 6c_r} \) for \( \gamma > 0 \), we also have \( \|u^1\| = 0 \). Thus we have \( u_{ij}^1 = 0, (i, j) \in \partial\omega \).

Now, we assume that \( u^k (1 < k \leq l \leq N - 1) \) has been determined, then we only need to prove the conclusion is also valid for \( k = l + 1 \). Firstly, considering the corresponding homogeneous linear system of equations

\[
\begin{align*}
\frac{1}{2\tau} A_h^{\alpha,\beta} u_{ij}^{k+1} - \frac{1}{2} (\nu + i\eta) A_h u_{ij}^{k+1} + \frac{1}{2} (\kappa + i\zeta) A_h^{\alpha,\beta} (|u_{ij}^k|^2 u_{ij}^{k+1}) - \gamma A_h^{\alpha,\beta} u_{ij}^{k+1} = 0 \\
(i, j) \in \omega, \tag{3.21}
\end{align*}
\]

\[
\begin{align*}u_{ij}^{k+1} = 0, \quad (i, j) \in \partial\omega. \tag{3.22}
\end{align*}
\]
Secondly, taking the inner product of (3.21) with \( u^{k+1} \) and taking the real part, we obtain
\[
\frac{1}{2\tau} \text{Re}(A_h^{\alpha,\beta} u^k + u^{k+1}) - \frac{1}{2} \text{Re} \left( (\nu + i\eta)(\Lambda_h u^k + u^{k+1}) \right) + \frac{1}{2} \text{Re} \left( (\kappa + i\zeta)(A_h^{\alpha,\beta}(|u|^2 u^k), u^{k+1}) \right) - \frac{2}{\tau} \text{Re}(A_h^{\alpha,\beta} u^k + u^{k+1}) = 0.
\]
With the help of Lemma 2.2 and Lemma 2.1, we have
\[
\frac{1}{6\tau} \|u^{k+1}\|^2 \leq -\frac{1}{2} \text{Re} \left( (\kappa + i\zeta)(A_h^{\alpha,\beta}(|u|^2 u^k), u^{k+1}) \right) + \frac{\gamma}{2} \text{Re}(A_h^{\alpha,\beta} u^k + u^{k+1}) \leq \left( c_r + \frac{\gamma}{2} \right) \|u^{k+1}\|^2.
\]
Therefore, we have
\[
\left( \frac{1}{6\tau} - \frac{\gamma}{2} - c_r \right) \|u^{k+1}\|^2 \leq 0.
\]
When \( \tau \leq \frac{1}{8c_r} \) for \( \gamma \leq 0 \), we have \( \|u^{k+1}\| = 0 \). When \( \tau \leq \frac{1}{8c_r + \gamma} \) for \( \gamma > 0 \), we also have \( \|u^{k+1}\| = 0 \). Thus we have \( u_{ij}^{k+1} = 0 \), \( (i, j) \in \bar{\omega} \). This completes the proof. \( \square \)

### 3.3. The boundedness of the compact scheme

**Theorem 3.2.** Let \( \{u_{ij}^k \mid (i, j) \in \bar{\omega}, 0 \leq k \leq N \} \) be the solution of the three-level linearized compact scheme (3.15)–(3.18). Denote
\[
c_i = \sqrt{\kappa^2 + \zeta^2 c_0^2} + |\gamma|, \quad c_3 = 3\sqrt{2}\exp(4cT).
\]
If \( \max\{0, \gamma\} \tau \leq \frac{1}{4} \), we have
\[
\|u^k\| \leq c_3 \|u^0\|, \quad 0 \leq k \leq N. \tag{3.23}
\]

**Proof.** We will use the mathematical induction. From (3.17), it is straightforward to obtain that (3.23) holds for \( k = 0 \). When \( k = 1 \), taking the inner product of (3.15) with \( u^0 \), we have
\[
(A_h^{\alpha,\beta}\delta_t u^0, u^0) - (\nu + i\eta)(\Lambda_h u^0, u^0) + (\kappa + i\zeta)(A_h^{\alpha,\beta}(|u|^2 u^0), u^0) - \gamma(A_h^{\alpha,\beta} u^0, u^0) = 0. \tag{3.24}
\]
From Lemma 2.2 and Lemma 2.1, we have
\[
\text{Re}(A_h^{\alpha,\beta}\delta_t u^0, u^0) = \frac{1}{2\tau} \left[ (A_h^{\alpha,\beta} u^1, u^0) - (A_h^{\alpha,\beta} u^0, u^0) \right] - (\Lambda_h u^1, u^0) = \|A_h^{\alpha,\beta} u^0\|^2. \tag{3.25}
\]
Taking the real part on both sides of (3.24) and noticing (3.26), we have
\[
\text{Re}(A_h^{\alpha,\beta}\delta_t u^0, u^0) + \nu \|A_h^{\alpha,\beta} u^0\|^2 + \text{Re}(i\zeta)(A_h^{\alpha,\beta}(|u|^2 u^0), u^0) - \gamma \text{Re}(A_h^{\alpha,\beta} u^0, u^0) = 0. \tag{3.27}
\]
Substituting (3.25) into (3.27), we have
\[
\frac{1}{2\tau} \left[ (A_h^{\alpha,\beta} u^1, u^0) - (A_h^{\alpha,\beta} u^0, u^0) \right] \leq - \text{Re}(i\zeta)(A_h^{\alpha,\beta}(|u|^2 u^0), u^0) + \gamma \text{Re}(A_h^{\alpha,\beta} u^0, u^0) \leq \sqrt{\kappa^2 + \zeta^2} \|u^0\|^2 + |\gamma| \cdot \|u^0\|^2 \leq (\sqrt{\kappa^2 + \zeta^2} c_0^2 + |\gamma|) \cdot \|u^0\|^2, \tag{3.28}
\]
Rearranging the above inequality (3.28), we have
\[(1 - c\tau)(A_h^{\alpha,\beta}u^1, u^1) \leq (1 + c\tau)(A_h^{\alpha,\beta}u^0, u^0).\]

When \(c\tau \leq \frac{1}{4}\), it gives
\[(A_h^{\alpha,\beta}u^1, u^1) \leq \left(1 + \frac{8}{3}c\tau\right)(A_h^{\alpha,\beta}u^0, u^0).\]

By the virtue of Lemma 2.2, we obtain
\[\|u^1\|^2 \leq 3 \left(1 + \frac{8}{3}c\tau\right)\|u^0\|^2 \leq 5\|u^0\|^2.\] (3.29)

In what follows, we suppose that (3.23) holds for \(1 < k \leq l \leq N - 1\). Then we only need to prove that (3.23) is valid for \(k = l + 1\).

Taking the inner product of (3.16) with \(\bar{u}_k\), we have
\[(A_h^{\alpha,\beta}\Delta_t u^k, \bar{u}_k) - (\nu + i\eta)(A_h^{\alpha,\beta}u^k, \bar{u}_k) + (\kappa + i\zeta)(A_h^{\alpha,\beta}(|u|^2u^k), \bar{u}_k) - \gamma(A_h^{\alpha,\beta}u^k, \bar{u}_k) = 0, \quad 1 \leq k \leq l.\] (3.30)

Similar to (3.25), we have
\[\text{Re} \left(A_h^{\alpha,\beta}\Delta_t u^k, \bar{u}_k\right) = \frac{1}{4\tau} (A^{k+1} - A^k),\] (3.31)

where
\[A^{k+1} = (A_h^{\alpha,\beta}u^{k+1}, u^{k+1}) + (A_h^{\alpha,\beta}u^k, u^k).\]

Taking the real part of (3.30) and combining (3.31) and Lemma 2.1 yield
\[\frac{1}{4\tau} (A^{k+1} - A^k) \leq \sqrt{\kappa^2 + \zeta^2\epsilon_0^2}\|u^k\|^2 + |\gamma| \cdot \|u^k\|^2 \leq \left(\sqrt{\kappa^2 + \zeta^2\epsilon_0^2} + |\gamma|\right) \cdot \|u^k\|^2 \leq \frac{c\tau}{2}(A_{k+1} + A^k),\]
amely,
\[(1 - 2c\tau)A^{k+1} \leq (1 + 2c\tau)A^k, \quad 1 \leq k \leq l.\]

When \(2c\tau \leq \frac{1}{2}\), we have
\[A^{k+1} \leq \exp(8c\tau) \left[ (A_h^{\alpha,\beta}u^1, u^1) + (A_h^{\alpha,\beta}u^0, u^0) \right] \leq \exp(8c\tau) \left(\|u^1\|^2 + \|u^0\|^2\right), \quad 1 \leq k \leq l.\]

Combining (3.29) with Lemma 2.2, we have
\[\|u^k\|^2 + \|u^{k+1}\|^2 \leq 18 \exp(8c\tau T)\|u^0\|^2, \quad 1 \leq k \leq l.\]

Using the inductive principle, it is easy to know that (3.23) holds for \(k = 0, 1, \ldots, N\). This completes the proof. \(\Box\)
3.4. The convergence and stability of the compact scheme

Subtracting (3.15)–(3.18) from (3.10), (3.12)–(3.14), we obtain the following error system

\[ A_h^{\alpha,\beta} \Delta_t e^{k}_{ij} - (\nu + i\eta)A_h e^{k}_{ij} + (\kappa + i\zeta)A_h^{\alpha,\beta} p^{k+1}_{ij} - \gamma A_h^{\alpha,\beta} e^{k}_{ij} = r^{k+1}_{ij}, \]

where \((i, j) \in \omega, 1 \leq k \leq N - 1,\)

\[ A_h^{\alpha,\beta} \delta_t e^{k}_{ij} - (\nu + i\eta)A_h e^{k}_{ij} + (\kappa + i\zeta)A_h^{\alpha,\beta} (|U^0|^2 e^{k}_{ij}) - \gamma A_h^{\alpha,\beta} e^{k}_{ij} = r^{k}_{ij}, \quad (i, j) \in \omega, \]

\[ e^{0}_{ij} = 0, \quad (i, j) \in \partial\omega, \quad 0 \leq k \leq N, \]

\[ e^{1}_{ij} = 0, \quad (i, j) \in \partial\omega, \quad 0 \leq k \leq N. \]

where \(p^{k+1}_{ij} = |U^{k}_{ij}|^2 U^{k}_{ij} - |U^{k}_{ij}|^2 u^{k}_{ij} = |U^{k}_{ij}|^2 e^{k}_{ij} + (e^{k}_{ij} U^{k}_{ij} + u^{k}_{ij} e^{k}_{ij}) u^{k}_{ij}.\)

**Theorem 3.3.** Let \(u(x,y,t)\) be the solution of the problems (1.1)–(1.3), \(\{u^{k}_{ij} \mid (i,j) \in \omega, 0 \leq k \leq N\}\) be the solution of (3.15)–(3.18). Denote

\[ c_4 = \sqrt{\kappa^2 + \zeta^2(c_0^2 + c_0 c_3 + c_3^2)} + |\gamma| + \frac{1}{2}, \]

\[ c_5 = 2\sqrt{3}\exp{(2(2c_4 + 1)T)c_2\sqrt{T(x_r - x_l)(y_u - y_d)}}. \]

When \(\tau < \frac{1}{c_5},\) we have

\[ \|e^{k}\| \leq c_5(r^2 + h_1^4 + h_2^4), \quad 0 \leq k \leq N. \]

_Proof._ The mathematical induction is used. Because of (3.34), we have \(\|e^{0}\| = 0.\) When \(k = 1,\) taking the inner product with \(e^\pm\) on both sides of (3.33), we have

\[ \langle A_h^{\alpha,\beta} \delta_t e^{\pm}_{ij}, e^\pm_{ij} \rangle - (\nu + i\eta)(A_h e^{\pm}_{ij}, e^\pm_{ij}) + (\kappa + i\zeta)(A_h^{\alpha,\beta} (|U^0|^2 e^{\pm}_{ij}), e^\pm_{ij}) - \gamma (A_h^{\alpha,\beta} e^{\pm}_{ij}, e^\pm_{ij}) = \langle r^{1}, e^\pm_{ij} \rangle. \]

Similar to (3.25) and (3.26), we have

\[ \text{Re}(A_h^{\alpha,\beta} \delta_t e^{\pm}_{ij}, e^\pm_{ij}) = \frac{1}{2\tau}(A_h^{\alpha,\beta} e^{\pm}_{ij}, e^\pm_{ij}), \]

\[ - (A_h e^{\pm}_{ij}, e^\pm_{ij}) = \|A_h^{\pm} e^{\pm}_{ij}\|^2. \]

Taking the real part of (3.37) and combining with (3.38) and (3.39) give

\[ \frac{1}{2\tau}(A_h^{\alpha,\beta} e^{\pm}_{ij}, e^\pm_{ij}) \leq -\text{Re}\left((\kappa + i\zeta)(A_h^{\alpha,\beta} (|U^0|^2 e^{\pm}_{ij}), e^\pm_{ij})\right) + \gamma \text{Re}(A_h^{\alpha,\beta} e^{\pm}_{ij}, e^\pm_{ij}) + \text{Re}(r^{1}, e^\pm_{ij}) \]

\[ \leq \sqrt{\kappa^2 + \zeta^2}\|U^0\|^2 \cdot \|e^\pm_{ij}\|^2 + |\gamma| \cdot \|e^\pm_{ij}\|^2 + \|r^{1}\| \cdot \|e^\pm_{ij}\| \]

\[ \leq c_4^2 \|e^{1}\|^2 + \frac{1}{2} \|r^{1}\| \cdot \|e^{1}\|. \]

Using Lemma 2.2 and noticing (3.11), we have

\[ (1 - 3\tau c_4)\|e^{1}\| \leq 3\tau\|r^{1}\|. \]

When \(3\tau c_4 < \frac{1}{2},\) we have

\[ \|e^{1}\| \leq 6\tau\|r^{1}\| \leq 6c_1\tau \sqrt{(x_r - x_l)(y_u - y_d)}(\tau + h_1^4 + h_2^4). \]
Now, we suppose (3.36) is valid for $1 < k \leq l \leq N - 1$. It is sufficient to prove that (3.36) is still valid for $k = l + 1$. Taking the inner product of (3.32) with $e^k$, we have

$$(A_h^{\alpha,\beta} \Delta e^k, e^k) - (\nu + i\eta)(A_h^{\alpha,\beta} e^k, e^k) + (\kappa + i\zeta)(A_h^{\alpha,\beta} p^{k+1}, e^k) - \gamma(A_h^{\alpha,\beta} e^k, e^k) = (r^{k+1}, e^k), \quad 1 \leq k \leq l. \quad (3.40)$$

Taking the real part of (3.40), we have

$$\text{Re} \left( A_h^{\alpha,\beta} \Delta e^k, e^k \right) + \nu \| A_h^{\alpha,\beta} e^k \|^2 + \text{Re} \left( (\kappa + i\zeta)(A_h^{\alpha,\beta} p^{k+1}, e^k) \right) = \gamma \text{Re} \left( A_h^{\alpha,\beta} e^k, e^k \right) + \text{Re}(r^{k+1}, e^k), \quad 1 \leq k \leq l. \quad (3.41)$$

Noticing

$$\text{Re} \left( A_h^{\alpha,\beta} \Delta e^k, e^k \right) = \frac{1}{4\tau} (B^{k+1} - B^k), \quad 1 \leq k \leq l. \quad (3.42)$$

where

$$B^{k+1} = (A_h^{\alpha,\beta} e^{k+1}, e^k) + (A_h^{\alpha,\beta} e^k, e^k).$$

Then, the third term on the left hand side in (3.41) can be estimated as

$$| \langle A_h^{\alpha,\beta} p^{k+1}, e^k \rangle | \leq \langle A_h^{\alpha,\beta} (U^k e^k), e^k \rangle + \langle A_h^{\alpha,\beta} (U^k U^k e^k), e^k \rangle + \langle A_h^{\alpha,\beta} (U^k |2 e^k|^2), e^k \rangle \leq c_0^2 \| e^k \|^2 + c_0 c_3 \| e^k \|^2 + c_3^2 \| e^k \|^2, \quad 1 \leq k \leq l. \quad (3.43)$$

For the two terms on the right hand side in (3.41), we have

$$\gamma \text{Re} \left( A_h^{\alpha,\beta} e^k, e^k \right) \leq |\gamma| \cdot \| e^k \|^2, \quad 1 \leq k \leq l, \quad (3.44)$$

$$\text{Re}(r^{k+1}, e^k) \leq \| r^{k+1} \| \cdot \| e^k \|, \quad 1 \leq k \leq l. \quad (3.45)$$

Substituting (3.42)–(3.45) into (3.41), we have

$$\frac{B^{k+1} - B^k}{4\tau} \leq \sqrt{\kappa^2 + \zeta^2 (c_0^2 + c_0 c_3 + c_3^2) + |\gamma|^2} \cdot \| e^k \|^2 + \frac{1}{2} \left( \| r^{k+1} \|^2 + \| e^k \|^2 \right), \quad 1 \leq k \leq l.$$

Rearranging the above formula yields

$$B^{k+1} - B^k \leq 4c_4 \tau \| e^k \|^2 + 2\tau (\| r^{k+1} \|^2 + \| e^k \|^2) \leq 2c_4 \tau (B_{k+1} + B_k) + 2\tau \| r^{k+1} \|^2, \quad 1 \leq k \leq l.$$

When $2c_4 \tau \leq \frac{1}{2}$, we have

$$B^{k+1} \leq (1 + 4(2c_4 + 1)\tau)B^k + 4\tau \| r^{k+1} \|^2, \quad 1 \leq k \leq l.$$

By Lemma 2.8, we have

$$\| e^k \|^2 + \| e^{k+1} \|^2 \leq 3 \exp \left( 4(2c_4 + 1)\tau \right) (4\tau \| r^{k+1} \|^2) \leq c_5^2 (r^2 + h_1^4 + h_2^4)^2, \quad 1 \leq k \leq l.$$
Using the inductive principle, it is easy to know that (3.36) holds for \(k = 0, 1, \ldots, N\). This completes the proof. \(\square\)

In the following, we will discuss the stability of the scheme (3.15)–(3.18). Let \(\{v^k_{ij} | (i, j) \in \omega, 1 \leq k \leq N\}\) be the solutions of

\[
A_h^{\alpha, \beta} \delta_i v^k_{ij} - (\nu + i\eta) \left( A_y^{\alpha, \beta} \delta_x v^k_{ij} + A_y^{\alpha, \beta} \delta_y v^k_{ij} \right) + (\kappa + i\zeta) A_h^{\alpha, \beta} (|v^0_{ij}|^2 v^k_{ij}) - \gamma A_h^{\alpha, \beta} v^k_{ij} = 0, \quad (i, j) \in \omega. 
\]

(3.46)

Denote \(\phi^k_{ij} = u^k_{ij} - v^k_{ij}\). Subtracting (3.15)–(3.18) from (3.46)–(3.49), we obtain

\[
A_h^{\alpha, \beta} \delta_i \phi^k_{ij} - (\nu + i\eta) \left( A_y^{\alpha, \beta} \delta_x \phi^k_{ij} + A_y^{\alpha, \beta} \delta_y \phi^k_{ij} \right) + (\kappa + i\zeta) A_h^{\alpha, \beta} (|u^0_{ij}|^2 \phi^k_{ij}) - (\kappa + i\zeta) A_h^{\alpha, \beta} (|u^0_{ij}|^2 v^k_{ij}) - \gamma A_h^{\alpha, \beta} \phi^k_{ij} = 0, \quad (i, j) \in \omega, 
\]

(3.47)

\[
\phi^0_{ij} = q(x_i, y_j), \quad (i, j) \in \omega, 
\]

(3.48)

\[
\phi^k_{ij} = 0, \quad (i, j) \in \partial \omega, \quad 0 \leq k \leq N. 
\]

(3.49)

Based on the boundedness in Theorem 3.2, and similar to the derivation of the convergence in Theorem 3.3, the almost unconditional stability (here almost unconditional stability means that there exists no step ratio limitation for spatial and temporal steps, but allows the limitations of the spatial step size or temporal step size independently) \(\|\phi^k\| \leq c_0\|q\|\) can be obtained for any \(k = 1, 2, \ldots, N\), where \(c_0\) is a positive constant, which is independent of \(h_1, h_2\) and \(\tau\). We omit the details for brevity.

4. A Fast Preconditioner for the Compact Scheme

In this section, a fast preconditioned Krylov subspace method is proposed to solve the linearized finite difference scheme (3.15)–(3.18). For ease of the notations, denote \(\hat{m}_1 = M_1 - 1\), \(\hat{m}_2 = M_2 - 1\), and \(\hat{m} = \hat{m}_1 \hat{m}_2\). Let \(u^k\) be the \(\hat{m}\)-dimensional vectors defined by

\[
u^k = [u^k_{1,1}, \ldots, u^k_{\hat{m}_1,1}, u^k_{1,2}, \ldots, u^k_{\hat{m}_1,\hat{m}_2}]^\top.
\]

Then the matrix form for the scheme (3.15)–(3.18) is expressed as follows

\[
((A_\beta \otimes A_\alpha) D_1^1 + A) u^1 = ((A_\beta \otimes A_\alpha) D_2^1 - A) u^0, \quad (4.1)
\]

\[
((A_\beta \otimes A_\alpha) D_1^{k+1} + A) u^{k+1} = ((A_\beta \otimes A_\alpha) D_2^{k+1} - A) u^{k-1}, \quad 1 \leq k \leq N - 1, \quad (4.2)
\]
where

$$D_1^1 = \text{diag} \left( \frac{2 + (\kappa + i\zeta)|u_{ij}^0|^2\tau - \gamma\tau}{2(\nu + i\eta)} \right), \quad D_2^1 = \text{diag} \left( \frac{2 - (\kappa + i\zeta)|u_{ij}^0|^2\tau + \gamma\tau}{2(\nu + i\eta)} \right),$$

$$D_1^k = \text{diag} \left( \frac{1 + (\kappa + i\zeta)|u_{ij}^k|^2\tau - \gamma\tau}{2(\nu + i\eta)} \right), \quad D_2^k = \text{diag} \left( \frac{1 - (\kappa + i\zeta)|u_{ij}^k|^2\tau + \gamma\tau}{2(\nu + i\eta)} \right),$$

are diagonal matrices, $A_\alpha$, $A_\beta$ are tridiagonal matrices

$$A_\alpha = \text{tridiag} \left( \frac{\alpha}{24}1 - \frac{\alpha}{12} \frac{\alpha}{24} \right), \quad A_\beta = \text{tridiag} \left( \frac{\beta}{24}1 - \frac{\beta}{12} \frac{\beta}{24} \right),$$

$$A = A_\beta \otimes \left( \frac{\tau}{2h_1^2}G_{\alpha,n_1} \right) + \left( \frac{\tau}{2h_2^2}G_{\beta,n_2} \right) \otimes A_\alpha,$$

where “$\otimes$” denotes the Kronecker product.

By Lemma 2.4, it is easily showed that the matrix

$$G_{\mu,m} = \begin{bmatrix}
  g_0^{(\mu)} & g_{-1}^{(\mu)} & g_{-2}^{(\mu)} & \cdots & g_{-m+2}^{(\mu)} & g_{-m+1}^{(\mu)} \\
  g_1^{(\mu)} & g_0^{(\mu)} & g_{-1}^{(\mu)} & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  g_{m-2}^{(\mu)} & \cdots & \cdots & \cdots & g_0^{(\mu)} & g_1^{(\mu)} \\
  g_{m-1}^{(\mu)} & g_{m-2}^{(\mu)} & \cdots & g_{m-3}^{(\mu)} & g_0^{(\mu)} & g_1^{(\mu)} \\
  g_{m}^{(\mu)} & g_{m-1}^{(\mu)} & \cdots & \cdots & g_0^{(\mu)} & g_1^{(\mu)}
\end{bmatrix}$$

is a symmetric positive definite matrix [3] with $\mu = \alpha$ or $\beta$ and also a Toeplitz matrix [4].

It can be seen that $A$ is a block Toeplitz matrix with Toeplitz blocks, and is called the BTTB matrix [4, 12]. We note that $A$ can be stored in $O(M_1M_2)$ of memory, and the matrix-vector multiplication $Av$ can be performed in $O(M_1M_2(\log M_1 + \log M_2))$ operations for some vector $v$ by the two-dimensional FFT; for more details, please refer to [4, 12]. It is worth noting that the coefficient matrix of linear systems (4.1) or (4.2) is non-Hermitian. The GMRES method [27] is a popular and effective iterative method for solving non-Hermitian linear systems. However, the drawback of the GMRES method is at its slow convergence rate, due to the fact that the condition number of the coefficient matrix is usually large. Therefore, the preconditioner technique could be exploited to accelerate the convergence rate of the GMRES method.

In this paper, a preconditioned GMRES (PGMRES) method [27] with Strang’s block circulant preconditioner is proposed in [4, 12] to solve the linear system in $O(M_1M_2(\log M_1 + \log M_2))$ operations. The detailed algorithm of the PGMRES method is given in Algorithm 4.1.

The Strang circulant matrix $S(B) = [s_{j-k}]_{0 \leq j,k < m}$ for a real Toeplitz matrix $B = [b_{j-k}]_{0 \leq j,k < m}$ is constructed by copying the central entries of $B$ and bringing them around to complete the circulant requirement [4, 5]. To be more specific, the first column of $S(B)$ are given by

$$s_j = \begin{cases} 
  b_j, & 0 \leq j < m/2, \\
  0, & j = m/2 \text{ if } m \text{ is even}, \\
  b_{j-m}, & m/2 < j < m, \\
  s_{j+m}, & 0 < -j < m.
\end{cases}$$
Algorithm 4.1. PGMRES for $Au = b$ with preconditioner $P$

1. Given initial guess $u_0$
2. Compute $r_0 = P^{-1}(b - Au_0)$, $\rho = ||r_0||_2$ and $z_1 = r_0/\rho$
3. For $j = 1, \ldots, m$ Do:
   4. Compute $w := P^{-1}A z_j$
   5. For $i = 1, \ldots, j$ Do:
      6. $h_{j,j} := z_i^H w$
      7. $w := w - h_{j,j} z_i$
   8. EndDo
9. Compute $h_{j+1,j} = ||w||_2$ and $z_{j+1} := w/h_{j+1,j}$
10. EndDo
11. Define $V_m = [z_1, \ldots, z_m]$, $\bar{H}_m = \{h_{i,j}\}_{1 \leq i \leq j+1; 1 \leq j \leq m}$
12. Compute $v_m = \arg\min_v ||\rho e_1 - \bar{H}_m v||_2$, and $u_m = u_0 + V_m v_m$
13. If $||r_m||_2/||r_0||_2 < \varepsilon$ satisfied, stop, else set $u_0 := u_m$ and GoTo 2

Noticing that
\[ A = A_\beta \otimes \left( \frac{\tau}{2h_1^2} G_{\alpha, \hat{m}_1} \right) + \left( \frac{\tau}{2h_2^2} G_{\beta, \hat{m}_2} \right) \otimes A_\alpha, \]
the Strang block circulant preconditioner of $A$ is
\[ S = S(A_\beta) \otimes S \left( \frac{\tau}{2h_1^2} G_{\alpha, \hat{m}_1} \right) + S \left( \frac{\tau}{2h_2^2} G_{\beta, \hat{m}_2} \right) \otimes S(A_\alpha). \tag{4.3} \]
It is noting that the discretized coefficient matrix of linear system of equations (4.1) or (4.2) is not BTTB matrix. However, standard block circulant preconditioners may not work for these systems. In the following, an effective preconditioner is proposed for such Toeplitz-like matrices. For the matrix $((A_\beta \otimes A_\alpha)D_1^1 + A) u^1$ in (4.1), let
\[ \tilde{d} = \frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}_1} \sum_{j=1}^{\hat{m}_2} 2 + (\kappa + 1\zeta)|u_0|_2^2 \tau - \gamma \tau \]
then using $S$ in (4.3), the preconditioner for the linear system (4.1) is given as follows
\[ P = \tilde{d} S(A_\beta) \otimes S(A_\alpha) + S. \]
It easily knows that $P$ is a block circulant matrix with circulant blocks. Note that the product of $P^{-1}$ and a vector requires only $O(M_1M_2(\log M_1 + \log M_2))$ operations by two-dimensional FFT; see [4, 12]. Because of Toeplitz structure, both the discretized coefficient matrix and the preconditioner can be implemented very efficiently by using FFT. Similarly, we can construct a preconditioner for linear systems (4.2). Thus Krylov subspace methods with the proposed preconditioner converge very fast.

5. Numerical Experiments

In this section, we will carry out some numerical examples, which have twofold objectives. All two-dimensional model problems are not only testified the stability and convergence with
the accurate convergence order $O(\tau^2 + h_1^4 + h_2^4)$ for our proposed three-level linearized difference method, but also illustrate computational effectiveness of the PGMRES algorithm. The stopping criterion of PGMRES is

$$\|r_k\|/\|r^0\| < 10^{-7},$$

where $r_k$ is the residual vector of the complex linear system of equations after $k$ times iterations, and the initial guess is chosen as the zero vector.

All numerical experiments were performed on a Windows 10, 64 bit PC-Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz, 16GB of RAM using MATLAB R2014a with machine epsilon $10^{-16}$ in double precision floating point arithmetic.

In the finite difference setting, we take the spatial step-size along with $x$- and $y$- directions to be equal ($h_1 = h_2 = h$ or $M_1 = M_2 = M$). Define the discrete $L_2$-norm for the numerical error as follows

$$E(h, \tau) = \sqrt{h^2 \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} |U_{ij}^N - u_{ij}^N|^2}.$$

• The exact solution is available, the convergence orders in temporal and spatial dimensions are defined by

$$\text{Ord}_1 = \log_2 \left( \frac{\|E(h, \tau)\|}{\|E(h/2, \tau/4)\|} \right), \quad \text{Ord}_2 = \log_2 \left( \frac{\|E(h, \tau)\|}{\|E(h/2, \tau/2)\|} \right),$$

respectively.

• The exact solution is unavailable, the convergence orders are tested by the following posterior error estimation:

$$\text{Ord}_1 = \log_2 \frac{\|u(h, \tau) - u(h/2, \tau/4)\|}{\|u(h/2, \tau/4) - u(h/4, \tau/16)\|},$$

$$\text{Ord}_2 = \log_2 \frac{\|u(h, \tau/2) - u(h, \tau/4)\|}{\|u(h, \tau/2) - u(h, \tau/4)\|}.$$ 

The following two examples come from [40].

**Example 5.1.** Consider the Ginzburg-Laudau equation as follows

$$\partial_t u - (\nu + i\eta)(\partial_x^2 u + \partial_y^2 u) + (\kappa + i\zeta)|u|^2 u - \gamma u = f(x, y, t),$$

$$(x, y) \in (-1, 1) \times (-1, 1), \quad t \in (0, 1],$$
where

\[ f(x, y, t) = -i \exp(-it) \cdot (1 + x)^4(1 - x)^4(1 + y)^4(1 - y)^4 + (\nu + i\eta)(1 + y)^4(1 - y)^4 \exp(-it) \]

\[
\times \left\{ \frac{\Gamma(9)}{\Gamma(9 - \alpha)} \left[(1 + x)^{8-\alpha} + (1 - x)^{8-\alpha}\right] - \frac{8\Gamma(8)}{\Gamma(8 - \alpha)} \left[(1 + x)^{7-\alpha} + (1 - x)^{7-\alpha}\right]
+ \frac{24\Gamma(7)}{\Gamma(7 - \alpha)} \left[(1 + x)^{6-\alpha} + (1 - x)^{6-\alpha}\right] - \frac{32\Gamma(6)}{\Gamma(6 - \alpha)} \left[(1 + x)^{5-\alpha} + (1 - x)^{5-\alpha}\right]
+ \frac{16(5)}{\Gamma(5 - \alpha)} \left[(1 + x)^{4-\alpha} + (1 - x)^{4-\alpha}\right] \right\} + (\nu + i\eta)(1 + x)^4(1 - x)^4 \exp(-it) \frac{4\cos \frac{\alpha}{2}}{2}.
\]

The exact solution is

\[ u(x, y, t) = (x + 1)^4(x - 1)^4(y + 1)^4(y - 1)^4 \exp(-it), \tag{5.1} \]

where \( \nu = \eta = \kappa = 1, \ \zeta = 2, \ \gamma = 3. \) The initial-boundary conditions are determined by (5.1).

Numerical results are reported in Tables 5.1 and 5.2. In Table 5.1, with reduced temporal step size and spatial grid mesh size, the spatial convergence order approaches to four order.

Table 5.1: Example 5.1: \( L^2 \)-norm error behavior with reduced temporal step size and spatial grid mesh size and their convergence orders when \( 1 < \alpha, \beta < 2 \) and compared CPU time (in seconds) as well as average numbers of iterations.

<table>
<thead>
<tr>
<th>( (\alpha, \beta) )</th>
<th>( (h, \tau) )</th>
<th>( E(h, \tau) )</th>
<th>Ord_{\Delta h}^4</th>
<th>Iter</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 50, 1/2)</td>
<td>1.0767e-1</td>
<td>*</td>
<td>21.0</td>
<td>0.234</td>
<td></td>
</tr>
<tr>
<td>(1, 100, 1/8)</td>
<td>5.4951e-3</td>
<td>4.2923</td>
<td>24.4</td>
<td>4.983</td>
<td></td>
</tr>
<tr>
<td>(1.2, 1.8)</td>
<td>3.0581e-4</td>
<td>4.1674</td>
<td>25.8</td>
<td>84.928</td>
<td></td>
</tr>
<tr>
<td>(1, 400, 1/128)</td>
<td>1.9467e-5</td>
<td>3.9736</td>
<td>22.0</td>
<td>1363.838</td>
<td></td>
</tr>
<tr>
<td>(1, 100, 1/2)</td>
<td>3.2120e-4</td>
<td>4.1587</td>
<td>13.9</td>
<td>36.195</td>
<td></td>
</tr>
<tr>
<td>(1.2, 1.8)</td>
<td>3.0581e-4</td>
<td>4.1674</td>
<td>25.8</td>
<td>84.928</td>
<td></td>
</tr>
<tr>
<td>(1, 400, 1/128)</td>
<td>1.9467e-5</td>
<td>3.9736</td>
<td>22.0</td>
<td>1369.474</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Table includes columns for error, order of accuracy, iteration count, and CPU time.
Table 5.2: Example 5.1: $L^2$-norm error behavior with temporal grid size reduction and temporal convergence orders when $1 < \alpha, \beta < 2$ with fixed spatial grid mesh size $h=1/128$ and compared CPU time (in seconds) as well as average numbers of iterations.

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$\tau$</th>
<th>$E(h, \tau)$</th>
<th>$\text{Ord}_{\tau}$</th>
<th>Iter</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1.2, 1.8)$</td>
<td>$1/2$</td>
<td>$1.0767e-1$</td>
<td>$*$</td>
<td>27.5</td>
<td>2.047</td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td>$2.5522e-2$</td>
<td></td>
<td>2.0768</td>
<td>27.0</td>
</tr>
<tr>
<td></td>
<td>$1/8$</td>
<td>$5.4929e-3$</td>
<td>$2.2161$</td>
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<td>7.358</td>
</tr>
<tr>
<td></td>
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<td>$2.1258$</td>
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<td>13.231</td>
</tr>
<tr>
<td></td>
<td>$1/32$</td>
<td>$3.0868e-4$</td>
<td>$2.0276$</td>
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<td>21.261</td>
</tr>
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<td>$1/8$</td>
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<td></td>
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<td>$2.1159$</td>
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</tr>
<tr>
<td></td>
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<td>10.277</td>
</tr>
<tr>
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<td>$1.0767e-1$</td>
<td>$*$</td>
<td>27.5</td>
<td>2.046</td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td>$2.5522e-2$</td>
<td></td>
<td>2.0768</td>
<td>27.0</td>
</tr>
<tr>
<td></td>
<td>$1/8$</td>
<td>$5.4929e-3$</td>
<td>$2.2161$</td>
<td>26.5</td>
<td>7.689</td>
</tr>
<tr>
<td></td>
<td>$1/16$</td>
<td>$1.2586e-3$</td>
<td>$2.1258$</td>
<td>24.6</td>
<td>13.415</td>
</tr>
<tr>
<td></td>
<td>$1/32$</td>
<td>$3.0868e-4$</td>
<td>$2.0276$</td>
<td>20.9</td>
<td>21.153</td>
</tr>
</tbody>
</table>

Fig. 5.1. Comparison the numerical error obtained by proposed numerical scheme with preconditioner for fixed $\tau$ but varying $h$.

which is consistent with our theoretical analysis. In Table 5.2, with fixed spatial step size $h = 1/128$ and reduced temporal step size, we find that temporal convergence order approaches two for PGMRES, which testifies that the temporal convergence orders are also consistent with theoretical convergence behavior. Moreover, in Fig. 5.1(a), by fixing temporal step size and varying $h$ with $\alpha = 1.2$ and $\beta = 1.8$, we see that the numerical error flatten out whether $h$ is larger or smaller than $\tau$. It indicates the proposed numerical scheme is almost unconditionally stable. Similar results can be observed with $\alpha = 1.5$ and $\beta = 1.5$ in Fig. 5.1(b).
A Fast Compact Difference Method for Space-Fractional Equations

$\alpha = 1.2, \beta = 1.8$, eig($[A_{\beta} \otimes A_{\alpha}]D_1^k + A$)

$\alpha = 1.2, \beta = 1.8$, eig($[A_{\beta} \otimes A_{\alpha}]D_1^k + A$)

$\alpha = 1.5, \beta = 1.5$, eig($[A_{\beta} \otimes A_{\alpha}]D_1^k + A$)

$\alpha = 1.5, \beta = 1.5$, eig($[A_{\beta} \otimes A_{\alpha}]D_1^k + A$)

Fig. 5.2. Spectrum of both original and preconditioned matrices at the final time level, respectively, when $M = 24$, $\alpha = 1.2$, $\beta = 1.8$. Red (upper) original matrix; Blue (lower) block circulant preconditioned matrix.

Fig. 5.3. Spectrum of both original and preconditioned matrices at the final time level, respectively, when $M = 24$, $\alpha = 1.5$, $\beta = 1.5$. Red (upper) original matrix; Blue (lower) block circulant preconditioned matrix.

It is observed that the numbers of iterations for our fast algorithm varied very little for the fixed $(\alpha, \beta)$, even when the coefficient matrices are very large. This can be explained in
some extent in Fig. 5.2 and numbers of iterations in Tables 5.1 and 5.2. Fig. 5.2 shows the
distribution of spectrum for original matrix \((A_n \otimes A_n)D_t^k + A\) in (4.2) and preconditioned
matrix \(P^{-1} ((A_n \otimes A_n)D_t^k + A)\) of PGMRES at the final time level, respectively, when \(M =
24, \alpha = 1.2, \beta = 1.8\). The blue points in Fig. 5.2 indicates that spectrum approaches to
one for the preconditioned matrix except very few outliers, while that of original matrix is
disorganized. These figures confirm that the block circulant preconditioner exhibits very nice
clustering properties while most of the eigenvalues are well separated away from zero. Similar
results are also referred to Fig. 5.3.

Example 5.2. Consider the Ginzburg-Landau equation as follows

\[
\partial_t u - (\nu + \eta)(\partial_x^2 u + \partial_y^2 u) + (\kappa + i \zeta)|u|^2 u - \gamma u = 0, \quad (x, y) \in \Omega, \ t \in (0, T],
\]

\[
u(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \ t \in (0, T],
\]

\[
u(x, y, 0) = \operatorname{sech}(x)\operatorname{sech}(y) \exp(i(x + y)), \quad (x, y) \in \Omega = \Omega \cup \partial \Omega,
\]

where \(\Omega = (-10, 10) \times (-10, 10), \nu = \eta = \kappa = \zeta = \gamma = 1\) and \(T = 0.1\). The exact is unknown.

The numerical results for PGMRES are reported in Tables 5.3 and 5.4. In Table 5.3, we
decrease temporal step size and the spatial mesh size. It demonstrates that spatial conver-

gence order approaches to four with \((\alpha, \beta) = (1.2, 1.8), (1.5, 1.5)\) and \((1.8, 1.2)\). Meanwhile,
we compare our numerical results with the second-order numerical results, which manifest that
our algorithm in the current paper is more efficient when the same accuracy is required. It
should be point out that CPU time for PGMRES is greatly reduced for the current problem.
For instance, when \(h = 5/72\), the scale of the coefficient matrix is \((M - 1)^2 = 287^2 = 82,369\),
which is beyond the computing power of our desktop, while PGMRES can go on computing

Table 5.3: Example 5.2: \(L^2\)-norm error behavior with reduced temporal step size and spatial grid mesh
size and their convergence orders when \(1 < \alpha, \beta < 2\) and compared CPU time(seconds) as well as
average numbers of iterations.

<table>
<thead>
<tr>
<th>((\alpha, \beta))</th>
<th>((h, \tau))</th>
<th>PGMRES – 4</th>
<th>PGMRES – 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>((10, 9, 1/20))</td>
<td>*</td>
<td>*</td>
<td>4.0</td>
</tr>
<tr>
<td>((5, 9, 1/80))</td>
<td>5.7124e – 2</td>
<td>*</td>
<td>3.0</td>
</tr>
<tr>
<td>((1.2, 1.8))</td>
<td>5.18, 1, 320</td>
<td>3.9889e – 3</td>
<td>3.8400e – 3</td>
</tr>
<tr>
<td>((5, 36, 1/1280))</td>
<td>2.4034e – 4</td>
<td>4.0528e – 4</td>
<td>3.0</td>
</tr>
<tr>
<td>((5, 72, 1/5120))</td>
<td>1.5713e – 5</td>
<td>3.9351e – 5</td>
<td>2.1</td>
</tr>
<tr>
<td>((10, 9, 1/20))</td>
<td>*</td>
<td>*</td>
<td>4.0</td>
</tr>
<tr>
<td>((5, 9, 1/80))</td>
<td>5.3407e – 2</td>
<td>*</td>
<td>3.0</td>
</tr>
<tr>
<td>((1.5, 1.5))</td>
<td>5.18, 1, 320</td>
<td>3.8886e – 3</td>
<td>3.7797e – 3</td>
</tr>
<tr>
<td>((5, 36, 1/1280))</td>
<td>2.3735e – 4</td>
<td>4.0341e – 4</td>
<td>2.7</td>
</tr>
<tr>
<td>((5, 72, 1/5120))</td>
<td>1.4989e – 5</td>
<td>3.9850e – 5</td>
<td>2.0</td>
</tr>
<tr>
<td>((10, 9, 1/20))</td>
<td>*</td>
<td>*</td>
<td>4.0</td>
</tr>
<tr>
<td>((5, 9, 1/80))</td>
<td>5.7124e – 2</td>
<td>*</td>
<td>3.0</td>
</tr>
<tr>
<td>((1.8, 1.2))</td>
<td>5.18, 1, 320</td>
<td>3.9889e – 3</td>
<td>3.8400e – 3</td>
</tr>
<tr>
<td>((5, 36, 1/1280))</td>
<td>2.4034e – 4</td>
<td>4.0528e – 4</td>
<td>3.0</td>
</tr>
<tr>
<td>((5, 72, 1/5120))</td>
<td>1.5713e – 5</td>
<td>3.9351e – 5</td>
<td>2.1</td>
</tr>
</tbody>
</table>
Table 5.4: Example 5.2: $L^2$-norm error behavior with temporal grid size reduction and temporal convergence orders when $1 < \alpha, \beta < 2$ with fixed spatial grid mesh size $h = 5/64$ and compared CPU time (in seconds) as well as average numbers of iterations.

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$\tau$</th>
<th>$E(h, \tau)$</th>
<th>$\text{Ord}_t^2$</th>
<th>Iter</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1.2, 1.8)$</td>
<td>1/20</td>
<td>$1.8297e-1$</td>
<td>*</td>
<td>8.5</td>
<td>0.515</td>
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<tr>
<td></td>
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<td>*</td>
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<td>0.687</td>
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<tr>
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<td>2.0158</td>
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<td>1.109</td>
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<td></td>
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<td>2.0187</td>
<td>4.1</td>
<td>1.625</td>
</tr>
<tr>
<td></td>
<td>1/320</td>
<td>$2.7640e-3$</td>
<td>2.0071</td>
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<td>3.091</td>
</tr>
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<td>1/640</td>
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<td>2.0031</td>
<td>3.0</td>
<td>4.843</td>
</tr>
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<td>$5.8344e-4$</td>
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<td>0.719</td>
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<tr>
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<td>$6.8953e-4$</td>
<td>2.0031</td>
<td>3.0</td>
<td>4.858</td>
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</table>

Fig. 5.4. Comparison the numerical error obtained by proposed numerical scheme with preconditioner for fixed $\tau$ but varying $h$.

with about 60 seconds. In Table 5.4, it testifies that temporal convergence order is two, at the same time it displays that the PGMRES works very well with a few numbers of iterations.

Similarly, to testify the almost unconditional stability, here we present two plots in Fig. 5.4. For a fixed temporal step size $\tau$, it can be seen that the errors asymptotically tend to
Fig. 5.5. Spectrum of both original and preconditioned matrices at the final time level, respectively, when \( M = 24, \alpha = 1.2, \beta = 1.8. \) \textcolor{red}{Red} (upper) original matrix; \textcolor{blue}{Blue} (lower) block circulant preconditioned matrix.

Fig. 5.6. Spectrum of both original and preconditioned matrices at the final time level, respectively, when \( M = 24, \alpha = 1.5, \beta = 1.5. \) \textcolor{red}{Red} (upper) original matrix; \textcolor{blue}{Blue} (lower) block circulant preconditioned matrix.
a constant, which implies that there exists no time-step restrictions dependent on the spatial mesh size $h$. Likewise, two eigenvalue plots for both original and preconditioned matrices are illustrated in Fig. 5.5 and Fig. 5.6 with $(\alpha, \beta) = (1.2, 1.8)$ and $(1.5, 1.5)$. It again displays that PGMRES exhibits very nice clustering properties.

Combining with Tables and Figures above, we have verified the theoretical results and validated the effectiveness and robustness of the designed block circulant preconditioner.

6. Conclusion and Future Work

In summary, we have proposed a superfast preconditioned linearized compact finite difference scheme, including the design, analysis, implementation, and application for solving two-dimensional nonlinear space-fractional complex Ginzburg-Landau equations. Numerical evidence well verifies the previous theoretical results. Other applications of our algorithm including space-fractional Schrödinger equations in high dimension and coupled space-fractional Ginzburg-Landau equations are also available, which will be our future work.

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